PRINCIPAL QUASIFIBRATIONS AND FIBRE HOMOTOPY EQUIVALENCE OF BUNDLES[1]

BY

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Introduction

Let H be a topological space with a continuous multiplication (an $\mathfrak{H}$-space) which is associative and has a two-sided unit e. In analogy to the case of a topological group we construct a universal principal quasifibration (= q.f.) $\mathfrak{E}_H = (E_H, p_H, B_H, H)$ with fibre H. As an application we get a classification of fibre bundles with respect to fibre homotopy equivalence (see 7.6).

The universal q.f. $\mathfrak{E}_H$ is obtained by iteration of a construction which is described in §2. This construction applies to any q.f. $p:E \to B$ (see §1) in which a (not necessarily associative) $\mathfrak{H}$-space H operates (see Definition 2.2). In a functorial way it embeds such a q.f. into a bigger q.f.

$$\begin{array}{ccc} & f & \\ E & \subset & \hat{E} \\ p\downarrow & & \downarrow\hat{p} \\ B & \subset & \hat{B} \end{array}$$

such that the inclusion f is nullhomotopic (see 2.3). In particular there exists always a q.f. whose fibre is homeomorphic with H and is contractible to a point in the total space; we have only to begin with the fibration $H \to P$ which sends all of H into a point P, and in which H operates by right translations.

We speak of a principal q.f. $\mathfrak{E} = (E, p, B, H)$ (see 3.1) if H is associative and $p:E \to B$ is a q.f. in which H operates such that $(yh)h' = y(hh')$ $(y \in E, h, h' \in H)$. Applying the construction of §2 to a principal q.f. $\mathfrak{E}$ gives a principal q.f. $\hat{\mathfrak{E}}$. Iteration gives a sequence $\mathfrak{E}_{n+1} = \hat{\mathfrak{E}}_n$ of principal q.f.s together with inclusion maps. By taking the limit (in a proper way; see 3.4) of this sequence one obtains a principal q.f. $\mathfrak{E}_\infty = (E_\infty, p_\infty, B_\infty, H)$ which is universal in the sense that all homotopy groups of E_∞ vanish (see 3.5). In particular there exists always a universal principal q.f. with fibre H; as above, one has only to start with a fibration $H \to P$ (= a point).

If $H = G$ is a topological group and we begin our construction with a principal bundle $\mathfrak{E}$ in the sense of Steenrod [6], then $\hat{\mathfrak{E}}$ and $\mathfrak{E}_\infty$ are principal bundles (see 4.1), and our construction coincides (see 4.2) essentially with Milnor's construction in [4].

In §§4–5 universal principal q.f.s are used for a partial homotopy classifi-

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cation of principal H-bundles (= locally trivial principal q.f.s; see 5.1). Two such bundles $\mathfrak{E}^i = \{E^i, p^i, B, H\}$, $i = 1, 2$, are homotopy equivalent (see 6.1) if there exist principal maps (see 3.1) $f:E^1 \to E^2$, $f^-:E^2 \to E^1$ such that the composites ff^-, f^-f are homotopic to the respective identity maps, by principal homotopies (see 3.1) which leave the fibres fixed (as a whole). If B is a polyhedron, then $\mathfrak{E}^1$ and $\mathfrak{E}^2$ are already homotopy equivalent if there exists a principal map in one direction, say $E^1 \to E^2$, which lies over the identity of the base (see 5.3). The classification theorem reads (see 6.2): Let $\mathfrak{E}_H = (E_H, p_H, B_H, H)$ be a universal principal q.f., and let B be a polyhedron. (1) There exist principal maps $g^i:E^i \to E_H$; (2) if $g^i:E^i \to E_H$ are principal maps, then $\mathfrak{E}^1$ and $\mathfrak{E}^2$ are homotopy equivalent if and only if the maps $\bar{g}^i:B \to B_H$, induced by g^i, are homotopic.

The usual notion of fibre homotopy equivalence (see [8], IV or 7.1) can be reduced to homotopy equivalence of H-bundles: Let $\mathfrak{E} = \{E, p, B, F\}$ be a fibre bundle with locally compact fibre F, and let H be the space of all homotopy equivalences $F \to F$. Define the associated principal H-bundle $\tilde{\mathfrak{E}} = \{\tilde{E}, \tilde{p}, B, H\}$ as follows (see 7.2): The fibre $\tilde{p}^{-1}(x)$ is the space of all homotopy equivalences $\varphi:F \to p^{-1}(x)$; the operation of H in $\tilde{E}$ is defined by composition $\varphi h = \varphi \circ h$ ($\varphi \in \tilde{E}$, $x \in B$, $h \in H$). Then two bundles $\mathfrak{E}^i$, $i = 1, 2$, are fibre homotopy equivalent if and only if $\tilde{\mathfrak{E}}^1$ and $\tilde{\mathfrak{E}}^2$ are homotopy equivalent (as H-bundles; see 7.4).

For bundles $\mathfrak{E} = \{E, p, B, F, G\}$ with topological structure group (see [6]) the classification with respect to fibre homotopy equivalence "factors" through the usual classification with respect to ordinary equivalence as follows: Let $\mathfrak{E}_G = \{E_G, p_G, B_G, G, G\}$ resp. $\mathfrak{E}_H = (E_H, p_H, B_H, H)$ be the universal principal G-bundle resp. q.f. as in §§3–4 (H the space of homotopy equivalences $F \to F$). There is a natural homomorphism $G \to H$ which induces a homomorphism (see 3.7–3.10)

$$\begin{array}{ccc} E_G & \xrightarrow{\xi} & E_H \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_H} \\ B_G & \xrightarrow[\bar{\xi}]{} & B_H . \end{array}$$

Now, the equivalence classes of bundles $\mathfrak{E}$ over B are in one-to-one correspondence with homotopy classes of maps $B \to B_G$ (see [6], 19). Two bundles $\mathfrak{E}^1$, $\mathfrak{E}^2$ which correspond to the homotopy classes of $\gamma^1, \gamma^2:B \to B_G$ are fibre homotopy equivalent if and only if $\bar{\xi}\gamma^1$, $\bar{\xi}\gamma^2$ are homotopic (see 7.5).

This shows that the image under $\bar{\xi}^*:H^*(B_H) \to H^*(B_G)$ consists of characteristic classes which are invariant under fibre homotopy equivalence.

1. Preliminaries on quasifibrations

We recall the definition and some properties of quasifibrations (see [3]).

1.1 DEFINITION. Let E, B be topological spaces. A continuous map $p: E \to B$ *onto* B is a *quasifibration* (= q.f.) if

$$(1) \qquad p_*: \pi_i(E, p^{-1}(x), y) \cong \pi_i(B, x) \qquad \text{for all } x \in B,\ y \in p^{-1}(x), \text{ and } i \geqq 0.$$

For $i = 0, 1$ this means that we have an isomorphism between sets with distinguished elements (see [3], 1.2). We define a group structure on $\pi_1(E, p^{-1}(x))$ by the requirement that (1) (for $i = 1$) should be an isomorphism of groups. E, p, B, $p^{-1}(x)$ in this order are the *total space*, the *projection*, the *base*, the *fibre* over x of the q.f.

As in the case of fibre bundles the isomorphisms (1) lead to the exact homotopy sequence of a q.f. (see [3], 1.4)

$$(2) \qquad \cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}(x)) \to \pi_i(E) \to \pi_i(B) \to \cdots .$$

If in a q.f. the base is arcwise connected, then any two fibres are of the same weak homotopy type (see [3], 1.10).

1.2 DEFINITION. Let $p: E \to B$, $p': E' \to B'$ be q.f.s. A map $f: E \to E'$ is called *fibrewise* if there exists a (continuous) application $\bar{f}: B \to B'$ such that commutativity holds in

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\bar{f}"'] & B'. \end{tikzcd}$$

We say $\bar{f}$ is *induced* by f or f *lies over* $\bar{f}$.

A fibrewise map induces a homomorphism of the exact homotopy sequence of p into that of p' (see [3], 1.8). If $p: E \to B$ is a continuous map, we call a subset $A \subset B$ *distinguished* with respect to p if $p_A: p^{-1}(A) \to A$, the restriction of p to $p^{-1}(A)$, is a q.f. Then we have the following criteria.

