# RANDOM WALKS

BY

SAMUEL KARLIN AND JAMES MCGREGOR[1]

## Introduction

An important class of discrete Markoff chains are the random walks $X_n$, $n = 0, 1, 2, \cdots$, whose state space is a set of consecutive integers and whose one-step transition probabilities

$$P_{ij} = \Pr\{X_{n+1} = j \mid X_n = i\}$$

form a Jacobi matrix; that is, $P_{ij} = 0$ if $|i - j| > 1$. As transitions can occur only to the neighboring states, we may regard such a process as a discrete version of a continuous diffusion model.

We shall use the notation $P_{ii-1} = q_i$, $P_{ii} = r_i$, $P_{ii+1} = p_i$ throughout. The $m$-step transition probabilities

$$P_{ij}^{(m)} = \Pr\{X_{n+m} = j \mid X_n = i\}$$

form a matrix $P^{(m)}$ which satisfies

$$P^{(0)} = I, \qquad P^{(1)} = (P_{ij}) \equiv P,$$

$$P^{(m+1)} = P^{(m)}P = PP^{(m)},$$

so that $P^{(m)}$ is simply the product of $m$ copies of $P$.

It is convenient to distinguish three cases according as the state space is the *finite* set $0, 1, \cdots, N$, or the *semi-finite* set $0, 1, \cdots, n, \cdots$, or the *doubly infinite* set $\cdots, -1, 0, 1, \cdots$. In the next two sections we discuss only the semi-infinite case, and the modifications necessary for the other two cases are presented in Section 3.

The "problem" of random walks may be described as follows: The fundamental matrix $P$ is given, and it is required to relate qualitative properties of the Markoff process to qualitative properties of $P$ and to compute various functionals of the process in terms of $P$.

In recent years numerous publications have appeared treating specialized aspects of random walk processes [5], [6], [7]. Our approach will be to obtain an integral representation for the transition matrix through which the probabilistic structure of the process may be analyzed. The integral representation involves a system of polynomials orthogonal with respect to a distribution $\psi(x)$ in the closed interval $[-1, 1]$.

A formalism which suggests the integral representation is as follows: Associ-

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ate with the $n^{\text{th}}$ state a polynomial $Q_n(x)$ of degree $n$, defined by the recurrence relations

$$
\begin{aligned}
xQ_n(x) &= p_n Q_{n+1}(x) + r_n Q_n(x) + q_n Q_{n-1}(x), \qquad n \geqq 0, \\
Q_0(x) &\equiv 1, \qquad Q_{-1}(x) \equiv 0.
\end{aligned} \tag{1}
$$

Note that the coefficient of $Q_m(x)$ on the right is precisely the probability of moving from state $n$ to state $m$ in one transition. In a similar manner we find that

$$x^k Q_n(x) = \sum_{m=0}^{\infty} P^k_{nm} Q_m(x).$$

By appealing to the theory of moments [10], it can be shown that $Q_n(x)$ constitute a system of polynomials orthogonal with respect to a distribution $\psi(x)$. Exploiting this orthogonality property of $Q_n$ we deduce that

$$P^k_{nm} = \int_{-1}^{1} x^k Q_n(x) Q_m(x)\, d\psi(x) \Big/ \int_{-1}^{1} Q_m^2(x)\, d\psi(x). \tag{2}$$

From (1), it also follows that

$$\int Q_m^2(x)\, d\psi(x) = \frac{q_1 q_2 \cdots q_m}{p_0 p_1 \cdots p_{m-1}}.$$

Although this formalism may be fully justified along the lines indicated, the natural meaning of the formula (2) is as a spectral representation of the linear operator $P$ acting on an appropriate Hilbert space. In Section 2 this point of view is employed.

The relevance of Hilbert space theory was also recognized by Feller [3] and McKean [9] in developing a general theory of diffusion operators. Feller dealt primarily with the problem of classifying diffusion models, while McKean was concerned with obtaining an abstract representation theory for such processes. Our results are refinements of this general theory for the special important case of discrete random walks.

The integral representation (2) exhibits in the simplest possible way the dependence of $P^k_{nm}$ on $k$, $n$, and $m$. By utilizing the properties of the components $Q_n(x)$ and $\psi(x)$ of the integral formula we are able to analyze the usual recurrence and absorption characteristics of the process. These results are described in Section 2 which also includes a discussion of the important problem of computing the distribution $\psi$. Several examples are also appended to illustrate the theory. In Section 3 we obtain a new ergodic ratio theorem. This result is considerably deeper than the classical Doeblin ratio theorem and is a property of random walks not common to general Markoff chains.

In the final section we develop the corresponding theory for random walks whose state space is either a finite set of integers or the full set of all positive and negative integers. The representation formula in this case involves a matrix of spectral functions and two systems of polynomials.

## 1. Representation formula

We assume that $p_i > 0$, $r_i \geqq 0$, $q_{i+1} > 0$ for $i \geqq 0$, and $p_i + q_i + r_i \leqq 1$. The possible inequality $p_i + q_i + r_i < 1$ may be interpreted in terms of an ignored state $i^*$ which is a permanent absorbing state of the process, the one-step transition probability from $i$ to $i^*$ being $1 - (p_i + q_i + r_i)$.

