CHARACTERIZATIONS OF THE ALGEBRA OF ALL REAL-VALUED CONTINUOUS FUNCTIONS ON A COMPLETELY REGULAR SPACE

BY

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Introduction

If X is a topological space, then we denote by $C(X)$ the set of all real-valued continuous functions on X. For X compact,[1] the set $C(X)$ has been characterized from a variety of topologico-algebraic points of view.[2] For X an arbitrary completely regular space, however, no such characterization of $C(X)$ has previously been given. The object of this paper is to obtain several such characterizations of $C(X)$. (For a partial result in this direction see Shirota [10, Theorem 12].)

The first section is preliminary in nature. In §2 we represent certain rings A as subrings of $C(X)$, where X is a completely regular space uniquely determined by A. Similar results are obtained in §3 for A an algebra[3] and X a Q-space [5] and in §4 for A an algebra and X compact.

In order to characterize $C(X)$ for X an arbitrary completely regular space, it suffices [5] to assume that X is a Q-space. In §§5 and 6 we obtain such characterizations of $C(X)$, regarding $C(X)$ as an algebra, as a lattice-ordered algebra [2], and as a vector lattice [1]. Moreover, for X compact, we give, in §5, new characterizations of $C(X)$ as an algebra.

1. Some separation conditions

In this section we introduce and investigate briefly some separation properties of certain subsets of $C(X)$.

Let A be a subset of $C(X)$. We shall adopt the following definitions:

(1) A is *weakly pseudoregular* in case X has a subbase $\mathcal{U}$ of open sets such that for $U \in \mathcal{U}$ and $x \in U$ there is an $\alpha > 0$ in R and an $f \in A$ such that $|f(x) - f(y)| \geqq \alpha$ whenever $y \notin U$.

(2) A is *pseudoregular* (*regular*) in case (i) A contains the identity e of $C(X)$, and (ii) whenever $x \in X$ and U is an open neighborhood of x, there is an $f \in A$ such that $f(x) = 0$ and $f(y) \geqq 1$ ($f(y) = 1$) for all $y \notin U$.[4]

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[1] We shall assume that all compact [6] spaces are Hausdorff.

[2] For references to results of this type see for example [3] and [7].

[3] By an algebra we shall always mean an algebra A over the real field R. If A has an identity, we shall denote it by e. We shall also adopt the convention that lower case Greek letters denote elements of R, unless otherwise specified.

[4] The most natural definitions of "pseudoregular" and "regular" would omit requirement (i). Its presence, however, does not affect the generality of our results; we include it merely as a matter of terminological convenience.

Clearly, a regular subset of $C(X)$ is pseudoregular, and a pseudoregular subset of $C(X)$ is weakly pseudoregular. However, the reverse implications do not hold.

Remark. A space X is completely regular if and only if it is T_1 and $C(X)$ is regular.

LEMMA 1.1. *If A is a weakly pseudoregular subalgebra of $C(X)$ which contains the identity, then A is pseudoregular.*

Proof. Let F be a closed set, and let $x \in X$ not be in F. Since A is a weakly pseudoregular subset of $C(X)$, there exists a finite set $U_1, \cdots, U_n$ of open neighborhoods of x such that $\bigcap_{i=1}^n U_i$ is disjoint from F, and there exist an $\alpha > 0$ in R and a set $f_1, \cdots, f_n$ in A such that, for each $i = 1, \cdots, n$, $|f_i(x) - f_i(y)| \geqq \alpha$ whenever $y \notin U_i$. Since A is an algebra, $g_i = [f_i - f_i(x)e]^2$ is in A for each $i = 1, \cdots, n$. If $y \in F$, then $g_i(y) \geqq \alpha^2$ for some i. Set $g = \alpha^{-2} \sum_{i=1}^n g_i$. Then $g \in A$, $g(x) = 0$, and $g(y) \geqq 1$ for all $y \in F$. Hence A is pseudoregular.

LEMMA 1.2. *If A is a weakly pseudoregular lattice-ordered subalgebra (or vector sublattice) of $C(X)$ which contains the identity, then A is regular.*

Proof. The case in which A is a lattice-ordered subalgebra is immediate from Lemma 1.1. Minor modifications in the proof of Lemma 1.1 yield the case in which A is a vector sublattice.

LEMMA 1.3. *If X is a T_1-space and if $C(X)$ contains a weakly pseudoregular subset A, then X is completely regular.*

Proof. Clearly $C(X)$ is a weakly pseudoregular lattice-ordered subalgebra of itself, and hence, by Lemma 1.2, $C(X)$ is regular. Therefore X is completely regular.

2. Subrings of $C(X)$

Let A be a ring. Our object in this section is to obtain conditions sufficient to represent A as a subring A^* of $C(X)$ for some uniquely determined completely regular space X. Theorem 2.1 shows that to insure uniqueness of X it suffices to insist that A^* be "point-determining" and weakly pseudoregular. Theorem 2.2 provides the desired representation.

If A is any ring, then an ideal M in A is said to be *real* in case A/M is isomorphic to R. We shall denote by $\mathfrak{M}_A$ the set of all maximal ideals of A and by $\mathfrak{R}_A$ the set of all real maximal ideals of A. For each $M \in \mathfrak{R}_A$ and each $f \in A$, we shall let $M(f)$ be the image of f in R under the homomorphism of A onto R with kernel M.

