OBSTRUCTION THEORY FOR FIBRE SPACES

BY

Robert Hermann[1]

Introduction

In Sections 1–6 we develop systematically the formal, elementary aspects of obstruction theory for the extension of cross-sections of Serre fibre spaces. The basis of our treatment is a definition due to W. Barcus of the obstruction cocycle that is phrased completely in terms of global, homotopy properties of the fibre space. (Recall that the classic definition given by Steenrod for fibre bundles uses also the local structure of the bundle.)

In the later sections we turn to the higher obstructions, and develop further a method for their computation that was sketched in [4], utilizing a "Postnikov decomposition" for the fibre space. Since in this paper we stay in the context of general fibre spaces, our answer is rather theoretical and approaches completeness only for the second obstruction case.

1. Notations

We shall use cubical singular homology [11] since we shall need a few simple facts about the homology of fibre spaces. Homology should be taken with integer coefficients unless mentioned otherwise. We could have used axiomatic homology theory [3] if it were coupled with an axiomatic homotopy group theory and perhaps an additional axiom concerning the homology of fibre spaces. However, the methods used are inspired by the axiomatic theory and do not explicitly refer to the way the homology groups are defined.

It will be assumed that all spaces occurring in this paper are arcwise connected and come with a definite basepoint. Homology and homotopy groups shall be taken with respect to this basepoint. All maps considered between spaces will map the basepoint on basepoint. If f is a map $X \rightarrow Y$, f will denote also the various maps induced on relative homotopy or homology groups by f with subscripts to identify the particular map. i will always denote an inclusion map. ∂ will always denote a boundary homomorphism, with subscripts to identify the various ones occurring in a diagram. $\Rightarrow$ will denote an onto mapping, $\rightarrowtail$ a one-to-one mapping, $\approx$ an isomorphism or homeomorphism. If a diagram involving homotopy or homology groups and mappings is written down, it will be assumed, unless mentioned otherwise, to be commutative. The proof is usually quite easy and is left to the reader.

A fibre space, denoted by (E, B, F, p), or $F \rightarrow E \xrightarrow{p} B$ will be defined by a

Received March 8, 1958; received in revised form March 20, 1959.

[1] This research was supported in part by OOR, U. S. Army.

map $p:E \Rightarrow B$, satisfying the covering homotopy condition with respect to finite polyhedra [11, p. 443]. The base B will be assumed simply connected. If b_0 resp. e_0 is the basepoint of B resp. E, $p(e_0) = b_0$, the fibre $F = p^{-1}(b_0)$, and e_0 is the basepoint of F.

2. First definitions

Let (E, B, F, p) be a fibre space. A map $f:B \to E$ is said to be a *cross-section* if pf is the identity on B. A cross-section f determines homomorphisms $\bar{f}:\pi_j(E) \to \pi_j(F)$, $j = 1, 2, \cdots$, in the following way: $\pi_j(E)$ is naturally isomorphic to $f(\pi_j(B)) \oplus i(\pi_j(F))$ [14, p. 92]. $\bar{f}$ is defined as the projection of $\pi_j(E)$ on $i(\pi_j(F))$ followed by i^{-1}.

The fibre space is said to be isomorphic to the product $B \times F$ if there is a fibre-preserving homeomorphism $E \to B \times F$. An obviously necessary and sufficient condition for this is that there is a map $q:E \to F$ such that $qi =$ identity on F. If such a q exists, one easily sees that

2.1 $$\bar{f} = q - qfp.$$

LEMMA 2.1. *Suppose $\pi_j(B) = 0$ for $j < n$ and $\pi_j(F) = 0$ for $j < m$, m and $n \geqq 1$, and that the fibre space (E, B, F) admits a cross-section f. Then*

$$H_j(E) = f(H_j(B)) \oplus i(H_j(F)), \qquad \text{for } 1 \leqq j \leqq m + n - 2.$$

$$i:H_j(F) \to H_j(E) \text{ is one-to-one} \qquad \text{for } j \leqq m + n - 2.$$

Proof. From spectral-sequence theory [12, p. 268],

$$p:H_j(E, F) \approx H_j(B, b_0) \qquad \text{for } 1 \leqq j \leqq m + n - 1.$$

For $x \in H_j(E)$, $fp(x) - x \in$ kernel $p =$ image i. f is one-to-one because f is a cross-section. To see that i is one-to-one, we have

$$\begin{array}{ccccccc} H_{j+1}(X) & \xrightarrow{i_1} & H_{j+1}(X, F) & \xrightarrow{\partial} & H_j(F) & \xrightarrow{i} & H_j(X) \\ \downarrow p_1 & & \approx \downarrow p_2 & & & & \\ H_{j+1}(B) & \approx & H_{j+1}(B, b_0) & & & & \end{array} \qquad \text{for } 1 \leqq j \leqq m + n - 2.$$

p_1 is onto; hence i_1 is onto, and $\partial(H_{j+1}(X, f)) = 0 =$ kernel i.

Using Lemma 2.1, we define a map $\bar{f}:H_j(E) \to H_j(F)$, $1 \leqq j \leqq m + n - 2$, such that $\bar{f}i =$ identity, and one has a diagram

$$\begin{array}{ccc} \pi_j(E) & \xrightarrow{\bar{f}} & \pi_j(F) \\ h \downarrow & & \downarrow h \\ H_j(E) & \xrightarrow{\bar{f}} & H_j(F) \end{array} \qquad \text{for } 1 \leqq j \leqq m + n - 2$$

(h will from now on denote the Hurewicz homomorphism from homotopy to homology groups.)

DEFINITION. Suppose $A \subset B$. Put $E_A = p^{-1}(A)$. Suppose $f: A \to E_A$ is a cross-section of the fibre space (E_A, A, F). *The homotopy obstructions to extending f to a cross-section over B are the homomorphisms*

$$w(f): \pi_j(B, A) \to \pi_{j-1}(F), \qquad j = 2, 3, \cdots,$$

each $w(f)$ defined as the composition $\bar{f}\partial p^{-1}$,

$$\pi_j(B, A) \overset{p}{\approx} \pi_j(E, E_A) \xrightarrow{\partial} \pi_{j-1}(E_A) \xrightarrow{\bar{f}} \pi_{j-1}(F).$$

If $H_j(B, A) = 0 = H_j(A)$ for $1 \leqq j < m$, and $\pi_j(F) = 0$ for $j < n$, then the *homology obstructions* are the homomorphisms

$$v(f): H_j(B, A) \to H_{j-1}(F), \qquad 2 \leqq j \leqq m + n - 2,$$

each $v(f)$ defined as the composition

$$H_j(B, A) \overset{p}{\approx} H_j(E, E_A) \xrightarrow{\partial} H_{j-1}(E_A) \xrightarrow{\bar{f}} H_{j-1}(F).$$

One has a diagram

$$\begin{array}{ccc} \pi_j(B, A) & \xrightarrow{w(f)} & \pi_{j-1}(F) \\ \downarrow h & & \downarrow h \\ H_j(B, A) & \xrightarrow{v(f)} & H_{j-1}(F). \end{array}$$

Example 2.1. Suppose $B = E_n$, the n-cell, $A = S_{n-1}$, and the fibre space (E, B, F) is isomorphic to the product, i.e., there is a map $q: E \to F$ with $qi =$ identity. We then have a diagram

$$\begin{array}{ccccccc} \pi_j(B, A) & \overset{p_1}{\approx} & \pi_j(E, E_A) & \xrightarrow{\partial_1} & \pi_{j-1}(E_A) & \xrightarrow{\bar{f}} & \pi_{j-1}(F) \\ \downarrow \partial_2 & & \downarrow \partial_1 & & \uparrow i & \swarrow\!\nearrow \text{id.} & \\ \pi_{j-1}(A) & \underset{p_2}{\overset{f}{\rightleftarrows}} & \pi_{j-1}(E_A) & \xrightarrow{q} & \pi_{j-1}(F) & & \\ & & \downarrow i_1 & & \uparrow \text{id.} & & \\ & & \pi_{j-1}(E) & \xrightarrow{q_1} & \pi_{j-1}(F). & & \end{array}$$

We prove that $w(f) = \bar{f}\partial_1 p_1^{-1} = qf\partial_2$. *This relation holds for any A and B.* By 2.1,

$$\bar{f} = q - qfp_2, \qquad qf\partial_2 p_1 = qfp_2 \partial_1 = \bar{f}\partial_1 + q\partial_1.$$

Now $q = q_1 i_1$; hence $q\partial_1 = qi_1 \partial_1 = 0$, Q.E.D.