The matrix $P$ determines a linear transformation in the space of all sequences $f = \{f(i)\}$, $i \geqq 0$, of complex numbers, by means of the formula

$$(Pf)(i) = \sum_{j=0}^{\infty} P_{ij} f(j).$$

The series on the right has at most three nonzero terms. The solution of $P\phi = x\phi$, where $x$ is a real or complex constant, is unique to within a constant factor. We normalize the sequence $\phi = \{\phi(i)\}$ by the relation $\phi(0) = 1$. The $n^{\text{th}}$ term in the sequence $\phi$ is then a polynomial $Q_n(x)$ in $x$ of exact degree $n$, and these polynomials satisfy the recurrence relations

$$\begin{aligned} Q_0(x) &= 1, \\ xQ_0(x) &= r_0Q_0(x) + p_0Q_1(x), \qquad (1) \\ xQ_n(x) &= q_n Q_{n-1}(x) + r_n Q_n(x) + p_n Q_{n+1}(x). \end{aligned}$$

We introduce the quantities $\{\pi_i\}$ defined as the solutions of the *symmetry equations* $P_{ij}\pi_i = P_{ji}\pi_j$ normalized by the condition $\pi_0 = 1$. We then have for $n \geqq 1$, $\pi_n = (p_0p_1 \cdots p_{n-1})/(q_1q_2 \cdots q_n)$. With these we form the Hilbert space $L_2(\pi)$ of all sequences $f = \{f(i)\}$ of complex numbers such that $\| f \|^2 = \sum_{i=0}^{\infty} | f(i) |^2\pi_i$ is finite, and note the following:

LEMMA 1. *The sequence transformation $f \to Pf$ induces in $L_2(\pi)$ a bounded self-adjoint linear operator $T$ of norm $\leqq 1$.*

*Proof.* The self-adjointness follows from the symmetry equations, and the norm inequality from the fact that $\sum_i \pi_i P_{ij} \leqq \pi_j$ for every $j$.

Thus $f \in L_2(\pi)$ implies $Pf = Tf$, and by iteration $P^{(n)}f = T^nf$. In order to express the components of the matrix $P^{(n)}$ in terms of the operator $T$, we introduce the sequences $e^{(i)} = \{e_j^{(i)}\}$ defined by $e_j^{(i)} = \delta_j^i/\pi_i$. Using the symmetry equations in the form $p_i\pi_i = q_{i+1}\pi_{i+1}$, we find by direct computation that

$$Te^{(i)} = q_i e^{(i-1)} + r_i e^{(i)} + p_i e^{(i+1)},$$

with an obvious modification for $i = 0$, and hence by an inductive argument based on the recurrence relation (1), $Q_n(T)e^{(0)} = e^{(n)}$, $n = 0, 1, 2, \cdots$. Now the inner product $(T^ne^{(j)}, e^{(i)})$ is equal to $P_{ij}^{(n)}/\pi_j$, and therefore

$$P_{ij}^{(n)} = \pi_j(T^nQ_j(T)e^{(0)}, Q_i(T)e^{(0)}).$$

The polynomials $Q_n(x)$ have real coefficients so the operator polynomials $Q_n(T)$ are self-adjoint, and since they commute with $T$,

$$P_{ij}^{(n)} = \pi_j(T^nQ_i(T)Q_j(T)e^{(0)}, e^{(0)}).$$

Consequently, if $\{E_x\}$ is the resolution of the identity corresponding to the self-adjoint operator $T$, and if $\psi(x) = (E_x e^{(0)}, e^{(0)})$, then

$$P_{ij}^{(n)} = \pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x) \, d\psi(x), \tag{2}$$

where the integral includes any jumps which may be present at $x = 1$ or at $x = -1$. This formula is the basic representation of the transition probability matrix of the random walk, and $\psi$ is called the *spectral measure function* of the random walk.

THEOREM 1. *There is a unique positive regular distribution* $\psi$ *on* $-1 \leqq x \leqq 1$ *such that* (2) *is valid for all* $i, j, n$.

*Proof.* The existence of one such $\psi$ has been demonstrated. But (2) with $i = j = 0$ determines all the moments of $\psi$ and hence determines $\psi$ uniquely.

Setting $n = 0$ in (2) we see that the polynomials $Q_n(x)$ are the orthogonal polynomials belonging to $\psi$, in fact,

$$\pi_j \int_{-1}^{1} Q_i(x) Q_j(x) \, d\psi(x) = \delta_{ij}. \tag{3}$$

An important interesting case arises when every $r_n$ is zero. If this is so, we say that the random walk is *symmetric*, the name being justified in part by the fact that the distribution $\psi$ is symmetric about $x = 0$. To see this, observe that when every $r_n$ is zero, it is clear from the probabilistic significance of $P_{ij}^{(n)}$ that $P_{00}^{(n)}$ is zero or positive according as $n$ is odd or even, and hence from (2), all the odd order moments of $\psi$ are zero so $\psi$ is symmetric. The name is further justified by the following converse statement:

LEMMA 2. *Every positive symmetric distribution function on* $-1 \leqq x \leqq 1$ *not supported by a finite set of points, and with total mass one, is the spectral measure function of a symmetric random walk. The random walk is unique if it is required in addition that* $p_0 = 1$ *and* $p_n + q_n = 1$ *for* $n \geqq 1$.

*Proof.* The system of polynomials orthogonal with respect to $\psi$ have all their zeros in the open interval $-1 < x < 1$. Let $Q_n(x)$ denote these polynomials normalized by the condition $Q_n(1) = 1$. The symmetry of $\psi$ implies $Q_n$ is even or odd with $n$, so the recurrence relation is of the form

$$xQ_n = q_n Q_{n-1} + p_n Q_{n+1}.$$

Since $Q_n(x) \geqq 1$ for $x \geqq 1$, it follows that all $p_n > 0$, and setting $x = 1$ gives $p_n + q_n = 1$. Multiplying the recurrence formula by $x^{n-1}$ and using the orthogonality gives $I_n = q_n I_{n-1}$, where $I_n = \int_{-1}^{1} x^n Q_n(x) \, d\psi(x)$. Since the coefficient of $x^n$ in $Q_n$ is greater than 0, we see that $I_n > 0$ and hence $q_n > 0$. Thus the recurrence relation determines a unique matrix $P$ (built from the $p_n$'s and $q_n$'s) which is the one-step transition probability matrix of a symmetric random walk, and it is clear that $\psi$ is the spectral measure function of this random walk.