1.3 LEMMA (see [3], 2.10). *Let $p: E \to B$ be a continuous map onto B, let $B' \subset B$ be a distinguished set, and put $E' = p^{-1}(B')$. Assume there is a "fibre-preserving" deformation of E into E', i.e., there are deformations*

$$D_t: E \to E, \qquad d_t: B \to B \qquad (t \in [0, 1])$$

with

$$D_0 = \mathrm{id}, \qquad D_t(E') \subset E', \qquad D_1(E) \subset E' \qquad (\mathrm{id} = \textit{identity map}),$$

$$d_0 = \mathrm{id}, \qquad d_t(B') \subset B', \qquad d_1(B) \subset B', \quad \textit{and} \quad pD_1 = d_1 p.$$

Assume further

$$D_{1*}:\pi_i(p^{-1}(x)) \cong \pi_i(p^{-1}(d_1(x))), \qquad \text{all } x \in B \text{ and } i \geqq 0.$$

Then B itself is distinguished, i.e., p is a q.f.

1.4 LEMMA ([3], 2.2). *Let $p:E \to B$ be a continuous map, and let $U, V \subset B$ be open sets. If U, V, and $U \cap V$ are distinguished with respect to p, then $U \cup V$ is distinguished.*

1.5 LEMMA ([3], 2.15). *Let $p:E \to B$ be a continuous map. Assume that B is the inductive limit of a sequence of subspaces $B_1 \subset B_2 \subset \cdots \subset B$, satisfying the first separation axiom (points are closed), and each B_ν is distinguished with respect to p. Then p is a q.f.*

2. The basic construction

Every q.f. $E \to B$ in which an $\mathfrak{H}$-space H operates (see Definition 2.2) is embedded in a q.f. $\hat{E} \to \hat{B}$ such that E is contractible to a point in $\hat{E}$. This is done by suitably attaching $CE \times H$ to E where CE is the cone over E.

2.1 DEFINITION (see [5], IV, 1). An *$\mathfrak{H}$-space* is a topological space H together with a continuous multiplication

$$H \times H \to H, \qquad (h, h') \to hh'$$

with two-sided unit e. We also require that the left translations

$$L_{h'}:H \to H, \qquad L_{h'}(h) = h'h$$

induce isomorphisms of all homotopy groups. (If H is arcwise connected, this follows from the existence of a unit.)

2.2 DEFINITION. Let $p:E \to B$ be a q.f., and H an $\mathfrak{H}$-space. An *operation* of H in this q.f. is a continuous map

$$\mu:E \times H \to E, \qquad \mu(y, h) = yh \qquad y \in E, \quad h \in H$$

such that

(1) $ye = y$

(2) $\mu(y \times H) \subset F_y = p^{-1}(p(y)) =$ fibre through y.

Define

(3)
$$\mu_y:H \to F_y, \qquad \mu_y(h) = yh$$
$$\mu_{y*}:\pi_i(H) \cong \pi_i(F_y) \qquad \text{for all } y \text{ and all } i \geqq 0.$$

This is obviously a generalization of the notion of a principal bundle. The word "principal" is reserved, however, for the case of an associative $\mathfrak{H}$-space. Note that we do not require $y(hh') = (yh)h'$.

Given a q.f. $p:E \to B$ in which H operates we shall embed it in a q.f. $\hat{p}:\hat{E} \to \hat{B}$ such that the inclusion map $E \subset \hat{E}$ is nullhomotopic. Roughly speaking our construction (patterned after Milnor [4]) runs as follows. Consider the diagram

$$\begin{array}{ccccc} E & \xleftarrow{\mu} & E \times H & \subset & CE \times H \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xleftarrow[p]{} & E & \subset & CE \end{array}$$

where CE is the cone over E and E is considered as a subspace (base) of CE. The map q is the natural projection, and μ is the operation of H in E. Attach $CE \times H$ to E and CE to B by the maps μ resp. p, i.e., form the topological sum $E + CE \times H$, and identify $(y, h) \in E \times H \subset CE \times H$ with $\mu(y, h) = yh \in E$; similarly form $B + CE$, and identify $y \in E \subset CE$ with $p(y) \in B$. Except for a slightly different (stronger) topology these quotients are $\hat{E}$ resp. $\hat{B}$, and $\hat{p}$ is induced by p resp. q.

More precisely we proceed as follows. A point of $\hat{E}$ is described by

(1) a real number t with $0 \leqq t \leqq 1$,

(2) a point $y \in E$ whenever $t \neq 0$,

(3) a point $h \in H$ whenever $t \neq 1$.

We denote this point by $y \mid t \mid h$ where y resp. h is omitted if $t = 0$ resp. 1. We use the following conventions:

$$y \mid 0 \mid h = 0 \mid h, \qquad y \mid 1 \mid h = yh \mid 1. \tag{4}$$

The topology in $\hat{E}$ is the strongest topology such that the "coordinate functions"

$$\begin{aligned} t&:\hat{E} \to [0, 1], & y \mid t \mid h &\to t, \\ yh&:t^{-1}(0, 1] \to E, & y \mid t \mid h &\to yh, \\ h&:t^{-1}[0, 1) \to H, & y \mid t \mid h &\to h, \\ y&:t^{-1}(0, 1) \to E, & y \mid t \mid h &\to y \end{aligned} \tag{5}$$

are continuous. I.e., taking counterimages with respect to these mappings of open sets in $[0, 1]$, E, H gives a subbasis for the open sets in $\hat{E}$. Therefore an application of a topological space into $\hat{E}$ will be continuous if and only if its compositions with the coordinate maps are continuous (where defined).

Define the *inclusion*

$$f:E \to \hat{E}, \qquad f(y) = y \mid 1 = y \mid 1 \mid e \qquad (y \in E). \tag{6}$$

The composition of f with the coordinate maps is clearly continuous; therefore f is continuous. The composition with the coordinate map yh is the identity map of E; therefore f is an inclusion map.

A *nullhomotopy* of f is given by

(7) $$E \times I \to \hat{E}, \qquad (y, t) \to y \mid t \mid e;$$

clearly $(y, 1) \to f(y)$ and $E \times 0 \to 0 \mid e$, a point.

We may also consider H as a subspace (fibre) of $\hat{E}$, using the inclusion

(8) $$i:H \to \hat{E}, \qquad i(h) = 0 \mid h,$$

or more generally

(8′) $$i_{y,t}:H \to \hat{E}, \qquad i_{y,t}(h) = y \mid t \mid h \qquad (t \neq 1).$$

(This time the composition with the coordinate map h gives the identity map of H.)

We now define $\hat{B}$ and the projection $\hat{p}:\hat{E} \to \hat{B}$. A point in $\hat{B}$ is denoted by a symbol $y \perp t$ where $y \in E$, $t \in [0, 1]$. The symbols $y \perp t$ and $y' \perp t'$ give the same point of $\hat{B}$ if

either (a) $t = t' = 0$; this point is denoted by $\bar{0}$;
or (b) $t = t' = 1$ and $p(y) = p(y')$.

The topology in $\hat{B}$ is the strongest topology such that the coordinate maps

(9) $$\begin{aligned} t:\hat{B} &\to [0, 1], & y \perp t &\to t, \\ y:t^{-1}(0, 1) &\to E, & y \perp t &\to y, \\ p(y):t^{-1}(0, 1] &\to B, & y \perp t &\to p(y) \end{aligned}$$

are continuous.

The *projection*

(10) $$\hat{p}:\hat{E} \to \hat{B}, \qquad p(y \mid t \mid h) = y \perp t$$

is continuous because its composition with the coordinate functions is continuous; the composite map $y \mid t \mid h \xrightarrow{\hat{p}} y \perp t \to p(y)$, for instance, is the same as the composition $y \mid t \mid h \to yh \xrightarrow{p} p(yh) = p(y)$, and hence is continuous. Let

(11) $$\bar{f}:B \to \hat{B}, \qquad \bar{f}(\bar{y}) = y \perp 1 \qquad \text{where } \bar{y} \in B \text{ and } y \in p^{-1}(\bar{y}).$$

As for f it follows that $\bar{f}$ is an inclusion map. From the definition it is clear that

(12) $$E = f(E) = \hat{p}^{-1}(\bar{f}(B)).$$

The main properties of $\hat{p}:\hat{E} \to \hat{B}$ are stated in the following

2.3 PROPOSITION. *The map $\hat{p}:\hat{E} \to \hat{B}$ is a q.f. There is a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{f} & \hat{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \hat{p}} \\ B & \xrightarrow[\bar{f}]{} & \hat{B} \end{array} \qquad (\textit{i.e., } f \textit{ is fibrewise})$$

where f, $\bar{f}$ are inclusion maps, f is nullhomotopic, and $f(E) = \hat{p}^{-1}(\bar{f}(B))$.

