EXTENSIONS AND OBSTRUCTIONS FOR RINGS

BY
SAUNDERS MAC LANE[1]

1. The setting of the problem

The cohomology theory of rings, in the form recently introduced [15], will here be shown appropriate to the systematic treatment of the general extension problem for rings.

The treatment is parallel to the known theory [5] of the extensions of groups. If G is a normal subgroup of a group E, the assignment to each $e \in E$ of the operation of conjugation by e in G induces a homomorphism θ of the quotient group $Q = E/G$ into the group of automorphism classes of G. The converse problem of group extensions therefore starts with the data: groups Q and G plus a homomorphism θ. These data are called a "Q-kernel" by Eilenberg-Mac Lane [5]. On the center C of G the homomorphism θ assigns to each element of Q a well defined automorphism of C; thus C may be regarded as a module over the integral group ring of Q, and the cohomology groups $H^n(Q; C)$ are then available. To θ one assigns an element of $H^3(Q, C)$ as "obstruction"; there exists an extension E of G by Q which realizes θ if and only if this obstruction is zero. When the obstruction is zero, the usual description of extensions by factor sets yields a one-one correspondence between $H^2(Q, C)$ and the set of those equivalence classes of extensions of G by Q which realize θ. These results [5] yield an algebraic interpretation of the two- and three-dimensional cohomology groups and provide a refinement of the usual extension theory (normally attributed to Schreier [17], but actually initiated[2] by Hölder [14]) in which the map θ and the factor sets are all treated together, in a somewhat indigestible lump.

There are subsequent and parallel studies for the extensions of associative algebras (Hochschild [11]) and of Lie algebras (Hochschild [12, 13]). In both cases, the algebras are taken over a field and hence have the additive structure of a vector space over that field. Consequently the extension problem for the additive structure involved is trivial, and only the multiplicative structure is substantially involved in the cohomology theory. The new cohomology for rings to be used here has as its object precisely the simultaneous treatment of additive and multiplicative structures. For example, Everett [10] has developed the analogue of the Schreier extension theory for the case of rings,

Received November 23, 1957.

[1] This research was supported by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command.

[2] The role of Hölder in the theory of group extensions was called to my attention by B. L. van der Waerden.

using factor sets for both addition and multiplication. His treatment is rendered perspicuous by the use of the cohomology of rings.

An essential point is the observation that the automorphisms of the group G must be replaced (§2) by the "bimultiplications" of a ring A. This notion is due to Hochschild [11], who called it a "multiplication". The same notion has later been called a "homothetie" by Redei and his co-workers ([17], [18]) who had apparently overlooked the work of Hochschild. This development leads to the consideration of the ring of bimultiplication classes of A.

Given these notions, the extension theory for rings becomes exactly parallel to that for groups. For example, for groups an extension of a centerless group is uniquely determined by the corresponding homomorphism θ. Similarly, let A be a ring with the property that $ax = 0 = xa$ for all x implies $a = 0$. Then (see the Corollary in §8) any ring extension $E \supset A$ is uniquely determined, up to isomorphism, by the quotient ring E/A and the induced homomorphism θ of E/A into the ring of bimultiplication classes of A. Furthermore, the obstruction theory carries over for rings.[3] The most difficult point is the demonstration (§10) that every three-dimensional cohomology class which satisfies an appropriate necessary condition can indeed be realized as the obstruction to a suitably constructed ring extension problem.

A few remarks on notation. Our rings are not fashionable: they do not need to have an identity element for multiplication. By a 1-ring we mean a ring which does have such an identity, and by a 1-homomorphism a homomorphism of 1-rings which carries the identity to the identity. If Λ is a ring, a Λ-bimodule K is as usual an abelian group which is simultaneously a left Λ-module and a right Λ-module in such wise that $(xk)y = x(ky)$ holds for all k in K and all x, y in Λ. In the 1-case, i.e., when Λ is a 1-ring, we require also that $1k = k = k1$ for all k in K.

2. Bimultiplications of a ring

A bimultiplication σ of a ring A is a pair of mappings $a \to \sigma a$, $a \to a\sigma$ of A into itself which satisfy the rules

$$\sigma(a + b) = \sigma a + \sigma b, \qquad (a + b)\sigma = a\sigma + b\sigma, \tag{2.1}$$

$$\sigma(ab) = (\sigma a)b, \qquad (ab)\sigma = a(b\sigma), \tag{2.2}$$

$$a(\sigma b) = (a\sigma)b, \tag{2.3}$$

for all elements a and b of A. The sum $\sigma + \tau$ and the product $\sigma\tau$ of two bimultiplications σ and τ are defined by the equations

$$(\sigma + \tau)a = \sigma a + \tau a, \qquad a(\sigma + \tau) = a\sigma + a\tau, \tag{2.4}$$

$$(\sigma\tau)a = \sigma(\tau a), \qquad a(\sigma\tau) = (a\sigma)\tau \tag{2.5}$$

[3] The relevance of trying to get an extension theory for rings which would go beyond H^2 and the extension of zero rings was pointed out to me by Professor W. Krull.

for all a in A. One verifies that $\sigma + \tau$ and $\sigma\tau$ are bimultiplications and that under these operations the set of all bimultiplications of A is a 1-ring, denoted by M_A.

For each element c of A a bimultiplication μ_c is defined by

$$\mu_c a = ca, \qquad a\mu_c = ac, \qquad a \in A; \tag{2.6}$$

we call μ_c an inner bimultiplication. Clearly $\mu: A \to M_A$ is a ring homomorphism. Since also

$$\sigma\mu_c = \mu_{\sigma c}, \qquad \mu_c \sigma = \mu_{c\sigma}, \tag{2.7}$$

the image μA of this homomorphism is a two-sided ideal in M_A. The quotient ring $P_A = M_A/\mu A$ is called the ring of *outer bimultiplications* or the ring of bimultiplication classes of A. The kernel of μ is that two-sided ideal K_A of A which consists of all those $c \in A$ with $cx = 0 = xc$ for every $x \in A$. We call K_A the *bicenter* of A. We thus have the exact sequence of rings

$$0 \to K_A \to A \xrightarrow{\mu} M_A \to P_A \to 0. \tag{2.8}$$

Observe that K_A is both a left and a right P_A-module, under the operations $k \to \sigma k$, $k \to k\sigma$, for the result of these operations does not depend upon the choice of the bimultiplication σ within its class. However, K_A need not be a P_A-bimodule.

The bimultiplications σ and τ are called *permutable* if $\sigma(a\tau) = (\sigma a)\tau$ and $\tau(a\sigma) = (\tau a)\sigma$ for every a in A. By (2.2), σ and any inner bimultiplication are permutable; hence we can speak of two permutable outer bimultiplications. In particular, σ is *self-permutable* if $\sigma(a\sigma) = (\sigma a)\sigma$ for all a. The self-permutable bimultiplications are exactly the double homotheties considered by Redei [16]. The set of all self-permutable bimultiplications of A need not be a ring.

For given rings A and Λ, an *extension* of A by Λ is an exact sequence

$$0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda \to 0$$

of rings and ring homomorphisms. In the 1-case, when Λ is a 1-ring, we require also that β be a 1-homomorphism. Since αA is an ideal in E, the assignment to each e in E of its inner bimultiplication μ_e yields a homomorphism $\mu: E \to M_A$; furthermore any two bimultiplications in μE are permutable. Since this μ carries the ideal αA into inner bimultiplications of A, it induces a homomorphism

$$\theta: \Lambda \to P_A$$

for which the image $\theta\Lambda$ again consists of permutable elements. The given extension thus determines θ; in the 1-case, θ is a 1-homomorphism. Given θ, A, and Λ, the "extension problem" is that of finding whether there is any corresponding extension and, if so, how many. The "how many" is taken in the sense of equivalence: two extensions E and E' are *equivalent* if there is a ring homomorphism $\varphi: E \to E'$ such that the diagram

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & \Lambda & \to 0 \\ & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle I} & \\ 0 \to & A & \xrightarrow{\alpha'} & E' & \xrightarrow{\beta'} & \Lambda & \to 0 \end{array}$$

(where each I is an identity map) is commutative. The commutativity of this diagram automatically implies that φ is an isomorphism of E to E' and hence, in the 1-case, that φ is a 1-homomorphism.

We have observed that K_A is a left and right P_A-module. If $\theta: \Lambda \to P_A$ has an image $\theta\Lambda$ which consists of mutually permutable outer bimultiplications, then K_A becomes a Λ-bimodule according to the operations

$$xk = (\theta x)k, \qquad kx = k(\theta x), \qquad k \in K_A, \quad x \in \Lambda. \tag{2.9}$$

In the group extension problem a special role is played by the extensions of abelian groups. The analogous case here is the extension of a zero ring K. A *zero ring* K is a ring in which the product of any two elements is zero. Each such K is its own bicenter, and $M_K = P_K$. For each $\theta: \Lambda \to M_K$ as above, K becomes a Λ-bimodule. Indeed, given K as an additive abelian group, to specify the structure of a Λ-bimodule on K is exactly the same as to specify that K is a zero ring and that $\Lambda \to M_K$ is a homomorphism in which the image consists of permutable bimultiplications.

3. The cubical complex

Eilenberg-Mac Lane has introduced a homology theory for an abelian group G. This theory, which may be described in many equivalent ways ([6], [9]) is for our present purposes best considered as the homology of a certain normalized "cubical" complex $Q(G)$, defined as in [6]. Specifically, $Q(G) = \sum Q_n(G)$ is a certain graded differential group, with $Q_n = 0$ for $n < 0$, and with Q_n for $n \geqq 0$ the free abelian group with generators all 2^n-tuples of elements of G, taken modulo a certain normalization. Each such 2^n-tuple of elements may conveniently be represented as an n-dimensional cube carrying an element of G as a "label" at each of its 2^n vertices. In the sequel we need only the low dimensions, for which we tabulate the free generators of Q_n together with their boundaries as follows:

Dimension	Generator	Boundary
0	(x)	0
1	(x, y)	$(x) + (y) - (x + y)$
2	$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$	$(x, y) + (z, t) - (x + z, y + t)$ $-(x, z) - (y, t) + (x + y, z + t)$

This is to mean, for example, that $Q_1(G)$ is the free abelian group with generators all pairs (x, y) for elements $x, y \in G$, and that $\partial = \partial_Q : Q_1 \to Q_0$ is that homomorphism for which $\partial_Q(x, y) = (x) + (y) - (x + y)$. Actually, $Q_1(G)$ is to be taken with a normalization; that is, modulo the subgroup generated by all pairs $(x, 0)$ and all pairs $(0, y)$, while $Q_0(G)$ is taken modulo the subgroup generated by (0). There is a simple normalization for Q_2, though we shall not actually need the form of the generators of Q_2 and their normalization, but only their boundary as given above.

In general $Q(G)$ is to be exactly the complex denoted by Q/Q_s in [6], i.e. with the normalization given by "slabs" and "diagonals". However the sign of the boundary formula (12.6) in [6] is to be taken corrected and changed: corrected by replacing $-S_i \sigma$ by $S_i \sigma$; changed by changing the total sign of ∂ by the factor -1.

Q has an "augmentation". Specifically, we may regard the group G as a graded differential group, with trivial grading and zero differential. A homomorphism $\eta : Q(G) \to G$ of graded differential groups is then defined by setting $\eta(x) = x$ for $x \in G$ and $\eta q = 0$ for $q \in Q_n$ and $n > 0$. We call η the *augmentation* of Q.