Returning to the case $B = E_n$, we see that the image of the generator under the map $w(f): \pi_n(E_n, S_{n-1}) \to \pi_{n-1}(F)$ is just the negative of the element of $\pi_{n-1}(F)$ represented by the map $qf: S_{n-1} \to F$.

To justify the name given to $w(f)$ we prove

THEOREM 2.2. *If f is extendable to B, all $w(f) = 0$.*

Proof. Suppose f_B denotes an extension of f to B.

$$\begin{array}{ccccc} \pi_j(E, E) & \xrightarrow{\partial_2} & \pi_{j-1}(E) & \xrightarrow{\bar{f}_B} & \pi_{j-1}(F) \\ \uparrow i_1 & & \uparrow i_2 & & \updownarrow \text{id.} \\ \pi_j(B, A) \approx \pi_j(E, E_A) & \xrightarrow{\partial_1} & \pi_{j-1}(E_A) & \xrightarrow{\bar{f}} & \pi_{j-1}(F), \end{array}$$

i.e., $\bar{f}\partial_1 = \bar{f}_B\, \partial_2\, i_1 = 0$.

We summarize now some known results on the homotopy groups of path spaces.

If X is an arcwise connected space with a basepoint x_0, $\Omega(X)$ will denote the space of loops based at x_0. From the space of paths over X, we derive an isomorphism $\phi: \pi_j(X) \approx \pi_{j-1}(\Omega X)$, $j = 1, 2, \cdots$. Further, if $A \subset X$, there is an isomorphism $\phi: \pi_j(X, A) \approx \pi_{j-1}(\Omega X, \Omega A)$ such that

$$\begin{array}{ccccccc} \pi_j(A) & \xrightarrow{i} & \pi_j(X) & \xrightarrow{i} & \pi_j(X, A) & \xrightarrow{\partial} & \pi_{j-1}(A) \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ \pi_{j-1}(\Omega A) & \xrightarrow{i} & \pi_{j-1}(\Omega X) & \xrightarrow{i} & \pi_{j-1}(\Omega X, \Omega A) & \longrightarrow & \pi_{j-2}(\Omega A) \end{array}$$

[9, p. 745].

If $y \in \Omega(X)$, y^{-1} is the path defined by $y^{-1}(x) = y(1 - x)$ for $0 \leqq x \leqq 1$. The map $y \to y \cdot y^{-1}$ (with multiplication that usually defined for paths) is inessential. If $\mathrm{In}: \Omega X \to \Omega X$ is the map $y \to y^{-1}$ and $m: \Omega X \times \Omega X \to \Omega X$ is the map $(y, z) \to y \cdot z$, then the induced maps

$$\mathrm{In}: \pi_j(\Omega X) \to \pi_j(\Omega X), \qquad m: \pi_j(\Omega X) \oplus \pi_j(\Omega X) \to \pi_j(\Omega X)$$

are defined by

$$\mathrm{In}\,(\alpha) = -\alpha, \qquad m(\alpha \oplus \beta) = m(\alpha) + m(\beta), \qquad \text{for } \alpha, \beta \in \pi_j(\Omega X).$$

If f_1, f_2 are maps $X \to Y$, f_1', f_2' the induced maps $\Omega X \to \Omega Y$, g the map $\Omega X \to \Omega Y$ defined by $g(x) = f_1'(x) \cdot f_2'(x^{-1})$, then we have that

$$g(\alpha) = f_1'(\alpha) - f_2'(\alpha) \qquad \text{for } \alpha \in \pi_j(\Omega X).$$

If $A \subset X$, and $f_1/A = f_2/A$, then $g(\Omega A)$ is contractable; hence there is a map $g': \pi_j(\Omega X, \Omega A) \to \pi_j(\Omega Y)$ such that

$$\begin{array}{ccc} \pi_j(\Omega X, \Omega A) & \xrightarrow{g'} & \pi_j(\Omega Y) \\ \uparrow i & \nearrow f_1' - f_2' & \\ \pi_j(\Omega X) & & \end{array} \quad , \qquad \begin{array}{ccc} & & \pi_j(\Omega Y) \\ & & \downarrow \approx \\ \pi_j(\Omega X, \Omega A) & \xrightarrow{g} & \pi_j(\Omega Y, g(\Omega A)). \end{array}$$

Now as application, if $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibre space, we can construct a fibre space $\Omega F \xrightarrow{i'} \Omega E \xrightarrow{p'} \Omega B$, p' and i' the maps induced by p and i. If $A \subset B$, and f is a cross-section $A \to E$, $f':\Omega A \to \Omega E$ is a cross-section of the loop-space fibering. We have

$$\begin{array}{ccc} \pi_j(B, A) & \xrightarrow{w(f)} & \pi_{j-1}(F) \\ \approx \downarrow \phi & & \approx \downarrow \phi \\ \pi_{j-1}(\Omega B, \Omega A) & \xrightarrow{w(f')} & \pi_{j-2}(\Omega F). \end{array}$$

If $f_1, f_2:B \to E$ are cross-sections such that $f_1/A = f_2/A$, construct the map $g:\Omega B \to \Omega E$ as above. For $b \in \Omega B$,

$$p'g(b) = p(f_1'(b) \cdot f_2'(b^{-1})) = p'f_1'(b) \cdot p'f_2'(b^{-1}) = b \cdot b^{-1},$$

i.e., $p'g$ is inessential, and

$g(\pi_j(\Omega B)) \subset \text{kernel } p' = \text{image } i'$, and $g'(\pi_j(\Omega B, \Omega A)) \subset \text{kernel } p = \text{image } i$.

Define $d(f_1, f_2):\pi_j(B, A) \to \pi_j(F)$ as the composite

$$\pi_j(B, A) \approx \pi_{j-1}(\Omega B, \Omega A) \xrightarrow{g'} \pi_{j-1}(\Omega E) \xrightarrow{i^{-1}} \pi_{j-1}(\Omega F) \approx \pi_j(F).$$

The collection of $d(f_1, f_2)$, $j = 1, 2, \cdots$, are called the *difference obstructions* to deforming f_1 onto f_2 over B.

