ON SUBDETERMINANTS OF DOUBLY STOCHASTIC MATRICES[1]

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In this note we obtain an inequality for the euclidean norm of an n-square complex matrix $A = (A_{ij})$, (Theorem 1). This is used to give lower bounds for the rank of A and in particular for the rank of a doubly stochastic matrix. We then distinguish (Theorems 2 and 3) a certain simple set of matrices among all doubly stochastic matrices in terms of possible values for the subdeterminants. In particular a characterization of the permutation matrices as a subclass of doubly stochastic matrices is given in terms of bounds on the subdeterminants.

We proceed to describe some notation to be used throughout. A typical r-square subdeterminant of A will be denoted by $d_r(A)$, det A will be the determinant of A. The sum over all $\binom{n}{r}^2$ choices of some function φ of the d_r will be denoted by

$$\sum \varphi(d_r(A)),$$

and the norm of A is given by

$$\| A \|^2 = \sum | d_1(A) |^2 = \sum_{i,j} | A_{ij} |^2.$$

The i^{th} row vector of A is $A_{(i)}$, and the j^{th} column vector is $A^{(j)}$. The rank of A is $\rho(A)$; I_k is the k-square identity matrix; 0_k is the k-square matrix of zeros; $A \dotplus B$ is the direct sum of A and B; and the conjugate transpose of A is A^*. A doubly stochastic (d.s.) matrix A is one which satisfies

$$\sum_{j=1}^{n} A_{ij} = 1, \qquad i = 1, \cdots, n$$

$$\sum_{i=1}^{n} A_{ij} = 1, \qquad j = 1, \cdots, n$$

$$A_{ij} \geqq 0, \qquad i, j = 1, \cdots, n.$$

The r^{th} symmetric function of the letters $a_1, \cdots, a_k$ is $E_r(a_1, \cdots, a_k)$.

In [4] H. Richter proved for an arbitrary n-square complex matrix A that[2]

$$\| (\det A) A^{-1} \|^2 \leqq n^{-(n-2)} \| A \|^{2(n-1)} \tag{1}$$

with equality if and only if AA^* is a scalar matrix.

The first result is an extension of (1).

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[2] The same result with a simpler proof appeared recently in a note of L. MIRSKY (Arch. Math., vol. 7 (1956), p. 276).

THEOREM 1. *If $\rho(A) = k$, then for $1 < r \leqq k$*

$$\sum | d_r(A) |^2 \leqq \binom{k}{r} k^{-r} \| A \|^{2r} \tag{2}$$

with equality if and only if $A = 0$ or

$$A = V(I_k \dot{+} 0_{n-k})W, \tag{3}$$

where $\alpha > 0$ and W and V are unitary.

Proof. The case $r = 1$ of (2) is obviously equality and is hence excluded from the proof. Let $C_r(A)$ denote the r^{th} compounded matrix of A. Then the elements of $C_r(A)$ are the numbers $d_r(A)$ arranged in doubly lexicographic ordering according to row and column selections of the subdeterminants. The eigenvalues of $C_r(A)$ are the $\binom{n}{r}$ products of the eigenvalues of A taken r at a time [5; p. 67]. Let the eigenvalues of AA^* be

$$\alpha_1^2 \geqq \alpha_2^2 \geqq \cdots \geqq \alpha_k^2 > 0 = \alpha_{k+1}^2 = \cdots = \alpha_n^2, \quad 1 \leqq k \leqq n.$$

We compute that

$$\begin{aligned} \sum | d_r(A) |^2 &= \sum | d_1(C_r(A)) |^2 = \| C_r(A) \|^2 \\ &= \text{trace } C_r(A)C_r^*(A) = \text{trace } C_r(AA^*) \\ &= E_r(\alpha_1^2, \cdots, \alpha_n^2) = E_r(\alpha_1^2, \cdots, \alpha_k^2) \\ &\leqq \binom{k}{r} k^{-r} E_1^r(\alpha_1^2, \cdots, \alpha_k^2) = \binom{k}{r} k^{-r} (\text{trace } AA^*)^r \\ &= \binom{k}{r} k^{-r} \| A \|^{2r}. \end{aligned} \tag{4}$$

The inequality sign in (4) is strict unless [3; p. 52]

$$\alpha_1^2 = \cdots = \alpha_k^2 = \alpha^2.$$

By the polar factorization theorem and the spectral theorem for Hermitian matrices, we conclude that

$$A = (A^*A)^{1/2}U = V(\alpha I_k \dot{+} 0_{n-k})V^{-1}U = V(\alpha I_k \dot{+} 0_{n-k})W,$$

where W and V are unitary.