## 2. Recurrence, absorption, and examples

The nonignored states of the random walk form a single communicating class. The representation formula (2) provides a tool for relating recurrence properties of the process to properties of the spectral measure $\psi$ on the one hand, and to properties of the basic matrix $P$ on the other hand. Problems of this sort have already been investigated in connection with birth and death processes (see [8]). With appropriate modifications the methods used for birth and death processes apply once more for random walks, and similar results can be obtained. Consider, for example, the first passage time distributions. For $i \neq j$, let $f_{ij}^n$ be the probability that if the initial state is $i$, then the state $j$ is reached for the first time on the $n^{\text{th}}$ transition, and let $f_{ii}^n$ be the probability that if the initial state is $i$, then the state $i$ is visited again for the first time on the $n^{\text{th}}$ transition. Then the generating functions

$$P_{ij}(s) = \sum_{n=0}^{\infty} P_{ij}^n s^n, \qquad F_{ij}(s) = \sum_{n=1}^{\infty} f_{ij}^n s^n$$

are connected by the well-known formulas [2]

$$\begin{aligned} P_{ij}(s) &= F_{ij}(s)P_{jj}(s), & i \neq j, \\ P_{ii}(s) &= 1 + F_{ii}(s)P_{ii}(s). \end{aligned} \tag{4}$$

These relations and (2) make it possible to express the $F_{ij}(s)$ in terms of spectral integrals. For example,

$$F_{00}(s) = 1 - 1 \Big/ \int_{-1}^{1} \frac{d\psi(x)}{1 - xs}. \tag{5}$$

Although the moments of the first passage distributions have been determined by other methods [5], it seems worth while to describe how the integral representation yields these same results. This discussion serves also as an introduction to the analysis involving the computation of $\psi$ presented below. The process is recurrent—i.e., the nonignored states form a recurrent class—if and only if $F_{00}(s) \to 1$ as $s \to 1$, and this is equivalent to the divergence of $\int_{-1}^{1} (1 - x)^{-1} d\psi(x)$. Using the analytical tools of the theory of orthogonal polynomials, integrals of the form

$$\int_{-1}^{1} \frac{Q_i(x)Q_j(x)}{(1 - x)^n} d\psi(x),$$

which arise in computing moments of first passage distributions, can be evaluated in terms of the constants $p_n$, $q_n$, $r_n$ of the process. The computations can usually be made to depend on the results of similar calculations carried out in [8]. For example, a necessary condition for recurrence is that $r_0 + p_0 = 1$ and for $n \geqq 1$, $q_n + r_n + p_n = 1$. If this condition is met, then the polynomials $R_n(x) = Q_n(1 - x)$ satisfy a recurrence formula

$$-xR_n = q_n R_{n-1} - (p_n + q_n)R_n + p_n R_{n+1}$$

and are orthogonal on $0 \leqq x \leqq 2$ with respect to a distribution $\theta$ obtained from $\psi$ by an obvious change of variable. Using results of [8],

$$\int_{-1}^{1} \frac{d\psi(x)}{1-x} = \int_{0}^{2} \frac{d\theta(x)}{x} = \sum_{n=0}^{\infty} \frac{1}{p_n \pi_n}.$$

Thus the random walk is recurrent if and only if $\sum 1/p_n \pi_n$ is divergent [5].

For a recurrent process the expected first passage times are all finite if and only if $\lim_{n\to\infty} P_{00}^{2n}$ is positive, and in this case the process is called ergodic. Since $x^{2n} \to 0$ monotonically on $-1 < x < 1$ as $n \to \infty$, it is seen from (2) that the process is ergodic if and only if $\psi$ has a jump at either $x = 1$ or $x = -1$. If $\psi$ has no jump at $x = 1$, then the amount of jump at $x = -1$ is

$$-\lim_{n\to\infty} \int_{-1}^{1} x^{2n+1}\, d\psi(x) = -\lim_{n\to\infty} P_{00}^{2n+1} \leqq 0,$$

so there is no jump at $x = -1$. Consequently, the process is ergodic if and only if $\psi$ has a jump at $x = 1$. Now a jump at $x = 1$ occurs if and only if $x = 1$ is an eigenvalue of the self-adjoint operator $T$, and this is the case if and only if the series

$$1/\rho = \textstyle\sum_{n=0}^{\infty} Q_n^2(1)\pi_n$$

is convergent, in which case $\rho$ is the amount of the jump. It can be deduced from the recurrence formula that the series diverges if $q_n + r_n + p_n < 1$ for some $n$. It is also clear probabilistically that this condition, which means positive probability of absorption at $n$, makes the process transient. If $r_0 + p_0 = 1$ and $q_n + r_n + p_n = 1$ for all $n \geqq 1$, then $Q_n(1) = 1$ for all $n$ and $\rho^{-1} = \sum_0^{\infty} \pi_n$, and the process is ergodic if and only if this series converges. We note in passing that if all $r_n$ vanish so that $\psi$ is symmetric, then any jump at $x = 1$ is matched by an equal jump at $x = -1$; but on the other hand if some $r_n$ is positive, then $\lim_{n\to\infty} P_{00}^{n}$ exists, and this implies that no jump at $x = -1$ is present.

Next suppose that $q_0 = 1 - (r_0 + p_0)$ is positive. Each time the zero state is visited there is a probability $q_0$ that absorption will occur on the next transition. Let $A_i^n$ denote the probability that when the initial state is $i$, absorption from the zero state takes place on the $n^{\text{th}}$ transition. Clearly $A_i^n = q_0 P_{i0}^{n-1}$ and hence

$$A_i(s) \equiv \sum_{n=1}^{\infty} A_i^n s^n = q_0 s \int_{-1}^{1} \frac{Q_i(x)}{1 - xs}\, d\psi(x).$$

If $q_n + r_n + p_n = 1$ for $n \geqq 1$, the computation of the moments of the absorption time distribution may be reduced to results of [8] by means of the device described above.

The $k^{\text{th}}$ associated random walk ($k \geqq 0$) belonging to the given random walk is defined as the process obtained by starting the given random walk at some state $i > k$ and stopping it when the state $k$ is first visited. In particular, therefore, the $0^{\text{th}}$ associated random walk has the zero state as an ignored absorbing state. The idea of the $k^{\text{th}}$ associated random walk is not new and has been implicitly used on numerous occasions [4], [5]. Let $\psi$ be the spectral measure function of a random walk and $\alpha$ the spectral measure function of the $0^{\text{th}}$ associated random walk. The event of first passage from state 1 to state 0 for the given random walk may be viewed as an absorption event for the $0^{\text{th}}$ associated process. Expressing the distribution of the time of occurrence of this event on the one hand in terms of $\psi$, and on the other hand in terms of $\alpha$, we obtain the identity

$$\frac{P_{10}(s)}{P_{00}(s)} = F_{10}(s) = q_1 s \int_{-1}^{1} \frac{d\alpha(x)}{1 - xs},$$

which on simplification yields

$$\int_{-1}^{1} \frac{d\psi(x)}{1 - xs} = 1 \Big/ \left(1 - r_0 s - q_1 p_0 \int_{-1}^{1} \frac{d\alpha(x)}{1 - xs}\right). \tag{6}$$

The Stieltjes transform of a finite measure function $\theta$ on $-1 \leqq x \leqq 1$ is defined by

$$B(z; \theta) = \int_{-1}^{1} \frac{d\theta(x)}{x - z}.$$

Equation (6) is equivalent to the identity

$$B(z; \psi) = -1/(z - r_0 + p_0 q_1 B(z; \alpha)) \tag{7}$$

between the Stieltjes transforms of $\psi$ and $\alpha$. This identity is frequently useful for computing spectral measure functions.

*Examples.* (i) The simplest semi-infinite random walk has $p_n = p$ for $n \geqq 0$, $q_n = q$ for $n \geqq 1$. The process is sometimes referred to as "gambling against an infinitely rich adversary" (see [2] for interpretations). The Stieltjes transform of the spectral measure $B(z; \psi) = B(z)$ satisfies (7) which reduces to

$$pqB^2(z) + zB(z) + 1 = 0,$$

and hence

$$B(z) = (-z + (z^2 - 4pq)^{1/2})/2pq, \tag{7a}$$

where the square root is determined by analytic continuation from positive values for real $z > 1$. The formula which gives $\psi$ in terms of $B(z; \psi)$ is

$$\int_{-1}^{x_0} d\psi(x) = \frac{1}{\pi} \lim_{\eta \to 0^+} \int_{-1-\varepsilon}^{x_0} \operatorname{Im} B(\xi + i\eta)\, d\xi,$$

where $\varepsilon > 0$ and $x_0$ is any point where $\psi$ has no jump. Consequently $\psi$ consists of the continuous density

$$\psi'(x) = (4pq - x^2)^{1/2}/2\pi pq$$

over the interval $-(4pq)^{1/2} < x < (4pq)^{1/2}$. The recurrence formula

$$xQ_n = qQ_{n-1} + pQ_{n+1}$$

can be reduced by transformation of variables to

$$\xi R_n = \tfrac{1}{2}R_{n-1} + \tfrac{1}{2}R_{n+1},$$

and one finds that

$$Q_n(x) = (q/p)^{n/2} U_n(x/(4pq)^{1/2})$$

where the $U_n(\xi)$ are the Chebycheff polynomials of the second kind [1].

(ii) To further illustrate the use of (7), consider the random walk with $q_n = q$, $p_n = p$ for $n \geqq 1$, but $p_0$ and $r_0$ arbitrary. If $\psi$ is the spectral measure function of the process, and $\alpha$ the spectral measure function of the $0^{\text{th}}$ associated process, then $\alpha$ is the same as the $\psi$ of example (i), and (7) becomes

$$B(z; \psi) = \frac{r_0 - (1 - p_0/2p)z + (p_0/2p)(z^2 - 4pq)^{1/2}}{(1 - p_0/p)z^2 - 2r_0(1 - p_0/2p)z + r_0^2 + p_0^2 q/p},$$

from which $\psi$ can be calculated.

We record two special cases:

(A) Let $r_0 = 0$. Then

$$\psi'(x) = \begin{cases} \dfrac{1}{2\pi} \dfrac{p_0(4pq - x^2)^{1/2}}{x^2(p - p_0) + p_0^2 q} & \text{if } \; -(4pq)^{1/2} < x < (4pq)^{1/2}, \\ 0 & \text{otherwise.} \end{cases}$$

Also, $\psi(x)$ has jumps located at $\pm\,(p_0^2 q/(p_0 - p))^{1/2}$ of magnitude

$$\tfrac{1}{2}(p_0 - 2p)/(p_0 - p)$$

provided $p_0 > 2p$. When $p_0 < 2p$, then the distribution $\psi$ is pure density as described above.

(B) Let $p_0 = p$ and $r_0 = q = 1 - p$. Then

$$\psi'(x) = \begin{cases} \dfrac{1}{2\pi(1 - p)} \dfrac{(4pq - x^2)^{1/2}}{1 - x} & \text{if } \; -(4pq)^{1/2} < x < (4pq)^{1/2}, \\ 0 & \text{elsewhere,} \end{cases}$$

and for $p < \tfrac{1}{2}$, $\psi(x)$ possesses an additional jump located at 1 of magnitude $(1 - 2p)/(1 - p)$.

By virtue of (7), either $\psi$ or $\alpha$ can be found if the other is known. By iteration of this relationship the spectral measure function of a random walk can be computed provided all but a finite number of the $p_n$, $r_n$, $q_n$ are equal to those of a random walk whose spectral measure function is known.