LEMMA 2.3. *We have*

$$\begin{array}{ccc} \pi_j(E) & \xrightarrow{\bar{f}_1 - \bar{f}_2} & \pi_j(F) \\ \downarrow & & \downarrow d(f_1, f_2) \\ \pi_j(E, E_A) & \overset{p}{\approx} & \pi_j(B, A). \end{array}$$

Proof. We start from the diagram

$$\begin{array}{ccccccc} \pi_j(\Omega E) & \xrightarrow{p_1'} & \pi_j(\Omega B) & \xrightarrow{g} & \pi_j(\Omega E) & \xrightarrow{i^{-1}} & \pi_j(\Omega F). \\ \downarrow & & \downarrow & \nearrow g' & & & \\ \pi_j(\Omega F, \Omega E_A) & \approx & \pi_j(\Omega B, \Omega A). & & & & \end{array}$$

One easily sees that it suffices to prove that $i^{-1}gpi_1 = \bar{f}_1 - \bar{f}_2$. To prove this, for $x \in \pi_j(\Omega E)$,

$$x = f_1' p_1'(x) - i\bar{f}_1'(x) = f_2' p_1(x) - i\bar{f}_2'(x)$$

i.e., $i\bar{f}_1'(x) - i\bar{f}_2'(x) = f_1' p_1(x) - f_2' p_2(x)$, and

$$g = f_1 - f_2 .$$

Then

$$\begin{aligned} i^{-1}gpi_1(x) &= i^{-1}gp_1(x) = i^{-1}gp_1(f_1' p_1(x) - i\bar{f}_1'(x)) \\ &= i^{-1}gp_1 f_1 p_1(x) = i^{-1}(f_1 p_1 - f_2 p_1)(x) = i^{-1}(i\bar{f}_1(x) - \bar{f}_2(x)) \\ &= f_1(x) - f_2(x). \end{aligned}$$

3. Functorial properties

Suppose (E, B, F) and (E', B', F') are fibre spaces, g a fibre-preserving mapping $E \to E'$ that covers a mapping $g_B : B \to B'$. Suppose that $A \subset B$, $A' \subset B'$, $g_B(A) \subset A'$, f and f' are cross-sections of E and E' over A and A' such that

$$\begin{array}{ccc} A & \xrightarrow{g_B} & A' \\ f \downarrow & & \downarrow f' \\ E_A & \xrightarrow{g} & E_{A'} . \end{array}$$

g induces a mapping $g_F : F \to F'$. One has a diagram

$$\begin{array}{ccc} \pi_j(B, A) & \xrightarrow{w(f)} & \pi_{j-1}(F) \\ g_B \downarrow & & \downarrow g_F \\ \pi_j(B', A') & \xrightarrow{w(f')} & \pi_{j-1}(F'). \end{array}$$

This is the functorial property of the obstructions.

4. Generalized cell complexes

DEFINITION. A sequence of subspaces $B_0 = (b_0) \subset B_1 \subset \cdots \subset B$ is called a *generalized cellular decomposition*, GCW complex for short, if $H_j(B_n, B_{n-1}) = 0$ for $j \neq n$, $n = 1, 2, \cdots$. A subspace A is said to be *adapted* to the decomposition if

$$H_j(A \cap B_n, A \cap B_{n-1}) = 0 \quad \text{for} \quad j \neq n,$$

$$H_j(A \cup B_n, A \cup B_{n-1}) = 0 \quad \text{for} \quad j \neq n.$$

If B_n is the n-skeleton of a CW cell decomposition of B, the above property is of course one of the main theorems of the theory of CW complexes. One recognizes that a CW subcomplex is adapted to the decomposition in the above sense. There exist other examples however: Let B be the space of all C^1 paths going from the north to the south pole of a unit 2-sphere. Let B_n be the space of all paths of length $\leqq \pi + (n - 1)2\pi$. The above property of the decomposition is one of the results of the Morse theory [10].

If A is a subspace adapted to a GCW decomposition, put $\hat{B}_n = B_n \cup A$.

$$C_n(B, A) = H_n(\hat{B}_n, \hat{B}_{n-1}).$$

Define $\partial : C_n(B, A) \to C_{n-1}(B, A)$ as the boundary operator of the triple $(\hat{B}_n, \hat{B}_{n-1}, \hat{B}_{n-2})$. One proves easily that $\partial^2 = 0$, i.e., $(C_n(E, A), \partial)$ is a chain complex. There is a standard process which we outline without proof (see [3, Chapter 3]), showing that the n^{th} homology group of this chain complex is isomorphic to $H_n(B, A)$.

Let $Z_n(B, A)$ be the cycles of this chain complex, $B_n(B, A)$ the boundaries. We have

$$H_n(\hat{B}, \hat{B}_0) \overset{i_3}{\Leftarrow} H_n(\hat{B}_n, \hat{B}_0) \overset{i_2}{\approx} H_n(\hat{B}_n, \hat{B}_{n-2}) \overset{i_1}{\to} H_n(\hat{B}_n, \hat{B}_{n-1}) \to H_{n-1}(\hat{B}_{n-1}, \hat{B}_{n-2}).$$

One proves that kernel $i_1 = 0$. Image i_1 = kernel $\partial = Z_n(B, A)$. Define $\theta = i_3 i_2^{-1} i_1^{-1} : Z_n(B, A) \Rightarrow H_n(B, \hat{B}_0)$. One proves that kernel $\theta = B_n(B, A)$ and that θ is onto, i.e., θ establishes the desired isomorphism.

Suppose now that f is a cross-section of the fibre space over $\hat{B}_{n-1}$. Consider the homotopy obstructions to extending f over $\hat{B}_n$, or rather consider just one, the first nontrivial one,

$$w(f) : \pi_n(\hat{B}_n, \hat{B}_{n-1}) \to \pi_{n-1}(F).$$

By the relative Hurewicz theorem [12],

$$C_n(B, A) = H_n(\hat{B}_n, \hat{B}_{n-1}) \approx \pi_n(\hat{B}_n, \hat{B}_{n-1}).$$

Definitively then, $w(f)$ will denote the map

$$C_n(B, A) \to \pi_{n-1}(F),$$

i.e., $w(f)$ is a cochain of the chain complex $(C_n(B, A), \partial)$ with coefficients in $\pi_{n-1}(F)$, and will be called the *obstruction to extending f, already given on B_{n-1}, to the n-skeleton.*

To justify this name, we must show that, in case the GCW decomposition of B is given by an ordinary simplicial decomposition of B and the fibre space is a fibre bundle, this definition gives a negative of the standard one [14, p. 149]. This is quite easy; in view of the functorial properties of both obstructions, it suffices to show that they coincide for the case where $B = E_n$, the n-cell, and the fibre space is isomorphic to the product. The discussion in Example 2.1 does precisely this.

LEMMA 4.1. *$\delta w(f) = 0$, i.e., $w(f)$ as a cochain is a cocycle.* (*See also* [1].)

Proof. We must prove $w(f) = 0$. We have

$$\begin{array}{ccccccccc}
C_n(B, A) & \overset{h_1}{\approx} & \pi_n(\hat{B}_n, \hat{B}_{n-1}) & \overset{p_1}{\approx} & \pi_n(\hat{E}_n, \hat{E}_{n-1}) & \xrightarrow{\partial_2} & \pi_{n-1}(\hat{E}_{n-1}) & \xrightarrow{\bar{f}} & \pi_{n-1}(F) \\
 & & \uparrow i_1 & & \uparrow i_2 & & \uparrow i_3 & & \\
 & & \pi_n(\hat{B}_n, b_0) & \overset{p_2}{\approx} & \pi_n(\hat{E}_n, F) & \xrightarrow{\partial_3} & \pi_{n-1}(F) & & \\
 & & \uparrow \partial_1 & & \uparrow \partial_4 & \nwarrow i_4 & & & \\
C_{n+1}(B, A) & \approx & \pi_{n+1}(\hat{B}_{n+1}, \hat{B}_n) & \overset{p_3}{\approx} & \pi_{n+1}(\hat{E}_{n+1}, \hat{E}_n) & \xrightarrow{\partial_5} & \pi_n(E_n). & &
\end{array}$$

Then $\partial_3 i_4 = 0$, which implies $w(f)\partial = \bar{f}\partial_2 p_1^{-1}h_1^{-1}\partial = \bar{f}i_3 \partial_3 i_4 \partial_5 p_3^{-1}h_2^{-1} = 0$.

