# ON THE CHARACTERISTIC LINEAR SYSTEMS OF ALGEBRAIC FAMILIES

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The completeness of the characteristic linear systems of algebraic families is one of the important and interesting problems in algebraic geometry. Recently it became clear that this is not true in general in the case of prime characteristic [1], but it is still an interesting question to ask what conditions are necessary or sufficient for the completeness of the characteristic systems. O. Zariski proposed to me the problem of extending to the abstract case the result that the vanishing of the geometric genus is sufficient for this purpose. This paper contains the affirmative answer to his conjecture. We shall prove in this paper the following: On the nonsingular surface we have an inequality $p_g \geqq h^{0,1} - q \geqq 0$ where $p_g$, $h^{0,1}$, and $q$ denote respectively the geometric genus, maximal deficiency, and the irregularity (the dimension of the Picard variety) of the original surface. In §1 we shall introduce the notion of the characteristic set of an algebraic family and prove the linearity of that set. Then we shall show the inequality $q \leqq h^{0,1}$ for the nonsingular variety of any dimension (§2). In §3 we shall prove an important property of an ample linear system on the nonsingular variety, and then the final results can be deduced from this property in the case of the nonsingular surfaces. The author was inspired very much by the works of F. Severi[1] during these researches.

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## I. The characteristic linear systems

Let $V$ be a projective variety of dimension $\geqq 2$ which is irreducible and free from singular subvarieties of codimension 1.[2] Let $\Sigma$ be a maximal algebraic family of positive divisors on $V$, and let $k$ be a common field of definition for $V$ and $\Sigma$. We shall assume in the following that the generic member $C$ of $\Sigma$ over $k$ is an irreducible variety and that any subvariety of $C$ of codimension 1 is simple, not only on $C$, but also on $V$. Let $C_0$ be a fixed generic member of $\Sigma$ over $k$, and $C$ a generic member of $\Sigma$ over $k(C_0) = k'$.[3] Since $C \not\succ C_0$, we

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[1] Cf. Zariski [14], pp. 82–84.

[2] The term "variety" is used only for an irreducible variety, and the terms "irreducible", "simple", and "normal" in this paper are used always in the absolute sense.

[3] We shall mean by $k(C)$ the smallest field containing $k$ over which $C$ is rational. This is equivalent to the field generated over $k$ by the Chow point of $C$, since $C$ is a $V$-divisor and $V$ is defined over $k$.

can define the intersection product $C \cdot C_0$ on $V$ which is rational over $k'(C)$. Let $\Gamma$ be a specialization of $C \cdot C_0$[4] over $k'$ extending the specialization $C \to C_0$. We shall denote by $\mathfrak{C}$ the set of all $C_0$ divisors $\Gamma$ obtained in the way described above. This set $\mathfrak{C}$ is called *the characteristic set of the algebraic family* $\Sigma$ (on $C_0$). Our object is to prove that the set $\mathfrak{C}$ is a linear system of dimension $r - 1$, where $r$ is the dimension of $\Sigma$.

Let $| E_m |$ be the complete linear system on $V$ which contains the sections of $V$ with the hypersurfaces of order $m$. Let $C$ be any member of $\Sigma$, then deg $C$ is the same for all $C$ in $\Sigma$. We call it *the degree of* $\Sigma$. We can see easily that if $m$ is not less than the degree of $\Sigma$, then for any member $C$ in $\Sigma$ we can find an element $E$ in $E_m$ such that $E \succ C$ and $E \not\succ 2C$ [9]. We shall fix an integer $m$ to satisfy the above condition. Let $C$ be a generic member of $\Sigma$ over $k$; then the linear system $| E_m | - C$ exists, and it is a complete linear system defined over $k(C)$. Let $D$ be a generic member of $| E_m | - C$ over $k(C)$; then $C + D$ is a member of $| E_m |$ and $k(C + D)$ is a regular extension of $k$ of dimension $r + d$, where $d = \dim | E_m - C |$. Let $\Delta$ be the set of all positive divisors which are specializations of $C + D$ over $k$; then $\Delta$ is an algebraic family on $V$, and any member of $\Delta$ can be written in the form $C' + D'$, where $C'$ is a member of $\Sigma$ and $D'$ is a member of $| E_m | - C'$.