The homology of Q in low dimensions is known. Observe first that, for each $x \in G$, Q has a two-dimensional cycle

$$\gamma(x) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}. \tag{3.1}$$

THEOREM 1. *There are isomorphisms*

$$H_0(Q(G)) \cong G, \qquad H_1(Q(G)) = 0, \qquad H_2(Q(G)) \cong G/2G,$$

induced respectively by η, 0, *and* $\gamma(x) \to x + 2G$.

Proof. It is known [6] that $H_q(Q(G))$ is for large n isomorphic to the stable homology $H_{n+q}(G, n)$ of an Eilenberg-Mac Lane space $K(G, n)$, and the above three groups G, 0, $G/2G$ are the known first three stable Eilenberg-Mac Lane groups. For H_2 the fact that $\gamma(x)$ does yield the explicit isomorphism may be verified either by direct calculations on Q, or by a translation according to [6] of the known explicit representation for $H_{n+2}(G, n)$ found by means of the bar construction in [8, Theorem 23.1].

Now let Λ be a ring. The cubical complex $Q(\Lambda)$ of the additive group of Λ may now be turned into a ring by a product which is defined in low dimensions as

$$(p)(x) = (px), \qquad (p)(x, y) = (px, py), \quad (p)\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} px & py \\ pz & pt \end{pmatrix},$$

$$(p, q)(x) = (px, qx), \quad (p, q)(x, y) = \begin{pmatrix} px & py \\ qx & py \end{pmatrix},$$

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}(x) = \begin{pmatrix} px & qx \\ rx & sx \end{pmatrix}.$$

The general definition (which was suggested by J. Dixmier) is tolerably apparent, and is given in detail in [15]. In particular, if Λ is a 1-ring, so is $Q(\Lambda)$, with identity (1), and $\eta: Q(\Lambda) \to \Lambda$ is a 1-homomorphism. In the usual terminology one can prove that $Q(\Lambda)$ is a graded differential ring; in particular this means that the product of elements u and v of dimensions m and n in Q is an element uv of the dimension mn and with the boundary

$$\partial(uv) = (\partial u)v + (-1)^m u\partial v.$$

4. The universal embedding

If Λ is a 1-ring, then each abelian group G gives rise to a left Λ-module $\Lambda \otimes G$, where the tensor product is taken over the ring of integers. The embedding $\tau: G \to \Lambda \otimes G$ defined by $\tau(g) = 1 \otimes g$ has the following universal property: if $f: G \to K$ is any homomorphism of G into the additive group of a left Λ-module K, then there exists a unique Λ-module homomorphism $f': \Lambda \otimes G \to K$ such that $f'\tau = f$. This property is really just a reflection of the familiar universal properties of the tensor product for bilinear maps.

We need a similar embedding for bimodules and when Λ is not necessarily a 1-ring. To fix the terminology, consider any subcategory $\mathfrak{K}$ of a category $\mathfrak{G}$. We then say that a covariant functor $T: \mathfrak{G} \to \mathfrak{K}$ and a natural homomorphism $\tau: G \to T(G)$ provide a *universal embedding* if for every $\mathfrak{G}$-map $f: G \to K$ of an object G of $\mathfrak{G}$ into an object K of $\mathfrak{K}$ there exists a unique $\mathfrak{K}$-map $f': T(G) \to K$ such that $f'\tau = f$.

In [15], Lemma 1, we have already observed such embeddings in the case when $\mathfrak{G}$ was the category of graded differential right Λ-modules and $\mathfrak{K}$ the subcategory of modules which are also left differential modules over a given graded differential ring. We now cite two more such cases.

Suppose first that Λ is a 1-ring, $\mathfrak{G}$ is the category of abelian groups, and $\mathfrak{K}$ the category of Λ-bimodules. Set $T(G) = \Lambda \otimes G \otimes \Lambda$, where the tensor product is taken over the ring of integers. Then $T(G)$ has a natural structure as a Λ-bimodule, in that a typical element $\mu \otimes g \otimes \nu$, with $\mu, \nu \in \Lambda$ and $g \in G$, is multiplied on the left and right by an element $\lambda \in \Lambda$ according to

$$\lambda(\mu \otimes g \otimes \nu) = \lambda\mu \otimes g \otimes \nu, \qquad (\mu \otimes g \otimes \nu)\lambda = \mu \otimes g \otimes \nu\lambda.$$

A map $\tau: G \to T(G)$ may be defined by $\tau g = 1 \otimes g \otimes 1$. To each $f: G \to K$, where K is a bimodule, there is a corresponding $f': T(G) \to K$, defined by $f'(\mu \otimes g \otimes \nu) = \mu f(g)\nu$. Hence T and τ in this case yield a universal embedding into bimodules.

Suppose second that Λ is a ring, $\mathfrak{G}$ is again the category of abelian groups, and $\mathfrak{K}$ the category of Λ-bimodules. This case differs from the preceding one in that Λ need not have an identity, while even if it does, its bimodules K need not satisfy $1 \cdot k = k \cdot 1$. We now set

$$T(G) = G + \Lambda \otimes G + G \otimes \Lambda + \Lambda \otimes G \otimes \Lambda,$$

a direct sum of tensor products (over the ring of integers) in which a typical element will have the form $a + \mu \otimes b + c \otimes \nu + \beta \otimes d \otimes \gamma$ for a, b, c, d

in G and μ, ν, β, γ, in Λ. Then $T(G)$ is a Λ-bimodule, where the left and right operations of $\lambda \in \Lambda$ are defined by

$$\lambda(a + \mu \otimes b + c \otimes \nu + \beta \otimes d \otimes \gamma) = \lambda \otimes a + \lambda\mu \otimes b + \lambda \otimes c \otimes \nu + \lambda\beta \otimes d \otimes \gamma,$$

$$(a + \mu \otimes b + c \otimes \nu + \beta \otimes d \otimes \gamma)\, \lambda = a \otimes \lambda + \mu \otimes b \otimes \lambda + c \otimes \nu\lambda + \beta \otimes d \otimes \gamma\lambda.$$

A map τ: $G \to T(G)$ is defined by $\tau(g) = g$ for $g \in G$. To each f: $G \to K$, where K is a bimodule, there is a unique corresponding f': $T(G) \to K$ given by

$$f'(a + \mu \otimes b + c \otimes \nu + \beta \otimes d \otimes \gamma) = f(a) + \mu f(b) + f(c)\nu + \beta f(d)\gamma.$$

Hence T and τ yield a universal embedding for bimodules.

If Hom (G, K) denotes as usual the group of homomorphisms of G into K while Hom_Λ denotes the group of bimodule homomorphisms, the universal property of $T(G)$ may be summarized in either case by the fact that $f \to f'$ yields an isomorphism

$$\mathrm{Hom}\,(G, K) \cong \mathrm{Hom}_\Lambda\,(T(G), K). \tag{4.1}$$

5. The bar construction

The homology of the ring Λ will be obtained from the cubical complex $Q(\Lambda)$ by a variant of the Eilenberg-Mac Lane bar construction. This construction $\bar{B}$ operates relative to a fixed base ring Λ, and provides a graded differential Λ-module $\bar{B}(Q, \eta)$ from the data η: $Q \to \Lambda$, where Q is any given differential ring and η a homomorphism of graded differential rings. Here the base ring is regarded as a graded differential ring with trivial grading (i.e., all elements are of degree zero) and trivial differential (i.e., $\partial x = 0$ for all $x \in \Lambda$).

The explicit definition of $\bar{B}$ is as follows. As a Λ-bimodule, $\bar{B}(Q, \eta)$ is the direct sum $\sum_{n=0}^{\infty} \bar{B}_n$, where $\bar{B}_0 = T(Z)$ and $\bar{B}_n = T(Q \otimes \cdots \otimes Q)$ for $n > 0$ is obtained by applying the universal embedding functor of §4 to the tensor product over the ring of integers of n factors Q. We write

$$[u_1 \mid \cdots \mid u_n] = \tau(u_1 \otimes \cdots \otimes u_n)\,, \qquad u_i \in Q \tag{5.1}$$

for the generators of $\bar{B}_n$. Let the grading of Q be denoted by writing du for the degree of a homogeneous element u of Q; then the grading of $\bar{B}$ is defined by the requirement that the element (5.1) is homogeneous when the u_i are homogeneous and has the degree

$$d[u_1 \mid \cdots \mid u_n] = n + du_1 + \cdots + du_n\,. \tag{5.2}$$

The boundary operator (differential) ∂: $\bar{B} \to \bar{B}$ has the form $\partial = \partial_r + \partial_s$, where the ∂_r and ∂_s are Λ-bimodule homomorphisms defined, in terms of the product and the differential ∂_Q of Q, by the formulas

$$\partial_r[u_1 | \cdots | u_n] = - \sum_{i=1}^{n} (-1)^{\varepsilon_{i-1}} [u_1 | \cdots | \partial_Q u_i | \cdots | u_n],$$

$$\partial_s[u_1 | \cdots | u_n] = \eta(u_1)[u_2 | \cdots | u_n] + \sum_{i=1}^{n-1} (-1)^{\varepsilon_i} [u_1 | \cdots | u_i u_{i+1} | \cdots | u_n] + (-1)^{\varepsilon_n}[u_1 | \cdots | u_{n-1}]\eta(u_n).$$

Here the signs ε_i are the following:

$$\varepsilon_i = d[u_1 | \cdots | u_i], \qquad i = 0, 1, \cdots, n.$$

One readily verifies that these definitions do give $\bar{B}$ the structure of a graded differential Λ-bimodule.

In the special case when Λ is the 1-ring Z of integers, this bar construction is exactly the original one discovered by Eilenberg-Mac Lane [7]. The general case, provided Λ is a 1-ring, is that discussed in more detail in [15]; there it is shown how $\bar{B}$ may be obtained from an acyclic construction $B(Q, \eta)$ resembling the acyclic bar construction of Cartan [2]. Note however that the direct application of Cartan's theory, which is formulated for augmented *algebras*, does not give the above construction, which must be formulated for *rings* "augmented" by the homomorphism η.

In case Λ is a 1-ring, we require also that Q be a 1-ring and η a 1-homomorphism in this construction. In this case, let C denote the subbimodule of $\bar{B}$ spanned by the elements $[u_1 | \cdots | u_n]$ with at least one term $u_i = 1$. From the coboundary formulas one verifies at once that C is a subcomplex of $\bar{B}$. The complex $\bar{B}$ is equivalent to the quotient $\bar{B}_N = \bar{B}/C$ in virtue of the following "normalization" theorem.

THEOREM 2. *The subcomplex C of $\bar{B}(Q, \eta)$ has homology groups zero.*

Proof. This theorem is a mild generalization of the corresponding normalization theorem of [7, Theorem 11.2]. To prove it, take C_m to be the submodule of $\bar{B}$ spanned by all elements $[u_1 | \cdots | u_n]$ with at least one term $u_i = 1$ for an index $i = 1, 2, \cdots, m$. Then $0 \subset C_1 \subset \cdots \subset C_m \subset C_{m+1} \subset \cdots$ is a chain of subcomplexes with C as union, and C_{m+1}/C_m is a Λ-bimodule spanned by the elements $[u_1 | \cdots | u_m | 1 | v_1 | \cdots | v_k]$. It suffices to define a contracting homotopy in each quotient C_{m+1}/C_m by the formula

$$D_m[u_1 | \cdots | u_m | 1 | v_1 | \cdots | v_k] = (-1)^{\varepsilon} [u_1 | \cdots | u_m | 1 | 1 | v_1 | \cdots | v_k],$$

where $\varepsilon = d[u_1 | \cdots | u_m]$. One then verifies readily that $\partial D_m + D_m \partial = I$, as required to complete the proof.

The homology of a ring Λ is now defined as the homology of the graded differential Λ-bimodule $R(\Lambda) = \bar{B}(Q(\Lambda), \eta)$; in case Λ is a 1-ring we may use equivalently the normalized construction $\bar{B}_N$. Correspondingly, the cohomology of Λ may be defined with any Λ-bimodule K as coefficients and is

$$H^n(\Lambda, K) = H^n(\mathrm{Hom}_\Lambda (\bar{B}(Q(\Lambda), \eta), K)), \tag{5.3}$$

where Hom_Λ (B, K) designates the graded differential group which in dimension n consists of all the Λ-bimodule homomorphisms of the n-dimensional part of $\bar{B}$ into K. Since $\bar{B}$ is defined by means of the universal embedding functor T of §4, we may consequently use (4.1) to express these bimodule homomorphisms.

In the sequel we need $R_q(\Lambda)$ through dimension $q = 4$ inclusive. Each such R_q has the form $T(F_q)$, where T is the embedding functor and F_q is a certain free abelian group. Upon consulting the definition of $\bar{B}$, we may list the free generators of these groups F_q as follows

F_0	generator 1				
F_1	$[(x)] = [x]$				$x \neq 0$ in Λ
F_2	$[u]$,	$[x \mid y]$,			$u \in Q_1$
F_3	$[v]$,	$[u \mid x]$,	$[x \mid u]$,	$[x \mid y \mid z]$,	$v \in Q_2$
F_4	$[w]$,	$[v \mid x]$,	$[x \mid v]$,	$[u \mid u']$,	
	$[x \mid y \mid u]$,	$[x \mid u \mid y]$,	$[u \mid x \mid y]$,	$[x \mid y \mid z \mid t]$.	

Here x, y, z, t are nonzero elements of Λ; u and u' are free generators of the group Q_1, so that u for example has the form $u = (s, t)$ with s, t not both zero in Λ; v is a generator of Q_2, and w is a generator of Q_3 (i.e., a 3-cube). To simplify the notation, we have written $[x, y]$ in place of the more correct $[(x) \mid (y)]$.

For each type of cell above, the boundary formulas may be written out explicitly, using the boundary formulas given in the definitions of Q and $\bar{B}$. These explicit formulas will be summarized below; for example in the next section we will use the following boundary formulas for R_3:

$$\begin{aligned}
\partial \begin{bmatrix} x & y \\ z & t \end{bmatrix} &= -(x, y) - (z, t) + (x + z, y + t) \\
&\qquad + (x, z) + (y, t) - (x + y, z + t), \\
\partial[x, y \mid t] &= -[\partial_Q(x, y) \mid t] + [xt, yt] - [x, y] \otimes t, \\
\partial[t \mid x, y] &= [t \mid \partial_Q(x, y)] + t \otimes [x, y] - [tx, ty], \\
\partial[x \mid y \mid z] &= x \otimes [y \mid z] - [xy \mid z] + [x \mid yz] - [x \mid y] \otimes z.
\end{aligned}$$

6. Extensions and H^2

A two-dimensional cochain g of the ring Λ with coefficients in a bimodule K is a function on the two-dimensional generators of $R(\Lambda)$ with values in K. There are two types of such generators, $q_1 = (z, t)$ and $[x \mid y]$; hence the cochain g can be regarded as a pair of functions $g(z, t) \in K$ and $g(x \mid y) \in K$, defined for all arguments z, t, x, y in Λ and satisfying the 0-normalization conditions $g(0, t) = 0 = g(z, 0)$ and $g(0 \mid y) = 0 = g(x \mid 0)$. The coboundary of this cochain is the function $\delta g = g\partial$ defined on the four types of three-dimensional generators of $R(\Lambda)$ as listed above. Thus for example

$$\delta g[x \mid z, t] = -g(x \mid z + t) + g(x \mid z) + g(x \mid t) + xg(z, t) - g(xz, xt).$$

The function g is a cocycle if $\delta g = 0$; that is, g satisfies the four identities $g\partial = 0$ given by the four types of boundary formulas (e.g., one is the identity obtained by setting the right-hand side of the equation displayed above equal to 0).

Given a cocycle g, construct the ring E_g of all pairs (k, x) for $k \in K$, $x \in \Lambda$, and define the sum and the product of such pairs by the equations

$$(k, x) + (l, y) = (k + l + g(x, y), x + y),$$

$$(k, x)\,(l, y) = (ky + xl - g(x \mid y), xy).$$

(Here the function g appears in the second formula with a negative sign in place of the more natural positive sign in order to make the choice of signs here agree with signs already chosen for the bar construction.) The 0-normalization condition on g implies that $(0, 0)$ is the zero element for the addition in E_g. The fact that g must vanish on the boundary of each $u \in Q_2$ gives, upon consultation of the boundary (§3) for such a u, the fact that addition in E_g satisfies

$$(x + y) + (z + t) = (x + z) + (y + t).$$

We call this law the commutassociative law because, in the presence of a zero for addition, it is equivalent to the commutative and associative laws together. Finally, $g\partial[u \mid x] = 0$ gives the right distributive law; $g\partial[x \mid u]$, the left distributive law; and $g\partial[x \mid y \mid z]$, the associative law for multiplication in E_g. In other words, E_g is a ring; the two types of two-dimensional generators of $R(\Lambda)$ correspond to the two operations (addition and multiplication) in a ring, while the four types of three-dimensional generators correspond exactly to the four identities which these operations must satisfy in a ring. In the 1-cases the 1-normalization condition on the cochain g means that $g(1 \mid y) = 0 = g(x \mid 1)$, which implies that $(0, 1)$ is an identity for the ring E_g.

By using the homomorphisms $k \to (k, 0)$ of K into E_g and $(k, x) \to x$ of E_g into Λ we see that the ring E_g is an extension of the zero ring K by the given ring Λ. We have thus constructed to each 2-cocycle g of $R(\Lambda)$ an extension E. From the boundary formulas for the two-dimensional generators of $R(\Lambda)$ one readily verifies that cohomologous cocycles give equivalent extensions. The usual arguments also show that every equivalence class of extensions is represented in this way by one and only one two-dimensional cohomology class of $R(\Lambda)$. We thus have the following theorem, already proved for the 1-case in [15].

THEOREM 3. *Let Λ be a ring, K a zero ring and $\theta:\Lambda \to M_K$ a homomorphism with image $\theta\Lambda$ consisting of permutable bimultiplications of K. Give K its* *induced structure as a Λ-bimodule. There is then a natural one-one correspondence between the two-dimensional cohomology group $H^2(\Lambda; K)$ and the set of equivalence classes of those ring extensions of K by Λ which induce θ. The same result holds in the* 1*-case; hence in particular any extension of a zero ring by a* 1*-ring which induces a* 1*-homomorphism θ is a* 1*-extension.*

As in the case of group extensions, this result implies that the manifold of equivalence classes of ring extensions realizing a given θ has a canonical abelian group structure. In §8 below we will give a direct description of this composition of extensions.

The last assertion about identity elements holds more generally even if the kernel of the extension is not a zero ring.

THEOREM 3a. *If $0 \to A \to E \xrightarrow{\beta} \Lambda \to 0$ is an exact sequence of rings such that Λ is a* 1*-ring and the homomorphism $\theta: \Lambda \to P_A$ induced by θ is a* 1*-homomorphism, then E has an identity element (and hence β is a* 1*-homomorphism).*

We give a direct proof. Since θ is a 1-homomorphism, we can choose for the identity 1 of Λ a representative t in E such that $\beta t = 1$ and bimultiplication in A by t is the identity. This means that

$$ta = a, \qquad at = a,$$

for all a in A, and that $t^2 = t + d$ for some d in A. Then $a = ta$, hence $ta = t^2a = (t + d)a = ta + da$, and therefore $da = 0$ for all a. Similarly $ad = 0$ for all a. Thus d lies in the bicenter of A, and

$$(t - d)^2 = t^2 - dt - td + 0 = t - 2d + d = t - d.$$

Thus if we set $s = t - d$, we have $\beta s = 1$, $s^2 = s$, and $sa = a = as$. Now consider any e in E. Since βs is the identity, there are elements b and c in A with

$$se = e + b, \qquad es = e + c.$$

Then $(es)s = (e + c)s = e + 2c$; on the other hand by the associative law $(es)s = es^2 = es = e + c$. Therefore $c = 0$. By similar argument $b = 0$. Thus $se = e = es$ for all e; in other words, s is the desired identity element.

7. The obstruction

For given maps A and Λ let $\theta: \Lambda \to P_A$ be a homomorphism whose image consists of permutable outer bimultiplications. We propose to assign to θ a three-dimensional cohomology class of $H^3(\Lambda, K_A)$ as "obstruction". To this end, choose to each $x \in \Lambda$ a bimultiplication $\sigma_x \in \theta x$, with $\sigma_0 = 0$ and, in the 1-case, with $\sigma_1 = I$, the identity bimultiplication. Since θ is a homomorphism, both $\sigma_x + \sigma_y - \sigma_{x+y}$ and $\sigma_x \sigma_y - \sigma_{xy}$ are then inner bimultiplications, so we may choose elements $h(x, y)$ and $h(x \mid y)$ in A such that

$$\sigma_x + \sigma_y = \mu h(x, y) + \sigma_{x+y}\,, \tag{7.1}$$

$$\sigma_x \sigma_y = -\mu h(x \mid y) + \sigma_{xy}\,. \tag{7.2}$$

In particular, since $\sigma_0 = 0$, we may choose h so that

$$h(0, y) = h(x, 0) = h(0 \mid y) = h(x \mid 0) = 0, \tag{7.3}$$

while in the 1-case we may require in addition that

$$h(1 \mid y) = 0 = h(1 \mid x) \tag{7.3a}$$

for all x, y in Λ.

We now assert that, for all x, y, t, z, in Λ the right-hand sides of the following four equations represent elements in K_A (here ∂_Q denotes the boundary operation in $Q(\Lambda)$):

$$f(v) = -h(\partial_Q v), \qquad v \in Q_2 \tag{7.4}$$

$$f(x, y \mid t) = -h(\partial_Q(x, y) \mid t) + h(xt, yt) - h(x, y)\sigma_t, \tag{7.5}$$

$$f(t \mid x, y) = h(t \mid \partial_Q(x, y)) + \sigma_t h(x, y) - h(tx, ty), \tag{7.6}$$

$$f(x \mid y \mid z) = \sigma_x h(y \mid z) - h(xy \mid z) + h(x \mid yz) - h(x \mid y)\sigma_z ; \tag{7.7}$$

indeed, upon application of μ to the right-hand sides and the use of the definition of h, one obtains exactly the four basic identities valid for the bimultiplications σ_x in M_A (namely, the commutassociativity of addition, the right and left distributive laws, and the associativity of multiplication). Since these right-hand sides thus lie in K_A, we may define a function f of the four types of generators of $R_3(\Lambda)$ as indicated by the left-hand sides of these equations. This function is then a three-dimensional cochain of $R(\Lambda)$ with coefficients in K_A; indeed the four formulas in its definition above are chosen so as to exactly parallel the boundary formulas for the four types of generators of R_3. In other words, this definition of f has the form $f = \delta h$. We call the cochain f an *obstruction* of $\theta:\Lambda \to P_A$. It follows easily that f is 0-normalized and, in the 1-case, 1-normalized.

THEOREM 4. *Each obstruction f of θ is a cocycle, and any two obstructions of the same homomorphism θ are cohomologous. If f is an obstruction of θ then any cocycle cohomologous to f is also an obstruction of θ.*

Proof. The definition of f has the form $f = \delta h$, except for the fact that the 2-cochain h has its values in the additive group of A, which is not quite a bimodule under the bimultiplications σ_x. Hence the proof that $\delta f = 0$ is essentially the same as the proof that $\delta\delta h = 0$, except for those terms which involve composite bimultiplications σ_{x+y} or σ_{xy}. For example, to prove that $\delta f[x, y \mid z \mid t] = f\partial[x, y \mid z \mid t] = 0$, one has two such terms, $\sigma_{x+y} h(z \mid t)$ and $h(x, y)\sigma_{zt}$. Upon using (7.1) and (7.2), these terms give

$$\sigma_x h(z \mid t) + \sigma_y h(z \mid t) - h(x, y)h(z \mid t) + h(x, y)\sigma_z\sigma_t + h(x, y)h(z \mid t).$$

The two "extra" terms (those not involving σ) cancel, and the rest of the terms cancel with others exactly as in the explicit proof that $\partial\partial[x, y \mid z \mid t] = 0$. The other seven cases of boundaries of four-dimensional generators of $R(\Lambda)$ are treated similarly.

We next show that any two obstructions f are cohomologous. The definition of f depends upon two choices, that of the σ_x and that of h. We assert that any change in the choice of the σ_x can be followed by a suitable new choice of h such as to leave the original obstruction f unchanged. Indeed, any different choice σ_x' of bimultiplications will have σ_x and σ_x' yielding the same outer bimultiplication θx, and hence $\sigma_x' = \mu b(x) + \sigma_x$, where $\mu b(x)$ is an inner bimultiplication, so that b can be chosen as a function on Λ to A with $b(0) = 0$ (and, in the 1-case, with $b(1) = 1$). From (7.1) and (7.2) we may then compute that

$$\sigma_x' + \sigma_y' = \mu[b(x) + b(y) - b(x + y) + h(x, y)] + \sigma_{x+y}',$$

$$\sigma_x' \sigma_y' = \mu[-b(x)b(y) - \sigma_x b(y) - b(x)\sigma_y + b(xy) + h(x \mid y)] + \sigma_{xy}';$$

hence we may and do choose the new functions h' as

$$h'(x, y) = b(x) + b(y) - b(x + y) + h(x, y),$$

$$h'(x \mid y) = -b(x)b(y) - \sigma_x b(y) - b(x)\sigma_y + b(xy) + h(x \mid y).$$

These equations state that $h' = \delta b + h$, except for the extra term $b(x)b(y)$ in the second equation and the proviso that A is not quite a Λ-bimodule under σ_x and σ_y. Thanks to this extra term one readily calculates that $\delta h' = \delta\delta b + \delta h = \delta h$, much as in the proof above that $\delta f = 0$.

We next assert that, for fixed choices of the σ_x, any change in the choice of h will change the obstruction f by adding a coboundary from $\mathrm{Hom}_\Lambda (R(\Lambda), K_A)$, and moreover that f may be changed by the addition of any such coboundary. This will complete the proof of the theorem. For suppose that h' is another function satisfying (7.1) and (7.2). Then $\mu h = \mu h'$, hence $h' - h$ lies in the kernel K_A of μ, so that $h' - h = g$, for g with values in K_A. In detail

$$h'(x, y) = h(x, y) + g(x, y), \qquad h'(x \mid y) = h(x \mid y) + g(x \mid y),$$

where g is a (normalized) 2-cochain of Λ. Conversely, if g is any such cochain, then $h + g$ is an allowable choice of h'. From the definition of f one now calculates that $f' = \delta(h + g) = \delta h + \delta g = f + \delta g$, so that f is indeed changed by a coboundary δg, which may be arbitrary.

In virtue of this theorem, the unique cohomology class of any obstruction f of θ may be called *the obstruction* of θ.

THEOREM 5. *A homomorphism $\theta{:}\Lambda \to P_A$ (in the case of a 1-ring Λ, a 1-homomorphism) can be realized by a ring extension (respectively, by a 1-extension) if and only if the obstruction of θ is zero.*

Proof. Suppose first that $\beta{:}E \to \Lambda$ is an extension of A by Λ which realizes θ. To each x in Λ choose u_x in E with $\beta u_x = x$, and in particular choose $u_0 = 0$ and (in the 1-case) $u_1 = 1$. Then in the definition of the obstruction of θ we may choose $\sigma_x = \mu u_x$ and hence $h(x, y) = u_x + u_y - u_{(x+y)}$, $h(x \mid y) = u_x u_y - u_{xy}$. From (7.4)–(7.7) one then computes at once that the resulting obstruction cocycle f is identically zero.

Conversely, suppose that θ has an obstruction f which is a coboundary. By Theorem 4 we can then choose h so as to give a new obstruction which is identically zero. We then apply the following existence theorem.

THEOREM 6. *For given rings A and Λ let the function σ assign to each x in Λ a bimultiplication σ_x of A such that any two bimultiplications σ_x and σ_y are permutable and such that $\sigma_0 = 0$ and (if Λ is a 1-ring) $\sigma_1 = I$. Let $h(x, y) \in A$ and $h(x \mid y) \in A$ be two functions related to σ by* (7.1) *and* (7.2) *and satisfying the normalization conditions* (7.3) *and* (7.3a). *Then if h satisfies the four equations obtained from* (7.4)–(7.7) *by setting f there equal to* 0, *there exists an extension $\beta: E \to \Lambda$ of A by Λ and elements u_x in E such that*

$$\beta u_x = x, \qquad \mu u_x = \sigma_x ,$$
$$u_x + u_y = h(x, y) + u_{x+y} , \qquad u_x u_y = -h(x \mid y) + u_{xy} .$$

This theorem is due to Everett [10]; we have rearranged it only by separating the ten conditions which he lists into (i) the conditions that the σ_x be permutable bimultiplications; (ii) the conditions (7.1) and (7.2) that $x \to \sigma_x$ be a homomorphism of Λ to outer bimultiplications; (iii) normalization conditions; (iv) the conditions (7.4)–(7.7) stating the four identities necessary in a ring. (In Everett's statement, (7.4) is replaced by separate conditions for the associative and commutative laws for addition.)

The proof of this theorem is direct. One constructs E as the ring of all pairs (x, a) for $x \in \Lambda$, $a \in A$ with addition and multiplication defined by the equations

$$(x, a) + (y, b) = (x + y, h(x, y) + a + b) ,$$
$$(x, a)(y, b) = (xy, -h(x \mid y) + \sigma_x b + a\sigma_y + ab).$$

In the proof that this set E is in fact a ring, the four equations (7.4)–(7.7) enter exactly to prove that E satisfies the four corresponding identities. In the 1-case, the normalization condition (7.3) serves to prove that (1, 0) is the identity of E.

8. The manifold of extensions

THEOREM 7. *If there exists a ring extension of A by Λ realizing $\theta: \Lambda \to P_A$, then the set of equivalence classes of such extensions is in one-one correspondence with the cohomology group $H^2(\Lambda, K_A)$, where the bicenter K_A of A is regarded as a Λ-bimodule under the operations induced by θ.*

Proof. This result follows readily by Everett's methods, by observing that any two solutions h_0 and h of the equations (7.1)–(7.7), with $f = 0$ in (7.4) to (7.7), must differ by a 2-cocycle g. Hence the correspondence "$h_0 + g$" $\leftrightarrow$ "cohomology classes of g" yields the desired one-one correspondence. We prefer to give a more conceptual proof of this theorem, so as to illustrate how notions such as the "Baer product" of group extensions will work for extensions of rings.

The *graph* Γ of $\theta:\Lambda \to P_A$ is that subring of the direct sum $\Lambda + M_A$ which consists of all pairs (x, σ), for $x \in \Lambda$, and $\sigma \in M_A$, such that $\sigma \in \theta_x$. Under the ring homomorphisms $\mu A \to \Gamma \to \Lambda$ given by $\mu a \to (0, \mu a)$ and $(x, \sigma) \to x$, this graph Γ is a ring extension of μA by Λ. In the 1-case, Γ is a 1-ring and the map $\Gamma \to \Lambda$ is a 1-homomorphism.

Any extension $A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda$ of A by Λ which realizes θ induces an extension of K_A by the graph Γ of θ. Indeed, we define $\psi:E \to \Gamma$ by setting $\psi e = (\beta e, \mu e)$ for each $e \in E$; then $K_A \to E \xrightarrow{\psi} \Gamma$ is the desired extension, and the diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & \Lambda & \to & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle I} & & \\ 0 & \to & \mu A & \to & \Gamma & \longrightarrow & \Lambda & \to & 0 \end{array}$$

is commutative. In particular, if $K_A = 0$, ψ is an isomorphism, so that in this case any solution of the extension problem is equivalent to the graph Γ.

To any two extensions E and E' of A by Λ which realize the same θ, we now construct an extension $0 \to K_A \xrightarrow{\kappa} F \xrightarrow{\lambda} \Lambda \to 0$ of the zero ring K_A by Λ which will realize the given Λ-bimodule structure of K_A. To this end, let $\psi:E \to \Gamma$ and $\psi':E' \to \Gamma$ be the homomorphisms constructed as just above, and let T be that subring of the direct sum $E + E'$ which consists of all pairs (e, e'), for $e \in E$ and $e' \in E'$, such that $\psi e = \psi' e'$. Let T_0 be the two-sided ideal in T which consists of all pairs $(\alpha a, \alpha' a)$ for $a \in A$, where $\alpha:A \to E$ and $\alpha':A \to E'$ are the given monomorphisms under which E and E' extend A. Then $F = T/T_0$ is a ring, and is an extension of K_A by Λ relative to the homomorphisms $K_A \to F$ given by $k \to (\alpha k, 0) + T_0$ and $F \to \Lambda$ given by $(e, e') \to \beta e = \beta e'$. One verfies that this extension F does in fact realize the given Λ-bimodule structure of K_A. We write $T^*(E, E') = F$ for the extension so constructed, and regard T^* as a function applying to equivalence classes of extensions.

Now conversely let $K_A \xrightarrow{\kappa} F \xrightarrow{\lambda} \Lambda$ and $A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda$ be given extensions which realize respectively the bimodule structure of K_A and the homomorphism θ. We construct a second extension E^* as follows. Let S be that subring of the direct sum $F + E$ which consists of all pairs (f, e) for $f \in F$ and $e \in E$ and such that $\lambda f = \beta e$. Let S_0 be the two-sided ideal in S which consists of all pairs $(\kappa k, - \alpha k)$—notice the sign—for $k \in K_A$. Then $E^* = S/S_0$ is a ring, and is an extension of A by Λ relative to the homomorphisms $A \to E^*$ given by $a \to (0, \alpha a)$ and $E^* \to \Lambda$ given by $(f, e) \to \lambda f = \beta e$. Furthermore E^* does in fact realize the given θ. We write $S^*(F, E) = E^*$ for the extension so constructed, and regard S^* as a function applying to equivalence classes of extensions.

Now let E_0 be any one extension of A by Λ which realizes the given θ.

Then one readily proves two equivalences of extensions

$$T^*[S^*(F, E_0), E_0] \cong F, \qquad S^*[T^*(E, E_0), E_0] \cong E$$

for E and F as above. Indeed, in the first equivalence any element in the extension on the left may be written as a coset of a coset, in the form $[(f, e_1) + S_0, e_2] + T_0$, for suitable e_1, e_2 in E_0. Then f, e_1 and e_2 may be so chosen that $e_1 = e_2$; with this choice, the map which carries the cited coset of a coset to f provides the desired equivalence. The second equivalence is treated similarly.

In virtue of these equivalences, the maps

$$F \to E = S^*(F, E_0), \qquad E \to F = T^*(E, E_0)$$

provide a one-one correspondence between the equivalence classes of extensions F and those of extensions E. Since the classes of extensions F correspond again to the elements $H^2(\Lambda, K_A)$, by Theorem 3, this result gives the desired Theorem 7. Note however that the correspondence thus exhibited is not "natural", since it depends on the *choice* of a particular extension E_0. In other words, just as in the case of extensions of groups, the manifold of all extensions of A by Λ realizing θ does not have the natural structure of a group $H^2(\Lambda, K_A)$, but has the natural structure of a "coset" to this group. This phenomenon has recently been investigated thoroughly, for the case of group extensions, by Cobbe and Taylor [4].

These constructions also contain a "Baer product" of extensions in the special case of extensions of a zero ring ($A = K_A$). The extensions of the type denoted above by E are then identical with the extensions of the type denoted by F. In this case the obstruction of θ is always zero, and in fact there is a canonical choice of an extension E_0; namely, we may choose E_0 as the semidirect product of K and Λ, defined as that extension E_g constructed as in §6 from the cocycle $g = 0$. The set of equivalence classes of extensions of A by Λ realizing θ is then a group under the binary operations $S^*(E_1, E_2)$; indeed, this operation corresponds to the addition of cohomology classes in the isomorphism (Theorem 3) of equivalence classes to H^2. The identity element of this group is the class of the semidirect product E_0. The inverse of an extension $A \xrightarrow{\alpha} E \xrightarrow{\beta} \Lambda$ is the ring $E^{\#}$ whose elements are in one-one correspondence $e^{\#} \leftrightarrow e$ with those of E, with the two operations $(e_1 + e_2)^{\#} = (e_1^{\#} + e_2^{\#})$, $(e_1 e_2)^{\#} = -e_1^{\#} e_2^{\#}$. Note that $E^{\#}$ is in fact still a ring, and that the correspondences $k \to (\alpha k)^{\#}$ and $e^{\#} \to -\beta e$ make $E^{\#}$ an extension of the zero ring A by Λ.

The operation S^* thus provides the "Baer product" of ring extensions; on the additive group of the ring it reduces exactly to the Baer product of group extensions (cf. [1]).

The simple case where the bicenter K_A is zero also deserves special statement, as follows.

COROLLARY. *Let A and Λ be rings, where A has the property that $ax = 0 = xa$ for all x in A implies $a = 0$. Then there is one-one correspondence between the equivalence classes of extensions $0 \to A \to E \to \Lambda \to 0$ and the homomorphisms θ of Λ into the ring of outer bimultiplications of Λ. Given E, the corresponding θ is the homomorphism induced by the map which carries each $e \in E$ into the corresponding bimultiplication of A. Given θ, the corresponding E is the graph of θ.*

9. An auxiliary ring

The next constructions to be made will apply to the ring $Q(\Lambda)$ and equally well to many other rings $U(\Lambda)$; the essential feature is the observation that in low dimensions $Q(\Lambda)$ is a free resolution of the additive group of Λ. First recall that $Q_0(\Lambda)$ is the free abelian group generated by all the symbols (x) for $x \neq 0$ in Λ, that one sets $(0) = 0$, and that Q_0 becomes a ring under the product defined by $(x)(y) = (xy)$ for $x, y \in \Lambda$. Furthermore $\eta(x) = x$ defines a ring homomorphism $\eta: Q_0(\Lambda) \to \Lambda$ which in the 1-case is a 1-homomorphism.

We next state the requirements to be placed on the rings $U(\Lambda)$ which may be used to replace $Q(\Lambda)$:

(i) $U(\Lambda)$ is a graded differential ring.

(ii) $U_0 = Q_0(\Lambda)$, as a ring.

(iii) In each dimension n, U_n is a free abelian group.

(iv) The sequence $U_2 \xrightarrow{\partial} U_1 \xrightarrow{\partial} U_0 \xrightarrow{\eta} \Lambda \to 0$ is exact.

Note in particular that (ii) means that η, defined originally in $Q_0 = U_0$, also gives a homomorphism $\eta: U(\Lambda) \to \Lambda$ of graded differential rings which is zero on U_n for $n > 0$. These conditions (i)–(iv) are all satisfied in the case $U = Q$; for example the exactness in (iv) results from the statements of Theorem 1 about the homology of $Q(\Lambda)$ in dimensions 0 and 1.

From $Q_0(\Lambda) = U_0$ alone we now construct a certain auxiliary ring $L = L(\Lambda)$ as the tensor ring over Q_0; that is, we set

$$L = Q_0 + Q_0 \otimes Q_0 + \cdots + Q_0 \otimes \cdots \otimes Q_0 + \cdots,$$

where all the tensor products are taken over the ring of integers. Then L is a ring under the product denoted by $\otimes$. Since this product is not the originally given product in Q_0, we write $q \to \langle q \rangle$ for the embedding of Q_0 in L. Then $q \to \langle q \rangle$ is an additive but *not* a multiplicative homomorphism, and we can write the product in L as an ordinary multiplication $\langle q \rangle \langle r \rangle = \langle q \rangle \otimes \langle r \rangle$.

This ring L is a universal ring to the additive group of Q_0 in the sense that whenever an (additive) group homomorphism $f: Q_0 \to A$ into the additive group of a ring A is given, there exists a unique ring homomorphism $f': L \to A$ with $f'\langle q \rangle = fq$. In particular, a ring homomorphism $S: L(\Lambda) \to Q_0(\Lambda)$ is defined by setting $S\langle q \rangle = q$. The composite ηS is then also a ring homomorphism $\eta S: L(\Lambda) \to \Lambda$.

In the 1-case, the ring L defined above must be taken modulo the ideal spanned by all the elements $\langle q\rangle\langle 1\rangle - \langle q\rangle$ and $\langle q\rangle - \langle 1\rangle\langle q\rangle$ for 1 the identity of Λ. Then $\langle 1\rangle$ becomes the identity of L, and S and ηS are 1-homomorphisms.

In either case we wish to determine the kernels

$$C = \ker [\eta S : L \to \Lambda], \qquad D = \ker [S : L \to Q_0].$$

Both C and D are ideals in L, with $L \supset C \supset D$. Furthermore, since $U_1 \to Q_0 \to \Lambda$ is exact, $C/D \cong \partial U_1$. Clearly D contains the elements $P(q_1, \cdots, q_n)$ defined as follows:

$$P(q, r) = \langle qr\rangle - \langle q\rangle\langle r\rangle, \qquad q, r \in Q_0,$$

$$P(q_1, \cdots, q_{2m}) = P(q_1, q_2) \cdots P(q_{2m-1}, q_{2m}), \qquad m = 1, 2, \cdots,$$

$$P(q_1, \cdots, q_{2m+1}) = P(q_1, \cdots, q_{2m}) \langle q_{2m+1}\rangle, \qquad m = 1, 2, \cdots,$$

for all q_i in Q_0. The larger ideal C contains in addition for each $u \in U_1$ the element $\langle \partial u\rangle$ which we denote as

$$S(u) = \langle \partial u\rangle.$$

These elements satisfy the relations $S(\partial w) = 0$ for each element $w \in U_2$.

LEMMA 1. *As an additive group, D is generated by the elements $P(q_1, \cdots, q_n)$ for $q_i \in Q_0$ and $n = 2, 3, \cdots$, while C is generated by these elements and the elements $S(u)$ for $u \in U_1$. The only relations between these generators P and S are the relation $S(\partial w) = 0$, the linearity of $S(u)$ in u, and the multilinearity of P. In the* 1-*case there are additional relations*

$$\text{(9.1a)} \qquad P(q_1, \cdots, q_{i-1}, 1, q_{i+1}, \cdots, q_{2m}) = 0, \qquad 1 \leqq i \leqq 2m,$$

$$\text{(9.1b)} \qquad P(q_1, \cdots, q_{i-1}, 1, q_{i+1}, \cdots, q_{2m+1}) = 0, \qquad 1 \leqq i \leqq 2m,$$

$$\text{(9.1c)} \qquad P(q_1, \cdots, q_{2m}, 1) = P(q_1, \cdots, q_{2m}).$$

In the proof, it is convenient to use the natural grading of the ring L, obtained by assigning to each generator $\langle q\rangle$ of L the degree 1 (exception: in the 1-case, $\langle 1\rangle$ has the degree 0). Then L is a graded ring, and $P(q_1, \cdots, q_n) = \pm\langle q_1\rangle \cdots \langle q_n\rangle +$ terms of lower degree. Thus $P(q_1, \cdots, q_n)$ has degree n if no q_i is zero (or, in the 1-case, if in addition no $q_i = 1$).

Suppose now that $c \in C$ is a sum of homogeneous terms of degrees at most n in L; we will show by induction on n that c is a linear combination of the elements stated. If $n = 1$, c has the form $\langle q\rangle$ for $q \in Q_0$ and $\eta Sc = \eta q = 0$; hence by the exactness condition (iv), $q = \partial u$ for some $u \in U_1$ and $c = \langle \partial u\rangle = S(u)$. If $n > 1$, c is a linear combination of terms $\langle q_1\rangle \cdots \langle q_n\rangle$ plus terms of lower degree. By subtracting the corresponding combination of $\pm P(q_1, \cdots, q_n)$ we then reduce c to terms of degree less than n and hence complete the induction. These arguments apply equally well to the generators of D and to the 1-case.

Suppose now that R denotes some linear relation between the generators $S(u)$ and $P(q_1, \cdots, q_r)$. By assigning to each letter $S(u)$ the degree 1 and to each letter $P(q_1, \cdots, q_n)$, whatever the arguments q, the degree n, we obtain some (maximum) degree n for the whole relation R. We now can show by induction on the degree n that this relation R is a consequence of the stated relations. If $n = 1$, R has the form $\sum_i \varepsilon_i S(u_i)$ for $\varepsilon_i = \pm 1$ and $u_i \in U_1$. Since R is a valid relation in C, we have

$$0 = \sum_i \varepsilon_i S(u_i) = \sum_i \varepsilon_i \langle \partial u_i \rangle = \langle \partial \sum_i \varepsilon_i u_i \rangle.$$

Hence $\sum \varepsilon_i u_i$ is a 1-cycle, and thus by exactness a 1-boundary, so there is some element $v \in U_2$ with $\partial v = \sum_i \varepsilon_i u_i$. Thus R is a consequence of the linearity of S and the relation $S(\partial v) = 0$.

Now consider a relation R of maximum degree n with $n > 1$; it has the form $0 = \sum_i P(q_{1i}, \cdots, q_{ni})$ + terms of lower degree. Replace each P by its definition; we get $0 = \sum_i \langle q_{1i} \rangle \otimes \cdots \otimes \langle q_{ni} \rangle$ + terms of lower degree. The highest degree term lies in the tensor product $Q_0 \otimes \cdots \otimes Q_0$ with n factors, which can be zero only by virtue of the multilinearity which also applies to the P. Hence the given relation is a consequence of multilinearity.

In the 1-case, the additive group Q_0 has the form of a direct sum $Z(1) + Q_0'$, where Q_0' is the free abelian group generated by all (x) for $x \neq 0, 1$. Thus in the relation each argument q can be written as a sum, $m1 + q'$ for $m \in Z$, $q' \in Q_0'$; by multilinearity and the relations (9.1) all the terms $m1$ can be removed; the rest of the argument, on the q', proceeds as before since in this case $L = Q_0' + Q_0' \otimes Q_0' + \cdots$.

This completes the proof of the lemma.

We now describe the multiplicative structure of the rings C and D, starting with the inner bimultiplications by $\langle q \rangle$ in L. Since C and D are ideals in L, this yields a bimultiplication ρ_q which is given on the generators of C and D by the following explicit formulas

$$\begin{aligned} &\rho_{q_0} P(q_1, \cdots, q_n) \\ &\qquad = \sum_{i=0}^{n} (-1)^i P(q_0, \cdots, q_i q_{i+1}, \cdots, q_n) + (-1)^n P(q_0, \cdots, q_n), \end{aligned} \tag{9.2}$$

$$P(q_1, \cdots, q_{2m})\rho_r = P(q_1, \cdots, q_{2m}, r), \tag{9.3a}$$

$$\begin{aligned} &P(q_1, \cdots, q_{2m+1})\rho_r \\ &\qquad = -P(q_1, \cdots, q_{2m+1}, r) + P(q_1, \cdots, q_{2m}, q_{2m+1} r), \end{aligned} \tag{9.3b}$$

$$\rho_q S(u) = -P(q, \partial u) + S(qu), \tag{9.4}$$

$$S(u)\rho_q = -P(\partial u, q) + S(uq). \tag{9.5}$$

Furthermore one shows easily that, for $q, r \in Q_0$,

$$\rho_{q+r} = \rho_q + \rho_r, \qquad \rho_{qr} = \rho_q \rho_r + \mu_{P(q,r)}, \tag{9.6}$$

and in the 1-case one has $\rho_1 = I$.

The products of any two generators of C are given explicitly by the formulas

$$P(q_1, \cdots, q_{2m})P(r_1, \cdots, r_n) = P(q_1, \cdots, q_{2m}, r_1, \cdots, r_n), \tag{9.7a}$$

$$\begin{aligned} &P(q_1, \cdots, q_{2m}, r_0)P(r_1, \cdots, r_n) \\ &= \sum_{i=0}^{n-1} (-1)^i P(q_1, \cdots, q_{2m}, r_0, \cdots, r_i r_{i+1}, \cdots, r_n) \\ &\qquad + (-1)^n P(q_1, \cdots, q_{2m}, r_0, \cdots, r_n), \end{aligned} \tag{9.7b}$$

$$S(u)S(u') = \rho_{\partial u} S(u'), \qquad u, u' \in U_1, \tag{9.8}$$

$$S(u)P(q_1, \cdots, q_n) = \rho_{\partial u} P(q_1, \cdots, q_n), \tag{9.9}$$

$$P(q_1, \cdots, q_n)S(u) = P(q_1, \cdots, q_n)\rho_{\partial u}. \tag{9.10}$$

10. The main existence theorem

THEOREM 8. *Let K be a bimodule over the ring Λ and f a three-dimensional cocycle of $R(\Lambda)$ with coefficients in K. Then there exists a ring A with bicenter K together with a homomorphism $\theta:\Lambda \to P_A$ whose image consists of mutually permutable outer bimultiplications and which induces on the bicenter K the given bimodule structure, all such that f is an obstruction of θ, if and only if, for all $x \in \Lambda$*

$$f\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} = 0.$$

This condition is manifestly necessary, for by the definition (7.4) of the obstruction one has $f(v) = 0$ on every 2-cycle v of θ, in particular on the basic 2-cycle $\gamma(x)$ (see (3.1)) of $Q(\Lambda)$. In fact, the assignment $f \to f(\gamma(x))$ yields a homomorphism

$$\bar{\gamma}:H^3(\Lambda, K) \to \operatorname{Hom}(\Lambda/2\Lambda, K);$$

hence we may state the necessary condition in the more invariant form that $\bar{\gamma}$ of the cohomology class of f is zero.

Conversely, to define a ring A which will realize a given f we simply "invert" the equations (7.1)–(7.7) which defined the obstruction f and use these equations instead to define a ring A generated by the bimodule K and by symbols $h(x, y)$ and $h(x \mid y)$ which satisfy the conditions (7.1)–(7.7) relative to suitable bimultiplications σ_x. Compare these conditions with the relations of §9 on the generators S and P, writing now $S(x, y)$ instead of $S(u)$ for $u = (x, y) \in Q_1$. Specifically, replace $h(x, y)$ by $S(x, y)$, $h(x \mid y)$ by $P(x, y)$ and ρ by σ. Now if f were zero, the condition (7.5) on h is exactly the definition (9.5) of ρ on the right, while (7.6) is similarly (9.4), and (7.7) combines (9.2) for $n = 2$ and (9.3a) for $m = 1$. We will therefore define the ring A by modifying the defining equations found in §9 for the ring C by adjoining suitable terms in the given cocycle f. The eight conditions stating that f is a cocycle will then all enter in the demonstration that what we get is indeed a ring.

It will simplify the formulas if we replace the cubical construction $Q(\Lambda)$ by any U as in §9, so that the construction is applied to a cocycle f of $\bar{B}(U, \eta)$

and not to one of the more special complex $R(\Lambda) = B(Q(\Lambda), \eta)$. We assume that U satisfies the conditions (i) – (iv) of §9, together with

(iv′) The sequence $U_3 \xrightarrow{\partial} U_2 \rightarrow U_1$ is exact.

To be sure, this last condition doesn't hold for Q, but it would hold if Q_3 were enlarged by new generators with boundary $\gamma(x)$. We don't care at all how or whether products are defined for these new generators, for these products don't enter—the three-dimensional cohomology of $\bar{B}(U, \eta)$ involves the multiplication in U only up through products which lie in U_2. The merit of adding condition (iv′) is that the necessary condition $f(\gamma(x)) = 0$ of our theorem now becomes part of the statement that f is a cocycle of $\bar{B}(U, \eta)$.

Explicitly, let now Λ, K, and f be given, and take U to satisfy (i), (ii), (iii), (iv), and (iv′). Introduce the abelian group

$$A = (K + U_1 + D)/M,$$

for D as in §9 and M the subgroup consisting of all $f(v) + \partial v$, for $v \in U_2$. Define a homomorphism $\psi : A \rightarrow C$, where C is the additive group of the ideal described in §9, by setting

$$\psi(k + u + d) = S(u) + d, \qquad k \in K, \quad u \in U_1, \quad d \in D.$$

LEMMA 2. *The map ψ yields an exact sequence of abelian groups*

$$0 \rightarrow K \rightarrow A \xrightarrow{\psi} C \rightarrow 0.$$

Proof. That ψ as defined annihilates the subgroup S follows from the relation $S(\partial w)$ known to hold in C. To show that the natural map $k \rightarrow k + M$ of K into A has kernel zero, we observe that an element $f(v) + \partial v$ of M can lie in K only if $\partial v = 0$, hence by (iv) only if $v = \partial w$ for some $w \in U_3$, hence $f(v) + \partial v = f(\partial w) + \partial\partial w = \delta f(w) = 0$. That ψ annihilates K is clear. Conversely, suppose that $k + u + d$ is in the kernel of ψ. Then $S(u) + d = 0$ in C. But Lemma 1 of §9 determines the additive structure of C, and implies here that $d = 0$ and that $S(u) = \langle \partial u \rangle = 0$, hence $\partial u = 0$ and $u = \partial v$ for some $v \in U_2$. Thus $k + u + d = k + \partial v \equiv k + f(-v) \pmod{M}$ is an element in the image of K. This proves the lemma.

LEMMA 3. *For each $q \in U_0 = Q_0$ an endomorphism $a \rightarrow \tau_q a$ of the abelian group A is defined by*

$$\tau_q k = (\eta q)k, \qquad k \in K,$$

$$\tau_q u = f(q \mid u) - P(q, \partial u) + qu, \qquad u \in U_1,$$

$$\tau_q P(r, s) = f(q \mid r \mid s) + \rho_q P(r, s), \qquad r, s \in Q_0$$

$$\tau_q P(r, s, t) = f(q \mid r \mid s)\eta t + \rho_q P(r, s, t), \qquad t \in Q_0,$$

$$\tau_q P(r_1, \cdots, r_n) = \rho_u P(r_1, \cdots, r_n), \qquad n > 3.$$

These endomorphisms also satisfy

$$(10.1) \qquad \tau_{q_1+q_2} = \tau_{q_1} + \tau_{q_2}, \qquad \psi(\tau_q a) = \rho_q \psi a.$$

In the 1-*case we also have* $\tau_1 = I$, *the identity endomorphism.*

Proof. Since all the expressions involved are multilinear in their arguments, it will follow that τ_q is an endomorphism if $\tau_q M \subset M$. Take then v in U_2. Then

$$\begin{aligned} \tau_q(f(v) + v) &= (\eta q)f(v) + f(q \mid \partial v) - P(q \mid \partial\partial v) + q\partial v \\ &\equiv (\eta q)f(v) + f(q \mid \partial v) - f(qv) \qquad \pmod{M} \\ &\equiv f\partial[q \mid v] \equiv 0, \end{aligned}$$

as desired. The remaining assertions of the lemma follow readily; in particular $\psi(\tau_q a)$ has the indicated form because of the parallel between the definitions above and those found for ρ_q in §9.

LEMMA 4. *For each* $q \in U_0$ *an endomorphism* $a \to a\tau_q$ *of the additive group* A *is defined by*

$$\begin{aligned} k\tau_q &= k(\eta q), && k \in K, \\ u\tau_q &= -f(u \mid q) - P(\partial u, q) + uq, && u \in U_1, \\ d\tau_q &= d\rho_q, && d \in D. \end{aligned}$$

These endomorphisms also satisfy

$$(10.2) \qquad \tau_{q_1+q_2} = \tau_{q_1} + \tau_{q_2}, \qquad \psi(a\tau_q) = (\psi a)_q,$$

and in the 1-*case one has* $\tau_1 = I$.

The proof is similar, and uses the identity $f\partial[w \mid q] = 0$.

LEMMA 5. *For all* q, r *in* Q_0 *and* a *in* A,

$$(10.3) \qquad (\tau_q a)\tau_r = \tau_q(a\tau_r).$$

Proof. For $a \in K$, this follows because $(\lambda a)\mu = \lambda(a\mu)$ in the bimodule K. For $a = u \in U_1$, it reduces by the above definition to the identity $f\partial[q \mid u \mid r] = 0$. For $a = d$ it reduces by the definitions of τ to zero.

LEMMA 6. *A homomorphism* $A \otimes A \to A$, *that is, a product in* A, *is defined by*

$$(10.4) \qquad (k + u + d)(k' + u' + d') = \tau_{\partial u} u' + \tau_{\partial u} d' + d\tau_{\partial u'} + dd'$$

for $k, k' \in K$; $u, u' \in U_1$; $d, d' \in D$; *and* dd' *the known product in the ring* D. *For this product* $\psi: A \to C$ *is a homomorphism.*

Proof. Since the formula (10.4) is bilinear, it will yield a homomorphism as desired if each product with one factor in M lies in M. This is readily verified:

$$d(f(v) + \partial v) = d(\partial v) = d\tau_{\partial\partial v} = 0,$$

$$(f(v) + \partial v)(u + d) = (\partial v)u + (\partial v)d = \tau_{\partial\partial v}u + \tau_{\partial\partial v}d = 0.$$

Finally, ψ is a homomorphism as claimed because

$$\psi(du') = \psi(d\tau_{\partial u'}) = d\rho_{\partial u'} = dS(u) = d\psi u$$

by (9.5) and (10.1), with a similar calculation for the other products.

LEMMA 7. *For $u \in U_1$ and $a \in A$, $ua = \tau_{\partial u}\, a$ and $au = a\tau_{\partial u}$.*

In terms of the inner multiplication μ_u of A, the result can be written more briefly as $\mu_u = \tau_{\partial u}$. The first assertion about ua is simply a part of the definition (10.4) of the product. The second assertion about au may be treated in three cases, according as a is in K, U_1, or D. The third case is part of the definition (10.4); the first case follows because $k\tau_{\partial u} = k(\eta\partial u) = k0 = 0 = ku$. Finally, if $a = u'$ is in U_1,

$$\begin{aligned} u'u - u'\tau_{\partial u} &= \tau_{\partial u'}u - u'\tau_{\partial u} \\ &= f(\partial u' \mid u) - P(\partial u', \partial u) + (\partial u')u + f(u' \mid \partial u) + P(\partial u', \partial u) \\ &\quad - u'(\partial u) \\ &\equiv f(\partial u' \mid u) + f(u' \mid \partial u) - f(u'u) \qquad \pmod{M} \\ &\equiv -f\partial[u' \mid u] = 0. \end{aligned}$$

LEMMA 8. *For $q, r \in Q_0$ and μ the inner bimultiplication in A,*

$$\tau_{qr} = \tau_q\tau_r + \mu_{P(q,r)}\,.$$

Proof. We first compute the expression $E = \tau_{qr}\, a - \tau_q\tau_r\, a - P(q, r)a$ for any a in A. If $a \in K$, E is zero by definition. If $a \in U_1$, E turns out to be zero by the identity $f\partial[q \mid r \mid a] = 0$. Finally, if $a \in D$, we consider several cases corresponding to different generators of D. If $a = P(s, t)$ for elements $s, t \in U_0$, then $E = 0$ by the identity $f\partial[q \mid r \mid s \mid t] = 0$. If $a = P(s, t, p)$, we may write $a = P(s, t)\tau_p$; the expression E then reduces to that of the previous case, if we use Lemma 4 and the readily established result that $P(q, r)[P(s, t)\tau_p] = [P(q, r)P(s, t)]\tau_p$. Finally, if $a = P(s_1, \cdots, s_n)$ with $s_i \in Q_0$ and $n > 3$, the bimultiplications τ are identical with the ρ, and the expression is zero in virtue of (9.6).

It remains only to show that $E' = a\tau_{qr} - a\tau_q\tau_r - aP(q, r)$ is zero. For $a \in U_1$ this follows by the identity $f\partial[a \mid q \mid r] = 0$; in the other cases it is immediate.

LEMMA 9. *For $q \in Q_0$ and $a, b \in A$ we have*

$$\tau_q(ab) = (\tau_q\, a)b, \qquad (a\tau_q)b = a(\tau_q\, b), \qquad (ab)\tau_q = a(b\tau_q).$$

Proof. After setting for a and b the various types of elements of A—from K, U_1 or D—these expressions may be expanded by the definitions; by suitable use of the previous lemmas they all turn out to be identities, and this without further use of identities valid on the cocycle f.

LEMMA 10. *Multiplication in A is associative.*

Proof. We must prove $(ab)c = a(bc)$ for all a, b, c in A. If all three arguments lie in the ring D, this is known. If any one argument lies in K, both sides are zero. We are then left with the cases in which at least one argument is an element $u \in U_1$. In this case we have respectively

$$u(bc) = \tau_{\partial u}(bc) = (\tau_{\partial u} b)c = (ub)c$$

by Lemma 7, Lemma 9, and Lemma 7 again,

$$a(uc) = a(\tau_{\partial u} c) = (a\tau_{\partial u})c = (au)c,$$

$$a(bu) = a(b\tau_{\partial u}) = (ab)\tau_{\partial u} = (ab)u,$$

as asserted.

We pause to observe that this sequence of Lemmas 2–10 has fully used the fact that f is a cocycle; i.e., that f vanishes on the boundary of each of the 8 types of 4-cells from R. Indeed, these 8 types appeared as follows, with the notation $q, r, s, t \in U_0 = Q_0$, $u \in U_1$, $v \in U_2$, $w \in U_3$:

$[w]$, Lemma 2; $[q \mid v]$, Lemma 3; $[v \mid q]$, Lemma 4;

$[q \mid u \mid r]$, Lemma 5; $[u' \mid u]$, Lemma 7; $[q \mid r \mid u]$ Lemma 8;

$[u \mid q \mid r]$, Lemma 8; $[q \mid r \mid s \mid t]$, Lemma 9.

LEMMA 11. *A is a ring, ψ is a homomorphism of rings, and*

$$0 \to K \to A \xrightarrow{\psi} C \to 0$$

is an exact sequence of ring homomorphisms in which K is mapped isomorphically onto the bicenter of A.

Proof. Lemma 10 shows A a ring, Lemma 6 shows ψ a ring homomorphism, and Lemma 3 gives the exactness. By the definition of the product, the image of K clearly lies in the bicenter of A. On the other hand, $\psi A = C$ is a subring of L, and L has no divisors of zero; hence K is exactly the bicenter of A.

LEMMA 12. *For $x \in \Lambda$ the correspondence $\theta x = \tau_{(x)}$ induces a homomorphism $\theta : \Lambda \to P_A$. The image of θ consists of mutually permutable outer bimultiplications, and in the 1-case θ is a 1-homomorphism.*

Proof. Set $(x) = q$. That τ_q is a bimultiplication of A is asserted by Lemmas 3, 4, and 9. That any two τ_q permute is the content of Lemma 5. That θ is a homomorphism for addition is stated in (10.1) and (10.2), while the fact that θ is a multiplicative homomorphism, modulo inner bimultiplications of A, is Lemma 8. In the 1-case $\tau_1 = I$, Q.E.D.

LEMMA 13. *An obstruction to θ is the given cocycle f.*

In keeping with our definition of the obstruction, this is proved only in the explicit case when U is Q. To apply the definition of the obstruction, choose for each x in Λ the bimultiplication $\sigma_x = \tau_{(x)}$ in the class θx. Then since $(x + y) + \partial(x, y) = (x) + (y)$ by the boundary formula for Q_1, and since $\mu_u = \tau_{\partial u}$ by Lemma 7, we have

$$\sigma_x + \sigma_y = \mu_{(x,y)} + \sigma_{x+y} .$$

Therefore we may choose $h(x, y) = (x, y)$ as in (7.1). Also, by Lemma 8

$$\sigma_x \sigma_y = \mu_{P(x,y)} + \sigma_{xy} ;$$

hence we may choose $h(x \mid y) = P(x, y) \in D \subset A$, as in (7.2). From the functions h thus chosen we now calculate the obstruction according to the definitions (7.4)–(7.7). We obtain exactly the original cocycle f; indeed (7.4) comes from the choice of M, (7.5) and (7.6) from the definitions of $u\tau_q$ and $\tau_q u$, and (7.7) is a combination of the definitions of $\tau_q P$ and $P\tau_q$.

With this lemma the main existence theorem is established.

11. Alternative forms for the cohomology of rings

We may now illustrate the general conditions considered in §§9 and 10 on the ring U by constructing a particular ring $V = V(\Lambda)$ which satisfies these conditions. Set $V_0 = Q_0(\Lambda)$. Since Q_0 as an additive group is free, the kernel V_1 of $\eta: V_0 \to \Lambda$ is also a free abelian group. Let the injection of this kernel in V_0 be written $\partial: V_1 \to V_0$; then

$$0 \to V_1 \xrightarrow{\partial} V_0 = Q_0 \xrightarrow{\eta} \Lambda \to 0 \tag{11.1}$$

is an exact sequence of abelian groups. Since the kernel of η is also an ideal in V_0, we can define the product of $v \in V_1$ and $q \in V_0$ by

$$vq = \partial^{-1}[(\partial v)q], \qquad qv = \partial^{-1}[q(\partial v)], \qquad vv' = 0.$$

With these products (together with those given in Q_0), $V = V_0 + V_1$ is a graded differential ring which satisfies the conditions (i)–(iv) and (iv′) placed on U in §9 and §10. Like $Q(\Lambda)$, $V(\Lambda)$ is a functor of Λ.

THEOREM 9. *There is a homomorphism $\chi: Q(\Lambda) \to V(\Lambda)$ of graded differential rings which induces isomorphisms*

$$H^n(\bar{B}(V(\Lambda), \eta); K) \cong H^n(\bar{B}(Q(\Lambda), \eta); K), \qquad n = 0, 1, 2, \tag{11.2}$$

$$H^3(\bar{B}(V(\Lambda), \eta); K) = \ker \{\bar{\gamma} H^3(\bar{B}(Q(\Lambda, \eta); K) \to \operatorname{Hom}(\Lambda/2\Lambda, K)\}. \tag{11.3}$$

Proof. To define χ, take $\chi = 1$ in dimension zero, $\chi u = \partial_V^{-1}(\partial_Q u)$ for $u \in Q_1$, and $\chi = 0$ in higher dimensions. Then χ is a homomorphism of graded differential rings. By the exactness of (11.1), χ induces also a homology isomorphism of Q to V in dimensions 0 and 1. If one adjoins to Q elements of dimension 3 to kill the cycles $\gamma(x)$ (i.e., elements with boundary $\gamma(x)$), then χ induces a homology isomorphism in dimensions 0, 1, and 2. Now in [7, Theorem 13.1] it was proved that a homology isomorphism $\chi:G \to G'$ between graded differential rings G and G' which are free as additive groups induces a homology isomorphism $\bar{B}(\chi):\bar{B}(G) \to \bar{B}(G')$. This theorem, in which the proof depended essentially upon a filtration of $\bar{B}$, applies *mutandis mutatis* to the present slight generalization of the bar construction. Since the application of the bar construction raises the dimensions of G by 1, we get a homology involving isomorphism $\bar{B}(Q(\Lambda)) \to \bar{B}(V(\Lambda))$ valid in dimensions 0, 1, 2, 3 (the latter provided the extra elements are adjoined to $Q(\Lambda)$). The asserted cohomology isomorphisms then follow. No trouble is caused by the products of the new elements adjoined to $Q(\Lambda)$, since in these dimensions only products lying in Q_2 and below matter.

We have used the cohomology theory for rings defined by the complex $\bar{B}(Q(\Lambda), \eta)$, and indeed this appears to be the proper complex for the relative homology for modules which was considered in [15]. The present theorem suggests that for the purposes of obstruction theory one might operate more generally, with the complex $\bar{B}(U(\Lambda), \eta)$ where $U(\Lambda)$ is any graded differential ring satisfying the conditions (i)–(iv) of §9 and (iv′) of §10. It is in fact the case that the whole theory of extensions and obstructions as presented in §§6–8 can be carried out directly with U replacing Q, though some little clarity is lost in this generalization. In particular, Lemma 13 as to the identification of this obstruction does hold in this generality.

The exactness conditions (iv) and (iv′) on U might more naturally be replaced by the stronger requirement that the whole sequence

$$\cdots \to U_n \xrightarrow{\partial} U_{n-1} \to \cdots \to U_1 \xrightarrow{\partial} U_0 \xrightarrow{\eta} \Lambda \to 0$$

be exact. Each such U may be described briefly as a free resolution of the additive group of Λ which has the structure of a graded differential ring and for which η is a homomorphism of such rings. In this sense then, the homology of a ring may be described as that obtained by applying the bar construction (or presumably any equivalent construction in the sense of Cartan [2]) to a free resolution of the additive group, this resolution being given a ring structure. However, the condition (ii) of §9 that U_0 be the specific ring Q_0, freely generated as an additive group by the elements (x) for $x \neq 0$, is a sharp restriction. This restriction appears to be an essential one. On the one hand, it enters essentially into the comparison made between Q and V (or some other U) in Theorem 9, since the choice $V_0 = Q_0$ was the starting point for the construction of a ring homomorphism χ. On the other hand, if the obstruction theory is carried out for a general U, as mentioned above, the assumption $U_0 = Q_0$ appears to be essential to the proof of the theorem that an extension realizing θ exists provided only that the obstruction of θ vanishes. Hence of all the resolutions which might be possible for U, only Q and the specific V exhibited in (11.1) above appear to be clearly of interest. Neither provides an easy means for the calculation of the cohomology groups of specific rings.

We mention in passing that still another ring U, satisfying (i)–(iv) but not (iv′), can be obtained by replacing the cubical complex $Q(\Lambda)$ by the complex $A(\Lambda)$ which is described in detail in [15] and which, like Q, is a complex for the homology of the abelian group Λ. Briefly, A is obtained by applying the iterated bar construction to the integral group ring $Z(\Lambda)$ of Λ. If $\eta: Z(\Lambda) \to \Lambda$ is the canonical augmentation of this group ring, then the exact definition is $A(\Lambda) = \bar{B}^{\infty}(Z(\Lambda, n)$—with suitable adjustment of the dimensions so that $\bar{B} \to \bar{B}(\bar{B}) \to \cdots$ preserves dimensions. The cohomology of Λ in dimensions 0, 1, 2, and 3 can then be obtained as that of the complex $\bar{B}(A(\Lambda), \eta)$ when in the bar construction one uses a product in $A(\Lambda)$ defined in low dimensions (up to products in A_2) by the formulas

$$(x)(y) = (xy), \qquad (x)(y, z) = (xy, xz), \qquad (x, y)(z) = (xz, yz),$$

$$(t)(x, y, z) = (tx, ty, tz), \qquad (x, y, z)(t) = (xt, yt, zt),$$

$$(t)[x \mid y] = [tx \mid ty], \qquad [x \mid y](t) = [xt \mid yt],$$

$$\begin{aligned}(x, y)(s, t) = (xs, ys, xt + yt) &- (xs, yt, ys + yt) \\ &+ (xt, ys, yt) - (ys, xt, yt) - [ys \mid xt].\end{aligned}$$

Here x, y, z, s, t are any elements of the ring Λ. I have not found any natural extension of this product to higher dimensions in $A(\Lambda)$.

The acyclic bar construction $B(Q(\Lambda), \eta)$, described as in [15], may informally be regarded as a sort of resolution of the graded differential ring $Q(\Lambda)$. Now B is a $Q(\Lambda)$-Λ-bimodule, and observe that η may be used to turn every Λ-bimodule K into a $Q(\Lambda)$-Λ-bimodule. The construction of $\bar{B}$ from B is then such that $\mathrm{Hom}_{\Lambda}(\bar{B}; K) = \mathrm{Hom}_{Q,\Lambda}(B; K)$; it follows that the cohomology groups $H^n(\Lambda, K)$ defined in (5.3) may be written as

$$H^n(\Lambda, K) = H^n(\mathrm{Hom}_{Q,\Lambda}(B(Q(\Lambda), \eta), K))$$

with B replacing $\bar{B}$. In other words, the cohomology theory of a ring arises by putting a "multiplicative" resolution B on top of an additive resolution Q of the ring Λ.

12. Two examples

To show that the obstruction to a ring extension problem is not always trivially zero, we construct two examples.

First, take Λ to be the ring generated by 1 and t, with $t^2 = 0$; thus the elements of Λ are all $a + bt$ for $a, b \in Z$ (the ring of integers). Take K to be the additive group Z of integers with the Λ-bimodule structure given by

$$(a + bt)k = ak, \qquad k(a + bt) = ka.$$

Define a three-dimensional cochain of $\bar{B}(Q(\Lambda), \eta)$ by setting

$$f[a_1 + b_1 t \mid a_2 + b_2 t \mid a_3 + b_3 t] = b_1 b_2 b_3 ,$$

$$f(\text{any other type of generator}) = 0.$$

This f is clearly multilinear in its argument. One proves that f is a cocycle by showing that $f\partial\beta = 0$ for each of the eight types of four-dimensional generators β of $\bar{B}$. For four of these types $f\partial\beta$ is identically zero, for three types $f\partial\beta = 0$ by virtue of the multilinearity of f, and finally we compute

$$\begin{aligned} f\partial(a_1 + b_1 t \mid a_2 + b_2 t \mid a_3 + b_3 t \mid a_4 + b_4 t) \\ = a_1 b_2 b_3 b_4 - a_1 b_2 b_3 b_4 - a_2 b_1 b_3 b_4 + b_1 a_2 b_3 b_4 + b_1 b_2 a_3 b_4 \\ - b_1 b_2 a_3 b_4 - b_1 b_2 b_3 a_4 + b_1 b_2 b_3 a_4 = 0. \end{aligned}$$

If f were the coboundary of some 2-cochain g, we would have in particular

$$f[t \mid t \mid t] = g\partial[t \mid t \mid t] = tg(t \mid t) - g(t^2 \mid t) + g(t \mid t^2) - g(t \mid t)t = 0,$$

since $t^2 = 0$. This contradiction shows that f gives a nontrivial cohomology class, and clearly one in the kernel of the homomorphism $\bar{\gamma}$. By the main existence theorem there therefore is a ring extension problem with obstruction $f \neq 0$.

The second example will construct explicitly a sample ring of outer bimultiplications. Take Z to be the ring of integers, p a prime, and set $A = pZ/p^3Z$. The bicenter of A is then $K = p^2Z/p^3Z$, and as an additive group A is a cyclic group of order p^2 with a generator which we write as $\bar{p} = p + p^3Z$. For each pair $\sigma = (a, c)$ with $a \in Z/p^2Z$, $c \in Z/pZ$, we define left and right endomorphisms $\sigma: A \to A$ by setting

$$\sigma(i\bar{p}) = ai\bar{p}, \qquad (i\bar{p})\sigma = (a - cp)i\bar{p}, \qquad i \in Z.$$

We verify easily that each such σ is a bimultiplication of A. They are in fact all the bimultiplications of A. For if τ is any bimultiplication, $\tau\bar{p} = a\bar{p}$ for some $a \in Z/p^2Z$, so $\tau - (a, 0) = \rho$ is a bimultiplication with $\rho'\bar{p} = 0$. If then $\bar{p}\rho = b\bar{p}$ for some integer b, we have $0 = \bar{p}(\rho\bar{p}) = (\bar{p}\rho)\bar{p} = b\bar{p}^2 = bp\bar{p}$, and hence $b \equiv 0 \pmod{p}$. Setting $b = cp$ we have $\rho = (0, c)$, as asserted.

Upon calculating sums and composites of these bimultiplications we conclude that the bimultiplication ring M_A consists of all pairs $\sigma = (a, c)$ and $\tau = (a', c')$ for $a \in Z/p^2Z$ and $c \in Z/pZ$ with

$$\sigma + \tau = (a + a', c + c'), \qquad \sigma\tau = (aa', ac' + a'c).$$

In particular $(1, 0)$ is the identity bimultiplication. Now the inner bimultiplication by $\bar{p} \in A$ is $(p, 0)$. Hence the ring P_A of outer bimultiplications may be represented as the ring of all pairs (a, c) of integers a and c each taken modulo p, with sum and product

$$(a, c) + (a', c') = (a + a', c + c')$$

$$(a, c)(a', c') = (aa', ac' + a'c).$$

Equivalently, P_A is a ring generated by $I = (1, 0)$ and $\gamma = (0, 1)$; each generator has additive order p, I is the identity, and $\gamma^2 = 0$.

We now take $\Lambda = P_A$ and assert that the identity homomorphism $\theta = I:\Lambda = P_A \to P_A$ cannot be realized by an extension. It will then follow that the obstruction cohomology class for θ in $H^3(\Lambda, K)$ is nonzero. For suppose that there did exist an extension

$$0 \to A \to E \xrightarrow{\beta} P \to 0$$

with A and P as above. Since β induces $\theta = I$ which is a 1-homomorphism, it follows by Theorem 3a that E has an identity 1_E and is a 1-homomorphism. Since β induces θ, we can choose in E an element r such that $\beta r = \gamma$ and such that the bimultiplication induced on A by r is the bimultiplication denoted by $(0, 1)$. This statement means that

$$rn = 0, \qquad nr = -(n + \cdots + n) = -pn \qquad (p \text{ terms}) \tag{12.1}$$

for all elements n of A. Since $p1_P = 0$ in P, there is a constant h in A such that $p1_E = h$ in E. Since 1_E is the identity, $rh = hr$; on the other hand $rh = 0$. However,

$$\begin{aligned} hr &= -ph && \text{(by (12.1))} \\ &= -\bar{p}h && \text{by the definition of product in } A, \\ &= -\bar{p}(p1_E), && \text{by definition of } h \\ &= -(p\bar{p})1_E, && \text{by distributive law} \\ &= -p\bar{p}, && \text{as } 1_E \text{ is identity.} \end{aligned}$$

But $-p\bar{p} \neq 0$ in $A = pZ/p^3Z$. Thus $0 = rh = hr \neq 0$, a contradiction, as asserted.

References

1. Reinhold Baer, *Erweiterung von Gruppen und ihren Isomorphismen*, Math. Zeit., vol. 38 (1934), pp. 375–416.
2. H. Cartan, *Algèbres d'Eilenberg-MacLane et homotopie*, Séminaire de l'École Normale Supérieure, Paris, 1955.
3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, Princeton University Press, 1956.
4. A. P. Cobbe and R. L. Taylor, *On Q-kernels with operators*, Quart. J. Math. Oxford Ser. (2), vol. 8 (1957), pp. 13–38.
5. S. Eilenberg and S. Mac Lane, *Cohomology theory in abstract groups. II, Group extensions with a non-Abelian kernel*, Ann. of Math. (2), vol. 48 (1947), pp. 326–341.
6. ———, *Homology theories for multiplicative systems*, Trans. Amer. Math. Soc., vol. 71 (1951), pp. 294–330.
7. ———, *On the groups $H(\Pi, n)$, I*, Ann. of Math. (2), vol. 58 (1953), pp. 55–106.
8. ———, *On the groups $H(\Pi, n)$, II, Methods of computation*, Ann. of Math. (2), vol. 60 (1954), pp. 49–139.
9. ———, *On the homology theory of abelian groups*, Canadian J. Math., vol. 7 (1955), pp. 43–53.

10. C. J. EVERETT, JR., *An extension theory for rings*, Amer. J. Math., vol. 64 (1942), pp. 363–370.
11. G. HOCHSCHILD, *Cohomology and representations of associative algebras*, Duke Math. J., vol. 14 (1947), pp. 921–948.
12. ———, *Lie algebra kernels and cohomology*, Amer. J. Math., vol. 76 (1954), pp. 698–716.
13. ———, *Cohomology classes of finite type and finite dimensional kernels for Lie algebras*, Amer. J. Math., vol. 76 (1954), pp. 763–778.
14. OTTO HÖLDER, *Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4*, Math. Ann., vol. 43 (1893), pp. 301–412, especially §18.
15. SAUNDERS MAC LANE, *Homologie des anneaux et des modules*, Colloque de topologie algébrique, Louvain, 1956, pp. 55–80.
16. L. REDEI, *Die verallgemeinerung der Schreierschen Erweiterungstheorie*, Acta Sci. Math. Szeged, vol. 14 (1952), pp. 252–273.
17. O. SCHREIER, *Über die Erweiterung von Gruppen I*, Monatsh. Math., vol. 34 (1926), pp. 165–180.
———, *Über die Erweiterung von Gruppen. II*, Abh. Math. Sem. Univ. Hamburg, vol. 4 (1926), pp. 321–346.
18. J. SZENDREI, *On Schreier extension of rings without zero-divisors*, Publ. Math. Debrecen, vol. 2 (1952), pp. 276–280.

THE UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS