EIGENFUNCTION EXPANSIONS IN L^p AND C

BY

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In L^p $(1 < p < \infty)$ [in L^1 or C] the Fourier series is (conditionally[1]) convergent [(C, θ) summable $(\theta > 0)$]. Known equiconvergence theorems extend this result to the eigenfunction expansions of operators of the form

$$T = D^u \qquad (u \geqq 2),$$

with u suitable boundary conditions. It will be shown that in cases where the sequence of eigenvalues of T also behaves reasonably, an operator $T + S$ will have the same type of spectral expansion as T. Here S can be

(i) any bounded linear operator;

(ii) in very restricted cases, any operator

$$B_{u-2} D^{u-2} + \cdots + B_1 D + B_0 ,$$

where the B_i are bounded linear operators.

The methods are those developed by Schwartz [4] and Kramer [2] for the L^2 case, but very much modified to deal with conditionally convergent or summable expansions.

1. Definitions and known results

$\mathfrak{B}$ will denote a Banach space with complex scalars. L^p will denote $L^p[0, 1]$ and C will denote $C[0, 1]$ (or one of its principal subspaces; see §5(1)). If T is a linear operator in $\mathfrak{B}$, the set of complex numbers λ such that $(T - \lambda I)^{-1}$ exists and is a bounded linear operator on $\mathfrak{B}$ to $\mathfrak{B}$ will be written $\rho(T)$ and called the resolvent set of T. The complement of $\rho(T)$ in the complex plane is $\sigma(T)$, the spectrum of T. $\mathfrak{D}(T)$ will denote the domain, and $\mathfrak{R}(T)$ the range, of T.

If G is a contour in $\rho(T)$, the following results are well known. (For our purposes it can be assumed that G is a circle or a rectangle.)

LEMMA 1.1. *The operator*

$$P = \frac{-1}{2\pi i} \int_G (T - \lambda I)^{-1} \, d\lambda$$

and its complement $I - P$ are bounded projections onto subspaces invariant under T. $\mathfrak{R}(P) \subseteqq \mathfrak{D}(T)$, and, as an operator in $\mathfrak{R}(P)$, T is bounded. The spectrum of T in $\mathfrak{R}(P)$ is the part of the spectrum of T lying inside G. If τ_1, τ_2 are disjoint components of $\sigma(T)$ and the curves G_1, G_2, G_3 contain respectively

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[1] See the examples on p. 212 of Zygmund [8]. (If $p = 2$, the convergence is unconditional.)

the components τ_1, τ_2, $\tau_1 \cup \tau_2$ *of* $\sigma(T)$, *then for the corresponding projections we have*

$$P_1 P_2 = 0; \qquad P_1 + P_2 = P_3. \tag{1.1}$$

Proof. See Taylor [6], especially Theorem 8.2 and the lines preceding the formulae (8.3).

If the contour G_i contains only one point of $\sigma(T)$, say λ_i, the dimension of $\mathfrak{R}(P_i)$ will be called the *generalised multiplicity* of λ_i. It is easily shown by induction on n that $\mathfrak{R}(P_i)$ contains the set

$$\mathfrak{B}(\lambda_i) = \bigcup_{n=1}^{\infty} \{x : (T - \lambda_i I)^n x = 0\},$$

the set of *proper vectors* of λ_i. Thus the dimension of $\mathfrak{B}(\lambda_i)$ (which will be called the algebraic multiplicity of λ_i) is at most the generalised multiplicity of λ_i. The term *eigenvector* will be reserved for a (nonzero) solution of $(T - \lambda I)x = 0$, and the term *eigenvalue* for a scalar λ for which eigenvectors exist. The *multiplicity* of λ is the dimension of its space of eigenvectors. The *point spectrum*, written $\pi(T)$, is the set of eigenvalues of T.

LEMMA 1.2. *If* $\mathfrak{R}(P)$ *is finite-dimensional, then* G *contains only a finite number of points of* $\sigma(T)$, *say* $\lambda_1, \lambda_2, \cdots, \lambda_n$, *and*

(i) *the generalised multiplicity of* λ_i, m_i *say, is finite and equal to its algebraic multiplicity and is the same whether we consider the operator* T *in* $\mathfrak{B}$ *or the operator* $T_{\mathfrak{R}(P)}$ *in* $\mathfrak{R}(P)$;

(ii) $\sum_1^n m_i = \dim \mathfrak{R}(P)$;

(iii) $\sum_1^n P_i = P$.

Proof. The part of $\sigma(T)$ inside G is the spectrum of T as an operator in the finite-dimensional space $\mathfrak{R}(P)$, which must be a finite set of points. Enclosing each of these points in a contour G_i we obtain (iii) from (1.1), and (ii) follows immediately. As $\mathfrak{R}(P_i)$ is finite-dimensional and the spectrum of T in $\mathfrak{R}(P_i)$ is the one-point set $\{\lambda_i\}$, we have

$$\mathfrak{R}(P_i) = \mathfrak{B}(\lambda_i),$$

which proves (i).

A result proved by Nagy [3] is

LEMMA 1.3. *Let* E, F *be bounded projections in a Banach space* $\mathfrak{B}$, *with* $\| E - F \| < 1$. *Then* $E\mathfrak{B}$ *and* $F\mathfrak{B}$ *have the same dimension if the dimension of either is finite.*

Proof. Let $(x_i)_{1 \leqq i \leqq m}$ be a basis for $E\mathfrak{B}$. We can write $Ex = \sum_1^m y_i(\,\cdot\,)x_i$ with $y_i \in \mathfrak{B}^*$ $(1 \leqq i \leqq m)$. If $\dim F\mathfrak{B} > m$, $F\mathfrak{B}$ contains a nonzero vector x orthogonal to $(y_i)_{1 \leqq i \leqq m}$. Thus $Fx = x$, $Ex = 0$, and

$$\| E - F \| \geqq \| (E - F)x \| / \| x \| = 1.$$

Thus $\dim F\mathfrak{B} \leqq m = \dim E\mathfrak{B}$, and the converse inequality is obtained by a similar argument.

We will say that an operator T in a Banach space $\mathfrak{B}$ ($= L^p$ or C) is *of the form D^u with the boundary conditions*

$$\sum_{j=0}^{u-1} \alpha_{ij} f^{(j)}(0) + \sum_{j=0}^{u-1} \beta_{ij} f^{(j)}(1) = 0, \tag{1.2}$$

where i runs over a finite index set I (possibly empty), if

(i) $\mathfrak{D}(T)$ is the set of functions f on $[0, 1]$ such that $f^{(j)}$ is absolutely continuous ($0 \leqq j \leqq u - 1$), $f^{(u)} \in \mathfrak{B}$, and (1.2) is satisfied for all $i \in I$;

(ii) for $f \in \mathfrak{D}(T)$, $Tf = f^{(u)}$.

LEMMA 1.4. *An operator T of the form D^u with the boundary conditions (1.2) is a closed linear operator.*

Proof. It is obvious from (i) and (ii) that T is linear. For the operator T_1 of the form D^u with boundary conditions

$$f^{(j)}(0) = f^{(j)}(1) = 0 \qquad (0 \leqq j \leqq u - 1)$$

it is easily seen that the convergence in $\mathfrak{B}$ norm of a sequence $T_1 f_n = f_n^{(u)}$ ($n \geqq 1$) in $\mathfrak{R}(T_1)$ implies the uniform convergence of $f_n^{(j)}$ ($0 \leqq j \leqq u - 1$) and the equation

$$(\lim f_n)^{(u)} = \lim f_n^{(u)}.$$

It follows that T_1 is closed, and, as the graph of T can be obtained from that of T_1 by removing a finite number of boundary conditions, T is closed. (This proof was adapted from that given by Schwartz [4].)

LEMMA 1.5. *For $f_1 \in C$, $g_1 \in L^1$ we have*

$$\| f_1 \| = \sup_{g \in L_1} | (f_1, g) | / \| g \|; \qquad \| g_1 \| = \sup_{f \in C} | (f, g_1) | / \| f \|,$$

where

$$(f, g) = \int_0^1 f(t) g(t)\, dt.$$

Proof. After approximating to g_1 with a continuous function the proof is obvious.

LEMMA 1.6. *The Fourier series of a function f in $\mathfrak{B} = L^1$ or C [$\mathfrak{B} = L^p$ ($1 < p < \infty$)] is (C, θ) summable ($\theta > 0$) [is convergent] to f in the norm of $\mathfrak{B}$.*

Proof. As regards C and L^p ($1 < p < \infty$) see Zygmund [8], §§3.3 and 7.3. As regards L^1: if $\theta > 0$ and $\mathfrak{B} = C$ or L^1, let $F_n = F_n(\mathfrak{B}) = F_n(\mathfrak{B}, \theta)$ be the operator in $\mathfrak{B}$ which maps each function in $\mathfrak{B}$ onto the (C, θ) mean of the first n partial sums of its Fourier series. We have $F_n(C)x \to x$ ($x \in C$) so that by Lemma 1.5 and the Banach-Steinhaus theorem,

$$\begin{aligned} \| F_n(L_1) \| &= \sup \{ | (F_n g, f) | : g \in L^1, f \in C, \| f \| = \| g \| = 1 \} \\ &= \sup \{ | (g, F_n f) | : g \in L^1, f \in C, \| f \| = \| g \| = 1 \} \\ &= \| F_n(C) \| \\ &= O(1) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $F_n g \to g$ on the dense subspace of L^1 consisting of trigonometric polynomials, $F_n g \to g$ for all g in L^1.

2. Axioms

All of the following axioms are satisfied by operators having the form D^u and domains restricted by u suitable boundary conditions. They have been separated into four groups needed for different purposes. Most of the axioms are relations between the distribution of the eigenvalues of T and the norms of its spectral projections.

The first group of axioms comprises those needed in showing the existence of the resolvent of T.

Condition (a).

(a_0) T is closed.

(a_1) $\pi(T)$ consists of a discrete sequence $(\lambda_n)_{n \geqq 1}$ of positive eigenvalues, each of multiplicity one.

(a_2) λ_n is monotone increasing and

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n \to \infty .$$

(a_3) There exist vectors $(\phi_j)_{j \geqq 1}$ in $\mathfrak{B}$ and $(\psi_i)_{i \geqq 1}$ in $\mathfrak{B}^*$ such that

$$(T - \lambda_j I)\phi_j \equiv 0 \quad \text{and} \quad \psi_i(\phi_j) \equiv \delta_{ij} .$$

(a_4) The eigenvectors $(\phi_j)_{j \geqq 1}$ form a total set.

(a_5) For the projections

$$E(\lambda_j) = \psi_j(\,\cdot\,)\phi_j ,$$

$$E_n = \textstyle\sum_1^n E(\lambda_j) = \sum_1^n \psi_j(\,\cdot\,)\phi_j ,$$

we have $\| E_n \| = o(\lambda_n)$ and

$$\sum_1^\infty \frac{\| E_n \| \, \Delta\lambda_n}{\lambda_{n+1} \cdot \lambda_n} = J < \infty .$$

The main difficulty in discussing the perturbation of T by a bounded linear operator B is to estimate $\| (T - \lambda I)^{-1} \|$ on suitable contours. For this purpose we need the following axioms:

Condition (b). There exists a real number δ such that

(b_1) $\quad 0 < \delta < \frac{1}{2}$;

(b_2) $$\sum_1^\infty \frac{\| E_n \| \, \Delta\lambda_n}{\lambda_{n+1} \cdot \lambda_n^\delta} = K < \infty ;$$

(b_3) $$\sum_1^\infty \frac{\| E_n \| \, \Delta\lambda_n}{| \mu_r - \lambda_{n+1} | \cdot | \mu_r - \lambda_n |^\delta} = H_r < H < \infty ,$$

where $\mu_r = \frac{1}{2}(\lambda_r + \lambda_{r+1})$ and H is independent of r.

The following condition simplifies the discussion of the relation of the new eigenvalues to the old.

Condition (c). There exists a number $l > 1$ such that, for $n > 1$,

(2.1) $$\Delta\lambda_{n-1} < l\Delta\lambda_n .$$

In discussing the perturbation of T by an operator S which is not necessarily bounded but whose domain includes that of T^ν, for some ν with

(2.2) $$0 < \nu < 1,$$

we must estimate not only $\| R_\lambda \|$ but also $\| T^\nu R_\lambda \|$. We therefore require the following additional axioms:

Condition (d). For some choice of ν and δ (satisfying (b) and (2.2)) there exists a number τ such that

(d_1) $0 < \tau < 1 - \delta$;

(d_2) $\sum_1^\infty \| E_n \| \cdot | \Delta(\lambda_n^{-\nu}) | = P < \infty$; $\quad \| E_n \| = o(\lambda_n^\nu)$;

(d_3) $$\sum_1^\infty \frac{\lambda_n^\nu \| E(\lambda_n) \|}{\lambda_n^\tau} = G < \infty;$$

(d_4) $$\sum_1^\infty \frac{\lambda_n^\nu \| E(\lambda_n) \|}{| \mu_r - \lambda_n |^\tau} = F_r < F < \infty,$$

where F is independent of r.

By (a_1) and (a_2) the following contours pass through no point λ_n :

The contour Γ_r ($r \geqq 1$) is the square with centre the origin and sides parallel to the axes, whose right-hand vertical side (written V_r) passes through

(2.3) $$\mu_r = \tfrac{1}{2}(\lambda_r + \lambda_{r+1}).$$

The contour formed by the upper, left-hand, and lower sides of Γ_r will be written L_r.

