INVERSION OF TOEPLITZ MATRICES

BY

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1. Introduction

This paper deals with the inversion of the Toeplitz matrix $T = (c_{j-k})$, $j, k = 0, 1, \cdots$. It will be assumed that the c_k are the Fourier coefficients of a function $\varphi(\theta)$,

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} \varphi(\theta)\, d\theta, \qquad k = 0, \pm 1, \cdots.$$

Since the inversion of T is equivalent to the solution of a system of equations of the form

$$\textstyle\sum_{k=0}^{\infty} c_{j-k}\, x_k = y_j, \qquad j = 0, \pm 1, \cdots,$$

we see that we are dealing with the discrete analogue of a Wiener-Hopf equation. It might be expected then that we shall look for a factorization of φ of the form $\varphi = \varphi_+ \varphi_-$, where $\varphi_+(\theta)$ and $\varphi_-(\theta)$ are boundary values of functions analytic inside and outside the unit circle, respectively. This, in fact, is the crux of the matter.

In Section 2 we consider the case $\sum_{-\infty}^{\infty} | c_k | < \infty$. Then T may be considered a bounded operator on the space l_∞^+ of bounded sequences $X = \{x_0, x_1, \cdots\}$ with $\| X \| = \sup | x_k |$, and a necessary and sufficient condition is found for the invertibility of T (Theorem I). In case T is invertible, a generating function is found for the entries of the matrix T^{-1} (Theorem III). As a consequence of the theory we obtain a theorem of Tauberian type: Certain sets are shown to be fundamental in l_1^+, the space of all $X = \{x_0, x_1, \cdots\}$ with $\| X \| = \sum_0^\infty | x_k | < \infty$ (Corollary of Theorem II).

In Section 3 the condition $\sum | c_k | < \infty$ is dropped, but it is still assumed that φ is bounded. In this case T may be considered an operator (bounded by the boundedness of φ using Parseval's relation) on the space l_2^+ of square summable sequences $X = \{x_0, x_1, \cdots\}$ with $\| X \|^2 = \sum_{k=0}^{\infty} | x_k |^2$, and we find a sufficient condition for the invertibility of T (Theorem IV).

Note added in proof. A substantial part of this paper (Theorems I and II and an analogue of Theorem III) was discovered independently by M. G. KREĬN in his paper, *Integral equations on the half-line with a difference kernel*, Uspehi Mat. Nauk, vol 13, no. 5 (1958), pp. 3–120 (Russian). Where the operator T is concerned, with $\sum | c_k | < \infty$, our paper is practically identical with Kreĭn's, in regard to both methods and results. Kreĭn has gone further

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in considering the continuous analogue of our problem, where T is an integral operator of the Wiener-Hopf type.

2. The l_∞^+ theory

Throughout this section we shall assume $\sum_{-\infty}^{\infty} | c_k | < \infty$, and consider T an operator on $l_\infty^+ : T\{x_j\} = \{ \sum_0^\infty c_{j-k} x_k \}$.

For convenience we introduce the larger space l_∞ of bounded doubly infinite sequences $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$. There is then a natural embedding of l_∞^+ into l_∞ given by $\{x_0, x_1, \cdots\} \to \{\cdots, 0, x_0, x_1, \cdots\}$, and a natural projection P of l_∞ onto l_∞^+ given by $P\{\cdots, x_{-1}, x_0, x_1, \cdots\} = \{x_0, x_1, \cdots\}$.

For a function $f(\theta) = \sum_{-\infty}^{\infty} b_k e^{ik\theta}$ with $\sum | b_k | < \infty$, we define the operator M_f on l_∞ by

$$M_f\{x_j\} = \{ \sum_{k=-\infty}^{\infty} b_{j-k} x_k \}.$$

It is clear that $T = PM_\varphi$; it should also be noted that $M_f M_g = M_{fg}$, and that if $b_k = 0$ for $k < 0$, then M_f leaves l_∞^+ invariant. (Note that we have identified the space l_∞^+ with its image in l_∞.)