On the other hand if (3) holds, then AA^* has eigenvalues $\alpha_1^2 = \cdots = \alpha_k^2 = \alpha^2$, $\alpha_{k+1}^2 = \cdots = \alpha_n^2 = 0$, and the inequality in the sequence (4) is equality.

COROLLARY 1. *If $1 < r \leqq n$, then*

$$\sum | d_r(A) |^2 \leqq \binom{n}{r} n^{-r} \| A \|^{2r}, \tag{5}$$

with equality if and only if either $A = 0$ or there exists an $\alpha > 0$ such that $\alpha^{-1}A$ is unitary.

Proof. Let $\rho(A) = k$, and note that $k < n$ and $r \leqq k$ imply that

$$\binom{k}{r} k^{-r} < \binom{n}{r} n^{-r}. \tag{6}$$

If $r > k$, the left side of (5) is 0, and the inequality is strict unless $A = 0$. The weak form of (5) then always follows from (2) and (6). If we have equality in (5), then

$$\sum | d_r(A) |^2 = \binom{n}{r} n^{-r} \| A \|^{2r} \geqq \binom{k}{r} k^{-r} \| A \|^{2r}$$
$$\geqq \sum | d_r(A) |^2.$$

Hence $k = n$, and by (3) we conclude that

$$AA^* = \alpha^2 I_n .$$

Richter's result follows immediately from Corollary 1 upon setting $r = n - 1$. Of course, if $A \neq 0$ is singular, then Theorem 1 implies the generally sharper estimate (since $n^{-(n-2)} > (n - 1)^{-(n-1)}$ for $n \geqq 3$):

$$\sum | d_{n-1}(A) |^2 \leqq (n - 1)^{-(n-1)} \| A \|^{2(n-1)},$$

with equality if and only if $\rho(A) = n - 1$ *and* AA^* *has* $n - 1$ *equal positive eigenvalues and one zero eigenvalue.*

COROLLARY 2. $\rho(A) \geqq (1 - 2\| A \|^{-4} \sum | d_2(A) |^2)^{-1}$.

Proof. If $\rho(A) = 1$, then $d_2(A) = 0$. If $\rho(A) \geqq 2$, then set $r = 2$ in (2).

We next state and prove two elementary results that will be used to obtain a characterization of a certain class of d.s. matrices (Theorem 2) by means of inequalities on the subdeterminants.

LEMMA 1. *If S is a d.s. n-square matrix and T is an r-square submatrix of S, then*

$$(\det T)^2 \leqq 1. \tag{7}$$

Proof. Let $\gamma_i(T)$ denote the sum of the elements in the i^{th} row of T. It is known that $\max \gamma_i(T) = \gamma$ is a bound for the absolute values of the eigenvalues of T. It follows from the definition of T that $\gamma \leqq 1$, and hence (7) is proved.

The question then arises: precisely how many r-square submatrices of a d.s. matrix can satisfy the equality in (7)? Before considering this we prove one further elementary fact.

LEMMA 2. *If S is an m-square d.s. matrix and $\rho(S) = 1$, then $S = m^{-1}J_m$, where J_m is the m-square matrix all of whose entries are* 1.

Proof. If $\rho(S) = 1$, then every column is a scalar multiple of the first column of S,

$$S^{(j)} = \alpha_j S^{(1)}, \qquad j = 1, \cdots, m.$$

But then

$$1 = \sum_{i=1}^{m} S_{ij} = \alpha_j \sum_{i=1}^{m} S_{i1} = \alpha_j .$$

Hence

$$S^{(j)} = S^{(1)}, \qquad j = 1, \cdots, m,$$

and

$$\sum_{j=1}^{m} S_{ij} = mS_{ij} = 1. \tag{8}$$

The result follows from (8).

THEOREM 2. *Let A be an n-square d.s. matrix with $\rho(A) = k$. Then for every r, $1 \leqq r \leqq k$,*

$$\sum d_r^2(A) \leqq \binom{k}{r}. \tag{9}$$

The equality in (9) holds for some r if and only if A is of the form

$$A = P(n_1^{-1}J_{n_1} \dotplus n_2^{-1}J_{n_2} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})Q,$$

where P and Q are permutation matrices and $n_1 \geqq \cdots \geqq n_k > 0$ are integers satisfying

$$\sum_{j=1}^{k} n_j = n.$$

Proof.[3] Set $T = AA'$. Now T is d.s., and it is known [1; p. 75] that any eigenvalue α_σ^2 of T satisfies

$$\alpha_\sigma^2 \leqq 1.$$

The inequality (9) then follows from (2) since $\rho(T) = \rho(A) = k$ implies that only k of the α_σ are different from zero;

$$\sum{}' d_r^2(A) \leqq \binom{k}{r}k^{-r} \| A \|^{2r} = \binom{k}{r}k^{-r}\left(\sum_{\sigma=1}^{k} \alpha_\sigma^2\right)^r \leqq \binom{k}{r}.$$

Now suppose equality holds in (9). If $0 < \alpha_\sigma^2 < 1$ for some σ, we conclude that

$$\| A \|^2 = \sum_{\sigma=1}^{k} \alpha_\sigma^2 < k,$$

and hence that

$$\binom{k}{r} = \sum d_r^2(A) < \binom{k}{r}.$$

Thus the k nonzero eigenvalues of T are 1. Now if $k = 1$, then by Lemma 2 $A = n^{-1}J_n$, and the proof is complete. Now assume $k > 1$. Then the dominant eigenvalue, 1, of T has multiplicity greater than 1, and by a theorem of Frobenius [2], proved in a simplified manner by Wielandt [6], T must be decomposable. That is, there exists a permutation matrix P_1 for which

$$P_1TP_1' = \begin{pmatrix} T_1 & & \cdots & \\ 0 & T_2 & & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & T_k \end{pmatrix}, \tag{10}$$

[3] The author is indebted to Dr. Morris Newman and Dr. Olga Taussky-Todd for several valuable conversations concerning this result.

where T_σ is n_σ-square indecomposable, $n_1 \geqq \cdots \geqq n_k$, and the elements in the lower triangle of P_1TP_1', not in any T_σ, are zero. But the symmetry of T implies from (10) that

$$P_1TP_1' = P_1AA'P_1' = T_1 \dotplus T_2 \dotplus \cdots \dotplus T_k . \tag{11}$$

Since the eigenvalues of each T_σ are 1 or 0, and since each T_σ is indecomposable, it follows that $\rho(T_\sigma) = 1$ for $\sigma = 1, \cdots, k$. Hence by Lemma 2

$$T_\sigma = n_\sigma^{-1} J_{n_\sigma} .$$

Thus (11) becomes

$$SS' = n_1^{-1}J_{n_1} \dotplus n_2^{-1}J_{n_2} \dotplus \cdots \dotplus n_k^{-1}J_{n_k} , \tag{12}$$

where we set $P_1A = S$, and S is d.s. since A is. We next show that the equation (12) can be satisfied for a d.s. matrix S if and only if

$$S = P_2(n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})Q_2 , \tag{13}$$

where P_2 and Q_2 are permutation matrices. We proceed by induction on k. We assume the result true for $k - 1$. Let r_1 be the largest integer m for which there exist m nonzero elements occurring in one of the first n_1 rows of S. Let row l_1, $1 \leqq l_1 \leqq n_1$, have r_1 nonzero elements, and by permuting the columns of S (multiplying S on the right by a permutation matrix P_3) we may assume that row l_1 of SP_3 has its first r_1 entries positive and every entry thereafter 0. Now choose Q_3, a permutation matrix, so that Q_3SP_3 has the l_1 row of SP_3 as its first row and Q_3 affects only the first n_1 rows of SP_3. However,

$$\begin{aligned}(Q_3SP_3)(Q_3SP_3)' &= Q_3SP_3P_3'S'Q_3' = Q_3SS'Q_3' \\ &= Q_3(n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})Q_3' .\end{aligned}$$

But Q_3 affects only the first n_1 rows, and Q_3' affects only the first n_1 columns, both of which operations leave J_{n_1} invariant. Hence Q_3SP_3 satisfies (12). Consider the form[4] of Q_3SP_3;

$$Q_3SP_3 = \left(\begin{array}{cccc|ccc} s_{11} & s_{12} & \cdots & s_{1r_1} & 0 & \cdots & 0 \\ s_{21} & s_{22} & \cdots & s_{2r_1} & s_{2(r_1+1)} & \cdots & s_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ s_{n_11} & \cdot & \cdots & s_{n_1r_1} & s_{n_1(r_1+1)} & \cdots & s_{n_1n} \\ \hline \vdots & \vdots & & \vdots & \vdots & & \vdots \end{array}\right) . \tag{14}$$

We assert first that the elements in rows $n_1 + j$, $j \geqq 1$, and columns $1, 2, \cdots$,

[4] We remark that although the first row of (14) has the largest number of nonzero elements among the first n_1 rows, it does not immediately follow from this alone that the upper right block in (14) is zero. As the proof shows, however, this is the case.

r_1 of Q_3SP_3 are 0. For, from the choice of r_1 we see that

$$\begin{aligned} ((Q_3SP_3)(Q_3SP_3)')_{1(n_1+j)} &= (SS')_{1(n_1+j)} \\ &= 0 \\ &= \sum_{t=1}^{r_1} s_{1t}\, s_{(n_1+j)t}\,, \end{aligned}$$

and $s_{1t} > 0$ for $t = 1, \cdots, r_1$. Hence $s_{(n_1+j)t} = 0$ for $1 \leqq t \leqq r_1$ and $j \geqq 1$, and the lower left block in (14) is 0. We next assert that $n_1 = r_1$; for since Q_3SP_3 is d.s.,

$$n_1 \geqq \sum_{i=1}^{n_1}\sum_{\sigma=1}^{r_1} s_{i\sigma} = \sum_{\sigma=1}^{r_1}\sum_{i=1}^{n_1} s_{i\sigma} = \sum_{\sigma=1}^{r_1} 1 = r_1\,.$$

On the other hand, since the 1, 1 element of SS' is $1/n_1$, it follows that

$$1 = \left(\sum_{\sigma=1}^{r_1} s_{i\sigma}\right)^2 \leqq r_1 \sum_{\sigma=1}^{r_1} s_{i\sigma}^2 = r_1/n_1\,,$$

and hence $n_1 = r_1$. Moreover we claim that $s_{\alpha t} = 0$ for $1 \leqq \alpha \leqq n_1$, $t > r_1$. For suppose not. Then for at least one α, $1 \leqq \alpha \leqq n_1$, it follows that

$$\sum_{\sigma=1}^{r_1} s_{\alpha\sigma} < 1,$$

and hence

$$n_1 > \sum_{i=1}^{n_1}\sum_{\sigma=1}^{r_1} s_{i\sigma} = r_1 = n_1\,.$$

Thus we see that the elements beyond the $r_1 = n_1$ column and in rows $1, \cdots, n_1$ must be zero.

Now let T be the n_1-square d.s. upper left block in (14), and let R be the d.s. lower right block. Then

$$\begin{aligned} SS' = (Q_3SP_3)(Q_3SP_3)' &= (T \dotplus R)(T' \dotplus R') = TT' \dotplus RR' \\ &= n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k}\,. \end{aligned} \tag{15}$$

Hence

$$TT' = n_1^{-1}J_{n_1}\,, \tag{16}$$

and

$$RR' = n_2^{-1}J_{n_2} \dotplus \cdots \dotplus n_k^{-1}J_{n_k}\,. \tag{17}$$

As above in the case $k = 1$, we conclude from (16) that

$$T = n_1^{-1}J_{n_1}\,. \tag{18}$$

Now by induction we obtain from (17) that

$$R = P_4(n_2^{-1}J_{n_2} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})Q_4\,, \tag{19}$$

where P_4 and Q_4 are $(n - n_1)$-square permutation matrices. Thus

$$Q_3SP_3 = T \dotplus R = n_1^{-1}J_{n_1} \dotplus P_4(n_2^{-1}J_{n_2} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})Q_4$$
$$= (I_{n_1} \dotplus P_4)(n_1^{-1}J_{n_1} \dotplus n_2^{-1}J_{n_2} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})(I_{n_1} \dotplus Q_4).$$

Hence

$$A = P(n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})Q. \tag{20}$$

It is clear from (20) that since P and Q are permutation matrices, we may conclude

$$\sum d_r^2(A) = \sum d_r^2(n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k}). \tag{21}$$

Now if T is an r-square submatrix of $n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k}$, then if $\det(T) \neq 0$, we must conclude that $T = T_{i_1} \dotplus \cdots \dotplus T_{i_l}$, where T_{i_σ} is a submatrix of $J_{n_{\sigma(i)}}$. It is also clear that if T_{i_σ} is a 2- or more-square matrix, then $\det(T_{i_\sigma}) = 0$, and if T_{i_σ} is a single element then $T_{i_\sigma} = (n_{i_\sigma}^{-1})$. Hence

$$\det{}^2(T) = \prod_{\sigma=1}^{l} n_{i_\sigma}^{-2},$$

and there are precisely $n_{i_1}^2 \cdots n_{i_r}^2$ such determinants possible. Hence

$$\sum d_r^2(A) = \sum_{1 \leq i_1 < \cdots < i_r \leq k} (n_{i_1}^2 \cdots n_{i_r}^2)\cdot(n_{i_1}^{-2} \cdots n_{i_r}^{-2}) = \sum_{1 \leq i_1 < \cdots < i_r \leq k} 1 = \binom{k}{r}, \tag{22}$$

and the equality in (9) holds. This completes the proof.

COROLLARY 3. *If A is d.s., then*

$$\rho(A) \geqq \tfrac{1}{2} + \tfrac{1}{2}\{1 + 8\sum d_2^2(A)\}^{1/2}. \tag{23}$$

Proof. If $\rho(A) = 1$, then $d_2(A) = 0$ and it is clear that (23) is equality. In general let $\rho(A) = k$. We conclude from Theorem 2 that

$$\sum d_2^2(A) \leqq \binom{k}{2},$$

or

$$k^2 - k - 2\sum d_2^2(A) \geqq 0.$$

Set

$$\varphi(z) = z^2 - z - 2\sum d_2^2(A),$$

with roots $\alpha < \beta$. Then if $\alpha < 2 \leqq k$, we can conclude that $k \geqq \beta$ since $\varphi(z) < 0$ in (α, β). But

$$\alpha = \tfrac{1}{2} - \tfrac{1}{2}\{1 + 8\sum d_2^2(A)\}^{1/2} < 2.$$

So

$$k \geqq \beta = \tfrac{1}{2} + \tfrac{1}{2}\{1 + 8\sum d_2^2(A)\}^{1/2}.$$

THEOREM 3. *Let A be an n-square d.s. matrix with $\rho(A) = k$. Then for any r, $1 \leqq r \leqq k$, A has at most $\binom{k}{r}$ r-square subdeterminants of value ± 1. There exists an r for which A has $\binom{k}{r}$ r-square subdeterminants of value ± 1 if and only if $k = n$ and A is a permutation matrix.*

Proof. From the inequality (9) it is clear that $d_r(A) = \pm 1$ can hold for at most $\binom{k}{r}$ r-square subdeterminants of A. On the other hand if the equality holds for $\binom{k}{r}$ values for some r, then by Theorem 2

$$A = P(n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k})Q; \qquad n_1 \geqq n_2 \geqq \cdots \geqq n_k > 0, \tag{24}$$

P and Q permutation matrices. It is clear from (24) that if there are to be *any* r-square subdeterminants of A with value ± 1, we must have

$$1 = n_k = n_{k-1} = \cdots = n_{k-\gamma},$$

where $\gamma \geqq r - 1$ and $\gamma \leqq k - 1$. Now if $\gamma = k - 1$, then $k - \gamma = 1$, and hence $k = n$, in which case we are finished. So assume $\gamma < k - 1$. Now there are precisely $\binom{\gamma+1}{r}$ r-square subdeterminants of value 1 in

$$n_1^{-1}J_{n_1} \dotplus \cdots \dotplus n_k^{-1}J_{n_k},$$

and every other one is strictly less than 1. But $\gamma < k - 1$, and hence $\binom{\gamma+1}{r} < \binom{k}{r}$, a contradiction. Hence $k = n$, and the proof is complete.

Added in proof. Professor Ky Fan communicated the following result to the author while the present paper was in press. It constitutes an extension of Theorem 2 above.

Let A be a nonnegative rectangular matrix of rank $k \geqq 1$. Then AA' has no eigenvalue different from 0 or 1 if and only if, after deleting all identically vanishing rows and columns of A, the remaining submatrix B can be brought by permutation of rows and columns to the form

$$B = B_1 \dotplus \cdots \dotplus B_k,$$

where each B_i is a rectangular positive matrix of rank 1 and norm 1.

We wish to thank Professor Fan for permission to publish this result here.

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