2.4 COROLLARY. *Every $\mathfrak{H}$-space H is fibre of a q.f. such that H is contractible to a point in the total space.*

Indeed, the projection $p:H \to P$ of H onto a point is a q.f. in which H operates (by right translations). Therefore $\hat{p}:\hat{H} \to \hat{P}$ is a q.f. in which the fibre H is contractible.

If we remark that $\hat{H}$ is essentially the join of H with itself, then 2.4 is essentially [7], Theorem 4.

Proof of 2.3. What remains to be proved is that $\hat{p}$ is a q.f. This follows from Lemma 1.4 if we show that the following open sets in $\hat{B}$ are distinguished:

(a) $U = t^{-1}(0, 1] = \{y \perp t \mid t > 0\}$

(b) $V = t^{-1}[0, 1) = \{y \perp t \mid t < 1\}$

(c) $W = t^{-1}(0, 1) = \{y \perp t \mid 0 < t < 1\} = U \cap V$.

Cases (b) *and* (c). We show that over V we have the product with H. Define

$$V \times H \to \hat{p}^{-1}(V), \qquad (y \perp t, h) \to y \mid t \mid h,$$
$$\hat{p}^{-1}(V) \to V \times H, \qquad y \mid t \mid h \to (y \perp t, h), \qquad y \in E, \quad t \in I, \quad h \in H.$$

These are reciprocal homeomorphisms which transform $\hat{p}$ into the natural projection $V \times H \to V$. Therefore V is distinguished, as well as every subset of V.

Case (a). Deform $\hat{p}^{-1}(U)$ into $E = \hat{p}^{-1}(\bar{f}(B))$ by

$$D_\tau:\hat{p}^{-1}(U) \to \hat{p}^{-1}(U), \qquad D_\tau(y \mid t \mid h) = y \mid t + \tau(1 - t) \mid h, \qquad 0 \leqq \tau \leqq 1. \tag{13}$$

Clearly $D_0 = \mathrm{id}$, $D_1(\hat{p}^{-1}(U)) = E$, $D_\tau \mid E = \mathrm{id}$.

A similar deformation in the base

$$d_\tau: U \to U, \qquad d_\tau(y \perp t) = y \perp t + \tau(1 - t) \tag{14}$$

shrinks U into B ($d_0 = \mathrm{id}$, $d_1(U) = B$, $d_\tau \mid B = \mathrm{id}$), and

$$\hat{p} D_\tau = d_\tau \hat{p}. \tag{15}$$

The mapping of the fibre $\hat{p}^{-1}(y \perp t)$ into $\hat{p}^{-1}(y \perp 1) = p^{-1}(p(y))$ under D_1 is given by $y \mid t \mid h \to y \mid 1 \mid h = yh$, and hence is a weak homotopy equivalence by assumption (3) in 2.2. Therefore Lemma 1.3 shows that U is distinguished.

2.3 *Remark.* Results corresponding to Lemmas 2.1–2.3 in [4] can be proved for $\hat{E}$, in particular (see [4], 2.3): If E is m-connected and H is n-connected, then $\hat{E}$ is $(m + n + 2)$-connected. Since this is not needed for the applications, no proof is given.

3. Principal q.f. and universal q.f.

We define principal q.f.s over an associative $\mathfrak{H}$-space. If we apply the construction of §2 to such a q.f., we obtain again a principal q.f. Iteration of the construction leads to a universal principal q.f.; we thus obtain a functor from principal q.f.s to universal principal q.f.s.

3.1 DEFINITION. Let H be an associative $\mathfrak{H}$-space (i.e., $h(h'h'') = (hh')h''$ holds). A q.f. $p:E \to B$ together with an operation of H in $p:E \to B$ (see 2.2) is called *principal q.f. over H* if

$$(1) \qquad y(hh') = (yh)h' \qquad \text{for every } y \in E, \quad h, h' \in H$$

(i.e., H operates on the right in E). A principal q.f. is denoted by (E, p, B, H).

If $\mathfrak{E} = (E, p, B, H)$ and $\mathfrak{E}' = (E', p', B', H)$ are principal q.f.s then a *principal map* from $\mathfrak{E}$ to $\mathfrak{E}'$ is a fibrewise map $f:E \to E'$ which satisfies

$$f(yh) = f(y)h \qquad \text{for} \quad y \in E, \quad h \in H.$$

A *principal homotopy* is a continuous map $F:E \times I \to E'$ ($I = [0, 1]$) such that

$$F_t:E \to E', \qquad F_t(y) = F(y, t)$$

is a principal map for all $t \in I$.

3.2 PROPOSITION. *If $\mathfrak{E} = (E, p, B, H)$ is a principal q.f. over H, then the construction $\hat{p}:\hat{E} \to \hat{B}$ of §2 can be given a structure as principal q.f.*

$$\hat{\mathfrak{E}} = (\hat{E}, \hat{p}, \hat{B}, H)$$

such that

$$(2) \qquad (y \mid t \mid h)h' = y \mid t \mid hh', \qquad y \mid t \mid h \in \hat{E}, \quad h' \in H.$$

The operation of H on $\hat{E}$ extends the given operation on $E \subset \hat{E}$.

Proof. We define an application $\hat{\mu}:\hat{E} \times H \to \hat{E}$ by (2), i.e.,

$$(y \mid t \mid h, h') \to y \mid t \mid hh'.$$

Since different symbols $y \mid t \mid h$ may denote the same point, we have to show that (2) is nonambiguous. The only case where this is not obvious is for $t = 1$. But if $y_1 \mid 1 \mid h_1 = y_2 \mid 1 \mid h_2$, then $y_1 h_1 = y_2 h_2$; hence

$$y_1 \mid 1 \mid h_1 h' = y_1(h_1 h')| 1 = (y_1 h_1)h'| 1 = (y_2 h_2)h'| 1 = y_2 \mid 1 \mid h_2 h'.$$

Continuity of the operation $\hat{\mu}:\hat{E} \times H \to \hat{E}$ follows as usual by composition with coordinate maps. We have to verify condition (3) of 2.2. But in $E \subset \hat{E}$ the operation is the original one, and on every fibre in $\hat{E} - E$ it can be identified (by §2, (8')) with right translations of H. Finally we have

$$((y \mid t \mid h)h')h'' = y \mid t \mid (hh')h'' = y \mid t \mid h(h'h'') = (y \mid t \mid h)(h'h''),$$

i.e., we have a principal q.f.

3.3 Definition (see [6], 19). A principal q.f. $\mathfrak{E} = (E, p, B, H)$ is called *universal* if E is aspherical, i.e., $\pi_i(E) = 0$ for all $i \geqq 0$.

3.4 *Construction of a universal principal q.f.* Let $\mathfrak{E} = (E, p, B, H)$ be a principal q.f. over H. Define $\mathfrak{E}_0 = \mathfrak{E}$, $\mathfrak{E}_1 = \hat{\mathfrak{E}}_0, \cdots, \mathfrak{E}_{n+1} = \hat{\mathfrak{E}}_n, \cdots$, i.e., $\mathfrak{E}_{n+1} = (E_{n+1}, p_{n+1}, B_{n+1}, H)$ is the principal q.f. which is obtained from $\mathfrak{E}_n$ by applying the construction of §2 and Proposition 3.2. We have a sequence of principal maps

$$\begin{array}{ccccccc} E & \to & E_1 & \to & \cdots \to & E_n & \to \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} & & & \downarrow{\scriptstyle p_n} & \\ B & \to & B_1 & \to & \cdots \to & B_n & \to , \end{array} \tag{3}$$

and all horizontal maps are inclusions.

We define a limit $\mathfrak{E}_\infty = (E_\infty, p_\infty, B_\infty, H)$ as follows. The sets of E_∞ resp. B_∞ are the direct limits of the sets E_n resp. B_n with respect to the maps (3), i.e., the set E_∞ is the union of the increasing sequence $E \subset E_1 \subset E_2 \subset \cdots$, and similarly for B_∞.

The topology in B_∞ is also the limit topology (inductive limit)

$$B_\infty = \lim_\nu B_\nu , \tag{4}$$

i.e., a set $A \subset B_\infty$ is closed if and only if $A \cap B_\nu$ is closed in B_ν for all ν. An application $f: B_\infty \to X$ is continuous if and only if $f \mid B_\nu$ is continuous for all ν.

The topology in E_∞ will be stronger than the limit topology (because we want H to operate *continuously* in E_∞). Define the projection

$$p_\infty : E_\infty \to B_\infty \quad \text{by} \quad p_\infty \mid E_n = p_n . \tag{5}$$

Then we shall take the strongest topology for which p_∞ and some applications R^n, still to be defined, are continuous.

Define open sets

$$B_k^n \subset B_k , \qquad E_k^n = p_k^{-1}(B_k^n) \subset E_k$$

and continuous maps

$$r_k^n : B_k^n \to B_n , \qquad R_k^n : E_k^n \to E_n$$

as follows (induction on $k - n$):

(1) $k \leqq n$. $B_k^n = B_k$, $E_k^n = E_k$, r_k^n and R_k^n are the inclusion mappings $B_k \subset B_n$, $E_k \subset E_n$.

(2) $k = n + 1$. Let $B_{n+1}^n \subset B_{n+1}$ consist of all points $y \perp t$ ($y \in E_n$, $t \in I$) with $t > 0$, and define $r_{n+1}^n(y \perp t) = p_n(y)$. Similarly $E_{n+1}^n \subset E_{n+1}$ consists of all points $y \mid t \mid h$ with $t > 0$ and $R_{n+1}^n(y \mid t \mid h) = yh$.

(3) $k > n + 1$. We put

$$B_k^n = (r_k^{k-1})^{-1}(B_{k-1}^n), \qquad E_k^n = (R_k^{k-1})^{-1}(E_{k-1}^n),$$

$$r_k^n = r_{k-1}^n \, r_k^{k-1}, \qquad R_k^n = R_{k-1}^n \, R_k^{k-1}.$$

We clearly have

$$p_n R_k^n = r_k^n p_k \tag{6}$$

$$R_k^n(yh) = R_k^n(y)h, \qquad y \in E_k^n, \quad h \in H, \tag{7}$$

i.e., R_k^n commutes with the operation of H. Also, if we consider B_k, E_k as subspaces of B_{k+1}, E_{k+1}

$$\begin{aligned} B_{k+1}^n \cap B_k &= B_k^n, \qquad & E_{k+1}^n \cap E_k &= E_k^n, \\ r_{k+1}^n \mid B_k^n &= r_k^n, \qquad & R_{k+1}^n \mid E_k^n &= R_k^n. \end{aligned} \tag{8}$$

We can therefore define sets $B^n \subset B_\infty$, $E^n = p_\infty^{-1}(B^n) \subset E_\infty$, and applications $r^n : B^n \to B_n$, $R^n : E^n \to E_n$ by

$$\begin{aligned} B^n &= \cup_k B_k^n, \qquad & E^n &= \cup_k E_k^n, \\ r^n \mid B_k^n &= r_k^n, \qquad & R^n \mid E_k^n &= R_k^n, \end{aligned} \tag{9}$$

and B^n is open, r^n is continuous (by definition of the topology of B_∞).

Now define the *topology in* E_∞ to be the strongest topology for which p_∞ and all R^n are continuous.

3.5 THEOREM. *Let* (E, p, B, H) *be a principal q.f., and let* E, B, H *be* T_1*-spaces (points are closed). Then* $p_\infty : E_\infty \to B_\infty$ *together with the operation*

$$\mu_\infty : E_\infty \times H \to E_\infty, \qquad \mu_\infty \mid E_n \times H = \mu_n \quad (= \textit{operation of } H \textit{ in } E_n) \tag{10}$$

is a universal principal q.f. $(E_\infty, p_\infty, B_\infty, H)$.

Proof. We show first that the set-theoretical inclusion $i_m : E_m \subset E_\infty$ is a topological inclusion. It follows then that $B_m \subset B_\infty$ (see [3], 2.11) is a distinguished set with respect to p_∞, and by Lemma 1.5 that p_∞ is a q.f. (It is easily proved that the B_ν are T_1-spaces if E, B, and H are.) Then we show that μ_∞ is continuous, and finally that E_∞ is aspherical.

(a) $i_m : E_m \subset E$ *is an inclusion map.* In order to prove continuity we have to show that $p_\infty i_m$ and $R^n i_m$ are continuous (where defined). But $p_\infty i_m = p_m$ and $R^n i_m = R_m^n$ (see (9) and (8)). Since $R_m^m = \mathrm{id}$, this proves also that i_m is topological.

(b) $\mu_\infty : E_\infty \times H \to E_\infty$ *is continuous.* Again we have to show that $p_\infty \mu_\infty$ and $R^n \mu_\infty$ are continuous. But $p_\infty \mu_\infty$ is the same as the natural projection $E_\infty \times H \to E_\infty$ followed by p_∞, and $R^n \mu_\infty$ (where defined) is the same (see (9) and (7)) as the composition

$$E^n \times H \xrightarrow{R^n \times \mathrm{id}} E_n \times H \xrightarrow{\mu_n} E_n .$$

(c) $\pi_i(E_\infty) = 0$. If $f : K \to E_\infty$ is any continuous map and K is compact, then $p_\infty f : K \to B_\infty$ maps K into some B_n (see [3], 2.12); hence $f(K) \subset E_n$. Since E_n is contractible to a point in $E_{n+1} \subset E_\infty$, the map f is nullhomotopic. Applying this to maps of spheres gives the asphericity of E_∞.

3.6 *Remark.* There is a stronger and simpler topology in E_∞ for which 3.5 is also true (but not the results of §4): One takes as subbase for closed sets (i) the sets $p^{-1}(A)$ where $A \subset B_\infty$ is closed; (ii) the closed subsets of E_n (considered as subsets of E_∞).

With every "homomorphism" $f:E \to E'$ between principal q.f.s we associate a homomorphism $f_\infty:E_\infty \to E'_\infty$ of the corresponding universal principal q.f.s, thus turning this construction into a functor.

3.7 Definition. Let $\mathfrak{E} = (E, p, B, H)$ and $\mathfrak{E}' = (E', p', B', H')$ be principal qf..s. A *homomorphism* $\mathfrak{E} \to \mathfrak{E}'$ is a pair (f, η) where $\eta:H \to H'$ is a homomorphism of $\mathfrak{H}$-spaces (in particular $\eta(e) = e'$), and $f:E \to E'$ is a fibrewise map satisfying

$$f(yh) = f(y)\eta(h), \qquad y \in E, \quad h \in H.$$

3.8 *Construction of* f_∞. Let (f, η) be a homomorphism as in 3.7. Define a map

$$(11) \qquad \hat{f}:\hat{E} \to \hat{E}', \qquad \hat{f}(y \mid t \mid h) = f(y) \mid t \mid \eta(h), \qquad y \in E, \quad t \in I, \quad h \in H.$$

It is easily verified that $\hat{f}$ together with η is a homomorphism of principal q.f.s. It lies over

$$(12) \qquad \hat{\bar{f}}:\hat{B} \to \hat{B}', \qquad \hat{\bar{f}}(y \perp t) = f(y) \perp t, \qquad y \in E, \quad t \in I.$$

Repeating the construction we obtain homomorphisms

$$f_n:E_n \to E'_n, \qquad f_n = \hat{f}_{n-1},$$

which fit together (i.e., $f_n \mid E_{n-1} = f_{n-1}$); we can therefore define an application

$$f_\infty:E_\infty \to E'_\infty, \qquad f_\infty \mid E_n = f_n .$$

3.9 Proposition. *Let $(f, \eta):(E, p, B, H) \to (E', p', B', H')$ be a homomorphism between principal q.f.s (see 3.7). Then (f_∞, η) is a homomorphism $(E_\infty, p_\infty, B_\infty, H) \to (E'_\infty, p'_\infty, B'_\infty, H')$. If*

$$(f', \eta'):(E', p', B', H') \to (E'', p'', B'', H'')$$

is a second homomorphism, then $((f'f)_\infty, \eta'\eta) = (f'_\infty f_\infty, \eta'\eta)$. If

$$(f, \eta) = (\mathrm{id}, \mathrm{id}),$$

then $f_\infty = \mathrm{id}$.

Proof. f_∞ clearly maps fibres into fibres and therefore induces an application $\bar{f}_\infty:B_\infty \to B'_\infty$. Since $\bar{f}_\infty \mid B_n = \bar{f}_n$ (induced by f_n), it follows that $\bar{f}_\infty$ is continuous (by definition of the limit topology in B_∞). Now it is clear that (f_∞, η) is a homomorphism if we show that f_∞ is continuous, i.e., that $p'_\infty f_\infty$ and $R'^n f_\infty$ are continuous (where defined). But $p'_\infty f_\infty = \bar{f}_\infty p_\infty$ and $R'^n f_\infty = f_n R^n$ (follows from (11)), and these are continuous maps. The two last statements of 3.9 are also easy to verify.

3.10 For every associative $\mathfrak{H}$-space H there exists a principal q.f., namely $\mathfrak{E} = (H, \pi, P, H)$ where P is a point and the operation of H in H is given by right translations. The corresponding universal principal q.f. $\mathfrak{E}_\infty$ will be denoted by $\mathfrak{E}_H = (E_H, p_H, B_H, H)$. A homomorphism $\eta: H \rightarrow H'$ between $\mathfrak{H}$-spaces induces by 3.8–3.9 a homomorphism $f_\infty = f_\eta: E_H \rightarrow E_{H'}$. This defines a *functor from associative $\mathfrak{H}$-spaces* to universal principal q.f.s.[2]

4. The case of a topological group: Milnor's construction

4.1 THEOREM. *Let G be a topological group and $\mathfrak{E} = (E, p, B, G)$ a principal q.f. If $\mathfrak{E}$ is a principal bundle (in the sense of Steenrod* [6]*), then $\mathfrak{E}_\infty$ (see* §3*) is a universal principal bundle.*

Proof. We show first (by induction) that $(E_{n+1}, p_{n+1}, B_{n+1}, G)$ is a principal bundle. Since $E_{n+1} - E_n = (B_{n+1} - B_n) \times G$ (see proof of 2.3, case (b)), it is sufficient to show that each coordinate neighborhood $U^n \subset B_n$ can be extended to an open set $U^n_{n+1} \subset B_{n+1}$ over which E_{n+1} is trivial. Let $\pi: p_n^{-1}(U^n) \rightarrow G$ be a principal map (i.e., $\pi(yg) = \pi(y)g$), let

$$U^n_{n+1} = (r^n_{n+1})^{-1}(U^n)$$

(i.e., U^n_{n+1} consists of points $y \perp t$ with $t > 0$, $p_n(y) \in U^n$), and define

$$\Phi = \Phi^n_{n+1}: U^n_{n+1} \times G \rightarrow p^{-1}_{n+1}(U^n_{n+1}), \qquad \Phi(y \perp t, g) = (y \mid t \mid \pi(y)^{-1}g),$$

$$\Psi = \Psi^n_{n+1}: p^{-1}_{n+1}(U^n_{n+1}) \rightarrow U^n_{n+1} \times G, \qquad \Psi(y \mid t \mid g) = (y \perp t, \pi(y)g).$$

Then Φ and Ψ are continuous, and $\Phi\Psi = \text{id}$, $\Psi\Phi = \text{id}$, i.e., U^n_{n+1} can be chosen as coordinate neighborhood, and $\mathfrak{E}_{n+1}$ is a principal bundle.

Iterating this extension of coordinate neighborhoods gives open sets $U^n_k \subset B_k$, $k > n$, and product representations

$$\Phi^n_k: U^n_k \times G \rightarrow p_k^{-1}(U^n_k), \qquad \Psi^n_k: p_k^{-1}(U^n_k) \rightarrow U^n_k \times G,$$

$$\Phi\Psi = \text{id}, \qquad \Psi\Phi = \text{id}.$$

The map Ψ^n_k can be defined by $\Psi^n_k(z) = (p_k(z), \pi R^n_k(z))$. Now define

$$U^n_\infty = \textstyle\bigcup_k U^n_k,$$

$$\Phi^n_\infty: U^n_\infty \times G \rightarrow p_\infty^{-1}(U^n_\infty), \qquad \Phi^n_\infty \mid U^n_k \times G = \Phi^n_k,$$

$$\Psi^n_\infty: p_\infty^{-1}(U^n_\infty) \rightarrow U^n_\infty \times G, \qquad \Psi^n_\infty \mid p_\infty^{-1}(U^n_k) = \Psi^n_k,$$

and we have $\Psi^n_\infty(z) = (p_\infty(z), \pi R^n(z))$. The latter shows continuity of Ψ^n_∞. Since $\Phi\Psi = \text{id}$, $\Psi\Phi = \text{id}$, it remains to show continuity of Φ^n_∞. But

$$\Phi^n_\infty(x, g) = \Phi^n_\infty(x, e)g,$$

[2] *Added in proof.* A similar result has been obtained by M. SUGAWARA, *A condition that a space is group-like*, Math. J. Okayama Univ., vol. 7 (1957), pp. 123–149.

the operation of G in E_∞ is continuous, and $\Phi_\infty^n(x, e)$ is a continuous function of $x \in B_\infty$ (definition of limit topology in B_∞). This completes the proof of 4.1.

4.2 *Remark.* It may be shown that in the case of a topological group G the bundle (E_G, p_G, B_G, G) is essentially (namely on finite levels) the construction of Milnor [4]. We content ourselves with a brief indication how the finite levels of the two constructions coincide, i.e., that for a given principal bundle (E, p, B, G) the construction $\hat{E}$ coincides with the join $E \circ G$.

The points in $\hat{E}$ are described by symbols $y \mid t \mid g$, the ones in $E \circ G$ by $t_0 y \oplus t_1 g$ where $y \in E$; $t, t_0, t_1 \in I$; $g \in G$, and $t_0 + t_1 = 1$. We define mappings

$$\hat{E} \to E \circ G, \qquad y \mid t \mid g \to t(yg) \oplus (1 - t)g,$$

$$E \circ G \to \hat{E}, \qquad t_0 y \oplus t_1 g \to yg^{-1} \mid t_0 \mid g.$$

The two composites are the respective identity maps which proves the desired equivalence.

5. Principal H-bundles and their maps

The problem of classifying fibre bundles with respect to fibre homotopy equivalence leads (see §7) to the following

5.1 Definition. Let H be an associative $\mathfrak{H}$-space. A *principal H-bundle* is a *locally trivial* principal q.f. $p:E \to B$ over H; i.e.,

(LT) for every $\bar{x} \in B$ there is a neighborhood $U(\bar{x}) \subset B$ and a local cross section $s:U \to E$ ($ps = \text{id}$) such that the map

$$(1) \qquad \Psi_U : U \times H \to p^{-1}(U), \qquad \Psi_U(x, h) = s(x)h, \qquad x \in U, \quad h \in H,$$

is a homeomorphism. Principal H-bundles are denoted by $\{E, p, B, H\}$.

A principal H-bundle is a fibre bundle in the sense of [6] but possibly without topological structure group.

A principal map between principal H-bundles is defined by 3.1. We do *not require* that these maps preserve the additional structure (LT); the fibres are therefore, in general, not mapped homeomorphically.

5.2 Proposition. *Let $\{E^i, p^i, B, H\}$, $i = 0, 1$, be principal H-bundles with a polyhedron B as a common base, let $\mathfrak{E} = (E, p, B', H)$ be a principal q.f., and $f^i:E^i \to E$, $i = 0, 1$, principal maps such that the induced maps $\bar{f}^i:B \to B'$ are homotopic, $\bar{f}^0 \simeq \bar{f}^1$. Then there exists a principal map $f:E^0 \to E^1$ which lies over the identity map of the base B.*

If $\mathfrak{E}$ is also a principal H-bundle, and if $\bar{f}^t:B \to B'$ is a homotopy between $\bar{f}^0$ and $\bar{f}^1$, then f can be chosen such that $f^0 \simeq f^1 f$ by a principal homotopy (*see* 3.1) *$g^t:E^0 \to E$ which lies over $\bar{f}^t$* (*i.e.*, $g^0 = f^0$, $g^1 = f^1 f$, $pg^t = \bar{f}^t p^0$).

Remark. In the second part of 5.2 the condition that $\mathfrak{E}$ should be a bundle can be weakened. It is sufficient that homotopies $d^t: P \to B'$ of polyhedrons can be lifted into E such that the lifting is "stationary" where d^t is stationary.

5.3 COROLLARY (cf. [2], §2, Satz 1). *Let $\mathfrak{E}^i = \{E^i, p^i, B, H\}$, $i = 0, 1$, be principal H-bundles with a polyhedron B as common base, and let $g:E^1 \to E^0$ be a principal map which lies over the identity map of B. Then there exists a principal map $f:E^0 \to E^1$ such that $fg \simeq$ id, $gf \simeq$ id ($\simeq$ means that there exists a principal homotopy D^t between the two maps such that each D^t lies over the identity of the base).*

Proof of 5.3. Take $\mathfrak{E} = \mathfrak{E}^0$, $f^0 = \text{id}$, $f^1 = g$, and apply 5.2. This gives a principal map $f:E^0 \to E^1$ with $gf \simeq \text{id}$. Apply the same argument to f and obtain a principal map $f':E^1 \to E^0$ with $ff' \simeq \text{id}$. This gives $f' \simeq (gf)f' = g(ff') \simeq g$; hence $fg \simeq \text{id}$.

Proof of 5.2. Let $p^S:E^S \to B'$ be the Serre-fibration (i.e., satisfying the covering homotopy property) which is associated with the map $p:E \to B'$. The points of E^S are the pairs (y, w) where $y \in E$ and $w:I \to B'$ is a path in B' starting at $p(y)$, i.e., $w(0) = p(y)$. The projection p^S maps (y, w) into the end point $w(1)$ of w. The inclusion

(2) $\quad j:E \to E^S, \qquad j(y) = (y, p(y)), \qquad p(y)$ the constant path $I \to p(y)$,

s a homotopy equivalence. It is fibrewise and induces isomorphisms $\pi_i(p^{-1}(b)) \cong \pi_i((p^S)^{-1}(b))$ on the homotopy groups of the fibres (see [3], 1.10). Therefore with respect to the operation $(y, w)h = (yh, w)$ of H in E^S the q.f. $p^S:E^S \to B'$ is principal over H, and the injection j is a principal map which induces the identity map of the base.

We can therefore replace the principal maps f^i by jf^i and thereby reduce the problem to the case where $p:E \to B'$ is a Serre-fibration. (If $\mathfrak{E}$ is an H-bundle, it already has the covering homotopy property, and we do *not* replace it.) This has the advantage that we can take induced fibrations and again have the covering homotopy property, whereas induced "fibrations" of q.f.s may not be q.f.s (see [2], 2.3). Let $\bar{F}:B \times I \to B'$ be the given homotopy between $\bar{f}^0$ and $\bar{f}^1$; i.e., $\bar{F}(x, t) = \bar{f}^t(x)$. Then we can replace $E \to B'$ by the principal q.f. (Serre-fibration) with base $B \times I$ which is induced from $\mathfrak{E}$ by the map $\bar{F}$. (If $\mathfrak{E}$ is a principal H-bundle, then the induced bundle will again be a principal H-bundle.) Hence we can assume

(a) $p:E \to B' = B \times I$, $\qquad$ (b) $\bar{f}^t:B \to B \times I$ is the map $\bar{f}^t(x) = (x, t)$.

The maps $f^i:E^i \to E$ are again principal maps.

Now take a cellular subdivision of B so fine that each cell V is contained in some neighborhood U over which both bundles $\{E^i, p^i, B, H\}$ are trivial. We shall then construct $f:E^0 \to E^1$ together with a principal homotopy $D:E^0 \times I \to E$ between f^0 and $f^1 f$ such that:

(1) $D((p^0)^{-1}(V) \times I) \subset p^{-1}(V \times I)$ $\qquad$ for every cell V of B,

(2) $pD(y, t) = \bar{f}^t(p^0(y)) = (p^0(y), t), \quad y \in E^0,$ $\qquad$ *if $\mathfrak{E}$ is an H-bundle.*

This is done step by step over the skeletons B^n of B:

For every vertex $x^0 \in B$, choose a homeomorphism of $(p^0)^{-1}(x^0)$ with H, and identify the fibre with H by this map. Now, by the covering homotopy theorem, we can choose a path $d(t)$ in $p^{-1}(x^0 \times I) \subset E$ starting at $f^0(e)$, e the unit of H, and ending at some point in $p^{-1}(x^0, 1)$. Since $f^1_* : \pi_0((p^1)^{-1}(x_0)) \to \pi_0(p^{-1}(x^0 \times 1))$ is an isomorphism, we can assume that this endpoint is $f^1(y^1)$ for some y^1 in E^1. If $\mathfrak{E}$ is an H-bundle, we can further assume that $pd(t) = (x^0, t)$ by applying the covering homotopy theorem in E to the deformation of $pd(t)$ into (x^0, t), the endpoints of the covering path remaining stationary during the deformation (see W. Huebsch, *On the covering homotopy theorem*, Ann. of Math. (2), vol. 61 (1955), pp. 555–563). Then define

$$f(h) = y^1 h \quad \text{and} \quad D(h, t) = d(t)h.$$

This takes care of the 0-skeleton B^0.

Now assume that f and the homotopy D are already defined over B^n, and let V be an $(n + 1)$-cell in B^{n+1}. Let $S = \partial V$ be its boundary n-sphere. Choose a homeomorphism $(p^0)^{-1}(V) \approx V \times H$. Using the covering homotopy theorem for $\mathfrak{E}$ we find a map $d : V \times I \to p^{-1}(V \times I)$ such that

(α) $d(x, 0) = f^0(x, e), \quad x \in V,$

(β) $d(z, t) = D(z, e, t), \quad z \in S = \partial V,$

(γ) $d(V \times 1) \subset p^{-1}(V \times 1).$

Now consider the map

$$\varphi : S \to (p^1)^{-1}(V), \qquad \varphi(z) = f(z, e).$$

It is nullhomotopic because $f^1\varphi : S \to p^{-1}(V \times 1)$ is nullhomotopic (d provides a nullhomotopy) and $f^1_* : \pi_n((p^1)^{-1}(V)) \to \pi_n(p^{-1}(V \times 1))$ is a monomorphism (actually an isomorphism). Therefore φ has an extension

$$\Phi : V \to (p^1)^{-1}(V), \qquad \Phi \mid S = \varphi.$$

The two maps

$$x \to f^1\Phi(x) \quad \text{and} \quad x \to d(x, 1) \quad \text{of } V \text{ into } p^{-1}(V \times 1) \tag{3}$$

agree on the boundary S and therefore define an element of $\pi_{n+1}(p^{-1}(V \times 1))$. If Φ runs through all possible extensions, then this element runs through all of $\pi_{n+1}(p^{-1}(V \times 1))$ because $f^1_* : \pi_{n+1}((p^1)^{-1}(V)) \to \pi_{n+1}(p^{-1}(V \times 1))$ is epimorphic. In particular this element is zero for a proper choice of Φ, i.e., the two maps (3) are homotopic rel S. After a deformation of d we can assume they are equal

$$f^1\Phi(x) = d(x, 1). \tag{4}$$

We can also assume that $p_1 \Phi(x) = x$ (by a covering homotopy in E^1). Finally, if $\mathfrak{E}$ is a bundle, we can assume that $pd(x, t) = (x, t)$ (using again a covering homotopy, now in E). Then we define

$$f(x, h) = \Phi(x)h, \qquad D(x, h, t) = d(x, t)h, \qquad x \in V, \quad h \in H, \quad t \in I.$$

Doing this for all $(n+1)$-cells extends f and D over B^{n+1}, and proves the proposition.

6. Homotopy classification of H-bundles

In classical theory principal bundles are classified by homotopy classes of maps into the base of a universal principal bundle (see [6], §19). We give a weak analogue of this result for the homotopy classification of principal H-bundles.