Now, suppose f_1 and $f_2 : \hat{B}_{n-1} \to E$ are cross-sections that agree on $\hat{B}_{n-2}$. Construct the difference obstruction (see Section 2) in dimension $n - 1$, $d(f_1, f_2) : \pi_{n-1}(\hat{B}_{n-1}, \hat{B}_{n-2}) \to \pi_{n-1}(F)$. It is an $(n - 1)$-cochain of B mod A, with coefficients in $\pi_{n-1}(F)$.

LEMMA 4.2. *$\delta d(f_1, f_2) = w(f_1) - w(f_2)$, i.e., $d(f_1, f_2)\partial = w(f_1) - w(f_2)$.*

Proof. Using Lemma 2.2, we write the diagram,

$$\begin{array}{ccccccccc}
C_n(B, A) & \approx & \pi_n(\hat{B}_n, \hat{B}_{n-1}) & \overset{p}{\approx} & \pi_n(\hat{E}_n, \hat{E}_{n-1}) & \to & \pi_{n-1}(\hat{E}_{n-1}) & \xrightarrow{\bar{f}_1 - \bar{f}_2} & \pi_{n-1}(F) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \downarrow i_1 & & \uparrow d(f_1, f_2) \\
C_{n-1}(B, A) & \approx & \pi_{n-1}(\hat{B}_{n-1}, \hat{B}_{n-2}) & \overset{p_1^{-1}}{\approx} & \pi_{n-1}(\hat{E}_{n-1}, \hat{E}_{n-2}) & & \overset{p_1}{\approx} & & \pi_{n-1}(\hat{B}_{n-1}, \hat{B}_{n-2}),
\end{array}$$

from which the proof easily follows.

Example. Suppose $B = S_n$. Define then $B_j = (b_0)$ for $j < n$, $B_n = S_n$, A the empty set. We have

$$w(f) : C_n(S_n) \approx \pi_n(S_n, b_0) \approx \pi_n(E, F) \to \pi_{n-1}(F).$$

$w(f)$, the obstruction to constructing a cross-section of the fibre space, is then determined by the homotopy transgression homomorphism, i.e., it is, in this simple case, describable in terms independent of the decomposition of the base.

5. The homology obstruction and the first homotopy obstruction

Lemmas 4.1 and 4.2 describe the known formal properties of the obstruction cocycles [14]. We pass from homotopy to homology by means of the Hurewicz homomorphism. The homology obstruction is to be a map

$$v(f) : C_n(B, A) \to H_{n-1}(F)$$

leading to a diagram

5.1
$$
\begin{array}{ccc}
 & & \pi_{n-1}(F) \\
 & \nearrow^{w(f)} & \\
C_n(B, A) & & \Big\downarrow h \\
 & \searrow_{v(f)} & \\
 & & H_{n-1}(F).
\end{array}
$$

To define such a map one must impose certain conditions on f.

DEFINITION. Suppose (E, B, F), A, $f: \hat{B}_{n-1} \to E$ are as before, $\pi_j(F) = 0$ for $j < m$, $H_j(\hat{B}_{n-1}) = 0$ for $0 < j < p$, and $n \leqq m + p - 2$. The *homology obstruction to extending f to B_n* is defined as the composite

$$v(f): C_n(B, A) = H_n(\hat{B}_n, \hat{B}_{n-1}) \overset{p}{\approx} H_n(\hat{E}_n, E_{n-1}) \to H_{n-1}(\hat{E}_{n-1}) \xrightarrow{\bar{f}} H_{n-1}(F).$$

Obviously the diagram 5.1 holds. One proves, easily, just as for the homotopy obstruction, that the cochain is a cocycle. $\bar{v}(f)$ denotes its cohomology class.

We investigate now to what extent $\bar{v}(f)$ is independent of the cellular decomposition of B.

There is a diagram

5.2
$$
\begin{array}{ccccccccc}
C_n(B,A) & = & H_n(\hat{B}_n, \hat{B}_{n-1}) & \overset{p_1}{\approx} & H_n(\hat{E}_n, \hat{E}_{n-1}) & \xrightarrow{\partial_1} & H_{n-1}(\hat{E}_{n-1}) & \xrightarrow{\bar{f}} & H_{n-1}(F) \\
\uparrow & & & & \uparrow & & \uparrow & \nearrow & \\
Z_n(B, A) & \approx & H_n(\hat{B}_n, \hat{B}_{n-2}) & \approx & H_n(\hat{E}_n, \hat{E}_{n-2}) & \xrightarrow{\partial_2} & H_{n-1}(\hat{E}_{n-2}) & & \\
\Big| & & \approx\uparrow & & \approx\uparrow & & \uparrow & & \\
\theta & & H_n(\hat{B}_n, \hat{B}_0) & \approx & H_n(\hat{E}_n, \hat{E}_0) & \xrightarrow{\partial_3} & H_{n-1}(\hat{E}_0) & \approx & H_{n-1}(E_A) \\
\downarrow & & \downarrow & & \downarrow & & \updownarrow \text{id.} & & \\
H_n(B, A) & = & H_n(\hat{B}, \hat{B}_0) & \approx & H_n(E, \hat{E}_0) & \xrightarrow{\partial_4} & H_{n-1}(\hat{E}_0) & \approx & H_{n-1}(E_A).
\end{array}
$$

Suppose now A is $(q - 1)$-connected. Let ϕ be the canonical homomorphism $H^n(B, A; H_{n-1}(F)) \to \text{Hom}\,(H_n(B, A); H_{n-1}(F))$. Then, if

$n \leqq m + q - 2$, there is a map $\bar{f}: H_{n-1}(E_A) \to H_{n-1}(F)$ which fits into diagram 5.2, i.e.

$$\begin{array}{ccc} H_{n-1}(\hat{E}_{n-1}) & & \\ \uparrow & \searrow \bar{f} & \\ H_{n-1}(E_A) & \xrightarrow{\bar{f}_1} & H_{n-1}(F). \\ \uparrow & \swarrow\!\!\nearrow \text{id.} & \\ H_{n-1}(F) & & \end{array}$$

As a result we have

THEOREM 5.1. *With the above notations, if* $n \leqq m + q - 2$ *and* $n + p - 2$, *then* $\phi(\bar{v}(f))$ *is given as the composite*

$$H_n(B, A) \approx H_n(E, E_A) \longrightarrow H_{n-1}(E_A) \xrightarrow{\bar{f}_1} H_{n-1}(F).$$

In particular, $\phi(\bar{v}(f))$ is independent of the GCW decomposition of B used in its definition and is describable in terms of the homology properties of the fibre space (E, B, F) and the map $f: A \to E$ only. For example, when A is the empty set, $\phi(\bar{v}(f)): H_n(B, b_0) \to H_{n-1}(F)$ is just the transgression homomorphism of the fibre space, a classical result [14, p. 178].