(iii) As a final application of (7) we determine the spectral measure function of a random walk with $p_n$, $q_n$, $r_n$ periodic. For simplicity, let

$$p_{2n} = p, \qquad q_{2n} = q = 1 - p \qquad n > 0$$

$$p_{2n+1} = p_1, \qquad q_{2n+1} = q_1 = 1 - p_1 \qquad n \geqq 0$$

and

$$p_0 = p, \qquad r_0 = 0.$$

By (7) iterated twice, we obtain

$$p_1qzB^2 + (z^2 - pq_1 + p_1q)B + z = 0,$$

and hence

$$B(z, \psi) = \frac{-(z^2 - pq_1 + p_1q) + ((z^2 - pq_1 + p_1q)^2 - 4p_1qz^2)^{1/2}}{2p_1qz}.$$

If $pq_1 - p_1q \neq 0$, then $(x^2 - pq_1 + p_1q)^2 - 4p_1qx^2$ is negative on the two intervals

$$J_1: \quad |(pq_1)^{1/2} - (p_1q)^{1/2}| < x < (pq_1)^{1/2} + (p_1q)^{1/2},$$

$$J_2: \quad -((pq_1)^{1/2} + (p_1q)^{1/2}) < x < -|(pq_1)^{1/2} - (p_1q)^{1/2}|,$$

and on these two intervals $\psi$ has the continuous density

$$\psi'(x) = \frac{(4p_1qx^2 - (x^2 - p_1q + pq_1)^2)^{1/2}}{2\pi p_1qx}.$$

There is in addition a jump at $x = 0$ of magnitude $1 - pq_1/p_1q$ if $pq_1 < p_1q$ but no jump at $x = 0$ if $pq_1 > p_1q$. The case $pq_1 = p_1q$ is Example (i).

In many cases useful random walks are generated by known distributions. As an example, consider the ultraspherical density

$$d\psi(x) = C(1 - x^2)^{\lambda - 1/2}\, dx$$

($C$ is a normalization constant). If $\lambda = 0$, the orthogonal polynomials of this distribution are the Chebycheff polynomials of the first kind. To $\lambda = \frac{1}{2}$ correspond the classical Legendre polynomials, and the Chebycheff polynomials of the second kind are determined by the parameter value $\lambda = 1$. If we normalize the polynomials so that $Q_n(1) = 1$, then

$$xQ_n = p_n\, Q_{n+1}(x) + q_n\, Q_{n-1}(x), \qquad n = 1,$$

$$Q_0(x) \equiv 1, \qquad Q_1(x) \equiv x,$$

where

$$p_n = \tfrac{1}{2}(1 + \lambda/(n + \lambda)), \qquad q_n = \tfrac{1}{2}(1 - \lambda/(n + \lambda)).$$

## 3. Ratio theorem

If the random walk is symmetric, the classical limit theorem on Markoff chains states that $P_{ij}^{2n}$ converges. This is also evident by virtue of formula (2) since $\psi$ is a symmetric distribution and $Q_k(x)$ are even or odd polynomials according as $k$ is even or odd. In the nonsymmetric case $\lim_{n\to\infty} P_{ij}^n$ exists and is different from zero if and only if the process is ergodic. If the process is null recurrent or transient, then $\lim_n P_{ij}^n = 0$, and further information relating to these probabilities may be derived by considering the ratios

$$(*) \qquad P_{ij}^n / P_{kl}^n .$$

These quantities are in general very difficult to study. As a recourse

Doeblin investigated and established various limit theorems by replacing numerator and denominator of (*) by their partial sums, viz.,

$$(**) \qquad \sum_{n=1}^{m} P_{ij}^{n} \Big/ \sum_{n=1}^{m} P_{kl}^{n}\,.$$

We shall demonstrate the intrinsic power of the integral representation formula by showing that under suitable conditions $P_{ij}^n/P_{kl}^n$ converges to a finite positive limit as $n \to \infty$. This may be viewed as a Tauberian theorem on top of the Doeblin ratio theorem.

Lemma 3. *If the random walk is recurrent and some $r_i$ is positive, then*

$$\lim_{n\to\infty}\left(\int_{-1}^{0} x^n\, d\psi \Big/ \int_0^1 x^n\, d\psi\right) = 0.$$

*Proof.* Since the process is recurrent, the integral $\int_{-1}^{1}(1-x)^{-1}\, d\psi$ diverges and hence $\int_{1-\varepsilon}^{1} d\psi$ is positive for every $\varepsilon > 0$. This implies that for fixed real $p \geqq 0$,

$$\lim_{n\to\infty}\left(\int_0^1 x^{n+p}\, d\psi \Big/ \int_0^1 x^n\, d\psi\right) = 1.$$

In fact, for any $\varepsilon$, $0 < \varepsilon < 1$,

$$\int_0^{1-\varepsilon} x^n\, d\psi \Big/ \int_{1-\varepsilon}^{1} x^n\, d\psi \leqq (1-\varepsilon)^n \Big/ (1-\tfrac{1}{2}\varepsilon)^n \int_{1-\varepsilon/2}^{1} d\psi \to 0 \quad \text{as} \quad n \to \infty;$$

so as $n \to \infty$,

$$\left|\frac{\int_0^1 x^{n+p}\, d\psi}{\int_0^1 x^n\, d\psi} - 1\right| = \left|\frac{\int_{1-\varepsilon}^{1} (x^{n+p} - x^n)\, d\psi}{\int_{1-\varepsilon}^{1} x^n\, d\psi}\cdot[1 + 0(1)]\right|$$
$$\leqq [1 - (1-\varepsilon)^p][1 + 0(1)],$$

and since $\varepsilon$ is arbitrary, the result follows. A similar argument shows that if $\{n_k\}$ is a subsequence of the sequence of positive integers such that

$$\lim_{k\to\infty}\left(\int_{-1}^{0} x^{n_k}\, d\psi \Big/ \int_0^1 x^{n_k}\, d\psi\right) = a,$$

then for each integer $p \geqq 0$,

$$\lim_{k\to\infty}\left(\int_{-1}^{0} x^{n_k+p}\, d\psi \Big/ \int_0^1 x^{n_k}\, d\psi\right) = (-1)^p a.$$

If the conclusion of the lemma is false, then there is a subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty}\left(\int_{-1}^{0} x^{n_k}\, d\psi \Big/ \int_0^1 x^{n_k}\, d\psi\right) = a \neq 0.$$

Any polynomial $Q(x)$ can be written as a sum of an even polynomial $R(x)$ and an odd polynomial $S(x)$, and we obtain

$$\lim_{k\to\infty} \frac{\int_{-1}^{1} x^{n_k} Q(x)\, d\psi}{\int_0^1 x^{n_k}\, d\psi} = \lim_{k\to\infty} \frac{\int_0^1 x^{n_k}(R+S)\, d\psi + \int_{-1}^{0} x^{n_k}(R+S)\, d\psi}{\int_0^1 x^{n_k}\, d\psi}$$

$$= R(1) + S(1) + a[R(1) - S(1)]$$

$$= Q(1) + aQ(-1),$$

and similarly

$$\lim_{k\to\infty} \left( \int_{-1}^{1} x^{n_k+1} Q(x)\, d\psi \Big/ \int_0^1 x^{n_k}\, d\psi \right) = Q(1) - aQ(-1).$$

By taking $Q(x) = Q_i(x)$ and noting that $\int_{-1}^{1} x^n Q_i(x)\, d\psi = P_{i0}^n \geqq 0$, it follows that $|a| \leqq Q_i(1)/|Q_i(-1)|$. If the process is recurrent, then $r_0 + p_0 = 1$, $q_n + r_n + p_n = 1$ for $n \geqq 1$, and the recurrence formula shows that $Q_i(1) = 1$ for every $i$. On the other hand, the constants $\alpha_i = (-1)^i Q_i(-1)$ satisfy

$$\alpha_0 = 1,$$

$$2r_0\alpha_0\pi_0 = p_0\pi_0(\alpha_1 - \alpha_0),$$

$$2r_n\, \alpha_n\, \pi_n = p_n\, \pi_n(\alpha_{n+1} - \alpha_n) - p_{n-1}\pi_{n-1}(\alpha_n - \alpha_{n-1}),$$

and hence

$$\alpha_{n+1} = 1 + 2 \textstyle\sum_{i=0}^{n} (1/p_i\, \pi_i) \sum_{k=0}^{i} r_k\, \pi_k\, \alpha_k\, .$$

This shows that $1 = \alpha_0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots$ and that if some $r_k$ is positive and the series $\sum_0^\infty (1/p_i\, \pi_i)$ diverges, then $\alpha_n \to \infty$ as $n \to \infty$. Since the divergence of the series follows from the recurrence of the process, we have $|Q_i(-1)| \to \infty$ as $i \to \infty$ and consequently $a = 0$. This proves the lemma.

The preceding argument also proves the following:

COROLLARY. *Subject to the same conditions as Lemma 3, if $f(x)$ is continuous on $[-1, 1]$, then*

$$\lim_{n\to\infty} \left( \int_{-1}^{1-\varepsilon} x^n f(x)\, d\psi(x) \Big/ \int_{1-\varepsilon}^{1} x^n\, d\psi(x) \right) = 0$$

*for every positive $\varepsilon$.*

THEOREM 2. *If the random walk is recurrent and some $r_n$ is positive, then*

$$\lim_{n\to\infty} (P_{ij}^n / P_{kl}^n) = \pi_j / \pi_l\, .$$

*Proof.* By Lemma 3,

$$\lim_{n\to\infty} \left( P_{ij}^n \Big/ \int_0^1 x^n\, d\psi \right) = \lim_{n\to\infty} \left( \pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x)\, d\psi \Big/ \int_0^1 x^n\, d\psi \right)$$

$$= \pi_j\, Q_i(1) Q_j(1) = \pi_j\, ,$$

and the rest is immediate.

If the process is not recurrent, the limit may still exist but does not necessarily have the value given by Theorem 2. That the limit may fail to exist is shown by means of Example (B) of Section 2 with $p_0 = p$, $r_0 = q = 1 - p$, and $p > \frac{1}{2}$.[2] In fact, it is easily verified that $P_{01}^{2n}/P_{00}^{2n}$ and $P_{01}^{2n+1}/P_{00}^{2n+1}$ converge to distinct positive limits. If every $r_n$ is zero, the limit fails to exist because $P_{ij}^n$ is zero if $i - j$ and $n$ are of different parity, but we have the following:

THEOREM 3. *For any symmetric random walk*

$$\lim_{n\to\infty} \frac{P_{ij}^{2n}}{P_{kl}^{2n}} = \frac{\pi_j\, Q_i(\alpha)Q_j(\alpha)}{\pi_l\, Q_k(\alpha)Q_l(\alpha)} \qquad \textit{if} \quad i - j,\, k - l \quad \textit{are even},$$

$$\lim_{n\to\infty} \frac{P_{ij}^{2n+1}}{P_{kl}^{2n+1}} = \frac{\pi_j\, Q_i(\alpha)Q_j(\alpha)}{\pi_l\, Q_k(\alpha)Q_l(\alpha)} \qquad \textit{if} \quad i - j,\, k - l \quad \textit{are odd},$$

*where* $-\alpha \leqq x \leqq \alpha$ *is the smallest interval containing the support of the spectral measure* $\psi$.

The proof makes use of the symmetry of the distribution $\psi$ and arguments similar to those used in Lemma 3.