6.1 DEFINITION. Two principal H-bundles $\mathfrak{E}^i = \{E^i, p^i, B, H\}$, $i = 0, 1$, (see 5.1) with the same base B are *homotopy equivalent* if there exist principal maps $f:E^0 \to E^1$ and $f^-:E^1 \to E^0$ such that $ff^- \simeq$ id, $f^-f \simeq$ id (see 5.3 for the meaning of $\simeq$).

6.2 THEOREM. *Let H be an associative $\mathfrak{H}$-space (satisfying T_1, i.e., points are closed), $\mathfrak{E}_H = (E_H, p_H, B_H, H)$ a universal principal q.f. (see 3.10), and let B be a polyhedron.*

(1) *Every principal H-bundle $\mathfrak{E} = \{E, p, B, H\}$ admits a principal map $g:E \to E_H$.*

(2) *If $\mathfrak{E}^i = \{E^i, p^i, B, H\}$, $i = 0, 1$, are principal H-bundles and $g^i:E^i \to E_H$ are principal maps, then $\mathfrak{E}^0$ and $\mathfrak{E}^1$ are homotopy equivalent if and only if the maps $\bar{g}^i:B \to B_H$, induced by g^i, are homotopic, $\bar{g}^0 \simeq \bar{g}^1$.*

6.3 *Remark.* There may exist mappings $\gamma:B \to B_H$ such that for every principal H-bundle $\mathfrak{E} = \{E, p, B, H\}$ and principal map $g:E \to E_H$ we have $\gamma \not\simeq \bar{g}$, i.e., not every homotopy class of maps $B \to B_H$ corresponds to a principal H-*bundle.* This is due to the fact that in general γ does not induce a bundle over B (possibly not even a q.f.; see [3], 2.3); for an example of such a γ see 7.7.

The proof of 6.2 is analogous to [6], §19. It uses the

6.4 LEMMA. *Let $\mathfrak{E} = \{E, p, B, H\}$ be a principal H-bundle over a polyhedron B, let $A \subset B$ be a subcomplex and $f:p^{-1}(A) \to E_H$ a principal map of the part of $\mathfrak{E}$ over A. Then there exists a (principal) extension $F:E \to E_H$ of f to the whole of E.*

Proof of 6.4. The extension is constructed step by step over the cells of $B - A$. If x^0 is a 0-cell in $B - A$, we choose a homeomorphism $p^{-1}(x^0) \approx H$ and a point $y \in E_H$. Then we define $F:p^{-1}(x^0) \to E_H$ by $F(h) = yh$, $h \in H \approx p^{-1}(x^0)$.

Assume now F is already defined over all n-cells of B, and let V be an $(n+1)$-cell in $B - A$, $S = \partial V$ its boundary sphere. Choose a homeomorphism $p^{-1}(V) = V \times H$. Then F is already defined on $S \times H$, in particular on $S \times e$ (e the unit in H). Since E_H is aspherical, there is a map $\varphi:V \to E_H$ such that $\varphi(z) = F(z, e)$ for $z \in S$. Then the extension F over $p^{-1}(V)$ is given by $F(x, h) = \varphi(x)h$, $x \in V$, $h \in H$. This proves the lemma.

Proof of 6.2. Part (1) of 6.2 is Lemma 6.4 with $A = \emptyset$.

Part (2). If $\bar{g}^0 \simeq \bar{g}^1$, then by 5.2 there is a principal map $f:E^0 \to E^1$ which lies over the identity map of the base. By 5.3 this implies that $\mathfrak{E}^0$ and $\mathfrak{E}^1$ are homotopy equivalent.

Conversely, let $g:E^0 \to E^1$ be a principal map. Define a principal H-bundle $\{E', p', B', H\}$ by $E' = E^0 \times I$, $B' = B \times I$, $p' = p^0 \times \text{id}$. Let $A \subset B'$ be the set $B \times 0 \cup B \times 1$, and define a principal map

$$f:p^{-1}(A) \to E_H, \qquad f(y, 0) = g^0(y), \qquad f(y, 1) = g^1g(y), \qquad y \in E^0.$$

By 6.4 there is an extension

$$F:E' \to E_H, \qquad F \mid E^0 \times 0 \cup E^0 \times 1 = f.$$

The map $\bar{F}:B \times I \to B_H$ which is induced by F is then a homotopy between $\bar{g}^0$ and $\bar{g}^1$, Q.E.D.

7. Fibre homotopy equivalence

We apply the results of §6 to classify fibre bundles with respect to fibre homotopy equivalence (see Thom [8], IV, I).

7.1 Definition. Let $\mathfrak{E} = \{E, p, B, F\}$, $\mathfrak{E}' = \{E', p', B', F\}$ be fibre bundles (in the sense of [6] but not necessarily with topological structure group). A fibrewise map

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[\bar{f}]{} & B' \end{array}$$

is *admissible* if for every $x \in B$ the restriction

$$f_x : p^{-1}(x) \to p'^{-1}(\bar{f}(x)), \qquad f_x(y) = f(y), \qquad y \in E,$$

is a homotopy equivalence.

A homotopy $d:E \times I \to E'$ is admissible if for every $t \in I$ the map

$$d_t:E \to E', \qquad d_t(y) = d(y, t)$$

is admissible.

Let now $B = B'$. A *fibre homotopy equivalence* (see [8], IV, I) between $\mathfrak{E}$ and $\mathfrak{E}'$ is a pair of admissible maps

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[rr, "f^-"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

such that $ff^- \simeq \text{id}$, $f^-f \simeq \text{id}$ by admissible homotopies which leave the base fixed.

We associate now with every fibre bundle $\mathfrak{E}$ whose fibre is locally compact a principal H-bundle (see 5.1) $\tilde{\mathfrak{E}}$ over the same base such that $\mathfrak{E}$, $\mathfrak{E}'$ are fibre homotopy equivalent if and only if $\tilde{\mathfrak{E}}$, $\tilde{\mathfrak{E}}'$ are homotopy equivalent (as H-bundles; see 6.1).

7.2 Definition. Let $\mathfrak{E} = \{E, p, B, F\}$ be a fibre bundle with locally compact fibre F. Let H be the space of all homotopy equivalences $F \to F$ (with the compact-open topology; see [1], §2, 5). Composition of maps defines a continuous associative multiplication with unit e in H. Define a principal H-bundle $\tilde{\mathfrak{E}} = \{\tilde{E}, \tilde{p}, B, H\}$ as follows. $\tilde{E}$ is a subspace of the space of all continuous mappings $\varphi : F \to E$ (with the compact open topology). A map φ is in $\tilde{E}$ if

(1) $\varphi(F)$ is contained in some fibre $p^{-1}(x)$,

(2) $\varphi : F \to p^{-1}(x)$ is a homotopy equivalence.

The projection $\tilde{p}$ is given by $\tilde{p}(\varphi) = p\varphi(F)$, and H operates in $\tilde{E}$ by composition $\varphi h = \varphi \circ h$, $\varphi \in \tilde{E}$, $h \in H$. Continuity of this operation and of $\tilde{p}$ follows from §2, 6 and §2, 5 in [1].

It is clear that for trivial $\mathfrak{E}$, i.e., $E \approx B \times F$, $\tilde{\mathfrak{E}}$ is also trivial, $\tilde{E} \approx B \times H$; therefore local triviality of $\mathfrak{E}$ implies local triviality of $\tilde{\mathfrak{E}}$, i.e., $\tilde{\mathfrak{E}}$ is a principal H-bundle, called the *associated* principal H-bundle.

7.3 Proposition. *Let $\mathfrak{E} = \{E, p, B, F\}$, $\mathfrak{E}' = \{E', p', B', F\}$ be fibre bundles with the same locally compact fibre F. With every admissible map $f : E \to E'$ associate the map*

$$\tilde{f} : \tilde{E} \to \tilde{E}', \qquad \tilde{f}(\varphi) = f\varphi, \qquad \varphi \in \tilde{E}.$$

Then $\tilde{f}$ is a principal map which induces the same map on the base as f. The functorial properties $(ff')^{\sim} = \tilde{f}\tilde{f}'$ and $(\mathrm{id})^{\sim} = \mathrm{id}$ hold. Moreover the assignment $f \to \tilde{f}$ establishes a one-to-one correspondence between admissible maps $E \to E'$ and principal maps $\tilde{E} \to \tilde{E}'$.

7.4 Corollary. *Two bundles $\mathfrak{E} = \{E, p, B, F\}$ and $\mathfrak{E}' = \{E', p', B, F\}$ with the same base and same locally compact fibre are fibre homotopy equivalent if and only if their associated principal H-bundles $\tilde{\mathfrak{E}}$, $\tilde{\mathfrak{E}}'$ are homotopy equivalent (as H-bundles; see* 6.1).

The corollary follows from 7.3 since fibre homotopy equivalence is defined in terms of admissible maps.

Proof of 7.3. Let $f : E \to E'$ be an admissible map. Consider the diagram

(1)
$$\begin{array}{ccc} \tilde{E} \times F & \xrightarrow{\tilde{f} \times \mathrm{id}} & \tilde{E}' \times F \\ \downarrow P & & \downarrow P' \\ E & \xrightarrow[f]{} & E' \end{array}$$

where $P(\varphi, y) = \varphi(y)$, $\varphi \in \tilde{E}$, $y \in F$, and similarly for P'. P, P' are continuous by [1], §2, 5, and (1) is commutative by definition of $\tilde{f}$. Since P is clearly *onto*, it follows from the commutativity of (1) that f is uniquely determined by $\tilde{f}$.

Assume now a principal map $g:\tilde{E} \to \tilde{E}'$ is given; we want to construct an admissible map $g':E \to E'$ such that $\tilde{g}' = g$. Since there is at most one g', the existence of g' is a local (with respect to B) problem; we can therefore assume that E is trivial, i.e., $E = B \times F$. Then $\tilde{E} = B \times H$, and we define

$$g':B \times F \to E', \qquad g'(b, y) = g(b, e)(y),$$
$$b \in B, \quad y \in F, \quad e \text{ the unit of } H.$$

In order to prove $\tilde{g}' = g$ it is sufficient to verify commutativity for (1), i.e., $g'(b, h(y)) = g(b, h)(y)$ for $h \in H$; this follows from $g(b, h) = g((b, e)h) = (g(b, e))h$ and the definition of g'.

Proposition 7.4 together with 6.2 gives a classification theorem for fibre bundles with respect to fibre homotopy equivalence. If we consider bundles with structure group, this classification "factors" through the usual classification with respect to ordinary equivalence as follows.

7.5 THEOREM. *Consider fibre bundles $\mathfrak{E} = \{E, p, B, F, G\}$ (see [6]) with a fixed polyhedron B as base, locally compact fibre F, and topological structure group G. Let H be the space of homotopy equivalences $F \to F$ (with the compact open topology), and let $\eta:G \to H$ be the natural homomorphism (each homeomorphism $g \in G$ is a homotopy equivalence). By 3.9–3.10, η induces a homomorphism*

$$\begin{array}{ccc} E_G & \xrightarrow{\xi} & E_H \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_H} \\ B_G & \xrightarrow[\bar{\xi}]{} & B_H \end{array}$$

where $\{E_G, p_G, B_G, G, G\}$ is a universal principal G-bundle (see §4) and (E_H, p_H, B_H, H) is a universal principal q.f. over H.

If we associate with every $\mathfrak{E}$ a classifying map $\gamma(\mathfrak{E}):B \to B_G$ (see [6], §19), then

(1) *two bundles $\mathfrak{E}$, $\mathfrak{E}'$ are equivalent if and only if $\gamma(\mathfrak{E}) \simeq \gamma(\mathfrak{E}')$ (see [6], §19),*

(2) *two bundles $\mathfrak{E}$, $\mathfrak{E}'$ are fibre homotopy equivalent if and only if $\bar{\xi}\gamma(\mathfrak{E}) \simeq \bar{\xi}\gamma(\mathfrak{E}')$.*

Proof. Only the assertion (2) has to be proved. It will follow from 6.2 and 7.4 if we show that for every classifying map $\gamma(\mathfrak{E})$: $B \to B_G$ there is a principal map $\zeta(\mathfrak{E}):\tilde{E} \to E_H$ which lies over $\bar{\xi}\gamma(\mathfrak{E})$ (where $\tilde{\mathfrak{E}}$ is the associated principal H-bundle). By definition of a classifying map there is a principal map $g(\mathfrak{E})\colon \bar{E} \to E_G$ lying over $\gamma(\mathfrak{E})$, where $\bar{\mathfrak{E}} = \{\bar{E}, \bar{p}, B, G, G\}$ is the principal G-bundle associated with $\mathfrak{E}$ (cf. [6], §8.9). Now $\tilde{\bar{E}}$ is obtained from $\bar{E}$ by "extending" the fibre. More precisely there is a homomorphism $\alpha\colon \bar{E} \to \tilde{\bar{E}}$ of principal bundles, associated with $\eta\colon G \to H$ (see 3.7), and the image of $\bar{E}$ generates $\tilde{\bar{E}}$ with respect to the operations of H. From this it is easy to see that there is a unique application $\zeta\colon \tilde{\bar{E}} \to E_H$ which commutes with the operations of H, and such that

$$\begin{tikzcd} \bar{E} \arrow[r, "\alpha"] \arrow[d, "g(\mathfrak{E})"'] & \tilde{\bar{E}} \arrow[d, "\zeta"] \\ E_G \arrow[r, "\xi"'] & E_H \end{tikzcd}$$

is commutative. Continuity of ζ follows from a local consideration as in the proof of 7.3.

Let $[B, B_G]$ be the set of homotopy classes of continuous maps of B into B_G. $\bar{\xi}\colon B_G \to B_H$ induces $\bar{\xi}_*\colon [B, B_G] \to [B, B_H]$ by $\bar{\xi}_*[\gamma] = [\bar{\xi}\gamma]$, $[\gamma] \in [B, B_G]$.

7.6 Corollary. *The fibre homotopy classes of fibre bundles* (*as in* 7.5) *with fixed base* B, *fibre* F, *and group* G *are in one-to-one correspondence with* $\bar{\xi}_*[B, B_G]$, *the image of* $[B, B_G]$ *in* $[B, B_H]$.

For example, if $B = S^n$, the n-sphere, it follows from 7.6 that the fibre homotopy equivalence classes of bundles $\mathfrak{E} = \{E, p, S^n, F, G\}$ are in one-to-one correspondence with the image $\bar{\xi}_*\, \pi_{n-1}(G)$ in $\overline{\pi_{n-1}(H)}$, where $\overline{\pi_{n-1}(H)}$ is the set of conjugacy classes of $\pi_{n-1}(H)$ under the operations of $\pi_0(H)$ (cf. [2]).

7.7 *Remark.* In general not every map $\gamma\colon B \to B_H$ can be factored, up to homotopy, through a B_G, i.e., there may exist maps γ whose homotopy class does not correspond to any bundle $\{E, p, B, F, G\}$. It may not even correspond to any bundle (base B, fibre F) at all, with or without topological structure group.

For an example let F consist of an isolated point plus a segment (topological sum). It is easy to see that every *bundle* with fibre F over the circle S^1 is fibre homotopy equivalent to the product $S^1 \times F$. The space H of homotopy equivalences $F \to F$ has two components, i.e., $\pi_0(H) = Z_2 =$ group with two elements; hence $\pi_1(B_H) = Z_2$. But an essential map $S^1 \to B_H$ cannot correspond to a bundle over S^1 with fibre F.

This example suggests that in order to obtain a better classification theorem one should consider more objects than *bundles* with a *fixed fibre*, e.g., admit different but homotopy equivalent fibres or/and use some notion between bundles and q.f.s.

7.8 *Remark.* Theorem 7.5 can be used to study the behavior of characteristic classes under fibre homotopy equivalences. Characteristic classes of bundles $\{E, p, B, F, G\}$ are defined by cohomology classes in $H^*(B_G)$. It follows from 7.5 that the subring $\bar{\xi}^*H^*(B_H) \subset H^*(B_G)$ consists of characteristic classes which are invariant under fibre homotopy equivalence.

In the case of orthogonal sphere-bundles it turns out that the (integral or rational) Pontrjagin classes $p_i \in H^*(B_G)$ are not in $\bar{\xi}^*H^*(B_H)$, and actually are not invariant under fibre homotopy equivalence, whereas the Stiefel-Whitney classes are, as is known from the work of Thom [8].

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