Supplement for cohomology. If

$$\phi: H^{n-1}(F; H_{n-1}(F)) \to \text{Hom}\,(H_{n-1}(F), H_{n-1}(F))$$

is onto, i.e., if there is a $u \in H^{n-1}(F, H_{n-1}(F))$ such that $\phi(u)$ is the identity map $H_{n-1}(F) \to H_{n-1}(F)$, then $\bar{v}(f)$ is $p^{-1*}\partial_4^* \bar{f}_1^*(u)$, where

$$H^n(B, A; H_{n-1}(F)) \overset{p^*}{\approx} H^n(E, E_A; H_{n-1}(F)) \xleftarrow{\partial_4^*} H^{n-1}(E_A; H_{n-1}(F)) \xleftarrow{\bar{f}_1^*} H^{n-1}(F; H_{n-1}(F)).$$

(The proof, left to the reader, follows upon writing down the dual cohomology diagram to diagram 5.2.)

If F is $(n - 2)$-connected, the above conditions are automatically satisfied; $v(f)$, isomorphic to $w(f)$, is called the *primary obstruction* cocycle mod A, $\bar{v}(f)$ the *characteristic cohomology class* of the fibre space mod A. The explicit description of $\bar{v}(f)$ given by the Supplement is just the generalization of the Hopf Extension Theorem [14, Part III]. One fundamental problem for the higher obstruction cohomology classes is to describe explicitly, if possible, their independence of the cellular decomposition of the base. Even for the fibre bundle case, this has been done partially only for the second obstruction [8], although it is possible by means of the theory of semisimplicial complexes to prove the independence, without however giving any hint as to computation.

6. Sufficient conditions for the existence of sections

LEMMA 6.1. *Suppose the base B of the fibre space is a CW complex* [17], *with A a subcomplex, $B_0 \subset B_1 \subset \cdots$ the skeleton subcomplexes. Suppose $f: B_{n-1} \cup A \to A$, $n \geqq 2$, $w(f) \in C^n(B, A; \pi_{n-1}(F))$ are as before. Then $w(f) = 0$ is also sufficient for the extension of f to $B_n \cup A$. Suppose, secondly, that $C \in C^n(B, A; \pi_{n-1}(F))$, is another cocyle cohomologous to $w(f)$. Then, f restricted to $B_{n-2} \cup A$ can be extended to a cross-section $f': B_{n-1} \cup A \to E$ such that $w(f') = C$.*

For the proof, see [1].

THEOREM 6.2. *(X, B, F) is a fibre space; F is j-simple, for $j \leqq n - 1$; B is a CW complex with subcomplexes $B_0 = (b_0) \subset B_1 \subset \cdots \subset B_n \subset \cdots$, with $H_j(B_n, B_{n-1}) = 0$ for $j \neq n$; $f: B_{n-1} \to E$ is a cross-section. Then, considering $B_0 \subset B_1 \subset \cdots$ as a GCW decomposition of B, if the homotopy obstruction $w(f)$ to extending f to B_n is zero, f is actually extendable to B_n.*

Proof. Let $B_{n,j}$ be the j-skeleton of B_n. Suppose that f can be extended to a cross-section

$$f_{j-1}: B_{n,j-1} \cup B_{n-1} \to E.$$

We prove that the obstruction cohomology class

$$\bar{w}(f_{j-1}) \in H^j(B_n, B_{n-1}; \pi_{j-1}(f))$$

is 0.

It will then follow from Lemma 6.1 that there is a cross-section

$$f'_{j-1}: B_{n,j-1} \cup B_{n-1} \to E,$$

agreeing with f_{j-1} on $B_{n,j-2} \cup B_{n-1}$, such that $w(f'_{j-1}) = 0$, extendable therefore over $B_{n,j} \cup B_{n-1}$. By induction on j, f can then be extended over B_n.

Now, $\bar{w}(f_{j-1})$ is automatically zero if $j \neq n$. We deal then with the case $j = n$. We have the diagram

$$\begin{array}{ccccc}
H_n(B_{n,n} \cup B_{n-1}, B_{n,n-1} \cup B_{n-1}) & \overset{h_1}{\approx} & \pi_n(B_{n,n} \cup B_{n-1}, B_{n,n-1} \cup B_{n-1}) & \xrightarrow{w(f_{n-1})} & \pi_{n-1}(F) \\
\uparrow i_1 & & \uparrow i_2 & & \\
H_n(B_{n,n} \cup B_{n-1}, B_{n-1}) & \xleftarrow{h_2} & \pi_n(B_{n,n} \cup B_{n-1}, B_{n-1}) & \longrightarrow & \pi_{n-1}(F) \\
\downarrow i_3 & & \downarrow & & \\
H_n(B_n, B_{n-1}) & \overset{h_3}{\approx} & \pi_n(B_n, B_{n-1}) & \xrightarrow{w(f)} & \pi_{n-1}(F).
\end{array}$$

If we succeed in proving that h_2 is an isomorphism, it is clear from the above diagram that $\overline{w}(f_{n-1})$ is determined by $w(f)$, i.e., is zero. (Every element of $H^n(B_n, B_{n-1}; \pi_{n-1}(F))$ is determined by the corresponding element of Hom $(H_n(B_n, B_{n-1}), \pi_{n-1}(F))$ since all $H_j(B_n, B_{n-1})$ are zero for $j \neq h$.)

To prove that h_2 is an isomorphism, we prove that

$$H_j(B_{n,n} \cup B_{n-1}, B_{n-1}) = 0$$

for $j < n$, i.e., we prove that $i: H_j(B_{n,n}) \to H_j(B_{n,n} B_{n-1})$ is an isomorphism for $j < n$. To prove this, we utilize the exact Mayer-Vietoris sequence associated with the proper triads $(B_{n,n} \cup B_{n-1}, B_{n,n}, B_{n-1})$ and $(B_{n,n}, B_{n,n}, B_{n-1,n})$ [3, p. 39]. (The triads are proper since they are formed of subcomplexes of CW complexes.)

$$\begin{array}{ccc} H_j(B_{n-1,n}) \to & H_j(B_{n-1,n}) \oplus H_j(B_{n,n}) \to \\ \downarrow i_2 & \downarrow i_1 \\ H_j(B_{n-1,n}) \to & H_j(B_{n-1}) \oplus H_j(B_{n,n}) \to \end{array}$$

$$\begin{array}{ccccc} H_j(B_{n,n}) & \to & H_{j-1}(B_{n-1,n}) & \to & H_{j-1}(B_{n-1,n}) \oplus H_{j-1}(B_{n,n}) \\ \downarrow i & & \downarrow i_3 & & \downarrow i_4 \\ H_j(B_{r,n} \cup B_{n-1}) & \to & H_{j-1}(B_{n-1}) & \to & H_{j-1}(B_{n-1}) \oplus H_{j-1}(B_{n,n}). \end{array}$$

Now, i_1, i_2, i_3, and i_4 are isomorphisms; hence by the "Five Lemma" so is i.

Remark. This result is a consistency condition; if this notion of GCW complex had any value in obstruction theory, one would expect Theorem 6.2 to be true.

7. The higher obstructions and Postnikov systems

$F \xrightarrow{i} E \xrightarrow{p} B$ will be a fixed fibre space with B a simply connected CW complex. F, the fibre, is supposed connected. B_k ($k \geqq 1$) will denote the k-skeleton of B. π_k will denote the abelian group $\pi_k(F)$.