Let $n = \dim | E_m |$. Then the members of $| E_m |$ are parametrized in 1-1 way by the points in a projective space $L^n$, and the set of points corresponding to the divisors in $\Delta$ is an irreducible subvariety $W$ of $L$ defined over $k$. Let $C_0$ be, as before, a generic member of $\Sigma$ over $k$; then we can find a divisor $D_0$ in $| E_m | - C_0$ such that $C_0 + D_0$ is a generic member of $\Delta$ over $k$. Let $t_0$ be the point on $W$ corresponding to $C_0 + D_0$. Then from the choice of the integer $m$, we have $D_0 \not\succ C_0$. Let $T$ be the tangent linear variety to $W$ at $t_0$, and $H^{n-d}$ a generic linear variety over $k(t_0)$ containing the point $t_0$. Then $H$ is a generic linear variety over $k$, and the intersection product $W_0 = W \cdot H$ is defined. It is known that $W_0$ is an irreducible variety, that $t_0$ is a simple point of $W_0$, and that $W_0$ is defined over a field $K$ which is a field of definition containing $k(t_0)$ [3], [6], [10]. We shall set $T_0 = T \cdot H$. Then we can see easily that $T_0$ is the tangent linear variety to $W_0$ at $t_0$. Let $t$ be a generic point of $W_0$ over $K$, and $C + D$ the corresponding divisor in $\Delta$. Then $t$ is a generic point of $W$ over $k(t_0)$, and hence $C + D$ is a generic member of $\Delta$ over $k(t_0)$.

Next we consider the linear system $C_0 + | E_m - C_0 |$. This linear system is represented by a linear subvariety $S_0^d$ contained in $W$. Moreover $S_0$ contains $t_0$, hence it is contained in the tangent linear variety $T$ to $W$ at $t_0$. Since $S_0$ is defined over $k(t_0)$, the intersection $S_0 \cap H$ is reduced to the point $t_0$. Thus we see that any member in $| E_m |$ corresponding to a point $s$ ($\neq t_0$) on $T_0$ does not contain $C_0$ as a component. We shall also remark here that the linear variety $S$ corresponding to the linear system $C + | E_m - C |$ intersects $H$ in only one point, namely in the point $t$.

[4] Concerning the theory of the specialization of cycles we refer to [7].

Let $E^0$ be the linear subsystem of $E_m$ defined by the tangent linear variety $T_0$ to $W_0$ at $t_0$. Then we can define the linear system $\mathrm{Tr}_{C_0} E^0$ which is the set of the intersections of $C_0$ with the members $E$ of $E^0$ such that $E \not\succ C_0$. Since any member in $E^0$, except $C_0 + D_0$, does not contain $C_0$, we see that $\dim \mathrm{Tr}_{C_0} E^0 = r - 1$.

We shall show that $\mathrm{Tr}_{C_0} E^0$ has the fixed component $C_0 \cdot D_0$ and consists of the $C_0$-divisors of the form $\Gamma + C_0 \cdot D_0$, where $\Gamma$ is a member of the characteristic set $\mathfrak{C}$.

We shall show first that $\Gamma + C_0 \cdot D_0 \supset \mathrm{Tr}_{C_0} E^0$. Let $A_0$ be a member of $E^0$, different from $C_0 + D_0$, and let $a_0$ be the point on $T_0$ corresponding to the divisor $A_0$. Let $F_0$ be the straight line connecting $a_0$ and $t_0$. Let $K'$ be a field of definition for $F_0$, containing $K$, and let $t$ be a generic point of $W_0$ over $K'$. Let $F_t$ be the straight line connecting $t_0$ and $t$. Then $F_t \to F_0$ is a specialization over the specialization $t \to t_0$ with reference to $K'$. Let $C + D$ be the divisor corresponding to the point $t$. Then we can extend the specialization $(t, F_t) \to (t_0, F_0)$ to the specialization