A final result in this direction is as follows: Suppose all $r_n \geqq \delta > 0$. It can be shown that if $[-\alpha, \alpha]$ defines the smallest symmetric interval containing the support of $\psi$, then in fact the spectrum of $\psi$ lies in the interval $[-\alpha + 2\delta, \alpha]$. However, because of the definition of $\alpha$, it follows that the distribution $\psi$ has measure in any interval $(\alpha - \varepsilon, \alpha)$ for $\varepsilon$ arbitrarily small and positive. This implies readily the truth of Lemma 3 from which we secure the existence of $\lim_{n\to\infty} (P_{ij}^n/P_{kl}^n)$. An extension of the method of Lemma 3 and Theorem 2 shows that the condition $\sum_{i=0}^{\infty} r_i/p_i = \infty$ implies the existence of $\lim_{n\to\infty} (P_{ij}^n/P_{kl}^n)$. We omit the proof.

## 4. Finite and doubly infinite cases

For a random walk with a finite set of states $0, 1, \cdots, N$ and fundamental matrix $P = (P_{ij})$ we define $\pi_0 = 1$, $\pi_n = (p_0 p_1 \cdots p_{n-1})/(q_1 q_2 \cdots q_n)$ for $1 \leqq n \leqq N$. The Hilbert space $L_2(\pi)$ is now $(N + 1)$-dimensional, and the matrix $P$ induces on $L_2(\pi)$ a self-adjoint operator $T$ as before. The recurrence formulas,

$$\begin{aligned} Q_0(x) &= 1 \\ xQ_0(x) &= r_0 Q_0(x) + p_0 Q_1(x) \\ xQ_n(x) &= q_n\, Q_{n-1}(x) + r_n\, Q_n(x) + p_n\, Q_{n+1}(x), \quad 1 \leqq n \leqq N - 1, \end{aligned}$$

determine a system of $N + 1$ polynomials $Q_0, Q_1, \cdots, Q_N$. In addition, we obtain from the vector equation $P\phi = x\phi$ one more equation

$$xQ_N(x) = q_N\, Q_{N-1}(x) + r_N\, Q_N(x), \tag{8}$$

[2] Examples, showing the limit does not exist for general Markoff chains even with a single recurrent class, were found by Chung and Dyson.

and the spectrum of $T$ is the set of values of $x$ which satisfy this equation. As before, we define $e^{(i)}$ to be the vector $\{\delta_j^i/\pi_j\}$ and deduce that

$$Q_i(T)e^{(0)} = e^{(i)}, \qquad i = 0, 1, \cdots, N. \tag{9}$$

It then follows that if $\psi(x)$ is the step function $(E_x e^{(0)}, e^{(0)})$ where $\{E_x\}$ is the resolution of the identity for $T$,

$$P_{ij}^n = \pi_j \int_{-1}^{1} x^n Q_i(x) Q_j(x)\, d\psi(x). \tag{10}$$

Each point where $\psi$ has a jump is in the spectrum of $T$, so there can be at most $N + 1$ jumps. On the other hand, the $N + 1$ functions $Q_i(x)$, $0 \leqq i \leqq N$ are orthogonal with respect to $\psi$ so there are at least $N + 1$ and therefore exactly $N + 1$ jumps.

A doubly infinite random walk has a fundamental matrix $P = (P_{ij})$, $-\infty < i, j < \infty$, where $P_{ij} = q_i, r_i, p_i$ according as $j = i - 1, i, i + 1$ and is zero otherwise. It is assumed that $p_i q_i > 0$ for every $i$. The equation $P\phi = x\phi$ has for each real or complex $x$ a two-dimensional family of solutions. Let $\phi^\alpha = \{\phi_i^\alpha\}$, $\alpha = 1, 2$, $i = 0, \pm 1, \cdots$ be the two solutions such that $\phi_0^1 = 1$, $\phi_{-1}^1 = 0$, $\phi_0^2 = 0$, $\phi_{-1}^2 = 1$, and let $Q_i^1(x) = \phi_i^1$, $Q_i^2(x) = \phi_i^2$. Then we have the recurrence relations

$$\begin{aligned} Q_0^1(x) &= 1, \qquad Q_0^2(x) = 0, \\ Q_{-1}^1(x) &= 0, \qquad Q_{-1}^2(x) = 1, \\ xQ_i^\alpha(x) &= q_i Q_{i-1}^\alpha(x) + r_i Q_i^\alpha(x) + p_i Q_{i+1}^\alpha(x). \end{aligned} \tag{11}$$

The $Q_i^\alpha(x)$ are polynomials in $x$. The solution of $P_{ij}\pi_i = P_{ji}\pi_j$ is unique to within a constant factor, and if we normalize by setting $\pi_0 = 1$, then for $n > 0$,

$$\pi_n = (p_0 p_1 \cdots p_{n-1})/(q_1 q_2 \cdots q_n),$$

$$\pi_{-n} = (q_0 q_{-1} \cdots q_{-n+1})/(p_{-1} p_{-2} \cdots p_{-n}).$$

In the Hilbert space $L_2(\pi)$ the matrix $P$ determines a bounded self-adjoint operator $T$ of norm $\leqq 1$. For each $i$ let

$$e^{(i)} = \{\delta_j^i/\pi_i\}.$$

Then $Te^{(i)} = q_i e^{(i-1)} + r_i e^{(i)} + p_i e^{(i+1)}$, and starting with the trivial cases $i = 0, -1$, we easily verify by induction based on (11) that

$$Q_i^1(T)e^{(0)} + Q_i^2(T)e^{(-1)} = e^{(i)}$$

for every $i$. Consequently,

$$\begin{aligned} P_{ij}^n &= \pi_j(T^n e^{(j)}, e^{(i)}) \\ &= \pi_j(T^n\{Q_j^1(T)e^{(0)} + Q_j^2(T)e^{(-1)}\}, Q_i^1(T)e^{(0)} + Q_i^2(T)e^{(-1)}). \end{aligned}$$

If now $\{E_x\}$ is the resolution of the identity for $T$, then it follows that

$$(12) \qquad P_{ij}^n = \pi_j \int_{-1}^1 x^n \sum_{\alpha,\beta=1}^2 Q_i^\alpha(x) Q_j^\beta(x)\, d\psi_{\alpha\beta}(x),$$

where

$$\psi_{11}(x) = (E_x e^{(0)}, e^{(0)}),$$
$$\psi_{12}(x) = \psi_{21}(x) = (E_x e^{(0)}, e^{(-1)}),$$
$$\psi_{22}(x) = (E_x e^{(-1)}, e^{(-1)}).$$

The $2 \times 2$ matrix $\Psi(x)$ with components $\psi_{\alpha\beta}(x)$ is a positive definite monotone-increasing function of $x$ which vanishes for $x < -1$. This is easily seen by noting that for any constants $c_1$, $c_2$,

$$\sum_{\alpha,\beta=1}^2 c_\alpha c_\beta^* \psi_{\alpha\beta}(x) = (E_x f, f)$$

where $f = c_1 e^{(0)} + c_2 e^{(-1)}$. Moreover, this shows that for $x > 1$, $\psi_{11}(x) = 1$, $\psi_{12}(x) = \psi_{21}(x) = 0$, $\psi_{22}(x) = 1/\pi_{-1}$. The matrix $\Psi$ is called the *spectral matrix* of the random walk.

It will be shown next that the spectral matrix can be expressed in terms of the spectral measure functions of two semi-infinite random walks. Let $f_{ij}^n$ be the first passage times and $F_{ij}(s)$ the corresponding generating functions (see Section 2). As before, these quantities are related to the transition probabilities by the identities (4). Let $\psi^+$ denote the spectral measure function of the random walk on the states $0, 1, 2, \cdots$ with one-step transition probabilities $P_{ij}^+$ given by

$$P_{ij}^+ = P_{ij}, \qquad i, j \geqq 0.$$

This is a process with an ignored absorbing state corresponding to the possible transition from 0 to $-1$ of the doubly infinite process. Let $P_{ij}^+(s)$, $F_{ij}^+(s)$ be the generating functions of the transition probabilities and first passage times of the $\psi^+$ process. Similarly, let $\psi^-$, $P_{ij}^-(s)$, $F_{ij}^-(s)$ be the corresponding quantities belonging to the semi-infinite random walk with states $-1, -2, -3, \cdots$ and one-step transition probabilities $P_{ij}^- = P_{ij}$, $i, j \leqq -1$. From the four identities

$$F_{00}(s) = F_{00}^+(s) + q_0 s F_{-10}(s),$$
$$F_{-10}(s) = p_{-1} s P_{-1-1}^-(s),$$
$$P_{00}(s) = 1 + F_{00}(s) P_{00}(s),$$
$$P_{00}^+(s) = 1 + F_{00}^+(s) P_{00}^+(s),$$

it is found that

$$(13) \qquad P_{00}(s) = \frac{P_{00}^+(s)}{1 - p_{-1} q_0 s^2 P_{00}^+(s) P_{-1-1}^-(s)}.$$

Similarly,

$$(14) \qquad P_{-1-1}(s) = \frac{P_{-1-1}^-(s)}{1 - p_{-1} q_0 s^2 P_{00}^+(s) P_{-1-1}^-(s)}.$$

Finally, $P_{-10}(s) = F_{-10}(s)P_{00}(s)$ gives

$$(15) \qquad P_{-10}(s) = \frac{p_{-1}sP^{+}_{00}(s)P^{-}_{-1-1}(s)}{1 - p_{-1}q_0s^2P^{+}_{00}(s)P^{-}_{-1-1}(s)}.$$

Now (13), (14), and (15) are equivalent to the Stieltjes transform relations

$$(16) \qquad B(z; \psi_{11}) = \frac{B(z; \psi^+)}{1 - p_{-1}q_0B(z; \psi^+)B(z; \psi^-)},$$

$$(17) \qquad \frac{q_0}{p_{-1}} B(z; \psi_{22}) = \frac{B(z; \psi^-)}{1 - p_{-1}q_0B(z; \psi^+)B(z; \psi^-)},$$

$$(18) \qquad B(z; \psi_{12}) = \frac{-p_{-1}B(z; \psi^+)B(z; \psi^-)}{1 - p_{-1}q_0B(z; \psi^+)B(z; \psi^-)},$$

which express $\Psi$ in terms of $\psi^+$ and $\psi^-$, in principle at least.

*Example.* The spectral matrix for the doubly infinite random walk with $p_n = p$, $r_n = 0$, $q_n = q$ $(p + q = 1)$ for all $n$, will be computed. With the notation of (16), (17), (18), we first note that $B(z; \psi^+)$ and $B(z; \psi^-)$ are both solutions of $pqB^2 + zB + 1 = 0$, so are both equal to the $B(z)$ given by (7a). Hence, from (16),

$$B(z; \psi_{11}) = -1/(z^2 - 4pq)^{1/2}$$

so the distribution $\psi_{11}$ consists of the continuous density

$$\psi'_{11}(x) = 1/\pi(4pq - x^2)^{1/2}$$

over the interval $-(4pq)^{1/2} < x < (4pq)^{1/2}$. Since $\psi^+ = \psi^-$, it follows from (16), (17) that $q_0\psi_{22} = p_{-1}\psi_{11}$. Finally, by substituting the values of $B(z; \psi^{\pm})$ in (18), one obtains

$$B(z; \psi_{12}) = \frac{1}{2q} \frac{z^2 - 4pq - z(z^2 - 4pq)^{1/2}}{z^2 - 4pq},$$

and consequently the distribution $\psi_{12}$ consists of the continuous density

$$\psi'_{12}(x) = x/2\pi q(4pq - x^2)^{1/2}$$

over the interval $-(4pq)^{1/2} < x < (4pq)^{1/2}$.

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STANFORD UNIVERSITY
STANFORD, CALIFORNIA