ON A THEOREM OF ERDÖS AND SZEKERES

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1. Introduction

Suppose h is a given positive integer greater than 1. Let $M(h)$ be the set of all positive integers n such that $p^h \mid n$ for every prime factor p of n. If x is a positive real number, let $N_h(x)$ be the number of elements of $M(h)$ not exceeding x. Erdös and Szekeres [3] proved that for h fixed

$$N_h(x) = x^{1/h} \prod \left(1 + \sum_{m=h+1}^{2h-1} p^{-m/h}\right) + O(x^{1/(h+1)}).$$

(In this paper an unspecified product is understood to be a product over all the primes p, while an O-relation is understood to be with respect to $x \to \infty$ and is not necessarily uniform in the parameters, such as h, that may be involved.) It is the purpose of this paper to point out that considerably more precise information may be easily obtained from known results in the theory of lattice-point problems. The general idea is the familiar one of expressing the given problem in terms of a "nearby" lattice-point problem whose solution is known, that is, we express the Dirichlet series corresponding to the given problem as the product of a Dirichlet series with a comparatively small abscissa of absolute convergence and the Dirichlet series corresponding to the known lattice-point problem.

More specifically, let $c_n = 1$ if $n \in M(h)$ and $c_n = 0$ if n is a positive integer not in $M(h)$, so that $N_h(x) = \sum_{n \leqq x} c_n$. Then, using the Euler product for the Riemann zeta-function, we have

$$(1)\qquad \begin{aligned} \sum c_n n^{-s} &= \prod \left(1 + \sum_{m=h}^{\infty} p^{-ms}\right) = \zeta(hs) \prod \left(1 + \sum_{m=h+1}^{2h-1} p^{-ms}\right) \\ &= \zeta(hs)\, \zeta((h+1)s) \prod \left(1 + \sum_{m=h+2}^{2h-1} p^{-ms} - \sum_{m=2h+2}^{3h} p^{-ms}\right). \end{aligned}$$

(Throughout this paper the letters m and n stand for positive integers and an unspecified sum is understood to be a sum over all the positive integers.) Continuing this process we obtain

$$(2)\qquad \sum c_n n^{-s} = \zeta(hs)\, \zeta((h+1)s) \cdots \zeta((2h-1)s)\, \zeta^{-1}((2h+2)s) \sum f_n n^{-s},$$

where $\sum f_n n^{-s}$ has abscissa of absolute convergence at most $1/(2h+3)$. (Actually $\sum f_n n^{-s} = 1$ if $h = 2$, while $\sum f_n n^{-s}$ has abscissa of absolute convergence exactly equal to $1/(2h+3)$ if $h > 2$. Repetition of the above procedure shows that $\sum c_n n^{-s}$ has a meromorphic continuation in the half-plane $\operatorname{Re} s > 0$.) Suppose now that we put

$$(3)\qquad \begin{aligned} \sum c_n n^{-s} &= \sum a_n n^{-s} \sum b_n n^{-s}, \\ \sum a_n n^{-s} &= \zeta(hs)\, \zeta((h+1)s) \cdots \zeta((h+r)s) \end{aligned}$$

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for a suitable r not exceeding $h - 1$. Then relatively good approximations for $\sum_{n \leqq x} a_n$ are known or can readily be derived from known results, while $\sum b_n n^{-s}$ has a comparatively small abscissa of absolute convergence. These facts enable us to get good approximations to $\sum_{n \leqq x} c_n = N_h(x)$.

2. Basic lemmas

Throughout this section we suppose that we have the Dirichlet series identity $\sum a_n n^{-s} \cdot \sum b_n n^{-s} = \sum c_n n^{-s}$ for complex s of sufficiently large real part, where a_n, b_n, and c_n need not have the specific meaning of the previous section. For positive real x write

$$A(x) = \sum_{n \leqq x} a_n , \qquad B(x) = \sum_{n \leqq x} b_n , \qquad C(x) = \sum_{n \leqq x} c_n .$$

The following (essentially known) lemmas are the basis of the method used in this paper.