We recall some known facts about Postnikov systems for spaces and generalizations to fibre spaces. For a given space F, the pair (F_n, g_n) of a space F_n and fibre map $g_n: F'_n \to F \to F_n$ will be called an *n-Postnikov map* for F providing

7.1. $g_n: \pi_k(F) \to \pi_k(F_n)$ is an isomorphism for $k \leqq n$, and $\pi_k(F_n) = 0$ for $k > n$.

It follows that

$$\begin{aligned} \pi_k(F'_n) &\approx 0 \qquad \text{if} \quad k \leqq n, \\ &\approx \pi_k(F) \quad \text{for} \quad k > n. \end{aligned}$$

Similarly, the triple (E_n, F_n, h_n), where

$$F_n \longrightarrow E_n \xrightarrow{p_n} B \quad \text{and} \quad F'_n \longrightarrow E \xrightarrow{h_n} E_n$$

are fibre spaces, leading to a commutative diagram

$$\begin{array}{ccccc} & & F & \xrightarrow{g_n} & F_n \\ & \nearrow & \downarrow & & \downarrow \\ F'_n & \longrightarrow & E & \xrightarrow{h_n} & E_n \\ & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_n} \\ & & B & \xleftrightarrow{\text{id.}} & B, \end{array}$$

will be called an *n-Postnikov map* for the fibre space E providing

7.2. h_n restricted to F, i.e., g_n in this diagram, is an n-Postnikov map for F.

It follows easily that

7.3. $h_n : \pi_k(E) \longrightarrow \pi_k(E_n)$ is an isomorphism for $k \leqq n$, and

$$\begin{aligned} \pi_k(F'_n) &\approx 0 \quad \text{if} \quad k \leqq n, \\ &\approx \pi_k \quad \text{if} \quad k > n. \end{aligned}$$

For the given n-Postnikov map $F'_n \longrightarrow F \xrightarrow{g_n} F_n$, the characteristic class of the fibre space g_n, $k^{n+2} \in H^{n+2}(F_n, \pi_{n+1})$, is called an *n-Postnikov invariant* of F. Similarly, for the given n-Postnikov map $F'_n \longrightarrow E \xrightarrow{h_n} E_n$ for the *fibre space E*, the characteristic class $\mathbf{k}^{n+2} \in H^{n+2}(E_n, \pi_{n+1})$ of the fibre space h_n will be called an *n-Postnikov invariant* for E. Then, $\mathbf{k}^{n+2}$ restricted to the fibre F is an n-Postnikov invariant for the space F, and $h_n^*(\mathbf{k}^{n+2}) = 0$. In certain cases this alone serves to calculate this invariant [4].

The calculation of all possible Postnikov maps for, and invariants for, the space F is equivalent to calculating the obstruction classes, in the classical sense of Eilenberg, of all maps of spaces into F. Our point of view will be to consider this calculation as known (although at present only fragmentary information is known) and try to see how the calculation of the obstruction classes for cross-sections of E can be reduced to the calculation of these invariants of F and additional "twisting" invariants of the fibre space.

With this meaning for h_n, E_n, and $\mathbf{k}^{n+2}$, let $f: B_{n+1} \to E$ be a cross-section of the fibre space, $w(f) \in H^{n+2}(B, \pi_{n+1})$ the obstruction cohomology class to extending it over B_{n+2}. $h_n f: B_{n+1} \to E_n$ is a cross-section, and, in view of the condition $\pi_k(F_n) = 0$ if $k > n$, it can be extended to a section $f': B \to E_n$. We have proved in [4] that

7.4. $w(f') = f'^*(\mathbf{k}^{n+2})$.

This gives us a method, already exploited in [4], for "calculating" $w(f)$. One assumes that E_n and h_n are given in terms of f, that their cohomology properties are accessible, and that $\mathbf{k}^{n+2}$ can be calculated in terms of these properties.

8. Second obstruction

We continue with a fibre space $F \xrightarrow{i} E \xrightarrow{p} B$, and in addition suppose that $\pi_k(F) = 0$ if $k < m$ or $m < k \leqq n$.

LEMMA 8.1. *Let $f: B_{n+1} \to E$ be a cross-section, G an abelian group, and $\alpha \in H^m(F, G)$. Then there is a unique $a \in H^m(E, G)$ such that $i^*(a) = \alpha$ and $f^*(a) = 0$. This class is the same for any two sections that agree on B_m.*

Proof. We have the exact sequence [11]

$$H^m(B, G) \xrightarrow{p^*} H^m(E, G) \xrightarrow{i^*} H^m(F, G) \xrightarrow{\tau} H^{m+1}(B, G),$$

where τ is the transgression. We know that $\tau = 0$ since f is a cross-section. There exists $a' \in H^m(E, G)$ with $i^*(a') = \alpha$. We have

$$H^m(B, G) \overset{i_2^*}{\approx} H^m(B_{n+1}, G) \overset{i_1^*}{\twoheadrightarrow} H^m(B_m, G).$$

Put

$$a = a' - p^* i_2^{-1*} f^*(a').$$

Clearly a satisfies *both* conditions. As for uniqueness, suppose $a'' \in H^m(E, G)$ also satisfies both conditions. Then, $i^*(a'' - a) = 0$, i.e., $a'' - a = p^*(b)$, for $b \in H^m(B, G)$. $0 = f^* p^*(b) = i_2^*(b)$; hence $b = 0$.

Using this lemma, we can construct an n-Postnikov map $h_n: E \to E_n$, with $E_n = B \times K(\pi_m, m)$. To see this, let $a(f) \in H^m(E, \pi_m)$ be the unique class such that $f^*(a(f)) = 0$ and $a(f)$ restricted to F is the fundamental class of $H^m(F, \pi_m)$ (i.e., the class which, as a homomorphism $H^m(F, Z) \to \pi_m$, is the inverse of the Hurewicz isomorphism). Let $\iota^m \in H^m(K(\pi_m, m), \pi_m)$ be the fundamental class of this Eilenberg-Mac Lane space, and let $g: E \to K(\pi_m, m)$ be a map such that $g^*(\iota^m) = a(f)$ [13]. Then g restricted to F is an n-Postnikov map for F. Let $h_n: E \to B \times K(\pi_m, m) = E_m$ be the map which is the product of p and g. h_n so constructed is an n-Postnikov map for the fibre space E.

Let $\mathbf{k}^{n+2} \in H^{n+2}(E_n, \pi_{n+1})$ be the corresponding Postnikov invariant. Let

$\lambda:B \to E_n$ be the inclusion map on the first factor, and let $f':B \to E_n$ be an extension of $h_n f$. Let $\sigma:E_n \to K(\pi_m, m)$ be the projection on the second factor. Then $(\sigma f')^* (\iota^m) = 0$, i.e., f' is homotopic to a point mapping. This means that f' and λ are homotopic, as cross-sections; hence

LEMMA 8.1′. *With the above notations,* $\lambda^*(\mathbf{k}^{n+2}) = w(f)$.

Further progress now is dependent on how $H^*(E_n, \pi_{n+1})$ is determined in terms of $H^*(B, \pi_{n+1})$ and $H^*(K(\pi_m, m), \pi_{n+1})$. In [4] we sketched how this can be done if π_{n+1} is a ring, and $H^*(E_n, \pi_{n+1})$ is a tensor product. This was applied to the case where F is an m-sphere. We give another

Example. $F = P_k(C)$, the complex projective space of real dimension $2k$, $n = 2k$, $m = 2$. It is well known that

$$\pi_j(P_k(C)) \approx Z \text{ (the integers)} \quad \text{if} \quad j = 2 \text{ or } 2k+1.$$

$$\approx 0 \qquad\qquad \text{if} \quad 2 < j < 2k+1.$$

Then $E_n = B \times K(Z, 2)$. $H^*(K(Z, 2), 2)$ is a polynomial ring generated by cup products of the fundamental class $\iota \in H^2(K(Z, 2), Z)$. The first Postnikov invariant of $P_k(C)$ is ι^{k+1} (cup product $k+1$ times, with the ring structure of Z).

LEMMA 8.2. *With the above notations,* $a(f) \in H^2(E, Z)$ *as in Lemma* 8.1, $\alpha \in H^2(P_k(C), Z)$ *a generator, define an additive homomorphism*

$$\phi:H^*(B, Z) \otimes H^*(P_k(C), Z) \to H^*(E, Z)$$

by

$$\phi(b \otimes \alpha^j) \to p^*(b) \cup a(f)^j, \qquad 0 \leqq j \leqq k.$$

Then, ϕ is an additive isomorphism.

Proof. Note that the fibre in the fibre space $P_k(C) \to E \to B$ is totally nonhomologous to zero [11, p. 472]. One can almost apply a result of Serre [11, p. 473], except that ϕ does not preserve cup products. But notice that Serre's proof does not actually require that ϕ do so.

Then $a(f)^{k+1} = \sum_{j=0}^{k} p^*(b_j) \cup a(f)^j$, where the classes $b_j \in H^{n+2-2j}(B, Z)$ can be considered as the "twisting invariants" of E. Note then that

$$\mathbf{k}^{n+2} = \iota^{k+1} - \sum_{j=0}^{k} b_j \otimes \iota^j$$

is the Postnikov invariant of E by Theorem 3.1 of [4]; hence, by Lemma 8.1,

$$w(f) = \lambda^*(\mathbf{k}^{n+2}) = b_0 .$$

Pulling back to E, we have

$$p^*(w(f)) = -a(f)^{k+1} + \sum_{j=1}^{k} p^*(b_j) \cup a(f)^j.$$

This is the formula for the second obstruction given by Kundert [6].

9. Reduction mod l

The computation of obstruction classes for a fibre space $F \to E \to B$ is of course dependent on the computation of the homotopy groups of F. At least since Serre's work [11], [12], when calculating the homotopy groups of a space, it has been most efficient to calculate the groups reduced modulo the prime numbers. One can ask then how the computation of the obstruction classes can be fitted into this program. We will not carry out the details required to adapt the $\mathcal{C}$-Theory paper [12], but sketch how the earlier results in [11] can be used.

If $f: B_{n+1} \to E$ is a cross-section, l a prime number, $w(f)_l$, the obstruction class reduced mod l, denotes the image of $w(f)$ in $H^{n+2}(B, \pi_{n+1}(F) \otimes Z_l)$ under the coefficient homomorphism $\pi_{n+1} \to \pi_{n+1} \otimes Z_l$. These classes have all the formal properties of $w(f)$ due to the functorial nature of the tensor product. In particular, one can prove the analogue of Theorem 5.1: If $\pi_j \otimes Z_l = 0$ for $j \leqq n$, $w(f)_l$ is the image under transgression of the "fundamental class mod l" of $H^{n+1}(F, \pi_{n+1} \otimes Z_l)$, i.e., the class corresponding to the Hurewicz isomorphism mod l: $H_{n+1}(F, Z_l) \to \pi_{n+1}(F) \otimes Z_l$ [11].

The method given in Section 8 works to calculate the "second obstruction mod l", i.e., the case where $\pi_j(F) \otimes Z_l = 0$ for $j < m$, $m < j \leqq n$.

We illustrate with the case $F = S_m$, an m-sphere, with m *odd*. (The case m even would require the machinery of primary cohomology operations mod a *power* of l [16], a refinement we will not go into here.) The relevant homotopy information is the following: For m odd, $l \neq 0$ [11],

$$\pi_j(S_m) \otimes Z_l = 0 \quad \text{for} \quad n < j < n + 2l - 3,$$

$$\approx Z_l \quad \text{for} \quad j = m + 2l - 3.$$

Put $n = m + 2l - 4$. The role played by the cohomology operation Sq^2 in the Liao formula for the secondary obstruction for a sphere bundle [7] is replaced here by the Steenrod reduced power operation $\mathcal{P}_l^1$ [15], defined mod Z_l.

Let $g: E \to K(Z, m)$ be defined as before, with $g^*(\iota) = a(f)$. Define $h_n: E \to E_n = B \times K(Z, m)$ as before. By means of the Gysin sequence [11, p. 470] for the fibre space $S_m \to E \to B$, one proves that

$$\mathcal{P}_l^1(a(f)) = p^*(b) + \psi(\mathcal{P}_l^1(a(f)) \smile a(f),$$

where all classes are reduced mod l, and $\psi: H^j(E, Z_l) \to H^{j-m}(B, Z_l)$ is the "integration over the fibre" homomorphism. Then the class

$$\mathbf{k} = \mathcal{P}_l^1(\iota) - b - \psi(\mathcal{P}_l^1(a(f)) \otimes \iota \in H^{n+2}(E_n, Z_l)$$

satisfies (a) $h_n^*(\mathbf{k}) = 0$, and (b) $\mathbf{k}$ restricted to $K(Z, m)$ is the "Postnikov invariant reduced mod l". The analogy with 7.4 and Theorem 8.1 leads to the result $w(f)_l = b$, i.e.,

$$p^*(w(f)_l) = \mathcal{P}_l^1(a(f)) - p^*\psi(\mathcal{P}_l^1(a(f)) \smile a(f).$$

10. The difference of two secondary obstructions

We work with the notations of Section 8. $f:B_{n+1} \to E$ is a cross-section; $E \xrightarrow{h_n = p \times g} B \times K(\pi_m, m) \xrightarrow{\sigma} K(\pi_m, m)$; $\mathbf{k}^{n+2} \in H^{n+2}(E_n, \pi_{n+1})$ is an n-Postnikov invariant for the fibre space.

Suppose $f_1:B_{n+1} \to E$ is *another* cross-section. Then $i^*(a(f) - a(f_1)) = 0$ and $a(f) - a(f_1) = p^*(b)$ for some $b \in H^m(B, \pi_m)$. Hence $f_1^*(a(f)) = b$. Let $\iota \in H^m(K(\pi_m, m), \pi_m)$ be the fundamental class. Then $(\sigma h_n)^*(\iota) = a(f)$.

$$b = f_1^*(a(f)) = f_1^* h_n^* \sigma^*(\iota) = (h_n f_1)^* \sigma^*(\iota).$$

Let $\gamma_b:B \to K(\pi_m, m)$ be a map such that $\gamma_b^*(\iota) = b$. Then $\sigma h_n f_1$ is homotopic to γ_b restricted to B_{n+1}. Let $G(\gamma_b):B \to E_n$ be the map which is the *graph* of γ_b.

Applying Lemma 8.1′, we get

10.1 $$w(f_1) = G(\gamma_b)^*(\mathbf{k}^{n+2}).$$

In order to make this formula explicit, we will suppose that the coefficient group π_{n+1} is a *ring*. If $\smile$ denotes cup product with respect to this ring, we suppose also that

10.2 $$\mathbf{k}^{n+2} = \sum_{j=0}^{N} p_n^*(b_j) \smile \sigma^*(\theta_j),$$

where b_j resp. θ_j are elements of $H^*(B, \pi_{n+1})$ resp. $H^*(K(\pi_m, m), \pi_{n+1})$, and dim $b_0 = 0$, dim $\theta_N = n + 2$.

Those b_j whose dimension is less than $n + 2$ are called "twisting invariants", since they are zero if the fibre space $F \to E \to B$ is isomorphic to a product. We have then

10.3 $$G(\gamma_b)^*(\mathbf{k}^{n+2}) = \sum_{j=0}^{N} b_j \smile \gamma_b^*(\theta_j),$$

10.4 $$w(f_1) - w(f) = \sum_{j=1}^{N} b_j \smile \gamma_b^*(\theta_j).$$

It is well known that the elements of $H^*(K(\pi_m, m), \pi_{n+1})$ are in one-to-one correspondence with primary cohomology operations [12], i.e., for each $\theta \in H^{m+j}(K(\pi_m, m), \pi_{n+1})$ and each space X, θ determines an operation, also denoted by $\theta:H^m(X, \pi_m) \to H^{m+j}(X, \pi_{n+1})$. With this notation, $\gamma_b^*(\theta_j) = \theta_j(b)$.

THEOREM 10.1. *With the above notations and with an n-Postnikov invariant* $\mathbf{k}^{n+2}$ *for E satisfying* 10.2, *a fixed cross-section* $f:B_{n+1} \to E$, *there is a cross-section over the* $(n + 2)$*-skeleton of the fibre space* $F \to E \to B$ *if and only if there is a cohomology class* $b \in H^m(B, \pi_m)$ *with*

10.5 $$w(f) + \sum_{j=1}^{N} b_j \smile \theta_j(b) = 0.$$

Proof. We have already proved the necessity. The argument can be retraced: Given such a b, let $\gamma:B \to K(\pi_m, m)$ be such that $\gamma^*(\iota) = b$. $G(\gamma):B \to E_n$ denotes the graph of γ. Since $h_n:E \to E_n$ maps the homotopy groups in dimension $\leqq n$ isomorphically, $G(\gamma)$ can be lifted to a cross-section $f_1: B_{n+1} \to E$, with $G(\gamma)$ restricted to B_{n+1} equal to $h_n f_1$. Thus 10.5 implies that $w(f_1) = 0$.

11. Second obstruction for fibre bundles

We continue with the notations of Sections 8 and 10. In order to decide whether the fibre space $F \to E \to B$ admits a cross-section over B_{n+2}, we have seen that it suffices to calculate the obstruction class and twisting invariants of just one cross-section $B_{n+1} \to E$. For the geometrically interesting examples one would expect then that there would be a way of singling out a cross-section. We go into the general aspects of this question in case the fibre space is an associated bundle to a principal bundle with structure group a compact Lie group G. For example, in case $F \to E \to B$ is a projective bundle associated to a complex vector bundle, Kundert has shown [6] that one can choose the cross-section so that the twisting invariants are the Chern classes.

Let $G \to E_G \to B_G$ be the universal principal bundle for G [2]. We suppose that G acts on F, and $F \to E_F \to B_G$ is the associated bundle with F as fibre, and that $\phi: B \to B_G$ is a cellular map that induces the given bundle $F \to E \to B$.

Let $K(\pi_m, m) \to B'_G \xrightarrow{q} B_G$ be a fibre space over B_G that kills off the characteristic class of the fibre space $F \to E_F \to B_G$, and let $F \to E'_F \to B'_G$ be the fibre space induced from the "standard" fibre space $F \to E_F \to B_G$ by q. We suppose that the first obstruction, i.e., characteristic class, vanishes. Hence ϕ admits a lifting $\phi': B \to B'_G$, which can again be taken as a cellular map. Hence $F \to E'_F \to B'_G$ plays the role of a fibre space that is "universal" for second obstruction problems, in the sense that the fibre space $F \to E \to B$ is induced from a map $\phi': B \to B'_G$. (These remarks are due, so far as I know, to W. Massey [8].)

Suppose $\pi_k(F) \approx 0$ for $k < m$, $m < k \leqq n$. A cross-section $f: B'_{G,n+1} \to E'_F$ obviously is pulled back via ϕ' to a cross-section $B_{n+1} \to E$, so the problem of deciding whether $F \to E \to B$ admits a cross-section over B_{n+2} is reduced to calculating the twisting invariants and obstruction class of f, finding how these are pulled back to B by ϕ', and seeing how certain primary cohomology operations act on $H^*(B)$.

If we are in the hypotheses of Theorem 10.1, i.e. $\pi_{n+2}(F)$ is a ring, $p'^*(w(f)) = \sum_{j=0}^{N} p'^*(b_j) \smile \theta_j(a(f))$, for $b_j \in H^*(B'_G, \pi_{n+2})$, notice that the b_j restricted to a fibre of $K(\pi_m, m) \to B'_G \to B_G$ are *zero*, since the fibre space $F \to E'_F \to B'_G$ restricted to such a fibre is isomorphic to a product. Thus we begin to see a general reason why Kundert and Liao found in special cases that these twisting invariants could be chosen to be characteristic classes of the principal G-bundle, i.e. classes arising from B_G.

Bibliography

1. W. Barcus, *Note on cross-sections over CW-complexes*, Quart. J. Math. Oxford Ser. (2), vol. 5 (1954), pp. 150–160.

2. A. BOREL, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2), vol. 57 (1953), pp. 115–207.
3. S. EILENBERG AND N. STEENROD, *Foundations of algebraic topology*, Princeton, 1952.
4. R. HERMANN, *Secondary obstructions for fibre spaces*, Bull. Amer. Math. Soc., vol. 65 (1959), pp. 5–8.
5. I. M. JAMES AND J. H. C. WHITEHEAD, *Note on fibre spaces*, Proc. London Math. Soc. (3), vol. 4 (1954), pp. 129–137.
6. E. G. KUNDERT, *Über Schnittflächen in speziellen Faserungen und Felder reeller und komplexer Linienelemente*, Ann. of Math. (2), vol. 54 (1951), pp. 215–246.
7. S. D. LIAO, *On the theory of obstructions of fiber bundles*, Ann. of Math. (2), vol. 60 (1954), pp. 146–191.
8. W. S. MASSEY, *On the cohomology ring of a sphere bundle*, J. Math. Mech., vol. 7 (1958), pp. 265–289.
9. H. SAMELSON, *A connection between the Whitehead and the Pontryagin product*, Amer. J. Math., vol. 75 (1953), pp. 744–752.
10. H. SEIFERT UND W. THRELFALL, *Variationsrechnung im Grossen (Theorie von Marston Morse)*, Leipzig und Berlin, 1938.
11. J-P. SERRE, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2), vol. 54 (1951), pp. 425–505.
12. ———, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2), vol. 58 (1953), pp. 258–294.
13. ———, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv., vol. 27 (1953), pp. 198–232.
14. N. STEENROD, *The topology of fibre bundles*, Princeton, 1951.
15. ———, *Reduced powers of cohomology classes*, Ann. of Math. (2), vol. 56 (1952), pp. 47–67.
16. E. THOMAS, *A generalization of the Pontrjagin square cohomology operation*, Proc. Nat. Acad. Sci. U. S. A., vol. 42 (1956), pp. 266–269.
17. G. W. WHITEHEAD, *Homotopy theory*, Massachusetts Institute of Technology, 1953.

HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS