PRODUCTS OF GENERALIZED MANIFOLDS

BY

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1. Introduction

Generalized manifolds were defined by Čech, and are studied in detail in R. L. Wilder's book, *Topology of Manifolds* [5]. It is the purpose of this paper to prove that the product of generalized manifolds is a generalized manifold, subject to restrictions of a dimension theoretic nature. A more general statement may be made: the bundle space of a fibre bundle whose base and fibre are in the class of generalized manifolds referred to is a generalized manifold.

The proof depends upon a formula for determining the local Betti numbers of a point in the product of two locally compact spaces. This formula, analogous to the Künneth formula for determining the homology of product spaces, is given in Theorem 1.

2. Preliminaries

Generalized manifolds are defined by conditions on the local homology of the space. In particular, the local Betti numbers are used, and defined as follows (see [5], p. 191).

Given a space S and a point $x \in S$, let $\{P_\alpha\}$ be a basis for the open sets containing x, and for each α let $\{Q_{\alpha\beta}\}$ be a basis for the open sets of S containing x and contained in P_α. The symbol $Z_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})$ represents the vector space of q-dimensional Čech cycles on S mod $(S - P_\alpha)$, with coefficients in a field. The symbol $B_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})$ denotes the subspace of the above consisting of those cycles which bound on S mod $(S - Q_{\alpha\beta})$.

The indices $\{\alpha\}$ of the $\{P_\alpha\}$ are ordered by inclusion, i.e. $\alpha_1 < \alpha_2$ if and only if $P_{\alpha_1} \supset P_{\alpha_2}$. In a similar manner the indices $\{\alpha\beta\}$ are ordered by the relation, $\alpha_1 \beta_1 < \alpha_2 \beta_2$ if and only if $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$.

The generalized limit

$$\lim_{\alpha\beta} \dim\ [Z_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})/B_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})],$$

which is induced by the order relation among the $\{\alpha\beta\}$, exists (or may consistently be called infinite), since

$$\dim\ [Z_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})/B_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})]$$

is nonincreasing for $\alpha\beta < \alpha_1 \beta_1$, i.e. $Q_{\alpha\beta} \supset Q_{\alpha_1\beta_1}$. The double limit

$$\lim_{\alpha} \lim_{\alpha\beta} \dim\ [Z_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})/B_q(x{:}S,\ S - P_\alpha\ ;\ S,\ S - Q_{\alpha\beta})]$$

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induced by the order relation on $\{\alpha\}$ exists (or may be called infinite), since the values for the first limit are nondecreasing. The *local Betti numbers* of the point x in the space S are defined as these double limits, and are denoted by $p_q(x)$, $q = 0, 1, \cdots$.

Generalized manifolds are defined using local Betti numbers $p_q(x)$ as follows (see [5], p. 244).

DEFINITION. A locally compact space S is a *generalized manifold* of dimension n, if

(1) S has Lebesque dimension n,

(2) $p_q(x) = 0$, for all $x \in S$, $q \neq n$,

(3) $p_n(x) = 1$, for all $x \in S$.

3. Homology theory of products

Let (X, A) be a compact pair of Hausdorff spaces, $\{(U_\alpha, U'_\alpha\}$ a cofinal collection of finite coverings of this pair. The nerve of a covering is denoted by $N(U_\alpha, U'_\alpha)$, and the q-dimensional Čech homology group $H_q(X, A)$ is, by definition, the inverse limit $\lim_{\leftarrow} H_q(N(U_\alpha, U'_\alpha))$, the inverse limit being with respect to all coverings (or a cofinal set). Considerations are limited here to homology with coefficient group a field.

The (topological) product of two spaces X, Y is denoted by $X \times Y$. The tensor product of two vector spaces G_1, G_2 is denoted by $G_1 \otimes G_2$. The relation $\dim (G_1 \otimes G_2) = \dim G_1 \cdot \dim G_2$ is used. To avoid confusion with the word dimension as used in homology, vector space dimension is called the rank.

The following theorem in the homology theory of products has grown out of the work of Künneth [2], and is the extended form of many theorems in the literature. In particular, the proof can be obtained from results in Eilenberg and Zilber [1].

THEOREM. *If the vector spaces* $H_r(X, A)$, $r = 0, 1, \cdots$, *and* $H_s(Y, B)$, $s = 0, 1, \cdots$, *are finitely generated, there exists an isomorphism onto*:

$$\eta : H_t(X \times Y, X \times B \cup A \times Y) \approx \textstyle\sum_{r+s=t} H_r(X, A) \otimes H_s(Y, B)$$

(where it is assumed that the same field of coefficients is used to compute the homology groups).

4. The product theorem for local Betti numbers

In order to apply the theorem in the preceding paragraph, the following lemma is needed.

LEMMA. *In order to compute the local Betti numbers of an* lc^n *space, it is sufficient to consider the ranks of the homology groups of certain compact pairs.*

Proof. Let S be a locally compact space, x a point in S, P an open set containing x whose closure is compact, and Q an open set containing x whose closure is contained in P. The excision axiom states that $H_q(S, S - Q)$ is isomorphic to $H_q(\bar{P}, \bar{P} - Q)$.

Wilder [4] has shown that in an lc^n (locally connected in dimensions $\leqq n$) space, the local Betti numbers are equal to the rank of the direct limit groups $\lim_{\rightarrow} H_q(S, S - P_\alpha)$, where $\{P_\alpha\}$ is a basis for the open sets in S which contain x.

If a space has finite local Betti numbers, then it is lc^n (see [5], p. 196]. All of the theorems in this paper are corollaries of the following theorem.

THEOREM 1. *If S, S' are locally compact spaces with finite local Betti numbers, $x \in S$, $y \in S'$, then the local Betti numbers of the point $(x, y) \in S \times S'$* are *determined by the formula*

$$p_t(x, y) = \textstyle\sum_{r+s=t} p_r(x)\, p_s(y). \tag{A}$$

Proof. Properties of compact pairs are used, as justified by the lemma. Let S be a compact space, $x \in S$, P_α an open set containing x, and $Q_{\alpha\beta}$ an open set whose closure is contained in P_α. For a second compact set S', $y \in S'$, let P'_α and $Q'_{\alpha\beta}$ be defined in a similar way.

By the theorem of section 3 there exist isomorphisms as indicated by the vertical maps in the following diagram.

$$\begin{array}{ccc}
H_t(S \times S', S \times S' - P_\alpha \times P'_\alpha) & \xrightarrow{\ j\ } & H_t(S \times S', S \times S' - Q_{\alpha\beta} \times Q'_{\alpha\beta}) \\
\updownarrow & & \updownarrow \\
\sum\limits_{r+s=t} H_r(S, S - P_\alpha) \otimes H_s(S', S' - P'_\alpha) & \xrightarrow{\ j_1\ } & \sum\limits_{r+s=t} H_r(S, S - Q_{\alpha\beta}) \otimes H_s(S', S' - Q'_{\alpha\beta}),
\end{array}$$

the horizontal map j in the top line being the one induced by inclusion; in the bottom line j_1 is the tensor product of the induced maps. This diagram is commutative, by the homology theory of products.

By definition, the Betti number

$$p_t((x, y) : S \times S', S \times S' - P_\alpha \times P'_\alpha : S \times S', S \times S' - Q_{\alpha\beta} \times Q'_{\alpha\beta})$$

is the rank of the vector space $H_t(S \times S', S \times S' - P_\alpha \times P'_\alpha)$ less the rank of the kernel of j in the diagram above. This can be computed by determining the rank of $\sum_{r+s=t} H_r(S, S - P_\alpha) \otimes H_s(S', S' - P'_\alpha)$ less the rank of the kernel of j_1.

The rank of $\sum_{r+s=t} H_r(S, S - P_\alpha) \otimes H_s(S', S' - P'_\alpha)$ is

$$\sum_{r+s=t} [\operatorname{rank} H_r(S, S - P_\alpha)]\, [\operatorname{rank} H_s(S', S' - P'_\alpha)].$$

An element $z_1 \otimes z_2$ of $H_r(S, S - P_\alpha) \otimes H_s(S', S' - P'_\alpha)$ is in the kernel of j_1 if and only if $j_1(z_1 \otimes z_2) = 0 \otimes z'_2$ or $j_1(z_1 \otimes z_2) = z'_1 \otimes 0$. Since $H_r(S, S - Q_{\alpha\beta}) = Z_r(S, S - Q_{\alpha\beta})/B_r(S, S - Q_{\alpha\beta})$, etc., this occurs if and only if z_1 maps into an element of $B_r(S, S - Q_{\alpha\beta})$, or z_2 maps into an element of $B_s(S', S' - Q'_{\alpha\beta})$ under the maps induced by inclusion.

Hence if the rank of $H_r(S, S - P_\alpha)$ is denoted by $p_r(S, S - P_\alpha)$, etc., and the kernel of $k_1 : H_r(S, S - P_\alpha) \to H_r(S, S - Q_{\alpha\beta})$ by $b_r(S, S - Q_{\alpha\beta})$, etc., then

$$p_t(S \times S', S \times S' - P_\alpha \times P'_\alpha) - b_t(S \times S', S \times S' - Q_{\alpha\beta} \times Q'_{\alpha\beta})$$
$$= \textstyle\sum_{r+s=t} [p_r(S, S - P_\alpha) - b_r(S, S - Q_{\alpha\beta})][p_s(S', S' - P'_\alpha) - b_s(S', S' - Q'_{\alpha\beta})]$$

since

$$b_t(S \times S', S \times S' - Q_{\alpha\beta} \times Q'_{\alpha\beta}) = \textstyle\sum_{r+s=t} [p_r(S, S - P_\alpha)\, b_s(S', S' - Q'_{\alpha\beta}) + b_r(S, S - Q_{\alpha\beta})\, p_s(S', S' - P'_\alpha) - b_r(S, S - Q_{\alpha\beta})\, b_s(S', S' - Q'_{\alpha\beta})].$$

Using the notation above, the local Betti numbers are defined as

$$p_r(x) = \lim_{\alpha} \lim_{\alpha\beta} [p_r(S, S - P_\alpha) - b_r(S, S - Q_{\alpha\beta})].$$

$$p_s(y) = \lim_{\alpha} \lim_{\alpha\beta} [p_s(S', S' - P'_\alpha) - b_s(S', S' - Q'_{\alpha\beta})],$$

$$p_t(x, y) = \lim_{\alpha} \lim_{\alpha\beta} [p_t(S \times S', S \times S' - P_\alpha \times P'_\alpha) - b_t(S \times S', S \times S' - Q_{\alpha\beta} \times Q'_{\alpha\beta})].$$

Since both limits in each expression are monotone, the theorem can be deduced from the formulas above.

COROLLARY 1. *The product of two locally compact spaces all of whose points have finite local Betti numbers is a space all of whose points have finite local Betti numbers.*

5. Products of generalized manifolds

The formula above is now applied to local Betti numbers in products of generalized manifolds.

COROLLARY 2. *If S and S' are generalized manifolds, of dimension m and n respectively, $x \in S$, $y \in S'$, $(x, y) \in S \times S'$, then $p_{m+n}(x, y) = 1$.*

Proof. The formula becomes $p_{m+n}(x, y) = \sum_{r+s=m+n} p_r(x)\, p_s(y)$. From the definition of generalized manifold, $p_r(x) = 0$, $r \neq m$, and $p_s(y) = 0$, $s \neq n$, while $p_m(x) = 1$ and $p_n(y) = 1$. Thus,

$$p_{m+n}(x, y) = p_m(x)\, p_n(y) = 1.$$

COROLLARY 3. *If S and S' are generalized manifolds, of dimension m and n respectively, $x \in S$, $y \in S'$, $(x, y) \in S \times S'$, then $p_t(x, y) = 0$, $t \neq m + n$.*

Proof. If $t \neq m + n$, and $r + s = t$, either $r \neq m$ or $t \neq n$. Hence the sum in the formula is zero.

THEOREM 2. *If S and S' are generalized manifolds (using the same field of coefficients) such that* $\dim (S \times S') = \dim S + \dim S'$, *then $S \times S'$ is a generalized manifold.*

Proof. This is an immediate consequence of the preceding two corollaries.

Fibre bundles, [3], are spaces such that each point in the bundle space has a neighborhood homeomorphic to a product of a neighborhood in the base and a neighborhood in the fibre. Hence the following generalization of Theorem 2 holds.

THEOREM 3. *If a fibre bundle $\mathfrak{B}$ has base X and fibre Y which are generalized manifolds using the same field of coefficients, and if* $\dim \mathfrak{B} = \dim X + \dim Y$, *then $\mathfrak{B}$ is a generalized manifold.*

As a consequence of Theorem 1, if the bundle space is a generalized manifold, and the base space X and fibre Y both lc^r (r their respective dimensions), and the dimension restriction is satisfied, then X and Y must be generalized manifolds.

BIBLIOGRAPHY

1. S. EILENBERG AND J. A. ZILBER, *On products of complexes*, Amer. J. Math., vol. 75 (1953), pp. 200–204.
2. H. KÜNNETH, *Über die Bettischen Zahlen einer Produktmannigfaltigkeit*, Math. Ann., vol. 90 (1923), pp. 65–85.
3. N. E. STEENROD, *The topology of fibre bundles*, Princeton University Press, 1951.
4. R. L. WILDER, *Concerning a problem of Alexandroff*, Michigan Math. J., vol. 3 (1955–1956), pp. 181–185.
5. ———, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 32, 1949.

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