The contour Ω_r (defined if (c) holds and $r > 1$) is a circle with centre λ_r and a radius ρ_r satisfying the condition

(2.4) $$\rho_r \leqq \Delta\lambda_{r-1}/2l.$$

Thus by (c), $\rho_r \leqq \inf \{\tfrac{1}{2}\Delta\lambda_{r-1}, \tfrac{1}{2}\Delta\lambda_r\}$.

LEMMA 2.1. *Let* (a_1) *and* (a_2) *be satisfied. Then*

(i) *there exists a number* $k > 0$ *such that for all* r, *all* $\lambda \in L_r$, *and all* n,

$$| \lambda - \lambda_n | > k\lambda_n , \qquad | \lambda - \lambda_n | > \tfrac{1}{2}\mu_r ;$$

(ii) *for all* r, *all* λ *on* V_r (*say* $\lambda = \mu_r + i\eta$) *and all* n,

$$| \lambda - \lambda_n | \geqq | \mu_r - \lambda_n | \geqq \tfrac{1}{2}\Delta\lambda_r , \qquad | \lambda - \lambda_n | > | \eta |.$$

Proof. The proof is obvious from (a_1).

LEMMA 2.2. *Let* (a) *and* (c) *be satisfied. Then for all* $r > 1$, *all* λ *on* Ω_r, *and all* $n \neq r$ *we have*

$$| \lambda - \lambda_n | \geqq \Delta\lambda_{r-1}/3l, \qquad | \lambda - \lambda_n | \geqq | \mu_{r-1} - \lambda_n |/3l.$$

Proof. By (c) we have

$$\begin{aligned}(1/2l)\Delta\lambda_{r-1} &\leqq \inf\left(\tfrac{1}{2}\Delta\lambda_{r-1}, \tfrac{1}{2}\Delta\lambda_r\right)\\ &\leqq \inf\left(\tfrac{1}{2}|\lambda - \lambda_{r-1}|, \tfrac{1}{2}|\lambda - \lambda_{r+1}|\right)\\ &\leqq |\lambda - \lambda_n|,\end{aligned}$$

by the definition of Ω_r and (a_2).

If $n < r$ we clearly have $|\lambda - \lambda_n| > |\mu_{r-1} - \lambda_n|$. If $n > r$,

$$\begin{aligned}|\mu_{r-1} - \lambda_n| &\leqq |\mu_{r-1} - \lambda_r| + |\lambda_r - \lambda| + |\lambda - \lambda_n|\\ &\leqq \tfrac{1}{2}\Delta\lambda_{r-1} + \tfrac{1}{2}\Delta\lambda_r + |\lambda - \lambda_n|\\ &\leqq \tfrac{1}{2}(l+1)\Delta\lambda_r + |\lambda - \lambda_n|\\ &\leqq 3l|\lambda - \lambda_n|.\end{aligned}$$

3. Operational calculus

Let $\mathfrak{B}_0$ be the normed linear space consisting of *finite* linear combinations of eigenvectors of T. Let T_0 be the restriction of T to $\mathfrak{B}_0$. We will set up a natural operational calculus for T_0 and extend it (subject to (a_4)) to an operational calculus for T.

We write the eigenvalues of T as $(\lambda_i)_{i\geqq 1}$ and write $E_0(\lambda_i)$ for the projection of $\mathfrak{B}_0$ onto the set of eigenvectors of λ_i parallel to the eigenvectors of all other eigenvalues. If $f(\lambda)$ is any function defined on the set $(\lambda_i)_{i\geqq 1}$ we may now define

$$f(T_0) = \textstyle\sum_1^\infty f(\lambda_i)E_0(\lambda_i). \tag{3.1}$$

If the functions $f(\lambda_i) = 1,\ 1/(\lambda_i - \mu),\ \lambda_i,\ \lambda_i^\nu,\ \lambda_i^{-\nu},\ \lambda_i^\nu(\lambda_i - \mu)^{-1}$, are defined, we write the corresponding operators $f(T_0) = I_0,\ (T_0 - \mu I_0)^{-1},\ T_0,\ T_0^\nu,\ T_0^{-\nu},\ T_0^\nu(T_0 - \mu I_0)^{-1}$.

THEOREM 3.1. *For the correspondence $f(\lambda) \to f(T_0)$ defined above,*

(i) *$\alpha f(\lambda) + \beta g(\lambda) \to \alpha f(T_0) + \beta g(T_0)$, for all scalars α, β;*

(ii) *$f(\lambda)g(\lambda) \to f(T_0)g(T_0)$;*

(iii) *$f(T_0)$ has an inverse (not necessarily bounded) on $\mathfrak{B}_0$ to $\mathfrak{B}_0$ if and only if $f(\lambda_i)$ never vanishes, in which case $1/f(\lambda) \to f(T_0)^{-1}$;*

(iv) *if $E_0(\lambda_i)$ is bounded for all i and*

$$\textstyle\sum_1^\infty |f(\lambda_i)| \cdot \|E_0(\lambda_i)\| < \infty, \tag{3.2}$$

the series (3.1) converges in operator norm to $f(T_0)$ (which is thus bounded);

(v) *if $|f(\lambda_i)| \cdot \|E_{0i}\| \to 0$ as $i \to \infty$ and E_{0i} is bounded for all i and*

$$\textstyle\sum_1^\infty |\Delta f(\lambda_n)| \cdot \|E_{0n}\| < \infty, \tag{3.3}$$

where $E_{0n} = \sum_1^n E_0(\lambda_i)$, then the series

$$-\textstyle\sum_1^\infty E_{0n}\,\Delta f(\lambda_n) \tag{3.4}$$

converges to $f(T_0)$ in operator norm.

Proof. Since eigenvectors of distinct eigenvalues are linearly independent, (i), (ii), and (iii) are obvious. If the conditions of (iv) are satisfied, we have

$$\begin{aligned}\| f(T_0) - \textstyle\sum_1^N f(\lambda_i)E_0(\lambda_i) \| &= \sup_{\|x\|=1, x\in\mathfrak{B}_0} \| \textstyle\sum_{N+1}^\infty f(\lambda_i)E_0(\lambda_i)x \| \\ &\leqq \textstyle\sum_{N+1}^\infty | f(\lambda_i) | \cdot \| E_0(\lambda_i) \| \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

This proves (iv), and the proof of (v) is similar. For if we write $x = \sum_1^m E_0(\lambda_i)x$ with $m = m(x, N) > N$ we have

$$\begin{aligned}f(T_0)x &= \textstyle\sum_1^m f(\lambda_i)E(\lambda_i)x \\ &= \textstyle\sum_1^N E_{0i}\, \Delta f(\lambda_i)x - \sum_{N+1}^{m-1} E_{0i}\, \Delta f(\lambda_i)x + f(\lambda_m)E_{0m}\, x.\end{aligned}$$

The following lemma contains our *definition* of $f(T)$.

LEMMA 3.1. *Let* (a_4) *be satisfied. Let* $f(T_0)$ *be bounded. Then the closure* $\overline{f(T_0)}$ *of* $f(T_0)$ *is the unique bounded linear operator* $f(T)$ *on* $\mathfrak{B}$ *to* $\mathfrak{B}$ *whose restriction to* $\mathfrak{B}_0$ *is* $f(T_0)$.

Proof. The proof is obvious.

LEMMA 3.2. *Let* (a) *be satisfied. Then*

(i) $\overline{E_0(\lambda_i)} = E(\lambda_i)$; $\quad \overline{E_{0i}} = E_i$;

(ii) if $\lambda \notin (\lambda_n)_{n\geqq 1}$, *then*

$$(T - \lambda I)^{-1} = \sum_1^\infty \frac{E_n\, \Delta\lambda_n}{(\lambda - \lambda_{n+1})(\lambda - \lambda_n)},$$

with convergence in operator norm;

(iii) $\sigma(T) = \pi(T) = (\lambda_n)_{n\geqq 1}$;

(iv) $\overline{(T_0 - \lambda I_0)^{-1}} = (T - \lambda I)^{-1}$;

(v) *for* $r > 1$,

$$E_r = \frac{-1}{2\pi i}\int_{\Gamma_r} (T - \lambda I)^{-1}\, d\lambda, \qquad E(\lambda_r) = \frac{-1}{2\pi i}\int_{\Omega_r} (T - \lambda I)^{-1}\, d\lambda.$$

Proof. (i) It is obvious from the definitions of $E_0(\lambda_i)$ (see p. 87) and $E(\lambda_i)$ (see (a_5)) that $E(\lambda_i)$ is bounded and $E_0(\lambda_i)$ is the restriction of $E(\lambda_i)$ to $\mathfrak{B}_0$. The analogous facts about E_i and E_{0i} are proved in the same way. Thus Lemma 3.1 gives both results.

(ii) By (a_5) the series converges to a bounded linear operator, S_λ say, on $\mathfrak{B}$ to $\mathfrak{B}$. By Theorem 3.1 (ii), (iii), and (iv), the restriction of S_λ to $\mathfrak{B}_0$ is $(T_0 - \lambda I_0)^{-1}$. Thus by Lemma 3.1,

$$S_\lambda = \overline{(T_0 - \lambda I_0)^{-1}},$$

so that as T is closed S_λ is contained in $(T - \lambda I)^{-1}$. Since $(T - \lambda I)^{-1}$ is single-valued by (a_1), and $\mathfrak{D}(S_\lambda) = \mathfrak{B}$, $S_\lambda = (T - \lambda I)^{-1}$. This proves (ii), (iii), and (iv).

(v) By (a_1) and (a_2), Γ_r contains $\lambda_1, \lambda_2, \cdots, \lambda_r$, and the set $(\lambda_n)_{n>r}$ lies outside Γ_r. Since (a_2) and (a_5) imply that the series in (ii) converges uniformly on Γ_r, we have

$$\begin{aligned}\frac{-1}{2\pi i}\int_{\Gamma_r}(T-\lambda I)^{-1}\,d\lambda &= \frac{-1}{2\pi i}\int_{\Gamma_r}\sum_1^\infty \frac{E_n\,\Delta\lambda_n}{(\lambda-\lambda_{n+1})(\lambda-\lambda_n)}\,d\lambda \\ &= \sum_1^\infty E_n\,\Delta\lambda_n\left(\frac{-1}{2\pi i}\int_{\Gamma_r}\frac{d\lambda}{(\lambda-\lambda_{n+1})(\lambda-\lambda_n)}\right) \\ &= E_r\,.\end{aligned}$$

The discussion for Ω_r is similar.

LEMMA 3.3. *Let* (a) *and* (b) *be satisfied. Then*
(i) *if* λ *lies on* L_r,

$$\|(T-\lambda I)^{-1}\| = O(|L_r|^{-1+\delta}),$$

independently of λ;
(ii) *if* λ *lies on* V_r, *say* $\lambda = \mu_r + i\eta$,

$$\|(T-\lambda I)^{-1}\| = O(|\eta|^{-1+\delta}), \qquad \|(T-\lambda I)^{-1}\| = O((\Delta\lambda_r)^{-1+\delta}),$$

independently of λ;
(iii) *if* (c) *holds and* λ *lies on the circle* Ω_r,

$$\|(T-\lambda I)^{-1}\| = O\left((\Delta\lambda_{r-1})^{-1+\delta} + \frac{\|E_{r-1}\| + \|E_r\|}{\rho_r}\right),$$

independently of λ *and* ρ_r.

Proof. (i) By Lemmas 3.2 (ii) and 2.1,

$$\begin{aligned}\|(T-\lambda I)^{-1}\| &\leqq \sum_1^\infty \frac{\|E_n\|\,\Delta\lambda_n}{|\lambda-\lambda_{n+1}|\cdot|\lambda-\lambda_n|} \\ &\leqq \frac{1}{\inf|\lambda-\lambda_n|^{1-\delta}}\sum_1^\infty \frac{\|E_n\|\,\Delta\lambda_n}{|\lambda-\lambda_{n+1}|\cdot|\lambda-\lambda_n|^\delta} \\ &\leqq \frac{2^{1-\delta}k^{-1-\delta}}{\mu_r^{1-\delta}}\sum_1^\infty \frac{\|E_n\|\,\Delta\lambda_n}{\lambda_{n+1}\,\lambda_n^\delta} \\ &= O(\mu_r^{-1+\delta}),\end{aligned}$$

by (b_2). And $|L_r| = 6\mu_r$ by the definition of L_r.
(ii) To the second line of the proof of (i) apply Lemma 2.1 (ii) to obtain

$$\begin{aligned}\|(T-\lambda I)^{-1}\| &\leqq \frac{1}{\sup(|\eta|, \frac{1}{2}\Delta\lambda_r)^{1-\delta}}\sum_1^\infty \frac{\|E_n\|\,\Delta\lambda_n}{|\mu_r-\lambda_{n+1}|\cdot|\mu_r-\lambda_n|^\delta} \\ &\leqq \frac{H}{\sup(|\eta|, \frac{1}{2}\Delta\lambda_r)^{1-\delta}},\end{aligned}$$

by (b_3).

(iii) Lemma 3.2 (ii) gives the series

$$\sum_{1}^{r-2} \frac{E_n \Delta\lambda_n}{(\lambda - \lambda_{n+1})(\lambda - \lambda_n)} - \frac{E_{r-1}}{\lambda - \lambda_{r-1}} + \frac{E_{r-1} - E_r}{\lambda - \lambda_r} + \frac{E_r}{\lambda - \lambda_{r+1}} + \sum_{r+1}^{\infty} \frac{E_n \Delta\lambda_n}{(\lambda - \lambda_{n+1})(\lambda - \lambda_n)}$$

for $(T - \lambda I)^{-1}$. Thus by Lemma 2.2 and the relations $|\lambda - \lambda_r| = \rho_r$, $|\lambda - \lambda_{r\pm 1}| \geqq \rho_r$ (see the definition of Ω_r),

$$\begin{aligned} &\| (T - \lambda I)^{-1} \| \\ &\leqq \left(\sum_{1}^{r-2} + \sum_{r+1}^{\infty} \right) \frac{\| E_n \| \Delta\lambda_n (3l)^2}{| \mu_{r-1} - \lambda_{n+1} | \cdot | \mu_{r-1} - \lambda_n |^{\delta} (\Delta\lambda_{r-1})^{1-\delta}} + 2 \frac{\| E_{r-1} \| + \| E_r \|}{\rho_r} \\ &\leqq \frac{(3l)^2 H_{r-1}}{(\Delta\lambda_{r-1})^2} + 2 \frac{\| E_{r-1} \| + \| E_r \|}{\rho_r} \qquad (r > 1), \end{aligned}$$

by (b_3).

LEMMA 3.4. *If* (d) *is satisfied,* $\overline{T_0^{-\nu}}$ *is bounded.*

Proof. Apply Theorem 3.1 (v) to (d_2).

We will *write* $T^{-\nu}$ *for* $\overline{T_0^{-\nu}}$ (cf. Lemma 3.1). Following Kramer [2] we prove

LEMMA 3.5. *If S is a closed linear operator in* $\mathfrak{B}$, *if* (d) *is satisfied, and if* $\mathfrak{D}(S) \supseteqq \mathfrak{R}(T^{-\nu})$, *then* $ST^{-\nu}$ *is bounded.*

Proof. $ST^{-\nu}$ is closed and $\mathfrak{D}(ST^{-\nu}) = \mathfrak{B}$ (by the previous lemma) so that $ST^{-\nu}$ is bounded by the Closed Graph Theorem.

LEMMA 3.6. *Let* (a), (b), *and* (d) *be satisfied. Let* $\mathfrak{D}(S) \supseteqq \mathfrak{R}(T^{-\nu})$. *Then*

(i) *if* λ *lies on* L_r,

$$\| S(T - \lambda I)^{-1} \| = O(| L_r |^{-1+\tau}),$$

independently of λ;

(ii) *if* λ *lies on* V_r, *say* $\lambda = \mu_r + i\eta$

$$\| S(T - \lambda I)^{-1} \| = O((\Delta\lambda_r)^{-1+\tau}), \qquad \| S(T - \lambda I)^{-1} \| = O(| \eta |^{-1+\tau}),$$

independently of λ;

(iii) *if* (c) *holds and* λ *lies on* Ω_r,

$$\| S(T - \lambda I)^{-1} \| = O((\Delta\lambda_{r-1})^{-1+\tau} + \rho_r^{-1} \| E(\lambda_r) \|),$$

independently of λ *and* ρ_r.

Proof. Since $S(T - \lambda I)^{-1}$ is closed, an argument on the lines of Lemma 3.1 shows that its order is the same as that of $S(T_0 - \lambda I_0)^{-1}$, if the latter is bounded. By Theorem 3.1,

$$S(T_0 - \lambda I_0)^{-1} = ST_0^{-\nu} T_0^{\nu} (T_0 - \lambda I_0)^{-1},$$

so that by Lemma 3.5 it is sufficient to estimate the order of

$$\| T_0^\nu (T_0 - \lambda I_0)^{-1} \| = \left\| \sum_1^\infty \frac{\lambda_n^\nu E_0(\lambda_n)}{\lambda_n - \lambda} \right\|.$$

(i) Using Lemma 2.1 (i) we obtain

$$\| T_0^\nu (T_0 - \lambda I_0)^{-1} \| \leqq \frac{2^{1-\tau}}{\mu_r^{1-\tau}} \sum_1^\infty \frac{\lambda_n^\nu \| E_0(\lambda_n) \|}{k^\tau \lambda_n^\tau} \leqq \frac{G 2^{1-\tau}}{k^\tau \mu_r^{1-\tau}},$$

by (d_3).

(ii) Using Lemma 2.1 (ii) we obtain

$$\| T_0^\nu (T_0 - \lambda I_0)^{-1} \| \leqq \frac{1}{\sup (|\eta|, \tfrac{1}{2}\Delta\lambda_r)^{1-\tau}} \sum_1^\infty \frac{\lambda_n^\nu \| E(\lambda_n) \|}{|\mu_r - \lambda_n|^\tau}$$
$$\leqq \frac{F}{\sup (|\eta|, \tfrac{1}{2}\Delta\lambda_r)^{1-\tau}}$$

by (d_4).

(iii) Using Lemma 2.2,

$$\| T_0^\nu (T_0 - \lambda I_0)^{-1} \| \leqq \frac{3}{(\Delta\lambda_{r-1})^{1-\tau}} \left(\sum_1^{r-1} + \sum_{r+1}^\infty \right) \frac{\lambda_n^\nu \| E(\lambda_n) \|}{|\mu_{r-1} - \lambda_n|^\tau} + \frac{\| E(\lambda_r) \|}{\rho_r}$$
$$\leqq \frac{3lF}{(\Delta\lambda_{r-1})^{1-\tau}} + \frac{\| E(\lambda_r) \|}{\rho_r} \qquad (r > 1)$$

by (d_4).

4. The perturbation theorems

THEOREM 4.1. *Let T satisfy* (a) *and* (b). *Let Γ_r $(r \geqq 1)$ be the contour defined on p. 86. Let S be any bounded linear operator on $\mathfrak{B}$ into $\mathfrak{B}$. Then*

(i) *$\sigma(T + S)$ consists of a sequence of eigenvalues ν_i $(i \geqq k + 1)$ of*[2] *generalised multiplicity $m_i < \infty$;*

and for r sufficiently large

(ii) *$\sum_{k+1}^r m_i = r$, $m_r = 1$ and ν_r lies between Γ_{r-1} and Γ_r (and if* (c) *is satisfied and $\| E_{r-1} \|^2 + \| E_r \|^2 = o(\Delta\lambda_{r-1})$,*

$$\nu_r = \lambda_r + O(\| E_{r-1} \|^2 + \| E_r \|^2)); \tag{4.1}$$

(iii) *the projection*

$$E_r' = \frac{-1}{2\pi i} \int_{\Gamma_r} (T + S - \lambda I)^{-1} \, d\lambda$$

exists;

(iv) *there are sets of linearly independent vectors $(\Phi_{pq})_{p \geqq k+1, 1 \leqq q \leqq m_p}$ in $\mathfrak{B}$*

[2] k should be regarded as the number of eigenvalues of T which disappear, by amalgamating with other eigenvalues, when S is added to T.

and $(\Psi_{ij})_{i\geqq k+1, 1\leqq j\leqq m_i}$ *in* $\mathfrak{B}^*$ *such that* $(T + S - \nu_p I)^{m_p}\Phi_{pq} \equiv 0$ *and* $\Psi_{ij}(\Phi_{pq}) \equiv \delta_{ip}\,\delta_{jq}$, *and*

$$E'_r = \textstyle\sum_{p\leqq r} \Psi_{pq}(\,\cdot\,)\Phi_{pq}\,;$$

(v) $\| E'_r - E_r \| \to 0$.

Proof. (iii) Write R_λ for $(T - \lambda I)^{-1}$. By (a_2) and Lemma 3.3, $\| SR_\lambda \| < \frac{1}{2}$ for λ on $\Gamma_r = L_r \cup V_r$, for r sufficiently large. Thus $(I + SR_\lambda)^{-1}$ exists and

$$\begin{aligned} \| (I + SR_\lambda)^{-1} \| &= \| I - (SR_\lambda) + (SR_\lambda)^2 - \cdots \| \\ &\leqq 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + \cdots = 2. \end{aligned} \tag{4.2}$$

With Lemma 3.2 (iii) this proves the existence on Γ_r of the right-hand side of the equation

$$(T + S - \lambda I)^{-1} = R_\lambda - R_\lambda(I + SR_\lambda)^{-1}SR_\lambda\,, \tag{4.3}$$

which can now be verified by multiplying the right-hand side (on the left and on the right) by $T + S - \lambda I$. Thus $(T + S - \lambda I)^{-1}$ exists on Γ_r. This proves (iii).

(v) By (iii), Lemma 3.2 (v), equations (4.3) and (4.2), and Lemma 3.3,

$$\begin{aligned} \| E'_r - E_r \| &= \frac{1}{2\pi}\left\| \int_{\Gamma_r} ((T + S - \lambda I)^{-1} - (T - \lambda I)^{-1})\, d\lambda \right\| \\ &\leqq \frac{1}{2\pi}\int_{\Gamma_r} \| S \| \cdot \| R_\lambda \|^2 \cdot \| (I + SR_\lambda)^{-1} \| \cdot | d\lambda | \\ &\leqq \frac{\| S \|}{\pi}\int_{\Gamma_r} \| R_\lambda \|^2\, | d\lambda | \\ &= \frac{\| S \|}{\pi}\left\{ \int_{L_r} \| R_\lambda \|^2 | d\lambda | + \int_{-\Delta\lambda_r}^{\Delta\lambda_r} \| R_{\mu_r + i\eta} \|^2\, d\eta \right. \\ &\qquad \left. + \left(\int_{-\mu_r}^{-\Delta\lambda_r} + \int_{\Delta\lambda_r}^{\mu_r} \right) \| R_{\mu_r + i\eta} \|^2\, d\eta \right\} \\ &= O(| L_r |^{-1+2\delta}) + O((\Delta\lambda_r)^{-1+2\delta}), \end{aligned}$$

since

$$\int_{\Delta\lambda_r}^{\mu_r} \eta^{-2+2\delta}\, d\eta = \Big[(-1 + 2\delta)^{-1}\eta^{-1+2\delta} \Big]_{\Delta\lambda_r}^{\mu_r}.$$

The result follows from (a_2) and (b_1).

(i), (ii), (iv). By (v) and Lemma 1.3, E_r and E'_r project onto subspaces of the same dimension—which by (a_5) is r—for all r sufficiently large. Thus Lemma 1.2 proves (ii) (apart from the bracket), (i), and the existence of a linearly independent set $(\Phi_{pq})_{k+1\leqq p, 1\leqq q\leqq m_p}$ such that $(T + S - \nu_p I)^{m_p}\Phi_{pq} \equiv 0$ and the Φ_{pq} with $p \leqq r$ span $E'_r\,\mathfrak{B}$. We can write any $x \in E'_r\,\mathfrak{B}$ in the form

$\sum_{p \leqq r} x_{pq} \Phi_{pq}$ where the x_{pq} are uniquely determined scalars. Since $\dim E_r' \mathfrak{B} < \infty$, the mappings $x \to x_{pq}$ are bounded linear functionals on $E_r' \mathfrak{B}$. Thus the linear functional Ψ_{pq} on $\mathfrak{B}$ defined by

$$\Psi_{pq}(x) = (E_r' x)_{pq}$$

for some chosen r greater than p has the required properties, for the following argument shows that the definition is independent of the choice of r. Take $p \leqq r \leqq r_0$; then

$$E_r' x = E_r'(E_{r_0}' x) = E_r' \sum_{p \leqq r_0} (E_{r_0}' x)_{pq} \Phi_{pq} = \sum_{p \leqq r} (E_{r_0}' x)_{pq} \Phi_{pq},$$

by Lemmas 1.1 and 1.2.

(ii) Let (c) be satisfied, and let ρ_r (subject to (2.4)) have any order strictly higher than $\| E_{r-1} \|^2 + \| E_r \|^2$. We can obtain (4.2) and (4.3) as before, but now with λ on the circle Ω_r of centre λ_r and radius ρ_r. Thus if we write

$$E'(r) = \frac{-1}{2\pi i} \int_{\Omega_r} (T + S - \lambda I)^{-1} \, d\lambda,$$

we obtain, by the argument for (v), and (2.4), (a_2) and (b_1),

$$\begin{aligned} \| E'(r) - E(\lambda_r) \| &\leqq \frac{\| S \|}{\pi} \int_{\Omega_r} \| R_\lambda \|^2 \, | d\lambda | \\ &= O\left(\rho_r (\Delta\lambda_{r-1})^{-2+2\delta} + \frac{\| E_{r-1} \|^2 + \| E_r \|^2}{\rho_r} \right) \\ &\to 0. \end{aligned}$$

Thus by Lemma 1.3, $E'(r)$ and $E(\lambda_r)$ project onto subspaces of the same dimension—which by (a_5) is one—for all r sufficiently large. Thus Ω_r contains an eigenvalue of $T + S$, which must be ν_r. In other words,

$$\nu_r = \lambda_r + O(\rho_r)$$

whenever ρ_r has a strictly higher order than $\| E_{r-1} \|^2 + \| E_r \|^2$. Thus

$$\nu_r = \lambda_r + O(\| E_{r-1} \|^2 + \| E_r \|^2).$$

COROLLARY 4.1. *Let T and S satisfy the conditions of the theorem, and let the vectors ϕ_j, ψ_i, Φ_{pq}, Ψ_{pq} be those defined in* (a_3) *and part* (iv) *of the theorem. Let x be a vector in $\mathfrak{B}$ such that*

$$\sum_1^\infty \psi_i(x) \phi_i = x,$$

in the sense of strong or weak convergence or strong or weak (C, θ) summability. Then

$$\sum_{p=k+1}^\infty \sum_{q=1}^{m_p} \Psi_{pq}(x) \Phi_{pq} = x,$$

in the same sense.

Proof. The proof is obvious from part (v) of the theorem (with the definitions in (a_5) and part (iv) of the theorem).

THEOREM 4.2. *Let T satisfy* (a), (b) *and* (d) (*for some ν, $0 < \nu < 1$*). *Let S be a finite sum of operators of the form $B_\alpha S_\alpha$, where B_α is a bounded linear operator on $\mathfrak{B}$ into $\mathfrak{B}$ and S_α is a closed linear operator in $\mathfrak{B}$ such that $\mathfrak{D}(S_\alpha) \geqq \mathfrak{R}(T^{-\nu}) \cup \mathfrak{D}(T)$. Then the conclusions* (i) *to* (v) *of Theorem* 4.1 *are true if we insert before* (4.1) *the words,*

$$\text{“and } \| E_{r-1} \| + \| E_r \| = o(\Delta\lambda_{r-1})^{1-\tau}\text{”}.$$

Proof. We can apply Lemma 3.6, with S_α replacing S.

(iii) Use the proof of Theorem 4.1 (iii), with Lemma 3.6 instead of Lemma 3.3.

(v) By (iii), Lemma 3.2 (v), equations (4.3) and (4.2), and Lemmas 3.3 and 3.6,

$$\begin{aligned}
\| E_r' - E_r \| &= \frac{1}{2\pi} \left\| \int_{\Gamma_r} ((T + S - \lambda I)^{-1} - (T - \lambda I)^{-1})\, d\lambda \right\| \\
&\leqq \frac{1}{2\pi} \int_{\Gamma_r} \| R_\lambda \| \cdot \| (I + SR_\lambda)^{-1} \| \cdot \| SR_\lambda \| \cdot |d\lambda| \\
&\leqq \frac{1}{\pi} \left\{ \int_{L_r} \| R_\lambda \| \cdot \| SR_\lambda \| \cdot |d\lambda| + \int_{-\Delta\lambda_r}^{\Delta\lambda_r} \| R_{\mu_r + i\eta} \| \cdot \| SR_{\mu_r + i\eta} \|\, d\eta \right. \\
&\qquad \left. + \left(\int_{-\mu_r}^{-\Delta\lambda_r} + \int_{\Delta\lambda_r}^{\mu_r} \right) \| R_{\mu_r + i\eta} \| \cdot \| SR_{\mu_r + i\eta} \|\, d\eta \right\} \\
&= O\left\{ |L_r| \cdot |L_r|^{-1+\delta} |L_r|^{-1+\tau} + \int_{-\Delta\lambda_r}^{\Delta\lambda_r} (\Delta\lambda_r)^{-1+\delta} (\Delta\lambda_r)^{-1+\tau}\, d\eta \right. \\
&\qquad \left. + \int_{\Delta\lambda_r}^{\mu_r} \eta^{-1+\delta} \eta^{-1+\tau}\, d\eta \right\} \\
&= O(|L_r|^{-1+\delta+\tau}) + O((\Delta\lambda_r)^{-1+\delta+\tau}) \\
&\to 0,
\end{aligned}$$

by (a_2) and (d_1).

(i), (ii), (iv). Since the resolvent set of $T + S$ is not empty, $T + S$ is closed. Thus we may follow the discussion of the corresponding parts of Theorem 4.1.

(ii) Use the first four lines of the proof of this part of Theorem 4.1. Then we obtain, by (2.4), (a_2) and (d_1), and the argument for (v),

$$\begin{aligned}
\| E'(r) - E(\lambda_r) \| &\leqq \frac{1}{\pi} \int_{\Omega_r} \| R_\lambda \| \cdot \| SR_\lambda \| \cdot |d\lambda| \\
&= O\left(\left\{ (\Delta\lambda_{r-1})^{-1+\delta} + \frac{\| E_{r-1} \| + \| E_r \|}{\rho_r} \right\} \left\{ (\Delta\lambda_{r-1})^{-1+\tau} + \frac{\| E(\lambda_r) \|}{\rho_r} \right\} \rho_r \right) \\
&= O((\Delta\lambda_{r-1})^{-1+\delta+\tau} + (\Delta\lambda_{r-1})^{-1+\tau} (\| E_{r-1} \| + \| E_r \|) \\
&\qquad + (\Delta\lambda_{r-1})^{-1+\delta} \| E(\lambda_r) \| + \| E(\lambda_r) \| (\| E_{r-1} \| + \| E_r \|) \rho_r^{-1}) \\
&\to 0.
\end{aligned}$$

We may now conclude with the last six lines of the proof of Theorem 4.1.

COROLLARY 4.2. *The statement is verbally the same as that of Corollary* 4.1.

Proof. The proof is the same as before.

5. Applications and extensions

(i) Let T be a Sturm-Liouville operator of the simplest type, i.e., an operator of the form $-D^2$ with boundary conditions

$$f(0) \cos \varphi + f'(0) \sin \varphi = 0, \qquad f(1) \cos \psi + f'(0) \sin \psi = 0,$$

where φ and ψ are real numbers. The following facts are well known (see for example Titchmarsh [7]): T has simple eigenvalues, and the eigenfunction expansion of any integrable function f is uniformly equiconvergent with its Fourier series expansion. If the boundary conditions satisfy $\sin \varphi \sin \psi \neq 0$ ($\sin \varphi = 0$, $\sin \psi \neq 0$) ($\sin \varphi \neq 0$, $\sin \psi = 0$) ($\sin \varphi = \sin \psi = 0$), the term "Fourier series" means an expansion in terms of $\cos n\pi x$ ($\sin (n + \frac{1}{2})\pi x$) ($\cos (n + \frac{1}{2})\pi x$) ($\sin n\pi x$). The space C is the space of all continuous functions on [0, 1] (all which vanish at 0) (all which vanish at 1) (all which vanish at both 0 and 1). Thus if f is in $\mathfrak{B} = C$ or L^1 [$\mathfrak{B} = L^p$ $(1 < p < \infty)$] its eigenfunction expansion is (C, θ) summable for all $\theta > 0$ [its expansion is convergent] in the norm of $\mathfrak{B}$. Thus we have, by the Banach-Steinhaus theorem and the limitation theorem for Cesàro summability (Hardy [1], Theorem 4.6),

$$\| E_n \| = O(n^\theta) \qquad (\theta > 0) \quad [\| E_n \| = O(1)], \tag{5.1}$$

where E_n is the projection in $\mathfrak{B}$ mapping each function onto the n^{th} partial sum of its Sturm-Liouville expansion. As we have also

$$\lambda_n \sim \pi^2 n^2, \qquad \Delta\lambda_n \sim 2\pi^2 n,$$

we take θ as any number between 0 and $\frac{1}{2}$, and the only possible divergence from the axioms (a), (b), and (c) is that some eigenvalues may be negative. This is easily corrected by operating in the space $E_N \mathfrak{B}$, where λ_N is the first positive eigenvalue. Let T' be any operator commuting with E_N and such that T' is zero in $(I - EN)\mathfrak{B}$ and, as an operator in $E_N \mathfrak{B}$, T' has a simple spectrum consisting of positive numbers less than λ_N. Then the operator $T - E_N T + T'$ will satisfy axioms (a), (b), and (c) and so can be used instead of T. (Since $E_N \mathfrak{B}$ is finite-dimensional there exist suitable operators T', and these are bounded.)

Thus if B is a bounded linear operator on $\mathfrak{B}$ to $\mathfrak{B}$ and f is an arbitrary element of $\mathfrak{B}$, the expansion of f in eigenvectors (and a finite number of other *proper vectors*, i.e., solutions of $(T + B - \nu_i I)^n x = 0$) of $T + B$ is, by Corollary 4.1, (C, θ) summable $(\theta > 0)$ [convergent] in the norm of $\mathfrak{B}$. Also if we know that the T-expansion of some continuous function is uniformly convergent, the same will be true of its $(T + B)$-expansion.

(ii) Let T be similar to a Sturm-Liouville operator, but with nonreal boundary conditions. Then by Schwartz [4], p. 439, the asymptotic behaviour of the eigenvalues will be satisfactory, but (a) the eigenvalues may be nonreal, and (b) T may have some proper vectors which are not eigenvectors. Difficulty (a) could be met by adapting our proof, but the analysis would become more complicated. Difficulty (b) is easily dealt with by the device used in (i) for negative eigenvalues.

(iii) Let T have the form D^u with u even ($u > 2$), and u linearly independent boundary conditions. Judging by Tamarkin [5] various possibilities exist; but it seems that in many cases—for example, if T is the $(u/2)^{\text{th}}$ power of one of the Sturm-Liouville operators discussed in (i)—we will have simple real eigenvalues λ_n such that

$$\lambda_n \sim i^u \pi^u n^u, \qquad \Delta\lambda_n \sim u i^u \pi^u n^{u-1}.$$

Thus, replacing T (if $u/2$ is odd) by $-T$, we may assume by the argument in (i) that all eigenvalues are positive. In the case of the $(u/2)^{\text{th}}$ power of a Sturm-Liouville operator, the expansion is simply that of the Sturm-Liouville operator; in more general cases an equiconvergence theorem (Tamarkin [5], p. 45) reduces the expansion problem to the Fourier series case. Thus (5.1) is obtained as before, and taking

$$\nu = (u - 2)/u, \qquad 1 - 1/4u < \tau < 1$$

and $0 < \delta < 1 - \tau$ we can verify (a), (b), (c), and (d) without difficulty. To show that S can have the form $B_{u-2}D^{u-2} + \cdots + B_1 D + B_0$ for arbitrary bounded linear operators B_i on $\mathfrak{B}$ to $\mathfrak{B}$, it is thus sufficient by Lemma 1.4 to show that $\mathfrak{R}(\overline{T_0^{-\nu}})$ consists of $(u - 2)$-times differentiable functions f with $f^{(u-2)}$ in $\mathfrak{B}$. Using the argument of Kramer [2], I can obtain this result but only where T is the $(u/2)^{\text{th}}$ power of a Sturm-Liouville operator R (with eigenvalues $\Lambda_i = \lambda_i^{2/u}$). Clearly R and T have the same normalised eigenvectors $(f_i)_{i \geqq 1}$. Thus

$$\begin{aligned} T_0 &= \sum \lambda_i f_i(\cdot) f_i, \\ T_0^\nu &= \sum \lambda_i^\nu f_i(\cdot) f_i = \sum \Lambda_i^{(u-2)/2} f_i(\cdot) f_i = \sum R^{(u-2)/2} f_i(\cdot) f_i \\ &\subseteqq R^{(u-2)/2} \end{aligned}$$

which is included in the operator of the form D^{u-2} (or $-D^{u-2}$) and no boundary conditions, which is a closed operator by Lemma 1.4. Thus $\overline{T_0^\nu}$ exists, and its domain (which is $\mathfrak{R}(\overline{T_0^{-\nu}})$ consists of $(u - 2)$-times differentiable functions.

(iv) In dealing with operators of the form D^u (u odd) I am handicapped by my ignorance of any equiconvergence theorem relating the expansions with different boundary conditions to one another. However, in the special case where the boundary conditions are

$$f^{(i)}(0) = f^{(i)}(1) \qquad (0 \leqq i \leqq u - 1) \tag{5.2}$$

the expansion is the Fourier series and the eigenvalues are simple. We have

$$\lambda_n = (2\pi i)^u n^u, \qquad \Delta\lambda_n = (2\pi i)^u u n^{u-1} + O(n^{u-2}),$$

where n runs through *all* the integers. Inspection of the proofs of our theorems shows that they can be adjusted to allow contours $\Gamma_{-m,n}$ which cut the spectrum of T twice instead of once, as is necessary in this case. Thus, as in (iii), we can allow a perturbation by

$$S = B_{u-2} D^{u-2} + \cdots + B_1 D + B_0 .$$

In this case we must take the space C to be the space of continuous functions with period 1.

(v) In the cases where the boundary conditions allow multiple eigenvalues of T—for example, D^u with u even and the boundary conditions (5.2)—our arguments hold as far as line 27 on page 92, at which point the fact that $\Delta \dim E_r$ is the (algebraic or generalised) multiplicity m_r of λ_r causes trouble. We can conclude that the annulus between Γ_{n-1} and Γ_n (or, at a later stage, the contour Ω_n) contains a finite number of eigenvalues of $T + S$ whose generalised multiplicities add up to m_n, and obtain an expansion theorem, but we will know only the convergence of a suitable subsequence of the partial sums, those corresponding to the contours Γ_n. The cases where, say, $\lambda_n \sim Kn^u$ but $\liminf \Delta\lambda_n/n^{u-1} = 0$ are similar: we cannot put useful contours through too narrow gaps in $\sigma(T)$.

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