Our first problem is the factorization of φ. Given a continuous function $f(\theta)$ on $[-\pi, \pi]$ with $f(\theta) \neq 0$, we set

$$I(f) = (1/2\pi)\Delta_{-\pi \leqq \theta \leqq \pi} \arg f(\theta).$$

LEMMA.[3] *If $\varphi(\theta) \neq 0$ and $I(\varphi) = 0$, any continuously defined* $\log \varphi(\theta)$ *has an absolutely convergent Fourier series.*

Proof. Letting $\arg \varphi(\theta)$ denote any continuous argument of φ we can find a trigonometric polynomial $p(\theta)$ such that

$$| \arg \varphi(\theta) - p(\theta) | < \pi/2, \qquad -\pi \leqq \theta \leqq \pi.$$

(Note that our assumption $I(\varphi) = 0$ is equivalent to $\arg \varphi(-\pi) = \arg \varphi(\pi)$.) Then if we set

$$\varphi_1(\theta) = e^{-ip(\theta)}\varphi(\theta),$$

$\varphi_1(\theta)$ has an absolutely convergent Fourier series, and its range lies in the half plane $\Re\varphi_1 > 0$. Therefore, by the Wiener-Lévy theorem, we can find a function $\psi_1(\theta)$ with absolutely convergent Fourier series such that $\varphi_1(\theta) = e^{\psi_1(\theta)}$. Then $\log \varphi(\theta)$ is, except for an additive constant, just $\psi_1(\theta) + ip(\theta)$, and so certainly has an absolutely convergent Fourier series.

With the hypothesis of the lemma holding, it is easy to obtain the desired factorization of φ. Choose any continuous $\log \varphi$ and write

$$(1) \qquad \log \varphi(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta},$$

$$(2) \qquad f_+(\theta) = \sum_{k=0}^{\infty} a_k e^{ik\theta}, \qquad f_-(\theta) = \sum_{k=-\infty}^{-1} a_k e^{ik\theta},$$

$$(3) \qquad \varphi_+(\theta) = \exp(f_+(\theta)), \qquad \varphi_-(\theta) = \exp(f_-(\theta)).$$

[3] This lemma follows from general results of R. Cameron and N. Wiener. The simple proof below was suggested by L. Welch.

Then the functions $\varphi_+(\theta)$, $\varphi_-(\theta)$, $\varphi_+(\theta)^{-1}$, $\varphi_-(\theta)^{-1}$ have absolutely convergent Fourier series, and $\varphi(\theta) = \varphi_+(\theta)\varphi_-(\theta)$. We are now in a position to prove half of the following.

THEOREM I. *A necessary and sufficient condition that T be invertible is that $\varphi(\theta) \neq 0$ and $I(\varphi) = 0$. Under these conditions $T^{-1} = M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}}$.*

We prove now that under the stated conditions T^{-1} exists and is what it is purported to be. Note that $M_{\varphi_+^{-1}}$ leaves l_∞^+ invariant, so $U = M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}}$ is a (bounded) operator on l_∞^+. Let $X \in l_\infty^+$. Then

$$\begin{aligned} TUX &= PM_\varphi M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}} X = PM_{\varphi_-} PM_{\varphi_-^{-1}} X \\ &= PM_{\varphi_-} M_{\varphi_-^{-1}} X - PM_{\varphi_-}(I - P)M_{\varphi_-^{-1}} X \qquad (4) \\ &= X - PM_{\varphi_-}(I - P)M_{\varphi_-^{-1}} X, \end{aligned}$$

where I is the identity operator. Since

$$M_{\varphi_-}(I - P) = (I - P)M_{\varphi_-}(I - P),$$

i.e., since M_{φ_-} leaves invariant the space of all $\{\cdots, x_{-1}, 0, 0, \cdots\}$, the second term of (4) vanishes, and we obtain $TUX = X$.

Similarly,

$$\begin{aligned} UTX &= M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}} PM_\varphi X \\ &= M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}} M_\varphi X - M_{\varphi_+^{-1}} PM_{\varphi_-}(I - P)M_\varphi X \\ &= X - M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}}(I - P)M_\varphi X. \end{aligned}$$

Again the second term vanishes since $M_{\varphi_-^{-1}}$ leaves invariant the space of all $\{\cdots, x_{-1}, 0, 0, \cdots\}$, and we obtain $UTX = X$.

