ON SOME ARITHMETICAL FUNCTIONS

BY
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In a paper on additive arithmetical functions,[1] P. Erdös incidentally states the following result:

($\mathcal{E}_1$) *Let* $\omega(n)$ *be the number of prime divisors of the positive integer* n, *and let* λ *be any irrational number.*

Then the numbers $\lambda\omega(n)$ *are uniformly distributed modulo* 1.

This means that, for $0 \leqq t \leqq 1$, the number of n's less than or equal to x and such that[2]

$$\lambda\omega(n) - I[\lambda\omega(n)] \leqq t$$

is $tx + o[x]$ as x tends to $+\infty$.

P. Erdös adds that the proof is not easy.

($\mathcal{E}_1$) can actually be deduced from a later result of Erdös, say ($\mathcal{E}_2$), concerning the number of integers $n \leqq x$ for which $\omega(n) = k$.[3]

Also a very short proof can be based on the following formula due to Atle Selberg:[4]

As x tends to $+\infty$,

$$\sum_{n \leqq x} z^{\omega(n)} = F(z)x(\log x)^{z-1} + O[x(\log x)^{\Re z-2}],$$

uniformly for $| z | \leqq R$, where R is any positive number and

$$F(z) = \frac{1}{\Gamma(z)} \prod \left[1 + \frac{z}{p-1}\right]\left[1 - \frac{1}{p}\right]^z.$$

We have only to take $z = \exp [2\pi q\lambda i]$, where q is any positive integer, and use a well known theorem of H. Weyl.[5]

However the proof of ($\mathcal{E}_2$) is not very simple, while the proof of Selberg's formula uses the properties of the Riemann Zeta-function in the critical strip.

In the present paper, we shall first give a simple proof of ($\mathcal{E}_1$) which uses only the nonvanishing of $\zeta(s)$ for $\Re s \geqq 1$. We shall also give some generalizations.

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[1] *On the distribution function of additive functions*, Ann. of Math. (2), vol. 47 (1946), pp. 1–20. See p. 2, lines 4 and 5.

[2] $I[u]$ denotes the greatest integer not exceeding u.

[3] *On the integers having exactly k prime factors*, Ann. of Math. (2), vol. 49 (1948), pp. 53–66.

[4] *Note on a paper by L. G. Sathe*, J. Indian Math. Soc., vol. 18 (1954), pp. 83–87.

[5] *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. vol. 77 (1916), pp. 313–352, Satz 1, p. 315.

1. We shall use the following alternative form of the classical tauberian theorem of Ikehara:

THEOREM A. *Let $\alpha(t)$ be a real function defined for $t \geqq 0$, nondecreasing and satisfying $\alpha(0) \geqq 0$.*

Suppose that the integral $\int_0^{+\infty} e^{-st}\alpha(t)dt$ is convergent for $\Re s > a > 0$ and equal to $f(s)$.

Suppose further that, for each real y other than zero, $f(s)$ tends to a finite limit as s tends to $a + iy$ in the half plane $\Re s > a$, and that, as s tends to a in this half plane,

$$f(s) - A/(s - a) = O[\,|\,s - a\,|^{-\omega}], \qquad \textit{where} \quad A > 0 \quad \textit{and} \quad 0 < \omega < 1.$$

Then, as t tends to $+\infty$,

$$\alpha(t) \backsim Ae^{at}.$$

The proof of Ikehara's theorem in Widder's book, *The Laplace Transform*, yields this alternative form as well.

1.1. From this we deduce the following result:

THEOREM B. *Consider the Dirichlet series $\sum_1^{+\infty} a_n/n^s$, where the a_n's are real or complex numbers satisfying $|\,a_n\,| \leqq 1$.*

Obviously this series is absolutely convergent for $\Re s > 1$.

Suppose that, for $\Re s > 1$,

$$\textstyle\sum_1^{+\infty} a_n/n^s = (s - 1)^{-\beta-i\gamma}g(s) + h(s),$$

where the functions g and h are regular for $\Re s \geqq 1$, β and γ are real numbers, $\beta < 1$, and $(s - 1)^{-\beta-i\gamma}$ has its principal value.

Then, as x tends to $+\infty$,

$$\textstyle\sum_{n\leqq x} a_n = o[x].$$

The proof is as follows: Set $a_n = u_n + iv_n$, where u_n and v_n are real, and

$$A(t) = \textstyle\sum_{1\leqq n\leqq e^t} [1 + u_n], \qquad B(t) = \sum_{1\leqq n\leqq e^t} [1 + v_n].$$

The functions A and B are nondecreasing for $t \geqq 0$; we have $A(0) \geqq 0$, $B(0) \geqq 0$ and, for $\Re s > 1$,

$$\sum_1^{+\infty} \frac{1 + u_n}{n^s} = s\int_0^{+\infty} e^{-st}A(t)dt,$$

and

$$\sum_1^{+\infty} \frac{1 + v_n}{n^s} = s\int_0^{+\infty} e^{-st}B(t)dt.$$

If D is a domain which is symmetric with respect to the real axis and contains the closed half plane $\Re s \geqq 1$, and in which f and g are regular, we may write in this domain

$$g(s) = g_1(s) + ig_2(s) \qquad \text{and} \qquad h(s) = h_1(s) + ih_2(s),$$

where g_1, g_2, h_1, and h_2 are regular in D and real for z real in D. Namely we have

$$g_1(s) = \frac{1}{2}[g(s) + \overline{g(\bar{s})}], \qquad g_2(s) = \frac{1}{2i}[g(s) - \overline{g(\bar{s})}],$$

$$h_1(s) = \frac{1}{2}[h(s) + \overline{h(\bar{s})}], \qquad h_2(s) = \frac{1}{2i}[h(s) - \overline{h(\bar{s})}],$$

where $\bar{z}$ denotes the conjugate of z.

We then see that, for s real and > 1, and hence for $\Re s > 1$,

$$\int_0^{+\infty} e^{-st}A(t)\,dt = \frac{1}{s}\zeta(s)$$

$$+ \frac{1}{s}(s-1)^{-\beta}\left\{g_1(s)\cos\left[\gamma\log\frac{1}{s-1}\right] - g_2(s)\sin\left[\gamma\log\frac{1}{s-1}\right]\right\} + \frac{1}{s}h_1(s),$$

and

$$\int_0^{+\infty} e^{-st}B(t)\,dt = \frac{1}{s}\zeta(s)$$

$$+ \frac{1}{s}(s-1)^{-\beta}\left\{g_1(s)\sin\left[\gamma\log\frac{1}{s-1}\right] + g_2(s)\cos\left[\gamma\log\frac{1}{s-1}\right]\right\} + \frac{1}{s}h_2(s).$$

Theorem A enables us to conclude that, as t tends to $+\infty$,

$$A(t) \backsim B(t) \backsim e^t,$$

so that

$$\sum\nolimits_{n \leqq e^t} u_n = o[e^t] \qquad \text{and} \qquad \sum\nolimits_{n \leqq e^t} v_n = o[e^t].$$

Therefore, as x tends to $+\infty$,

$$\sum\nolimits_{n \leqq x} u_n = o[x] \qquad \text{and} \qquad \sum\nolimits_{n \leqq x} v_n = o[x].$$

2. Now the result of Erdös which we stated at the beginning, and the similar result for $\Omega(n)$, are immediate consequences of the already mentioned theorem of H. Weyl and of the following theorem, which we proved in detail in our paper, *Sur la distribution des entiers ayant certaines propriétés*:[6]

THEOREM C. *There exist two functions $\mathcal{G}_1(s, z)$ and $\mathcal{G}_2(s, z)$ with the following properties*:

(α) *They are regular in s and z for $|z| < \sqrt{2}$ and s belonging to a certain domain Δ, which contains the closed half plane $\Re s \geqq 1$;*

[6] Ann. Sci. École Norm. Sup. (3), t. 73 (1956), pp. 15–74. In the following we shall denote this paper by "paper A". We also sketched the proof in *Quelques théorèmes taubériens relatifs à l'intégrale de Laplace et leurs applications arithmétiques*, Univ. e Politec. Torino. Rend. Sem. Mat., vol. 14 (1954–55), pp. 87–103 (§§3, 4, 5, 6, 7).

(β) *For* $\mathcal{R}s > 1$ *and* $|z| \leqq 1$,

(1) $$\sum_1^{+\infty} z^{\omega(n)}/n^s = \mathcal{G}_1(s, z)(s-1)^{-z}$$

and

(2) $$\sum_1^{+\infty} z^{\Omega(n)}/n^s = \mathcal{G}_2(s, z)(s-1)^{-z},$$

where $(s-1)^{-z}$ *has its principal value.*

In fact, λ being any irrational number, for any positive integer q we see, by taking $z = \exp[2\pi q\lambda i]$ in (1) and (2), that we have for $\mathcal{R}s > 1$

$$\sum_1^{+\infty} \frac{\exp[2\pi iq\lambda\omega(n)]}{n^s} = \mathcal{G}_1[s, \exp(2\pi q\lambda i)](s-1)^{-\exp(2\pi q\lambda i)}$$

and

$$\sum_1^{+\infty} \frac{\exp[2\pi iq\lambda\Omega(n)]}{n^s} = \mathcal{G}_2[s, \exp(2\pi q\lambda i)](s-1)^{-\exp(2\pi q\lambda i)}.$$

Since $\mathcal{R}[\exp(2\pi q\lambda i)] < 1$, we may conclude by Theorem B that, as x tends to $+\infty$,

$$\sum_{n \leqq x} \exp[2\pi iq\lambda\omega(n)] = o[x] \quad \text{and} \quad \sum_{n \leqq x} \exp[2\pi iq\lambda\Omega(n)] = o[x].$$

3. It is to be noticed that the results we proved for $\omega(n)$ and $\Omega(n)$ can be extended to other functions:

Let $f(n)$ *be an integral valued arithmetic function, and suppose that we have for* $|z| \leqq 1$ *and* $\mathcal{R}s > 1$

(3) $$\sum_1^{+\infty} z^{f(n)}/n^s = \mathcal{G}(s, z)(s-1)^{\alpha-1-\alpha z} + \mathcal{H}(s, z),$$

where α *is a real positive number and, for* $|z| \leqq 1$, *the functions* $\mathcal{G}(s, z)$ *and* $\mathcal{H}(s, z)$ *are regular in* s *for* s *belonging to a certain domain* Δ *which contains the closed half plane* $\mathcal{R}s \geqq 1$.

Then, λ *being any irrational number, the numbers* $\lambda f(n)$ *are uniformly distributed modulo* 1.

In fact, for any positive integer q we have for $\mathcal{R}s > 1$

$$\sum_1^{+\infty} \frac{\exp[2\pi iq\lambda f(n)]}{n^s} = \mathcal{G}[s, \exp(2\pi q\lambda i)](s-1)^{\alpha-1-\alpha\exp(2\pi q\lambda i)},$$

and, since $\mathcal{R}[-\alpha + 1 + \alpha\exp(2\pi q\lambda i)] < 1$, Theorem B enables us to conclude that, as x tends to $+\infty$,

$$\sum_{n \leqq x} \exp[2\pi iq\lambda f(n)] = o[x].$$

3.1. We thus see that the uniform distribution modulo 1 of $\lambda f(n)$ for any irrational λ holds for all functions of the family ($\mathcal{F}$) we consider in our paper, *Sur la distribution des valeurs de certaines fonctions arithmétiques.*[7]

[7] *Colloque sur la théorie des nombres*, Bruxelles 19, 20 et 21 décembre 1955, pp. 147–161. In the following we shall denote this paper by "paper C".

Let us recall the definition of this family.

E being a given set of primes, we define the two functions $\omega_E(n)$ and $\Omega_E(n)$ as follows:

$\omega_E(n)$ is the number of prime divisors of n which belong to the set E, and $\Omega_E(n)$ is the total number of factors belonging to E in the factorization of n. In other words, we have

$$\omega_E(1) = \Omega_E(1) = 0,$$

and, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} m$, where $p_1, p_2, \cdots, p_k$ are distinct primes of E, $\alpha_1, \alpha_2, \cdots, \alpha_k$ are positive integers, and m is a positive integer which is not divisible by any prime of E,

$$\omega_E(n) = k \qquad \text{and} \qquad \Omega_E(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k .$$

Then the family $(\mathfrak{F})$ consists of all the functions $\omega_E(n)$ and $\Omega_E(n)$ corresponding to the sets E which have the following property:

There exist a real positive number $\alpha \leqq 1$ and a function $\delta(s)$ regular for $\mathcal{R}s \geqq 1$, such that we have for $\mathcal{R}s > 1$

$$\textstyle\sum_{p \in E} 1/p^s = \alpha \log \{1/(s - 1)\} + \delta(s),$$

(where $\log \{1/(s - 1)\}$ of course has its principal value).

The set of all primes has this property. So does the set of all primes belonging to a given arithmetic progression, or to the union of two or more arithmetic progressions with the same difference.[8]

4. We may consider the modulo 1 distribution of the numbers $\lambda f(n)$ for n running through a certain infinite set A of positive integers, distinct from the set of all positive integers.

4.1. A being an infinite set of positive integers, and $\{u_n\}$ a sequence of real numbers, it is natural to say that *the numbers u_n are uniformly distributed modulo 1 when n runs through the set A if, for $0 \leqq t \leqq 1$, the number of the n's not greater than x and such that*

$$u_n - I[u_n] \leqq t$$

is $t\nu(x) + o[\nu(x)]$ as x tends to $+\infty$, where $\nu(x)$ is the total number of the n's belonging to A and not greater than x.

It is seen very easily that H. Weyl's theorem can be extended as follows:

In order that the u_n's be uniformly distributed modulo 1 when n runs through the set A, it is necessary and sufficient that, for every positive integer q, we have

$$\textstyle\sum_{n \leqq x, n \in A} \exp (2\pi i q u_n) = o[\nu(x)]$$

as x tends to $+\infty$.

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[8] See paper A, §§3.1 to 3.2.1.

If A has a positive density, $o[\nu(x)]$ may obviously be replaced by $o[x]$.

4.2. *$f(n)$ being an integral valued arithmetic function, Theorem B enables us to prove that, for any irrational λ, the numbers $\lambda f(n)$ are uniformly distributed modulo 1 when n runs through the set A of positive density, if we have for $\Re s > 1$ and $|z| \leqq 1$*

$$(4) \qquad \sum_{n \in A} z^{f(n)}/n^s = \mathcal{G}(s, z)(s-1)^{\alpha-1-\alpha z} + \mathcal{H}(s, z),^{9}$$

where α is a positive number and, for $|z| \leqq 1$, the functions $\mathcal{G}(s, z)$ and $\mathcal{H}(s, z)$ are regular in s for s belonging to a certain domain Δ, which contains the closed half plane $\Re s \geqq 1$.

We thus see that, *if f is a function of the family $(\mathfrak{F})$, for any irrational λ, the numbers $\lambda f(n)$ are uniformly distributed modulo 1 when n runs through the set of all squarefree positive integers.*[10]

Similarly, *if $f(n) = \omega(n)$ or $\Omega(n)$, and if k is an integer > 1 and l any integer coprime to k, then, for any irrational λ, the numbers $\lambda f(n)$ are uniformly distributed modulo 1 when n runs through the set of all positive integers satisfying*

$$n \equiv l \pmod{k},$$

or even when n runs through the set of all positive squarefree integers satisfying this congruence.[11]

It can be proved that *these last two results still hold if we do not suppose that k and l are coprime, provided that in the latter case we assume that the greatest common divisor of k and l is squarefree*[12] (otherwise no integer satisfying the congruence could be squarefree).

5. We shall add that it is possible to consider simultaneously two functions f and g of the family $(\mathfrak{F})$. Then the property to be proved is the uniform distribution of the points

$$(\xi_n, \eta_n) = (\lambda f(n) - I[\lambda f(n)], \mu g(n) - I[\mu g(n)])$$

in the square $0 \leqq \xi \leqq 1$, $0 \leqq \eta \leqq 1$, λ and μ being irrational numbers.

To deal with this question, one has only to use the formulas we give in §6.2 of paper C, Theorem B, and the two-dimensional form of H. Weyl's theorem.[13]

The desired property holds for any irrational λ and μ, when n runs either through the set of all positive integers or through the set of positive square-

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[9] We take this opportunity to note that the assertion at the end of §5.4 of paper C is false with formula (2) as it is there (p. 152). This formula has to be replaced by formula (4) here. This needs only obvious changes in §§5.1 to 5.1.2.

[10] See paper A, §5.3.

[11] See paper A, §§3.4, 3.5, and 3.6.

[12] For this, one has to use arguments similar to those of paper A in §§3.10, 3.10.1, 3.10.3, 3.10.4, 3.10.5.

[13] Loc. cit., Satz 3, p. 319.

free integers, if f and g correspond to two sets of primes E_1 and E_2 with no common element.

If $f(n)$ and $g(n)$ are the functions $\omega_E(n)$ and $\Omega_E(n)$ corresponding to the same set E, the property holds when n runs through the set of all positive integers, for λ and μ so chosen that $q\lambda + q'\mu$, where q and q' are rational integers, can never be a rational integer unless $q = q' = 0$.

It is also possible to prove that, if $f(n)$ and $g(n)$ are $\omega(n)$ and $\Omega(n)$, the property holds for λ and μ satisfying the latter condition, when n runs through the set of positive integers belonging to a given arithmetic progression.

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