If X is a topological space, if A is a subring of $C(X)$, and if $x \in X$, then we shall set

$$M_x = \{f \in A; f(x) = 0\}.$$

We say that A is *point-determining* in case $M \in \mathcal{R}_A$ if and only if $M = M_x$ for some unique $x \in X$. We note that if $C(X)$ contains a point-determining subring, then X must be T_1. Observe also that if A is pseudoregular and if X is T_0, then the uniqueness requirement of this definition is automatic.

A completely regular space X is a *Q-space* [5, p. 85] in case $C(X)$ is itself point-determining; that is, $M \in \mathcal{R}_{C(X)}$ (if and) only if $M = \{f \in C(X); f(x) = 0\}$ for some (necessarily unique) $x \in X$. It is known that X is a Q-space if and only if X is homeomorphic to a closed subset of a product of real lines ([5, Theorem 60] and [10, Theorem 1]).

The following theorem generalizes Theorem 57 of [5] to the effect that if X is a Q-space, then X is characterized by $C(X)$.

THEOREM 2.1. *If X is a topological space and if A is a weakly pseudoregular point-determining subring of $C(X)$, then A characterizes X.*

Proof. We shall show that $\mathcal{R}_A$, with a suitable (algebraically invariant) topology, is homeomorphic to X. Since A is point-determining, the mapping $x \to M_x$ is one-one from X onto $\mathcal{R}_A$. For each $M_x \in \mathcal{R}_A$, each $\varepsilon > 0$, and each $f \in A$, set $\mathfrak{U}_{M_x}(f, \varepsilon) = \{M_y \in \mathcal{R}_A ; |M_x(f) - M_y(f)| < \varepsilon\}$. The family of all $\mathfrak{U}_{M_x}(f, \varepsilon)$ forms an open subbase for a topology on $\mathcal{R}_A$. Since A is weakly pseudoregular, the family of all sets $U_x(f, \varepsilon) = \{y \in X; |f(x) - f(y)| < \varepsilon\}$ is an open subbase for X. Now since the mapping $f \to f(x)$ is a homomorphism of A into R with kernel M_x, it follows that $M_x(f) = f(x)$ for all $x \in X$ and all $f \in A$. Thus the mapping $x \to M_x$ is clearly a homeomorphism.

If A is a ring such that $\mathcal{R}_A$ is not empty, and if $f \in A$, we shall define the real-valued function f^* on $\mathcal{R}_A$ by $f^*(M) = M(f)$ for all $M \in \mathcal{R}_A$. Then the mapping $f \to f^*$ is a homomorphism of A onto a subring A^* of the ring of all real-valued functions on $\mathcal{R}_A$.

We observe that the topology on $\mathcal{R}_A$ introduced in the proof of Theorem 2.1 is the weakest topology on $\mathcal{R}_A$ which makes each $f^* \in A^*$ continuous. In the sequel we shall always assume, unless otherwise specified, that $\mathcal{R}_A$ is endowed with this topology; we shall call it the *weak topology* on $\mathcal{R}_A$ determined by A.[5]

If A is a ring with the property that $\bigcap \mathcal{R}_A = 0$, then the homomorphism $f \to f^*$ of A onto A^* is, in fact, an isomorphism.

(We remark that if $\mathcal{R}_A$ is endowed with its discrete topology and if $\bigcap \mathcal{R}_A = 0$, then A^* is clearly a point-determining subring of $C(\mathcal{R}_A)$; however, Theorem 2.1 shows that in this case A^* need not be a weakly pseudoregular subring of $C(\mathcal{R}_A)$.)

THEOREM 2.2. *A ring A is isomorphic to a weakly pseudoregular point-determining subring of $C(X)$ for some topologically unique completely regular space X if and only if $\bigcap \mathcal{R}_A = 0$.*

[5] Cf. [8, p. 10].

Proof. The necessity of this condition is clear. Conversely, endow $\mathfrak{R}_A$ with the weak topology determined by A. Let M and N be distinct elements of $\mathfrak{R}_A$. Then there is an $f \in M$ with $f \notin N$. Since f^* is continuous on $\mathfrak{R}_A$ and since $f^*(M) = 0$ and $f^*(N) \neq 0$, it follows that $\mathfrak{R}_A$ is Hausdorff. By the choice of the topology for $\mathfrak{R}_A$, it is evident that A^* is a weakly pseudoregular subring of $C(\mathfrak{R}_A)$; hence, by Lemma 1.3, $\mathfrak{R}_A$ is completely regular. Finally, since A^* is a point-determining subring of $C(\mathfrak{R}_A)$, an application of Theorem 2.1 completes the proof.

If X is a completely regular space, then there exists a topologically unique space vX characterized by the following three properties [5, Theorems 56 and 58]: (1) X is (homeomorphic to) a dense subset of vX, (2) vX is a Q-space, and (3) every $f \in C(X)$ has a unique continuous extension over vX. It follows that $C(X)$ is isomorphic to $C(vX)$.

Let X be a completely regular space. If $A = C(X)$, then $\bigcap \mathfrak{R}_A = 0$, so that, by the preceding theorem, A is isomorphic to a weakly pseudoregular point-determining subring of $C(Y)$ for some topologically unique completely regular space Y. In fact, since $C(X)$ is also isomorphic to $C(vX)$, it is clear that $Y = vX$. However, the following example shows that if A is merely a subring of $C(X)$ with $\bigcap \mathfrak{R}_A = 0$, then the space Y need not be a Q-space; thus, in particular, in Theorem 2.2 "completely regular" cannot be replaced by "Q-space".

Example 1. Let Ω be the first uncountable ordinal, let $T_{\Omega+1}$ be the chain of all ordinals $\delta \leqq \Omega$ with the interval topology, and let $T_\Omega = T_{\Omega+1} - \{\Omega\}$. We recall the following facts:[6] (i) $T_{\Omega+1} = \beta T_\Omega = vT_\Omega$ (where, as usual, βT_Ω denotes the Stone-Čech compactification of T_Ω) so that T_Ω is not a Q-space, and (ii) every $f \in C(T_\Omega)$ is eventually constant. Now let A be the subring of $C(T_{\Omega+1})$ consisting of all f such that $f(\Omega)$ is integral. Then for each $\delta < \Omega$, M_δ belongs to $\mathfrak{R}_A$, so that clearly $\bigcap \mathfrak{R}_A = 0$. On the other hand, suppose that M is a maximal ideal in A such that $M \neq M_\delta$ for every $\delta < \Omega$. Note first that M contains an element which does not vanish at Ω, since otherwise $M \subsetneqq M_\Omega$ while M_Ω is a nonmaximal ideal of A. Thus, since $T_{\Omega+1}$ is compact, there is an element $f \in M$ such that $f(\delta) \neq 0$ for all $\delta \in T_{\Omega+1}$. Suppose now that n is the constant (integral-valued) function in A such that $n(\Omega) = f(\Omega)$. By (ii), there exists a $\delta < \Omega$ such that $n(\gamma) = f(\gamma)$ for all $\gamma > \delta$. Now there is a $g \in A$ such that $(gf)(\beta) = 1$ for all $\beta \leqq \delta$ and $g(\gamma) = 0$ for all $\gamma > \delta$. Let $h \in A$ be such that $h(\beta) = 0$ for all $\beta \leqq \delta$ and $h(\gamma) = 1$ for all $\gamma > \delta$. Since $n = hn + gfn = hf + gfn$, it follows that $n \in M$. The field A/M is therefore of finite characteristic,[7] and hence $M \notin \mathfrak{R}_A$. Finally, A is isomorphic to the ring obtained by restricting each $f \in A$ to T_Ω. Consequently, A is isomorphic to a point-determining subring of $C(T_\Omega)$.

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[6] See [5] and [6, p. 167].

[7] We note in passing that such maximal ideals M do exist. For example, $M = \{f \in A\,;\, f(\Omega) \text{ is even}\}$ is a maximal ideal of A such that A/M is a field of characteristic two.

3. Subalgebras of $C(X)$

We continue the investigations of the preceding section with the restriction that A be an algebra with identity. In this case, if $\bigcap \mathfrak{R}_A = 0$, then A is representable as a pseudoregular point-determining subalgebra of $C(X)$ for a unique Q-space X.

LEMMA 3.1. *If $C(X)$ contains a subalgebra A which is weakly pseudoregular, point-determining, and which contains the identity, then X is a Q-space.*

Proof. By Theorem 2.2 and its proof, it follows that X and $\mathfrak{R}_A$ are homeomorphic. Since $\mathfrak{R}_A$ is endowed with the weak topology determined by A, we can embed $\mathfrak{R}_A$, and hence X, homeomorphically in the product space $Y = \prod_{f \in A} R_f$, where $R_f = R$ for each $f \in A$. Then the closure $\bar{X}$ of X in Y is a Q-space [10, Theorem 1]. For each $y \in Y$, let $y(f)$ be the f-component of y. Now let $y \in \bar{X}$. By standard arguments (cf. [8, p. 53]) the mapping $f \to y(f)$ is a homomorphism of A onto R. Let K be the kernel of this homomorphism. Then $K \in \mathfrak{R}_A$, so that $K = M_x$ for some $x \in X$, which clearly implies that $y = x$. That is, $X = \bar{X}$, and hence X is a Q-space.

THEOREM 3.2. *An algebra A is isomorphic to a pseudoregular point-determining subalgebra of $C(X)$ for some topologically unique Q-space X if and only if A has an identity and $\bigcap \mathfrak{R}_A = 0$.*

Proof. This follows immediately from Theorem 2.2, Lemma 1.1, and Lemma 3.1.

Motivated by the preceding theorem we say that an algebra A is *pseudoregular* in case A has an identity and $\bigcap \mathfrak{R}_A = 0$. It is of importance in later sections to have a characterization of those pseudoregular algebras A for which A^* is a regular subalgebra of $C(\mathfrak{R}_A)$. An algebra A is said to be *regular* in case (i) A is pseudoregular, and (ii) for every $M \in \mathfrak{R}_A$ and every $f \in M$ there is a $g \in M$ such that $N \in \mathfrak{R}_A$, $\alpha \in R$, and $(g - e)^2(e - f) + \alpha^2 e \in N$ together imply $\alpha = 0$. This terminology is justified by the following result:

LEMMA 3.3. *A pseudoregular algebra A is regular if and only if A^* is a regular subalgebra of $C(\mathfrak{R}_A)$.*

Proof. Suppose first that A^* is a regular subalgebra of $C(\mathfrak{R}_A)$. Let $M \in \mathfrak{R}_A$, let $f \in M$, and set $\mathfrak{U} = \{N \in \mathfrak{R}_A ; (e - f)^*(N) > 0\}$. Then $\mathfrak{U}$ is an open neighborhood of M. Since A^* is regular, there is a $g \in M$ such that $g^*(N) = 1$ for all $N \notin \mathfrak{U}$. Now clearly $[(g - e)^2(e - f)]^*$ is nonnegative on $\mathfrak{R}_A$; hence, if $N \in \mathfrak{R}_A$, and if $(g - e)^2(e - f) + \alpha^2 e \in N$, then $\alpha = 0$.

Conversely, if A is regular, if $M \in \mathfrak{R}_A$, and if $\mathfrak{U}$ is an open neighborhood of M, then there is an $f \in M$ such that $f^*(N) > 1$ for all $N \notin \mathfrak{U}$. Let $g \in M$ such that $(g - e)^2(e - f) + \alpha^2 e \in N$ implies $\alpha = 0$. Then $(g^* - e^*)^2(e^* - f^*)$ is nonnegative on $\mathfrak{R}_A$; hence $(g^* - e^*)^2(M) = 1$ and $(g^* - e^*)^2(N) = 0$ for all $N \notin \mathfrak{U}$. That is, A^* is regular.

Remark. (*Added in proof.*) The notion of the $\mathcal{S}$-spectrum of an element

$f \in A$ (see definition preceding Theorem 4.2) provides a second characterization of regular algebras. Let A be a pseudoregular algebra. Then it is easy to see that A is regular if and only if for each $\mathcal{S} \subseteq \mathcal{R}_A$ and for each $M \in \mathcal{R}_A$, $\bigcap \mathcal{S} \subseteq M$ implies $\inf S(f, \mathcal{S}) \leqq M(f)$ for all $f \in A$.

Recall that if A is a commutative ring with identity and if $\mathcal{S}$ is a set of maximal ideals of A, then the Stone (hull-kernel) topology [11] on $\mathcal{S}$ is described as follows: For $\mathfrak{J} \subseteq \mathcal{S}$ the closure of $\mathfrak{J}$ in $\mathcal{S}$ is $\{M \in \mathcal{S};\ \bigcap \mathfrak{J} \subseteq M\}$. We conclude this section with the following easily proved result (cf. [8, p. 57]).

LEMMA 3.4. *If A is a regular algebra, then the Stone topology on $\mathcal{R}_A$ coincides with the weak topology on $\mathcal{R}_A$ determined by A. Conversely, if A is pseudoregular and if these topologies on $\mathcal{R}_A$ coincide, then A is regular.*

4. Subalgebras of $C(X)$ for X compact

In this section we specialize the results of the preceding section to obtain representations of algebras as subalgebras of $C(X)$ for X compact. Our first result provides such a representation for regular algebras.

THEOREM 4.1. *If A is a regular algebra such that $\mathfrak{M}_A = \mathcal{R}_A$, then A is isomorphic to a regular point-determining subalgebra of $C(X)$ for some topologically unique compact space X.*

Proof. Since A has an identity and since $\mathfrak{M}_A = \mathcal{R}_A$, it follows that $\mathcal{R}_A$ is compact in its Stone topology. But, by Lemma 3.4, the Stone and weak topologies on $\mathcal{R}_A$ coincide. The desired result therefore follows from Theorem 3.2 and Lemma 3.3.

We do not know whether or not the converse of Theorem 4.1 (as well as that of Theorem 4.5 below) holds.

It is reasonable to inquire whether or not Theorem 4.1 remains true if "regular" is replaced by "pseudoregular" throughout. We settle this question in the negative by giving an example of a pseudoregular point-determining subalgebra A of $C(R)$ such that $\mathfrak{M}_A = \mathcal{R}_A$.

Example 2. Let A be the subalgebra of $C(R)$ generated by the set of all polynomial functions on R together with the inverses of those polynomials which have no (real) zeros. Then each element of A is of the form

$$p + \sum_{i=1}^{n} p_i\, q_i^{-1}$$

where p, p_i, q_i $(i = 1, \cdots, n)$ are polynomials and each q_i has no zeros. We claim that A is pseudoregular, point-determining, and that $\mathfrak{M}_A = \mathcal{R}_A$. Since it is clear that A is pseudoregular and that for each $x \in R$, $M_x \in \mathcal{R}_A$, it will suffice to show that for each $M \in \mathfrak{M}_A$, $M = M_x$ for some $x \in R$. Suppose on the contrary that there exists an $M \in \mathfrak{M}_A$ such that $M \neq M_x$ for every $x \in R$. Then for each $x \in R$ there is an element

$$f_x = p + \sum_{i=1}^{n} p_i\, q_i^{-1}$$

in M such that $f_x(x) \neq 0$. Set

$$g_x = (f_x \prod_{i=1}^{n} q_i)^2.$$

Now let $x_0 \in R$. Since g_{x_0} is a polynomial, g_{x_0} has (finitely many) zeros, say $x_1, \cdots, x_n$. But then

$$g = \sum_{i=0}^{n} g_{x_i}$$

is a polynomial in M with no zeros. This is a contradiction and we conclude that each $M \in \mathfrak{M}_A$ is of the form M_x for some $x \in R$.

If A is a ring, if $\mathcal{S} \subsetneq \mathcal{R}_A$, and if $f \in A$, then we define the $\mathcal{S}$-*spectrum* of f to be the set

$$S(f, \mathcal{S}) = \{M(f); M \in \mathcal{S}\}.$$

For brevity we shall set $S(f) = S(f, \mathcal{R}_A)$.