It will be convenient to defer the proof of the rest of Theorem I until after our next result, which tells us what happens when $\varphi(\theta) \neq 0$ but $I(\varphi) = n \neq 0$. In this case $I(e^{-in\theta}\varphi(\theta)) = 0$, and we again introduce functions $\varphi_+(\theta)$, $\varphi_-(\theta)$ by means of (2) and (3), but now the coefficients a_k are obtained from (1) with $\varphi(\theta)$ replaced by $e^{-in\theta}\varphi(\theta)$. The factorization now becomes

$$e^{-in\theta}\varphi(\theta) = \varphi_+(\theta)\varphi_-(\theta).$$

THEOREM II. *Assume $\varphi(\theta) \neq 0$.* (a) *If $n = I(\varphi) > 0$, T is one-one, and its range is a subspace of l_∞^+ of deficiency n; T has the bounded left inverse $M_{e^{-in\theta}\varphi_+^{-1}} PM_{\varphi_-^{-1}}$.* (b) *If $n = I(\varphi) < 0$, T is onto and has null space of dimension $|n|$; T has the bounded right inverse $M_{e^{-in\theta}\varphi_+^{-1}} PM_{\varphi_-^{-1}}$.*

Proof. (a) $T = PM_\varphi = PM_{e^{-in\theta}\varphi} M_{e^{in\theta}}$. Since $I(e^{-in\theta}\varphi(\theta)) = 0$, we conclude from (what has been proved of) Theorem I that $PM_{e^{-in\theta}\varphi}$ is one-one and onto l_∞^+. Since $M_{e^{in\theta}}\, l_\infty^+$ is a subspace of deficiency n, T is one-one and has range of deficiency n. Since $PM_{e^{-in\theta}\varphi}$ has inverse $M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}}$, we have

$$M_{e^{-in\theta}\varphi_+^{-1}} PM_{\varphi_-^{-1}} TX = M_{e^{-in\theta}}(M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}})(PM_{e^{-in\theta}\varphi})M_{e^{in\theta}} X = X.$$

(b) In this case $T = PM_{e^{-in\theta}\varphi} M_{e^{in\theta}}$ shows that T, when restricted to a subspace of l_∞^+ of deficiency $|n|$ (namely $M_{e^{-in\theta}} l_\infty^+$) is one-one and onto l_∞^+. Thus T is onto and has null space of dimension $|n|$. Moreover

$$TM_{e^{-in\theta}\varphi_+^{-1}} PM_{\varphi_-^{-1}} X = PM_{e^{-in\theta}\varphi} M_{e^{in\theta}} M_{e^{-in\theta}} M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}} X$$
$$= (PM_{e^{-in\theta}\varphi})(M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}})X = X.$$

COROLLARY. *Denote by C_k the element $\{c_{-k}, c_{-k+1}, \cdots, c_0, c_1, \cdots\}$ of l_1^+. Assume $\varphi(\theta) \neq 0$. Then a necessary and sufficient condition that the set $C_0, C_1, \cdots$ be fundamental in l_1^+ is that $I(\varphi) \leqq 0$.*

In the case $c_{-1} = c_{-2} = \cdots = 0$ the conditions $\varphi(\theta) \neq 0$, $I(\varphi) \leqq 0$ are equivalent to the assertion that the function $\Phi(z) = \sum_0^\infty c_k z^k$, which is analytic in the unit circle, has no zero on $|z| \leqq 1$. The result in this case was proved by Nyman [2]. (It is easy to see, in this special case, that the condition $\varphi(\theta) \neq 0$ is certainly necessary.)

When we pass to the proof, the Hahn-Banach theorem tells us that a necessary and sufficient condition that $C_0, C_1, \cdots$ is not fundamental is the existence of a nonzero vector $X = \{x_j\} \in l_\infty^+$ such that

$$\sum_{j=0}^\infty c_{k-j} x_j = 0, \qquad k = 0, 1, \cdots.$$

Thus $C_0, C_1, \cdots$ is not fundamental if and only if the Toeplitz matrix corresponding to $\overline{\varphi(\theta)}$ has a null space, and by Theorems I and II this is equivalent to $I(\bar\varphi) < 0$, i.e., to $I(\varphi) > 0$.

We proceed now to the completion of the proof of Theorem I. Now that we have Theorem II, we need only show that if $\varphi(\theta) = 0$ for some θ, then T is not invertible. We use the fact that the invertible elements of a Banach algebra form an open set. (See, for example, Loomis [1], Theorem 22B.) It suffices therefore to show that T is the limit of noninvertible Toeplitz matrices. Note that if T_ψ denotes the Toeplitz matrix of $\psi(\theta) = \sum_{-\infty}^\infty d_k e^{ik\theta}$, then $\|T_\psi\| = \sum |d_k|$. We may assume, without loss of generality, that $\varphi(0) = 0$. We make first some additional assumptions concerning φ, these being removed in stages.

(A) We assume, in addition to $\varphi(0) = 0$, that (i) $\varphi(\theta) = 0$ only for $\theta = 0$; (ii) $\varphi'(0) \neq 0$; (iii) for some $R_0 > 1$ the series $\sum_{-\infty}^\infty |c_k| R^{|k|}$ converges for $0 \leqq R < R_0$. Then the function $\Phi(z) = \sum_{-\infty}^\infty c_k z^k$ is analytic in the annulus $R_0^{-1} < |z| < R_0$, $\Phi(z)$ has exactly one zero on $|z| = 1$, this being at $z = 1$, and $\Phi'(1) \neq 0$. Choose $r > 0$ so small that firstly $r < 1 - R_0^{-1}$, and secondly that $\Phi(z)$ has no zero inside or on the circle C: $|z - 1| = r$ except for the one at $z = 1$. Set

$$\delta = \min\,(\min_C |\Phi(z)|, \min_{\{|z|=1\}\cap\{|z-1|\geqq r\}} |\Phi(z)|).$$

Let $\varepsilon > 0$ be so small that $\varepsilon < r$, $|\Phi(1 \pm \varepsilon)| < \delta$. Then the functions

$$\Phi_\varepsilon(z) = \Phi(z) - \Phi(1 + \varepsilon) \quad\text{and}\quad \Phi_{-\varepsilon}(z) = \Phi(z) - \Phi(1 - \varepsilon)$$

have, firstly, no zeros on $\{|z| = 1 \cap |z - 1| \geqq r\}$, secondly, no zeros on C, and thirdly (by Rouche's theorem), exactly one (of multiplicity one) inside C, the zero of $\Phi_\varepsilon(z)$ being of course at $1 + \varepsilon$ and that of $\Phi_{-\varepsilon}(z)$ at $1 - \varepsilon$. In particular, Φ_ε and $\Phi_{-\varepsilon}$ are not zero on $|z| = 1$. We claim

$$(5) \qquad \Delta_{|z|=1} \arg \Phi_{-\varepsilon}(z) - \Delta_{|z|=1} \arg \Phi_\varepsilon(z) = 2\pi,$$

the unit circle being traversed in the positive direction. The left side of (5), being continuous in ε and always an integral multiple of 2π, must be a constant. It suffices therefore to show that its limit as $\varepsilon \to 0+$ is 2π. We have

$$\Delta_{\{|z|=1\}\cap\{|z-1|\geqq r\}} \arg \Phi_{-\varepsilon}(z) - \Delta_{\{|z|=1\}\cap\{|z-1|\geqq r\}} \arg \Phi_\varepsilon(z) = \Delta_{\{|z|=1\}\cap\{|z-1|\geqq r\}} \arg \frac{\Phi(z) - \Phi(1 - \varepsilon)}{\Phi(z) - \Phi(1 + \varepsilon)}$$

which tends to 0 as $\varepsilon \to 0+$. Let $C^+ = C \cap \{|z| \geqq 1\}$ and $C^- = C \cap \{|z| \leqq 1\}$, the directions on these arcs being that of the positive direction on C. Since the one zero of $\Phi_{-\varepsilon}(z)$ is inside $|z| = 1$, it has no zero between $|z| = 1$ and C^+, so

$$\Delta_{\{|z|=1\}\cap\{|z-1|\leqq r\}} \arg \Phi_{-\varepsilon}(z) = \Delta_{C^+} \arg \Phi_{-\varepsilon}(z) \to \Delta_{C^+} \arg \Phi(z)$$

as $\varepsilon \to 0+$. Similarly,

$$\Delta_{\{|z|=1\}\cap\{|z-1|\leqq r\}} \arg \Phi_\varepsilon(z) = -\Delta_{C^-} \arg \Phi_\varepsilon(z) \to -\Delta_{C^-} \arg \Phi(z)$$

as $\varepsilon \to 0+$. Thus

$$\lim_{\varepsilon\to 0+}(\Delta_{|z|=1} \arg \Phi_{-\varepsilon}(z) - \Delta_{|z|=1} \arg \Phi_\varepsilon(z)) = \Delta_C \arg \Phi(z) = 2\pi,$$

and (5) is established.

Let $\varphi_\varepsilon(\theta) = \Phi_\varepsilon(e^{i\theta})$, $\varphi_{-\varepsilon}(\theta) = \Phi_{-\varepsilon}(e^{i\theta})$. Then $\varphi_{\pm\varepsilon}(\theta) \neq 0$, and, by (5), $I(\varphi_\varepsilon) \neq I(\varphi_{-\varepsilon})$. Thus at least one of $I(\varphi_\varepsilon)$, $I(\varphi_{-\varepsilon})$ is not zero. If, to be specific, $I(\varphi_\varepsilon) \neq 0$, we know from Theorem II that T_{φ_ε} is not invertible. Since $\| T_\varphi - T_{\varphi_\varepsilon} \| = |\Phi(1 + \varepsilon)| \to 0$ as $\varepsilon \to 0$, we conclude that T $(= T_\varphi)$ is not invertible.

(B) We now drop the restriction (i) of case (A), keeping (ii) and (iii). Now $\Phi(z)$ may have zeros on $|z| = 1$ other than the simple zero at $z = 1$. Denote these other zeros by α_k, the corresponding multiplicities being m_k. Then

$$\Phi(z) = (z - 1)\prod(z - \alpha_k)^{m_k}\Psi(z)$$

where $\Psi(z) \neq 0$ on $|z| = 1$. If $0 < r < 1$,

$$\Phi_r(z) = (z - 1)\prod(z - r\alpha_k)^{m_k}\Psi(z),$$

then $\varphi_r(\theta) = \Phi_r(e^{i\theta})$ satisfies the conditions of (A), so that T_{φ_r} is not invertible. Moreover, since $\Phi_r(z) \to \Phi(z)$ as $r \to 1-$ uniformly in an annulus around $|z| = 1$, it is easily seen that $\| T_\varphi - T_{\varphi_r} \| \to 0$, so T is not invertible.

(C) We drop restriction (ii), keeping only (iii). If $\varphi'(0) = 0$, then for $\varepsilon \neq 0$, $\varphi_\varepsilon(\theta) = \varphi(\theta) + \varepsilon \sin \theta$ satisfies (ii) and (iii), so by (B), T_{φ_ε} is not invertible. Moreover $\| T_\varphi - T_{\varphi_\varepsilon} \| = \varepsilon$. Therefore T is not invertible.

(D) Finally we drop (iii). For $0 < r < 1$, set

$$\varphi_r(\theta) = \sum_{-\infty}^{\infty} c_k r^{|k|}(e^{ik\theta} - 1).$$

Then $\varphi_r(0) = 0$, and (iii) is satisfied with $R_0 = r^{-1}$. We have

$$\| T_\varphi - T_{\varphi_r} \| \leqq \sum\nolimits'^{\infty}_{-\infty} | c_k(1 - r^{|k|}) | + | c_0 + \sum\nolimits'^{\infty}_{-\infty} c_k r^{|k|} |,$$

where the prime means that the term corresponding to $k = 0$ is to be omitted. The first sum certainly tends to 0 as $r \to 1-$, and so also does the second term since $\sum_{-\infty}^{\infty} c_k = 0$.

This completes the proof of Theorem I.

THEOREM III. *Assume* $\varphi(\theta) \neq 0$, $I(\varphi) = 0$, *and let* $T_{j,k}^{-1}$ *denote the* j, k *entry of the matrix* T^{-1}. *Then*

$$(6)\qquad \begin{aligned}&(1 - st) \sum_{j,k=0}^{\infty} s^j t^k T_{j,k}^{-1} \\ &= \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - st}{1 - se^{-i\theta} - te^{i\theta} + st} \log \varphi(\theta)\, d\theta\right), \qquad |s|, |t| < 1.\end{aligned}$$

Proof. We introduce the analytic functions

$$F_+(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad \Phi_+(z) = \exp(F_+(z)), \qquad |z| < 1,$$
$$F_-(z) = \sum_{k=-\infty}^{-1} a_k z^k, \qquad \Phi_-(z) = \exp(F_-(z)), \qquad |z| > 1.$$

We have

$$(7)\qquad \begin{aligned}\sum_{k=0}^{\infty} t^k T_{j,k}^{-1} &= j^{\text{th}} \text{ component of } T^{-1}\{1, t, t^2, \cdots\} \\ &= j^{\text{th}} \text{ component of } M_{\varphi_+^{-1}} P M_{\varphi_-^{-1}}\{1, t, t^2, \cdots\}.\end{aligned}$$

Now $M_{\varphi_-^{-1}}\{1, t, t^2, \cdots\}$ is the sequence of Fourier coefficients of

$$\frac{\varphi_-(\theta)^{-1}}{1 - te^{i\theta}},$$

so $PM_{\varphi_-^{-1}}\{1, t, t^2, \cdots\}$ is the sequence of Fourier coefficients of

$$\begin{aligned}\sum_{j=0}^{\infty} e^{ij\theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\varphi_-(\theta')^{-1}}{1 - te^{i\theta'}} e^{-ij\theta'}\, d\theta' &= \sum_{j=0}^{\infty} e^{ij\theta} \frac{1}{2\pi i}\int_{|z|=1} \frac{\Phi_-(z)^{-1}}{1 - tz} \frac{dz}{z^{j+1}} \\ &= \sum_{j=0}^{\infty} e^{ij\theta} \Phi_-(t^{-1})^{-1} t^j = \frac{\Phi_-(t^{-1})^{-1}}{1 - te^{i\theta}}.\end{aligned}$$

Therefore $M_{\varphi_+^{-1}} PM_{\varphi_-^{-1}}\{1, t, t^2, \cdots\}$ is the sequence of Fourier coefficients of

$$\Phi_-(t^{-1})^{-1} \frac{\varphi_+(\theta)^{-1}}{1 - te^{i\theta}}.$$

Consequently (7) gives

$$\begin{aligned}\sum_{j,k=0}^{\infty} s^j t^k T_{j,k}^{-1} &= \Phi_-(t^{-1})^{-1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\varphi_+(\theta)^{-1}}{(1 - te^{i\theta})(1 - se^{i\theta})}\, d\theta \\ &= \Phi_-(t^{-1})^{-1} \frac{1}{2\pi i}\int_{|z|=1} \frac{\Phi_+(z)^{-1}}{(1 - tz)(z - s)}\, dz \\ &= \frac{\Phi_-(t^{-1})^{-1}\Phi_+(s)^{-1}}{1 - st}.\end{aligned}$$

It remains to equate $\Phi_-(t^{-1})^{-1}\Phi_+(s)^{-1}$ with the expression on the right of (6). But this is easy since

$$F_+(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log \varphi(\theta)}{1 - ze^{-i\theta}}\, d\theta, \qquad |z| < 1,$$

(8)

$$F_-(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\log \varphi(\theta)}{ze^{-i\theta} - 1}\, d\theta, \qquad |z| > 1.$$

We should like to point out that in case T is symmetric, i.e., $\varphi(\theta)$ is even, the right side of (6) may be written in a different form. We have, in this circumstance,

$$\begin{aligned} F_+(s) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - s\cos\theta}{1 - 2s\cos\theta + s^2}\log\varphi(\theta)\, d\theta \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\log\varphi(\theta)\, d\theta + \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{1 - s^2}{1 - 2s\cos\theta + s^2}\log\varphi(\theta)\, d\theta, \\ F_-(t^{-1}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t\cos\theta - t^2}{1 - 2t\cos\theta + t^2}\log\varphi(\theta)\, d\theta \\ &= -\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\varphi(\theta)\, d\theta + \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{1 - t^2}{1 - 2t\cos\theta + t^2}\log\varphi(\theta)\, d\theta, \end{aligned}$$

and so

$$\Phi_-(t^{-1})^{-1}\Phi_+(s)^{-1} = \{G(s, \varphi)G(t, \varphi)\}^{-1/2},$$

where

$$G(r, \varphi) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1 - r^2}{1 - 2r\cos\theta + r^2}\log\varphi(\theta)\, d\theta\right).$$

3. The l_2^+ theory

Here we drop the assumption $\sum |c_k| < \infty$, replacing it by

$$\varphi(\theta) \in L_\infty(-\pi, \pi).$$

The Toeplitz matrix T may now be considered a bounded operator on the space l_2^+ of square summable sequences $\{x_0, x_1, \cdots\}$. Our results here go in only one direction; we find a sufficient condition for the invertibility of T.

THEOREM IV. *Assume* $\varphi(\theta)^{-1} \in L_\infty(-\pi, \pi)$, *and that there exists a determination of* arg φ (*belonging to* $L_2(-\pi, \pi)$) *whose conjugate function*

$$\frac{1}{2\pi}\,\mathrm{PV}\int_{-\pi}^{\pi}\cot\tfrac{1}{2}(\theta - \theta')\arg\varphi(\theta')\, d\theta'$$

belongs to $L_\infty(-\pi, \pi)$. *Then T is an invertible operator on* l_2^+, *and the entries of the matrix* T^{-1} *are determined by* (6).

Proof. With $\log\varphi$ determined by the arg φ in the hypothesis, we define the functions $f_\pm(\theta)$, $\varphi_\pm(\theta)$ by (1)–(3), the series converging in L_2. If we can show that the functions $\varphi_\pm(\theta)$, $\varphi_\pm(\theta)^{-1}$ are in $L_\infty(-\pi, \pi)$, the proofs of the relevant part of Theorem I and of Theorem III can be modified for the present situation; all we need do is replace ∞ by 2 on occasion.

The required boundedness is equivalent to the boundedness of $\mathcal{R}f_+(\theta)$ and $\mathcal{R}f_-(\theta)$; we shall give the proof for $\mathcal{R}f_+$, the proof for $\mathcal{R}f_-$ being analogous. We have, almost everywhere,

$$f_+(\theta) = \lim_{r\to 1-} \sum_{k=0}^{\infty} a_k r^k e^{ik\theta} = \lim_{r\to 1-} F_+(re^{i\theta}),$$

where, recall, $F_+(z)$ is given by (8). A simple computation gives

$$\mathcal{R}F_+(re^{i\theta}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log |\varphi(\theta')| \, d\theta' + \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2} \log|\varphi(\theta')|\, d\theta' - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r\sin\theta'}{1-2r\cos(\theta-\theta')+r^2} \arg\varphi(\theta')\, d\theta'.$$

The second integral converges almost everywhere to $\frac{1}{2}\log|\varphi(\theta)|$, and the third to

$$\frac{1}{2\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} \cot\tfrac{1}{2}(\theta-\theta') \arg\varphi(\theta')\, d\theta'$$

(see, for example, [3, §3.34]). Therefore, almost everywhere,

$$\mathcal{R}f_+(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log|\varphi(\theta')|\, d\theta' + \tfrac{1}{2}\log|\varphi(\theta)| - \frac{1}{2\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} \cot\tfrac{1}{2}(\theta-\theta')\arg\varphi(\theta')\, d\theta',$$

and the result is clear.

COROLLARY. *The matrix T is invertible under any of the following conditions*:

(i) $\sum_{-\infty}^{\infty} |c_k| < \infty$, $\varphi(\theta) \neq 0$, *and* $I(\varphi) = 0$;

(ii) $c_k = 0$ *for* $k < 0$ *and the function* $\Phi(z) = \sum_0^{\infty} c_k z^k$ (*defined for* $|z| < 1$ *and almost everywhere on* $|z| = 1$) *is* (*essentially*) *bounded away from zero on* $|z| \leqq 1$;

(ii′) $c_k = 0$ *for* $k > 0$ *and the function* $\Phi(z) = \sum_{-\infty}^{0} c_k z^k$ (*defined for* $|z| > 1$ *and almost everywhere on* $|z| = 1$) *is* (*essentially*) *bounded away from zero on* $|z| \geqq 1$;

(iii) $\varphi(\theta)$ *is real* (*i.e.*, $c_{-k} = \bar{c}_k$), *and either* $\varphi(\theta) \geqq m > 0$ *or* $\varphi(\theta) \leqq -m < 0$.

In the cases (i)–(ii′) it is easier to verify directly the boundedness of $\varphi_+(\theta)$, $\varphi_+(\theta)^{-1}$ (for suitable choice of $\log\varphi$) than to use the criterion of the theorem. In case (i), the lemma of Section 2 shows that the Fourier series for any continuous $\log\varphi$ converges absolutely, so the functions $f_\pm(\theta)$ are bounded. In case (ii) we have, clearly, $\varphi_+(\theta) = \Phi(e^{i\theta})$, $\varphi_-(\theta) = 1$, and the required boundedness is immediate. Case (ii′) is similar. (It follows easily from Theorem I that in cases (ii) and (ii′) we have $T^{-1} = T_{\varphi^{-1}}$.) In case (iii) we may take $\arg \varphi \equiv 0$ or $\equiv \pi$, and the invertibility follows from Theorem IV.

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