LEMMA 1. *Suppose that*

$$A(x) = \alpha_0 x^{\lambda_0} + \alpha_1 x^{\lambda_1} + \cdots + \alpha_r x^{\lambda_r} + O(x^{\lambda} \log^{\mu} (x + 1))$$

and

$$\sum_{n \leqq x} | b_n | = O(x^{\nu}),$$

where λ, μ, ν are nonnegative real constants and α_0, λ_0, $\cdots$, α_r, λ_r are complex constants. Then

$$C(x) = \gamma_0 x^{\lambda_0} + \gamma_1 x^{\lambda_1} + \cdots + \gamma_r x^{\lambda_r} + O(x^{\max(\lambda,\nu)} \log^{\mu'} (x + 1)),$$

where $\gamma_i = 0$ if $\operatorname{Re} \lambda_i \leqq \nu$ and $\gamma_i = \alpha_i \sum b_n n^{-\lambda_i}$ if $\operatorname{Re} \lambda_i > \nu$ $(i = 0, 1, \cdots, r)$; further $\mu' = \mu$ if $\lambda > \nu$, $\mu' = \mu + 1$ if $\lambda = \nu$, $\mu' = 0$ if $\lambda < \nu$ and $\nu \neq \operatorname{Re} \lambda_i$ for all i, while $\mu' = 1$ if $\lambda < \nu$ but $\nu = \operatorname{Re} \lambda_i$ for some i.

Proof. Clearly

$$\begin{aligned} C(x) &= \sum_{n \leqq x} b_n A(x/n) \\ &= \alpha_0 x^{\lambda_0} \sum_{n \leqq x} b_n n^{-\lambda_0} + \cdots + \alpha_r x^{\lambda_r} \sum_{n \leqq x} b_n n^{-\lambda_r} \\ &\qquad + O(x^{\lambda} \log^{\mu} (x + 1) \sum_{n \leqq x} | b_n | n^{-\lambda}). \end{aligned}$$

Now if η is real, we have for positive x by partial summation

$$\begin{aligned} \sum_{n \leqq x} | b_n | n^{-\eta} &= O(x^{\nu - \eta}) \quad \text{if} \quad \eta < \nu, \\ \sum_{n \leqq x} | b_n | n^{-\eta} &= O(\log (x + 1)) \quad \text{if} \quad \eta = \nu, \\ \sum_{n > x} | b_n | n^{-\eta} &= O(x^{\nu - \eta}) \quad \text{if} \quad \eta > \nu. \end{aligned}$$

Hence our result follows. (When $\lambda < \nu$, it is helpful to begin by replacing the error term $O(x^{\lambda} \log^{\mu} (x + 1))$ in the formula for $A(x)$ by $O(x^{\lambda'})$, where $\lambda < \lambda' < \nu$.)

If $\nu \leqq \lambda$, the result of Lemma 1 is easily seen to be best possible. However if $\lambda < \nu$, it can be substantially improved if we know somewhat more about the behavior of $B(x)$. For example, if $B(x) = o(x^{\nu})$ as $x \to \infty$, the following result is better. (The assumption $\mathrm{Re}\, \lambda_i > \nu$ is not essential but was added to avoid making the conclusion too complicated.)

LEMMA 2. *Suppose that*

$$A(x) = \alpha_0 x^{\lambda_0} + \cdots + \alpha_r x^{\lambda_r} + O(x^{\lambda} \log^{\mu} (x + 1)),$$
$$\textstyle\sum_{n \leqq x} | a_n | = O(x^{\kappa}), \qquad \sum_{n \leqq x} | b_n | = O(x^{\nu}),$$

where κ, λ, μ, ν *are nonnegative real constants and* $\alpha_0, \lambda_0, \cdots, \alpha_r, \lambda_r$ *are complex constants such that*

$$0 \leqq \lambda < \nu < \mathrm{Re}\, \lambda_i \leqq \kappa \qquad (i = 0, 1, \cdots, r).$$

Then for $1 \leqq y \leqq x$ *we have, uniformly in* y,

$$C(x) = \gamma_0 x^{\lambda_0} + \gamma_1 x^{\lambda_1} + \cdots + \gamma_r x^{\lambda_r} + O(x^{\nu} y^{\lambda - \nu} \log^{\mu} (y + 1)) + O(x^{\nu} y^{\kappa - \nu} \max\nolimits_{u \geqq x/y} \{B^*(u) u^{-\nu}\}),$$

where $\gamma_i = \alpha_i \sum b_n n^{-\lambda_i}$ *and* $B^*(u) = \max_{1 \leqq v \leqq u} | B(v) |$.

Proof. Let $z = x/y$. Then

$$C(x) = \textstyle\sum_{mn \leqq x} a_m b_n = \sum_{n \leqq z} b_n A(x/n) + \sum_{m \leqq y} a_m \{B(x/m) - B(z)\}.$$

Put $A_0 (x) = \sum_{n \leqq x} | a_n |$, and suppose $A_0 (x) \leqq K x^{\kappa}$ for all positive x. Then, since B^* is a nondecreasing function, we have

$$\begin{aligned}
\tfrac{1}{2} | \textstyle\sum_{m \leqq y} a_m \{B(x/m) - B(z)\} | &\leqq \textstyle\sum_{m \leqq y} | a_m | B^*(x/m) \\
&= \textstyle\sum_{m \leqq y} A_0 (m) \{B^*(x/m) - B^*(x/(m + 1))\} + A_0 (y) B^*(x/[y + 1]) \\
&\leqq \textstyle\sum_{m \leqq y} K m^{\kappa} \{B^*(x/m) - B^*(x/(m + 1))\} + K [y]^{\kappa} B^*(x/[y + 1]) \\
&= \textstyle\sum_{m \leqq y} K \{m^{\kappa} - (m - 1)^{\kappa}\} B^*(x/m) = \sum_{m \leqq y} K \kappa \int_{m-1}^{m} B^*(x/m) u^{k-1}\, du \\
&\leqq K \kappa \int_0^y B^*(x/u) u^{\kappa - 1}\, du = K \kappa x^{\kappa} \int_z^{\infty} B^*(u) u^{-\kappa - 1}\, du,
\end{aligned}$$

the convergence of the integrals being justified by the estimate $B^*(u) = O(u^{\nu})$. Further

$$\textstyle\sum_{n \leqq z} b_n A(x/n) = \alpha_0 x^{\lambda_0} \sum_{n \leqq z} b_n n^{-\lambda_0} + \cdots + \alpha_r x^{\lambda_r} \sum_{n \leqq z} b_n n^{-\lambda_r} + O\big(\sum_{n \leqq z} | b_n | (x/n)^{\lambda} \log^{\mu} (x/n + 1)\big).$$

Now for $i = 0, 1, \cdots, r$ we have

$$\begin{aligned}
\alpha_i x^{\lambda_i} \textstyle\sum_{n \leqq z} b_n n^{-\lambda_i} &= \alpha_i x^{\lambda_i} \sum b_n n^{-\lambda_i} - \lambda_i \alpha_i x^{\lambda_i} \int_z^{\infty} \{B(u) - B(z)\} u^{-\lambda_i - 1}\, du, \\
&= \gamma_i x^{\lambda_i} + O\left(x^{\mathrm{Re}\lambda_i} \int_z^{\infty} B^*(u) u^{-\mathrm{Re}\lambda_i - 1}\, du\right),
\end{aligned}$$

where convergence is guaranteed by the estimate $B^*(u) = O(u^\nu)$. Also if $f(u) = D_u\{u^\lambda \log^\mu (u + 1)\}$ we have

$$\begin{aligned}\textstyle\sum_{n\leqq z} | b_n | (x/n)^\lambda \log^\mu (x/n + 1) &- y^\lambda \log^\mu (y + 1)\textstyle\sum_{n\leqq z} | b_n | \\ &= \textstyle\sum_{n\leqq z} | b_n | \displaystyle\int_y^{x/n} f(u)\, du = \int_y^x f(u)\{\textstyle\sum_{n\leqq x/u} | b_n |\}\, du \\ &= O\left(x^\nu \int_y^x f(u)u^{-\nu}\, du\right) = O\left(x^\nu \int_y^\infty u^{\lambda-\nu-1} \log^\mu (u + 1)\, du\right) \\ &= O(x^\nu y^{\lambda-\nu} \log^\mu (y + 1)).\end{aligned}$$

Combining these estimates we obtain

$$\begin{aligned}C(x) &= \textstyle\sum_{i=0}^r \gamma_i\, x^{\lambda_i} + \sum_{i=0}^r O\left(x^{\mathrm{Re}\lambda_i} \displaystyle\int_z^\infty B^*(u)u^{-\mathrm{Re}\lambda_i-1}\, du\right) \\ &\quad + O\left(x^\kappa \int_z^\infty B^*(u)u^{-\kappa-1}\, du\right) + O(x^\nu y^{\lambda-\nu} \log^\mu (y + 1)).\end{aligned} \tag{4}$$

Using (4) and the inequality

$$\begin{aligned}x^\eta \int_z^\infty B^*(u)u^{-\eta-1}\, du &\leqq x^\eta(\max_{u\geqq z}\{B^*(u)u^{-\nu}\}) \int_z^\infty u^{\nu-\eta-1}\, du \\ &= (\eta - \nu)^{-1}x^\nu y^{\eta-\nu} \max_{u\geqq x/y} \{B^*(u)u^{-\nu}\},\end{aligned}$$

valid for $\eta > \nu$, we immediately obtain the assertion of the lemma.

The argument used in the proof of Lemma 2 goes back in essence to Axer. For example, §6 of [1] contains a special case of formula (4), namely the case $r = 0$, $\alpha_0 = 1$, $\lambda_0 = \kappa$, $\lambda = 0$, $\mu = 0$, $\nu = 1$.

3. Elementary results

By an elementary argument similar to that used by Dirichlet in the divisor problem, Landau [6] proved that if α and β are fixed positive numbers and if $\alpha \neq \beta$, then

$$\textstyle\sum_{m^\alpha n^\beta \leqq x} 1 = \zeta(\beta/\alpha)x^{1/\alpha} + \zeta(\alpha/\beta)x^{1/\beta} + O(x^{1/(\alpha+\beta)}) \tag{5}$$

for positive x. (See also [2] or Lemma 3 of [10].) The only information about the zeta-function required in this proof of (5) is the formula

$$\zeta(s) = s/(s - 1) + s \int_1^\infty ([y] - y)y^{-s-1}\, dy \qquad (s > 0,\ s \neq 1).$$

If $s > 1$, this is merely a disguised form of the definition $\zeta(s) = \sum n^{-s}$. For $0 < s < 1$ it can be regarded as the definition of $\zeta(s)$.

From Lemma 1 and formula (5) we easily derive the following theorem. (The result for $h = 2$ was given in [2].)

THEOREM 1. *There exist constants γ_{0h} and γ_{1h} such that for positive x*

$$N_h(x) = \gamma_{0h}\, x^{1/h} + \gamma_{1h}\, x^{1/(h+1)} + O(x^{1/(h+2)}).$$

When $h = 2$, we may replace the error term by $O(x^{1/5})$.

Proof. Take $r = 1$ in (3). Then we have $\sum c_n n^{-s} = \sum a_n n^{-s} \cdot \sum b_n n^{-s}$, where $\sum a_n n^{-s} = \zeta(hs)\,\zeta((h + 1)s)$ and $\sum b_n n^{-s}$ is equal to the infinite product in the final line of (1). By formula (5) with $\alpha = h$ and $\beta = h + 1$, we have for x positive

$$\begin{aligned}\sum\nolimits_{n \leqq x} a_n &= \sum\nolimits_{m^h n^{h+1} \leqq x} 1 \\ &= \zeta((h + 1)/h) x^{1/h} + \zeta(h/(h + 1)) x^{1/(h+1)} + O(x^{1/(2h+1)}).\end{aligned}$$

Since b_n does not exceed the coefficient of n^{-s} in the Dirichlet series for $\zeta((h + 2)s)\,\zeta((h + 3)s) \cdots \zeta(3hs)$, we have by a crude estimation

$$\sum\nolimits_{n \leqq x} |\, b_n \,| = O(x^{1/(h+2)}).$$

When $h = 2$, we have $\sum b_n n^{-s} = \zeta^{-1}(6s)$, and so in this case

$$\sum\nolimits_{n \leqq x} |\, b_n \,| = O(x^{1/6}).$$

Thus the theorem follows from Lemma 1 with

$$\begin{aligned}\gamma_{0h} &= \zeta((h + 1)/h) \sum b_n n^{-1/h} = \prod \left(1 + \sum\nolimits_{m=h+1}^{2h-1} p^{-m/h}\right), \\ \gamma_{1h} &= \zeta(h/(h + 1)) \sum b_n n^{-1/(h+1)} \\ &= \zeta(h/(h + 1)) \prod \left(1 + \sum\nolimits_{m=h+2}^{2h-1} p^{-m/(h+1)} - \sum\nolimits_{m=2h+2}^{3h} p^{-m/(h+1)}\right).\end{aligned}$$

Note that $\gamma_{02} = \zeta(3/2)/\zeta(3)$ and $\gamma_{12} = \zeta(2/3)/\zeta(2)$.

We remark that the argument of Erdös and Szekeres amounts to using (3) with $r = 0$. The application of Lemma 1 in that case would be based on the trivial estimate

$$\sum\nolimits_{n^h \leqq x} 1 = x^{1/h} + O(1).$$

4. More precise results for $h > 2$

By standard complex variable methods in the theory of lattice-point problems it can be proved that if $d_0, d_1, \cdots, d_r$ are given positive numbers and if $d_0 < d_1 < \cdots < d_r$, then for positive x

$$\begin{aligned}\sum\nolimits_{n_0^{d_0} n_1^{d_1} \cdots n_r^{d_r} \leqq x} 1 &= \rho_0\, x^{1/d_0} + \rho_1\, x^{1/d_1} + \cdots + \rho_r\, x^{1/d_r} \\ &\quad + O(x^{r/\{d_0(r+2)\}} \log^r (x + 1)),\end{aligned} \tag{6}$$

where

$$\rho_i = \prod\nolimits_{0 \leqq j \leqq r, j \neq i} \zeta(d_j/d_i) \qquad (i = 0, 1, \cdots, r).$$

In fact this follows at once if we apply Landau's colossal lattice-point theorem [7] with $Z(s) = \zeta(d_0 s)\zeta(d_1 s) \cdots \zeta(d_r s)$, $\beta = 1/d_0$, $H = d_0 + d_1 + \cdots + d_r$,

$\eta = (r+1)/2$, $A = r$, $\kappa = r/\{d_0(r+2)\}$. In this application the numbers l_n of Landau's theorem are the distinct numbers of the form $n_0^{d_0} n_1^{d_1} \cdots n_r^{d_r}$, where $n_0, n_1, \cdots, n_r$ are positive integers, while $\lambda_n = \pi^H l_n$; further

$$c_n = \sum_{n_0^{d_0} n_1^{d_1} \cdots n_r^{d_r} = l_n} 1, \qquad \pi^\eta e_n = \sum_{n_0^{d_0} n_1^{d_1} \cdots n_r^{d_r} = l_n} (n_0 n_1 \cdots n_r)^{-1}.$$

The essential fact which makes Landau's theorem applicable here is the functional equation

$$\Gamma(\tfrac{1}{2}d_0 s) \cdots \Gamma(\tfrac{1}{2}d_r s) Z(s) = \Gamma(\tfrac{1}{2} - \tfrac{1}{2}d_0 s) \cdots \Gamma(\tfrac{1}{2} - \tfrac{1}{2}d_r s) \sum e_n \lambda_n^s,$$

valid for Re $s < 0$. The results of [8] show that, on the other hand, the error term in (6) cannot be sharpened to $o(x^{r/(2d_0+2d_1+\cdots+2d_r)})$.

Using (6) and Lemma 1 we obtain the following result.

THEOREM 2. *If r is a positive integer greater than 1 and*

$$r^2/2 < h \leqq (r+1)^2/2,$$

then there exist constants $\gamma_{0h}, \gamma_{1h}, \cdots, \gamma_{rh}$ such that for positive x

$$N_h(x) = \gamma_{0h}\, x^{1/h} + \gamma_{1h}\, x^{1/(h+1)} + \cdots + \gamma_{rh}\, x^{1/(h+r)} + \Delta_h(x),$$

where

$$\Delta_h(x) = O(x^{r/\{h(r+2)\}} \log^r (x+1)) \qquad \text{if } r^2/2 < h < r(r+1)/2,$$

$$\Delta_h(x) = O(x^{r/\{h(r+2)\}} \log^{r+1} (x+1)) = O(x^{1/(h+r+1)} \log^{r+1}(x+1)) \qquad \text{if } h = r(r+1)/2,$$

$$\Delta_h(x) = O(x^{1/(h+r+1)}) \qquad \text{if } r(r+1)/2 < h \leqq (r+1)^2/2.$$

When $r = 2$ and $h = 3$ we may replace the factor $\log^3 (x+1)$ by $\log^2 (x+1)$.

Proof. We apply (3). By (6) we have

$$\sum_{n \leqq x} a_n = \sum_{n_0^h n_1^{h+1} \cdots n_r^{h+r} \leqq x} 1$$

$$= \alpha_{0h}\, x^{1/h} + \alpha_{1h}\, x^{1/(h+1)} + \cdots + \alpha_{rh}\, x^{1/(h+r)} + O(x^{r/\{h(r+2)\}} \log^r (x+1)),$$

where

$$\alpha_{ih} = \prod\nolimits_{0 \leqq j \leqq r, j \neq i} \zeta((h+j)/(h+i)) \qquad (i = 0, 1, \cdots, r).$$

(Note that, since $r^2 < 2h$, the error term here is of lower order of magnitude than the other terms.) From (2) and (3) it readily follows that

$$\sum\nolimits_{n \leqq x} |\, b_n \,| = O(x^{1/(h+r+1)})$$

if $h > 3$, while $\sum_{n \leqq x} |\, b_n \,| = O(x^{1/8})$ if $h = 3$. Now $1/(h+r+1)$ is less than, equal to, or greater than $r/\{h(r+2)\}$ according as h is less than, equal to, or greater than $r(r+1)/2$. Hence our result follows from Lemma 1 with

$$\gamma_{ih} = \prod_{0 \leq j \leq r, j \neq i} \zeta\left(\frac{h+j}{h+i}\right) \sum b_n n^{-1/(h+i)}$$
$$= \prod_{0 \leq j \leq h-1, j \neq i} \zeta\left(\frac{h+j}{h+i}\right) \zeta^{-1}\left(\frac{2h+2}{h+i}\right) \sum f_n n^{-1/(h+i)}.$$

It can be verified that if $\sum a_n n^{-s}$ were chosen to contain either more or fewer of the factors in (2), then a poorer error term would be obtained.

We remark that $(-1)^i \gamma_{ih} > 0$ for $i = 0, 1, \cdots, r$. For it is easily seen from the procedure by which we pass from (1) to (2) that if $\operatorname{Re} s > 1/(2h+3)$, then

$$\sum f_n n^{-s} = \prod \left(1 + \sum_{m=2h+3}^{\infty} a_m p^{-ms}\right),$$

where $1 + \sum_{m=2h+3}^{\infty} a_m x^m$ is a power series with integral coefficients which converges inside the unit circle and which takes only positive values when $0 < x < 1$.

5. The case $h = 2$

By means of the delicate theory of exponent pairs (due to J. G. van der Corput and Eric Phillips) H.-E. Richert [10] has recently proved that if $2\alpha > \beta > \alpha > 0$, then

$$(7) \qquad \sum_{m^\alpha n^\beta \leq x} 1 = \zeta(\beta/\alpha) x^{1/\alpha} + \zeta(\alpha/\beta) x^{1/\beta} + O(x^{2/(3\alpha+3\beta)})$$

for positive x. Note that (6) would give only an error term $O(x^{1/(3\alpha)} \log (x+1))$ in this case. If we were to use (6) and argue as in the previous two sections, we would obviously get

$$N_2(x) = x^{1/2}\zeta(3/2)/\zeta(3) + x^{1/3}\zeta(2/3)/\zeta(2) + O(x^{1/6} \log^2 (x+1)).$$

However, if we use Richert's result (7) instead of (6) and Lemma 2 instead of Lemma 1, we get the following sharper result.

THEOREM 3. *There is a positive absolute constant a such that for large x*

$$N_2(x) = x^{1/2}\zeta(3/2)/\zeta(3) + x^{1/3}\zeta(2/3)/\zeta(2) + O(x^{1/6} e^{-a\omega(x)}),$$

where $\omega(x) = (\log x)^{4/7} (\log \log x)^{-3/7}$.

Proof. Take $h = 2$ in (1) or (2). Then we have

$$\sum c_n n^{-s} = \sum a_n n^{-s} \sum b_n n^{-s},$$

where $\sum a_n n^{-s} = \zeta(2s)\zeta(3s)$, $b_n = 0$ if n is not a sixth power, and $b_n = \mu(n^{1/6})$ if n is a sixth power, μ denoting the Möbius function. By (7) we have

$$(8) \qquad \sum_{n \leq x} a_n = \sum_{m^2 n^3 \leq x} 1 = \zeta(3/2) x^{1/2} + \zeta(2/3) x^{1/3} + O(x^{2/15})$$

for positive x, while $\sum_{n \leq x} b_n = \sum_{n \leq x^{1/6}} \mu(n) = M(x^{1/6})$ in the usual notation (cf. [5]). Put

$$\Delta_2(x) = N_2(x) - x^{1/2}\zeta(3/2)/\zeta(3) - x^{1/3}\zeta(2/3)/\zeta(2).$$

Then if $1 \leqq y \leqq x$, we obtain from Lemma 2 (taking $\kappa = \frac{1}{2}$, $\lambda = \frac{2}{15}$, $\mu = 0$, $\nu = \frac{1}{6}$)

$$\Delta_2(x) = O(x^{1/6}y^{1/3} \max_{u \geqq x/y}\{M^*(u^{1/6})u^{-1/6}\}) + O(x^{1/6}y^{-1/30}), \tag{9}$$

where

$$M^*(v) = \max_{1 \leqq n \leqq v} | M(n) |.$$

Now if we have a nontrivial estimate for $M(n)/n$, i.e., one which tends to zero as $n \rightarrow \infty$, we can get an error term in (9) which is sharper than $x^{1/6}$ merely by taking y as a function of x which tends to infinity sufficiently slowly as $x \rightarrow \infty$. In particular the simple estimate $M(n) = o(n)$, which is equivalent to the prime number theorem, gives an error term $o(x^{1/6})$. However, from the information on p. 114 of [11], it follows by standard arguments (cf. §164 of [5] and pp. 157–159 of [9]) that for large n

$$M(n) = O(ne^{-67a\omega(n)}),$$

where $67a$ is a positive absolute constant. To use this to advantage in (9), suppose x is large and take $y = e^{30a\omega(x)}$. Then $x/y > x^{66/67}$, and so

$$\max_{u \geqq x/y} M^*(u^{1/6})u^{-1/6} = O(e^{-67a\omega((x/y)^{1/6})}) = O(e^{-67a\omega(x^{11/67})}) = O(e^{-11a\omega(x)}).$$

Thus the result of our theorem follows from (9), so that the proof of Theorem 3 is complete.

If it were known that the least upper bound of the real parts of the zeros of the Riemann zeta-function is less than $1 - \delta$, where $0 < \delta < \frac{1}{2}$, then a classical argument of Littlewood would give $M(n) = O(n^{1-\delta})$ (cf. pp. 161–166 of [9] or pp. 315–316 of [11]). Taking $y = x^{5\delta/(11+5\delta)}$ in (9), we could then obtain $\Delta_2(x) = O(x^{1/6-\delta/(66+30\delta)})$. On the other hand, the formula

$$\int_1^\infty \Delta_2(x)x^{-s-1}\,dx = \frac{\zeta(2s)\zeta(3s)}{s\zeta(6s)} - \frac{\zeta(\frac{3}{2})/\zeta(3)}{s - \frac{1}{2}} - \frac{\zeta(\frac{2}{3})/\zeta(2)}{s - \frac{1}{3}}$$

shows that if ρ is any zero of the Riemann zeta-function such that $\rho/2$ and $\rho/3$ are not zeros, then $\Delta_2(x) \neq o(x^{\mathrm{Re}\rho/6})$. (The details of the argument are as in the proof of Theorem 1.3 in [4] or the proof of Theorem 14.26 (B) in [11]). Thus one can expect to get an estimate of the form $\Delta_2(x) = O(x^\alpha)$, α fixed, $\alpha < \frac{1}{6}$, if and only if the least upper bound of the real parts of the zeros of the zeta-function is less than unity.

The above results about $\Delta_2(x)$ exhibit an obvious parallelism with (and are proved in much the same way as) the corresponding results concerning $R_h(x) = Q_h(x) - x/\zeta(h)$, where $Q_h(x)$ is the number of integers not exceeding x which are *not* divisible by the h^{th} power of any prime. (These corresponding results are:

$$R_h(x) = O(x^{1/h} e^{-b\omega(x)}),$$

where b is a positive constant depending only on h;

$$R_h(x) = O(x^{1/(h+\delta)}),$$

if the least upper bound of the zeros of the zeta-function is less than $1 - \delta$;

$$R_h(x) \neq o(x^{\mathrm{Re}\rho/h}),$$

if ρ is a zero of the zeta-function such that ρ/h is not a zero. Cf. §7 of [1] and §1 of [4].)

6. Concluding remarks

Let us return for a moment to the general situation described at the beginning of §2. Our lemmas deal only with very special cases of the general question of deducing an approximate formula for $C(x)$ from approximate formulas for $A(x)$ and $B(x)$ and upper bounds for $\sum_{n \leqq x} | a_n |$ and $\sum_{n \leqq x} | b_n |$. This question is discussed in much greater generality in a forthcoming paper by J. P. Tull. Our particular lemmas give good results only when the abscissa of absolute convergence of $\sum b_n n^{-s}$ is comparatively small (i.e., when $\nu \leqq \lambda$) or when the approximate formula for $B(x)$ is simply an upper estimate of its order of magnitude. Thus our results on $N_h(x)$ when $h > 2$ can be greatly improved by the use of somewhat less special instances of Tull's theorems.

In particular, Tull has proved the following result, which contains our Lemma 1, some of the results of §§214–217 of [5], and some of the arguments of [10]. (Note also that equation (5) is an immediate corollary, in view of the obvious formulas $\sum_{n^\alpha \leqq x} 1 = x^{1/\alpha} + O(1)$ and $\sum_{n^\beta \leqq x} 1 = x^{1/\beta} + O(1)$.)

Suppose that (with the notation of §2)

$$A(x) = \alpha_0 x^{\lambda_0} + \cdots + \alpha_r x^{\lambda_r} + O(x^\lambda \log^\mu (x + 1)),$$

$$B(x) = \beta_0 x^{\rho_0} + \cdots + \beta_t x^{\rho_t} + O(x^\rho \log^\sigma (x + 1)),$$

$$\sum_{n \leqq x} | a_n | = O(x^\kappa), \qquad \sum_{n \leqq x} | b_n | = O(x^\nu),$$

where $\kappa, \lambda, \mu, \nu, \rho, \sigma$ *are nonnegative real constants such that*

$$\min (\kappa, \nu) > \max (\lambda, \rho)$$

and $\alpha_0, \lambda_0, \cdots, \alpha_r, \lambda_r, \beta_0, \rho_0, \cdots, \beta_t, \rho_t$ *are complex constants such that*

$$\lambda < \mathrm{Re}\, \lambda_i \leqq \kappa, \qquad \rho < \mathrm{Re}\, \rho_j \leqq \nu, \qquad \lambda_i \neq \rho_j$$

for $i = 0, \cdots, r$ *and* $j = 0, \cdots, t$. *For* $\mathrm{Re}\, s > \lambda$ *put*

$$f(s) = \alpha_0 s/(s - \lambda_0) + \cdots + \alpha_r s/(s - \lambda_r) + s \int_1^\infty \{A(y) - \alpha_0 y^{\lambda_0} - \cdots - \alpha_r y^{\lambda_r}\} y^{-s-1}\, dy,$$

so that $f(s) = \sum a_n n^{-s}$ *for* Re $s > \kappa$. *Similarly, for* Re $s > \rho$ *put*

$$g(s) = \beta_0 s/(s - \rho_0) + \cdots + \beta_r s/(s - \rho_r) + s \int_1^\infty \{B(y) - \beta_0 y^{\rho_0} - \cdots - \beta_t y^{\rho_t}\} y^{-s-1}\, dy,$$

so that $g(s) = \sum b_n n^{-s}$ *for* Re $s > \nu$. *Then*

$$C(x) = \alpha_0 g(\lambda_0) x^{\lambda_0} + \cdots + \alpha_r g(\lambda_r) x^{\lambda_r} + \beta_0 f(\rho_0) x^{\rho_0} + \cdots + \beta_t f(\rho_t) x^{\rho_t} + O(x^\tau \log^\omega (x + 1)),$$

where

$$\tau = \frac{\kappa\nu - \lambda\rho}{(\kappa + \nu) - (\lambda + \rho)} = \lambda \frac{\kappa - \rho}{(\kappa - \rho) + (\nu - \lambda)} + \kappa \frac{\nu - \lambda}{(\kappa - \rho) + (\nu - \lambda)} = \nu \frac{\kappa - \rho}{(\kappa - \rho) + (\nu - \lambda)} + \rho \frac{\nu - \lambda}{(\kappa - \rho) + (\nu - \lambda)},$$

and

$$\omega = \mu \frac{\kappa - \rho}{(\kappa - \rho) + (\nu - \lambda)} + \sigma \frac{\nu - \lambda}{(\kappa - \rho) + (\nu - \lambda)}.$$

(*Note that* max $(\lambda, \rho) < \tau <$ min (κ, ν) *and that* min $(\mu, \sigma) \leqq \omega \leqq$ max (μ, σ), *with strict inequality in both places if* $\mu \neq \sigma$.)

If we apply the preceding result to $N_3(x)$, for example, the following results are obtained. Suppose that for some positive η and some nonnegative μ we have

$$\sum_{l^3 m^4 \leqq x} 1 = \zeta(4/3) x^{1/3} + \zeta(3/4) x^{1/4} + O(x^{1/5 - \eta} \log^\mu (x + 1)). \tag{10}$$

Using Tull's theorem in connection with (10) and the obvious formula $\sum_{m^5 \leqq x} 1 = x^{1/5} + O(1)$, we obtain

$$\sum_{l^3 m^4 n^5 \leqq x} 1 = \zeta(4/3)\zeta(5/3) x^{1/3} + \zeta(3/4)\zeta(5/4) x^{1/4} + \zeta(3/5)\zeta(4/5) x^{1/5} + O(x^{1/(5+15\eta)} \log^{\mu/(1+3\eta)} (x + 1)).$$

Using (2) and Lemma 1, we then immediately obtain

$$N_3(x) = \gamma_{03}\, x^{1/3} + \gamma_{13}\, x^{1/4} + \gamma_{23}\, x^{1/5} + O(x^{1/(5+15\eta)} \log^{\mu/(1+3\eta)} (x + 1)). \tag{11}$$

Now if we use the results of the form (10) given respectively by (5), (6), and (7), we obtain for the error term in (11) the following: $O(x^{7/41})$ from elementary methods, $O(x^{3/19} \log^{15/19} (x + 1))$ from classical complex variable methods, and $O(x^{7/46})$ from Richert's results.

However, although the results in §§3 and 4 for $h > 2$ are accordingly not definitive, there does not seem to be any obvious way of substantially improving the results of §5 for the case $h = 2$.

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