THEOREM 4.2. *An algebra A is isomorphic to a pseudoregular point-determining subalgebra of $C(X)$ for some topologically unique compact space X if and only if* (i) *A is pseudoregular and* (ii) *$S(f)$ is bounded for each $f \in A$.*

Proof. The necessity of these conditions is obvious. To prove their sufficiency it will suffice, in view of Theorem 3.2, to prove that $\mathcal{R}_A$ is compact. But $\mathcal{R}_A$ is homeomorphic to a closed subset of the compact space $\prod_{f \in A} S(f)^-$, where $S(f)^-$ is the closure of $S(f)$ in R (see the proof of Lemma 3.1).

It is clear from the preceding theorem and from the Stone-Weierstrass theorem that conditions (i) and (ii) of Theorem 4.2 are sufficient in order that an algebra A be isomorphic to a uniformly dense subalgebra of $C(X)$ for some compact space X. It is also clear, conversely, that if X is compact and if A is a uniformly dense subalgebra of $C(X)$ which contains the identity, then A satisfies condition (i). However, A need not satisfy condition (ii), as is shown by the following example of a uniformly dense subalgebra A of $C([0, 1])$ which contains the identity but which has the property that $S(f)$ is unbounded for some $f \in A$.

Example 3. For any subset S of R, let $P(S)$ be the algebra of all polynomial functions on S. We shall show first that $P(R)$ is a point-determining subalgebra of $C(R)$. To do this it will clearly suffice to prove that if $M \in \mathcal{R}_{P(R)}$, then $M = M_x$ for some $x \in R$. Suppose, on the contrary, that $M \in \mathcal{R}_{P(R)}$ and that $M \neq M_x$ for all $x \in R$. Then clearly M contains a nonvanishing polynomial

$$p = \sum_{i=0}^{n} \alpha_i u^i,$$

where $u \in P(R)$ satisfies $u(x) = x$ for all $x \in R$. Then

$$p[M(u)] = \sum_{i=0}^{n} \alpha_i [M(u)]^i = M(p) = 0,$$

which is a contradiction since $M(u)$ is real. Now let $I = [0, 1]$ so that, by the Weierstrass theorem, $P(I)$ is uniformly dense in $C(I)$. If $p \in P(I)$, then p has a unique polynomial extension $\bar{p} \in P(R)$; hence $P(I)$ and $P(R)$ are isomorphic. From this and from the fact that $P(R)$ is point-determining we conclude that

$$S(p) = \{\bar{p}(x); x \in R\}$$

for each $p \in P(I)$. Hence $S(p)$ is unbounded for each nonconstant $p \in P(I)$.[8]

Now let X be compact and let A be a dense subalgebra of $C(X)$ which contains the identity. As the preceding example shows, $S(f) = S(f, \mathcal{R}_A)$ may be unbounded for some $f \in A$. However, there is clearly a subset of $\mathcal{R}_A$, namely $\mathcal{F} = \{M_x \in \mathcal{R}_A ; x \in X\}$, such that $S(f, \mathcal{F})$ is bounded for every $f \in A$. Moreover, it is clear that $\bigcap \mathcal{F} = 0$. This proves half of the following theorem:

THEOREM 4.3. *Let A be an algebra with identity. Then A is isomorphic to a uniformly dense subalgebra of $C(X)$ for some compact space X if and only if there exists a subset $\mathcal{S}$ of $\mathcal{R}_A$ such that* (i) $\bigcap \mathcal{S} = 0$ *and* (ii) $S(f, \mathcal{S})$ *is bounded for each $f \in A$.*[9]

Proof. In view of the above remarks, we need only prove the sufficiency of the stated conditions. Let $\bar{\mathcal{S}}$ be the closure of $\mathcal{S}$ in $\prod_{f \in A} S(f, \mathcal{S})^-$. Since $\mathcal{R}_A \cap \prod_{f \in A} S(f, \mathcal{S})^-$ is closed in $\prod_{f \in A} R_f$ (see the proof of Lemma 3.1), it follows that $\bar{\mathcal{S}}$ is a compact subspace of $\mathcal{R}_A$. Moreover, since $\bigcap \mathcal{S} = 0$, the mapping $f \to f^*|\bar{\mathcal{S}}$, where $f^*|\bar{\mathcal{S}}$ denotes the restriction of f^* to $\bar{\mathcal{S}}$, is an isomorphism of A onto a pseudoregular subalgebra A' of $C(\bar{\mathcal{S}})$. But, by the Stone-Weierstrass theorem, A' is uniformly dense in $C(\bar{\mathcal{S}})$, and the proof is complete.

We note next that if X is compact and if A is a uniformly dense subalgebra of $C(X)$ which contains the identity, then A need not be point-determining. (For, in Example 3, $P(I)$ is uniformly dense in $C(I)$ but is not point-determining.) Indeed, the situation with respect to point-determination seems to be rather pathological. For example, A can be isomorphic to a uniformly dense subalgebra of $C(Y)$, where Y is compact but not homeomorphic to X. In fact, if Y is any infinite subset of R, then the polynomial algebras $P(I)$ and $P(Y)$ are isomorphic; and if in addition, Y is compact, then $P(Y)$ is uniformly dense in $C(Y)$.

LEMMA 4.4. *If X is a completely regular space and if $C(X)$ contains a pseudo-regular point-determining subalgebra A such that* (i) $\mathcal{M}_A = \mathcal{R}_A$ *and* (ii) $S(f)$ *is closed for each $f \in A$, then X is compact.*

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[8] Example 3 and the example preceding Lemma 4.4 are simpler than the corresponding examples included in an earlier version of this paper; these simplifications were suggested to us by C. W. Kohls. In this earlier version we raised the question of whether or not the condition that every maximal ideal of A be real is necessary in order that an algebra A with identity be isomorphic to a uniformly dense subalgebra of $C(X)$ for some compact space X. We are indebted to Kohls for pointing out that Example 3 answers this question in the negative.

[9] Theorem 4.3 includes Theorem 4.1 (1) of [7]; we remark, however, that in a footnote Kohls announces a substantial improvement of his Theorem 4.1. We wish to thank Dr. Kohls for making the manuscript of [7] available to us prior to its publication.

Proof. Each $f \in C(X)$ may be regarded as a continuous function on X to the one-point compactification $R \cup \{\infty\}$ of R; as such, f has a unique continuous extension $f^\wedge$ over βX.[10] Suppose that X is not compact so that there exists a point $y \in \beta X - X$. We note first that if $f \in A$, then $f^\wedge(y) \in R$. For suppose that $f^\wedge(y) = \infty$. If $g = f^2 + e$, then $g^{-1} \in A$ since $\mathfrak{M}_A = \mathfrak{R}_A$ and since $g \notin M$ for every $M \in \mathfrak{R}_A$. Moreover, $(g^{-1})^\wedge(y) = 0$. Thus, since $S(g^{-1})$ is closed, $0 \in S(g^{-1})$, which is contrary to $g^{-1}(x) \neq 0$ for every $x \in X$. It follows now that if $f \in A$ with $f^\wedge(y) = 0$, then $(fg)^\wedge(y) = 0$ for every $g \in A$. Therefore the set $M_y = \{f \in A; f^\wedge(y) = 0\}$ is a proper ideal of A, and hence $M_y \subseteq M_x$ for some $M_x \in \mathfrak{R}_A$. Since A is pseudoregular and $x \neq y$, there is an $f \in A$ such that $f(x) = 0$ and $f^\wedge(y) = \alpha \neq 0$. Then $f - \alpha e \in M_y$ while $f - \alpha e \notin M_x$, a contradiction. That is, $X = \beta X$.

The final result of this section now follows at once from Theorem 3.2 and the preceding lemma.

THEOREM 4.5. *If A is a pseudoregular algebra such that* (i) $\mathfrak{M}_A = \mathfrak{R}_A$ *and* (ii) *$S(f)$ is closed for each $f \in A$, then A is isomorphic to a pseudoregular point-determining subalgebra of $C(X)$ for some topologically unique compact space X.*

5. Characterizations of $C(X)$

In this section we use variants of Fan's notion of "direct extension" [3], together with the representation theorems of the preceding sections, to obtain characterizations of all of $C(X)$.

Let A be a pseudoregular algebra. An algebra B is an $\mathfrak{R}$-*extension* of A in case B is pseudoregular with the property that A can be embedded isomorphically in B in such a fashion that the mapping $M \to M \cap A$ is one-one from $\mathfrak{R}_B$ onto $\mathfrak{R}_A$.

We note that if B is an $\mathfrak{R}$-extension of A, then $M \to M \cap A$ is automatically continuous in the weak topologies of $\mathfrak{R}_A$ and $\mathfrak{R}_B$.

The following lemma is an analogue of Lemma 8.1 of [3]. The proof depends upon Theorem 2.1 and Lemma 3.1; we omit the details.

LEMMA 5.1. *If A is a pseudoregular point-determining subalgebra of $C(X)$ such that A is isomorphic to $C(X)$, then $A = C(X)$.*

LEMMA 5.2. *Let A be a pseudoregular algebra, and suppose that $\mathcal{E}$ is a class of $\mathfrak{R}$-extensions of A such that* (i) $C(\mathfrak{R}_A) \in \mathcal{E}$, *and* (ii) *for each $B \in \mathcal{E}$, $M \to M \cap A$ is an open mapping from $\mathfrak{R}_B$ onto $\mathfrak{R}_A$. Then A is isomorphic to $C(\mathfrak{R}_A)$ if and only if A is isomorphic to B for every $B \in \mathcal{E}$.*

Proof. The sufficiency is obvious. Since the mapping $M \to M \cap A$ is one-one and continuous, the necessity follows from Theorem 3.2 and Lemma 5.1.

By making suitable choices of $\mathcal{E}$ in the preceding lemma, and by combining the results of §§3 and 4 with Lemma 5.2, we obtain various characterizations of $C(X)$. We begin with the case where X is compact.

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[10] Since $R \cup \{\infty\}$ is compact, the existence of $f^\wedge$ follows from [11, Theorem 88]. For the essential properties of $f^\wedge$ see [4].

THEOREM 5.3. *An algebra A is isomorphic to $C(X)$ for some topologically unique compact space X if and only if* (i) *A is regular and $\mathfrak{M}_A = \mathcal{R}_A$, and* (ii) *$A$ is isomorphic to every regular $\mathcal{R}$-extension B of A for which $\mathfrak{M}_B = \mathcal{R}_B$.*

Proof. If A satisfies (i), then, by Theorem 4.1, $C(\mathcal{R}_A)$ is an $\mathcal{R}$-extension of A such that $\mathfrak{M}_{C(\mathcal{R}_A)} = \mathcal{R}_{C(\mathcal{R}_A)}$. Hence if A also satisfies (ii), then A is isomorphic to $C(\mathcal{R}_A)$. Conversely, suppose that A is isomorphic to $C(X)$ for some compact space X and let $\mathcal{E}$ be the class of all regular $\mathcal{R}$-extensions B of A for which $\mathfrak{M}_B = \mathcal{R}_B$. Clearly A satisfies (i) so that $C(\mathcal{R}_A) \in \mathcal{E}$. Moreover, if $B \in \mathcal{E}$, then $\mathcal{R}_B$ is compact so that $M \rightarrow M \cap A$ is a homeomorphism of $\mathcal{R}_B$ onto $\mathcal{R}_A$. Finally, since $C(X)$ is isomorphic to $C(\mathcal{R}_A)$, an application of Lemma 5.2 shows that A satisfies (ii), and the proof is complete.

The proof of the following theorem is analogous to that of Theorem 5.3.

THEOREM 5.4. *An algebra A is isomorphic to $C(X)$ for some topologically unique compact space X if and only if* (i) *A is pseudoregular and $S(f, \mathcal{R}_A)$ is bounded for each $f \in A$, and* (ii) *A is isomorphic to every $\mathcal{R}$-extension B of A for which $S(f, \mathcal{R}_B)$ is bounded for each $f \in B$.*[11]

We note that a variety of other characterizations of $C(X)$, for X compact, can be obtained in a similar way by using the results of the preceding section. For example, two further characterizations are obtained by simply interchanging condition (i) of Theorem 5.3 and condition (i) of Theorem 5.4.

It appears plausible from Theorems 3.2 and 5.4 that an algebra A is isomorphic to $C(X)$ for some Q-space X if and only if A is pseudoregular and A is isomorphic to every pseudoregular $\mathcal{R}$-extension of A. On similar grounds, Lemma 3.3 and Theorem 5.3 suggest that the above statement might hold if "pseudoregular" were replaced throughout by "regular". However, as the following example shows, neither of these conjectures is tenable.

Example 4. Let R_D denote the set R with the discrete topology. Clearly $C(R)$ and $C(R_D)$ are regular algebras and, since R and R_D are both Q-spaces [10, Corollary 2, p. 28], $C(R_D)$ is an $\mathcal{R}$-extension of $C(R)$. But $C(R)$ is not isomorphic to $C(R_D)$.

Thus, for the purpose of characterizing $C(X)$ for X a Q-space, we strengthen the concept of $\mathcal{R}$-extension by introducing the notions of "$\mathcal{R}_s$-extension" and "$\mathcal{R}_w$-extension". An $\mathcal{R}_s$-extension B of A will have the property that $\mathcal{R}_B$ and $\mathcal{R}_A$ are homeomorphic in their Stone topologies; an $\mathcal{R}_w$-extension B of A will have the property that $\mathcal{R}_B$ and $\mathcal{R}_A$ are homeomorphic in their weak topologies.

If A is a pseudoregular algebra, then an algebra B is an *$\mathcal{R}_s$-extension* of A in case (i) B is an $\mathcal{R}$-extension of A, and (ii) if $\mathcal{S} \subseteq \mathcal{R}_B$ and if $M \in \mathcal{R}_B$ with $(\bigcap \mathcal{S}) \cap A \subseteq M$, then $\bigcap \mathcal{S} \subseteq M$ (cf. [9]).

[11] Theorem 5.4 includes [7, Theorem 4.1]. (Cf. Footnote 9.)

LEMMA 5.5. *If A is a regular point-determining subalgebra of $C(X)$, then $C(X)$ is a regular $\mathfrak{R}_s$-extension of A.*

Proof. Since, by Lemma 3.1, X is Q-space, it is clear that $C(X)$ is a regular $\mathfrak{R}$-extension of A. Thus let $\mathcal{S} \subseteq \mathfrak{R}_{C(X)}$ and suppose that $M_x \in \mathfrak{R}_{C(X)}$ with $(\bigcap \mathcal{S}) \cap A \subseteq M_x$. Since X is a Q-space, $\mathfrak{R}_{C(X)}$, endowed with its Stone topology, is homeomorphic to X. Hence $F = \{y \in X; \bigcap \mathcal{S} \subseteq M_y\}$ is closed in X. Since A is regular, if $x \notin F$, then there is an $f \in A$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in F$. But then $f \in \bigcap \mathcal{S}$ so that $f \in M_x$, a contradiction. Therefore $x \in F$. That is, $\bigcap \mathcal{S} \subseteq M_x$ and $C(X)$ is an $\mathfrak{R}_s$-extension of A.

THEOREM 5.6. *An algebra A is isomorphic to $C(X)$ for some topologically unique Q-space X if and only if* (i) *A is regular and* (ii) *A is isomorphic to every regular $\mathfrak{R}_s$-extension of A.*

Proof. Suppose that A satisfies (i) so that, by Lemma 5.5, $C(\mathfrak{R}_A)$ is a regular $\mathfrak{R}_s$-extension of A. Hence if A also satisfies (ii), then A is isomorphic to $C(\mathfrak{R}_A)$. Conversely, suppose that A is isomorphic to $C(X)$ for some Q-space X and let $\mathcal{E}_s$ be the class of all regular $\mathfrak{R}_s$-extensions of A. Clearly A satisfies (i) so that, again by Lemma 5.5, $C(\mathfrak{R}_A) \subseteq \mathcal{E}_s$. Moreover, if $B \in \mathcal{E}_s$, then $M \to M \cap A$ is clearly a homeomorphism of $\mathfrak{R}_B$ onto $\mathfrak{R}_A$ in their Stone topologies. But, by Lemma 3.4, the Stone and weak topologies on $\mathfrak{R}_B$ coincide for every $B \in \mathcal{E}_s$. Finally, since $C(X)$ is isomorphic to $C(\mathfrak{R}_A)$, an application of Lemma 5.2 shows that A satisfies (ii), and the proof is complete.

If A is a pseudoregular algebra, then an algebra B is an *$\mathfrak{R}_w$-extension* of A in case (i) B is an $\mathfrak{R}$-extension of A, and (ii) for every $M \in \mathfrak{R}_B$, $f \in M$, and $\varepsilon > 0$ in R there exist a $g \in M \cap A$ and a $\delta > 0$ such that $N \in \mathfrak{R}_B$ and $N(g) < \delta$ together imply $N(f) < \varepsilon$.

The following lemma implies that if A is pseudoregular, then $C(\mathfrak{R}_A)$ is an $\mathfrak{R}_w$-extension of A.

LEMMA 5.7. *Let X be a completely regular space and let A be a weakly pseudoregular subalgebra of $C(X)$ which contains the identity. If $x \in X$, if $f \in C(X)$ with $f(x) = 0$, and if $\varepsilon > 0$, then there exist a $g \in A$ and a $\delta > 0$ such that $g(x) = 0$ and that $g(y) < \delta$ implies $f(y) < \varepsilon$.*

Proof. Set $U = \{y \in X; f(y) < \varepsilon\}$ so that U is an open neighborhood of x. Since, by Lemma 1.1, A is pseudoregular, there exists a $g \in A$ such that $g(x) = 0$ and $g(y) \geqq 1$ for every $y \notin U$. Then any positive $\delta < 1$ satisfies the requirements of the lemma.

THEOREM 5.8. *An algebra A is isomorphic to $C(X)$ for some topologically unique Q-space X if and only if* (i) *A is pseudoregular and* (ii) *A is isomorphic to every $\mathfrak{R}_w$-extension of A.*

Proof. In view of the remark preceding Lemma 5.7, the sufficiency of these conditions is clear. Conversely, let A be isomorphic to $C(X)$ for some Q-space X, and let $\mathcal{E}_w$ be the class of all $\mathcal{R}_w$-extensions of A. Clearly A satisfies (i) and $C(\mathcal{R}_A) \in \mathcal{E}_w$. To complete the proof it will suffice, in view of Lemma 5.2, to prove that if $B \in \mathcal{E}_w$, then $M \to M \cap A$ is an open mapping from $\mathcal{R}_B$ onto $\mathcal{R}_A$. Thus let $B \in \mathcal{E}_w$, let $M \in \mathcal{R}_B$, and let $\mathcal{U}$ be an open neighborhood of M. Since B is pseudoregular, there exists an $f \in M$ such that $N(f) \geqq 1$ for every $N \notin \mathcal{U}$; and thus there exist a $g \in M \cap A$ and a $\delta > 0$ such that $N(g) < \delta$ implies $N(f) < 1$, which in turn implies $N \in \mathcal{U}$. Then $\{N \cap A; N \in \mathcal{R}_B \text{ and } N(g) < \delta\}$ is an open neighborhood of $M \cap A$ whose inverse image is contained in $\mathcal{U}$.

6. Characterizations of $C(X)$ as a lattice-ordered algebra

Characterizations of $C(X)$ as a lattice-ordered algebra (or as a vector lattice) may be obtained by using techniques similar to those used in §5. Since the modifications required are slight, we shall, in this final section, merely indicate how this can be done.

By an *ideal* of a lattice-ordered algebra (vector lattice) A we shall mean a ring ideal (subspace) I of A with the following additional property: If $g \in I$ and if $f \in A$ with $|f| \leqq |g|$, then $f \in I$. The set of all real maximal ideals of A (in the appropriate sense) will again be denoted by $\mathcal{R}_A$. Suitable definitions of $\mathcal{R}$-extension and $\mathcal{R}_s$-extension are then obtained by merely replacing, in the former definitions, "algebra" by "lattice-ordered algebra" ("vector lattice"). Moreover, Theorem 3.2 and Lemmas 3.1, 3.4, 5.2, and 5.5 hold if, in their statements, we make these same replacements and if, in the case of a vector lattice A, we assume, instead of an identity, the existence of an element $e \in A$ such that $e \notin M$ for every $M \in \mathcal{R}_A$.[12] On the basis of these observations, and in view of Lemma 1.2, we obtain the following theorems.

THEOREM 6.1. *A lattice-ordered algebra A is isomorphic to the lattice-ordered algebra $C(X)$ for some topologically unique Q-space X if and only if* (i) *A is pseudoregular and* (ii) *A is isomorphic to every $\mathcal{R}_s$-extension of A.*

THEOREM 6.2.[13] *A vector lattice A is isomorphic to the vector lattice $C(X)$ for some topologically unique Q-space X if and only if* (i) *there is an element $e \in A$ such that $e \notin M$ for every $M \in \mathcal{R}_A$,* (ii) $\bigcap \mathcal{R}_A = 0$, *and* (iii) *$A$ is isomorphic to every $\mathcal{R}_s$-extension of A which satisfies* (i) *and* (ii).

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[12] We note also that a pseudoregular lattice-ordered algebra is, in the sense of [2], an f-algebra whose identity is a positive weak order unit.

[13] That conditions (i) and (ii) imply that A is isomorphic to a regular vector sublattice of $C(X)$ for a unique Q-space X is due to Shirota [10, Theorem 12].

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