$$(t, F_t, C, (C + D)\cdot C_0) \to (t_0, F_0, C_0, \Gamma + G), \tag{1}$$

where $\Gamma$ is a specialization of $C \cdot C_0$ (hence a member of $\mathfrak{C}$) and $G$ is a specialization of $D \cdot C_0$. Since $C_0, D_0$ is the uniquely determined specialization of $C, D$ over the specialization $t \to t_0$ and $D_0 \not\succ C_0$, $C_0 \cdot D_0$ is the uniquely determined specialization of $C_0 \cdot D$ over the above specialization. Hence we have $G = C_0 \cdot D_0$. Since $a_0$ is a point of $F_0$, there exists a point $a$ on $F_t$ such that $a \to a_0$ is a specialization compatible with the specialization (1) [11]. Let $A$ be the divisor corresponding to the point $a$. Then since $A$ is a member of the linear pencil determined by the divisors $C_0 + D_0$ and $C + D$, we have $A \cdot C_0 = (C + D)\cdot C_0$. Thus we see that

$$(t, F_t, C, (C + D)\cdot C_0, A, A \cdot C_0) \to (t_0, F_0, C_0, \Gamma + C_0 \cdot D_0, A_0, A_0 \cdot C_0)$$

is a specialization over $K'$. Since $A_0 \not\succ C_0$, the intersection product $A_0 \cdot C_0$ is defined, and this is the uniquely determined specialization of $A \cdot C_0$ over the specialization $A \to A_0$. On the other hand, $\Gamma + C_0 \cdot D_0$ is also a specialization of $A \cdot C_0 = (C + D)\cdot C_0$ over the specialization $A \to A_0$, hence we must have $\Gamma + C_0 \cdot D_0 = A_0 \cdot C_0$. Since $A_0 \cdot C_0$ is an arbitrary member of $\mathrm{Tr}_{C_0} E^0$, the above arguments show that $\Gamma + C_0 \cdot D_0 \supset \mathrm{Tr}_{C_0} E^0$.

Let, as before, $t$ be a generic point of $W_0$ over $K$, and let $C + D$ be the divisor corresponding to $t$. Then $C$ is also a generic member of $\Sigma$ over $K$. In fact assume that $\dim_K K(C) < r$; then, since $\dim_K K(C + D) = r$, $K(D)$ has dimension $\geqq 1$ over $K(C)$. Hence the set of divisors which are the specializations of $D$ over $K(C)$ form an algebraic system of dimension $\geqq 1$ contained in $H$ and $S$. This contradicts the remark mentioned before. Thus we see that $C$ is a generic member of $\Sigma$, not only over $k$, but also over $K$.

We shall now proceed to the proof of the inverse inclusion relation. Let $\Gamma$ be an arbitrary member of the characteristic set $\mathfrak{C}$. Then $\Gamma$ is a specialization of $C \cdot C_0$ over the specialization $C \to C_0$ with reference to $K(C_0)$. But since $C$ is also a generic member of $\Sigma$ over $K$, $\Gamma$ is a specialization of $C \cdot C_0$ over the specialization $C \to C_0$ with reference to $K$. We shall extend this specialization to the specialization

$$(t, C, C \cdot C_0, D, D \cdot C_0) \to (t_0, C_0, \Gamma, D_0, D_0 \cdot C_0)$$

over $K$. Let $F_t$ be the straight line connecting $t_0$ and $t$, and let $F_0$ be a specialization of $F_t$ over the above specialization with reference to $K$. Then $F_0$ is a straight line in $T_0$ passing through $t_0$. We can see by the same reasoning as before that $\Gamma + C_0 \cdot D_0 = C_0 \cdot A_0$, where $A_0$ is the divisor corresponding to a point in the straight line $F_0$ different from $t_0$. This proves that $\Gamma + C_0 \cdot D_0$ is contained in the linear system $\mathrm{Tr}_{C_0} E^0$. Thus we get the following

THEOREM 1. *Let $V$ be an irreducible variety of dimension $\geqq 2$, without singular subvarieties of codimension* 1, *and let $\Sigma$ be an algebraic family of positive divisors on $V$ such that the generic member $C$ of $\Sigma$ is an irreducible variety and such that any subvariety of $C$ of codimension* 1 *is simple, not only on $C$, but also on $V$. Let $r$ be the dimension of the algebraic family $\Sigma$. Then the characteristic set $\mathfrak{C}$ of the algebraic family $\Sigma$ on its generic member is a linear system of dimension* $r - 1$.

Note the following corollary:

$$\mathfrak{C} \subset | \mathrm{Tr}_C | C | |.$$

## 2. The inequality $q \leqq h^{0,1}$

Let $V$ be a projective variety defined over $k$. We shall assume that $k$ is algebraically closed and that $V$ has no singular point. Let $\Sigma$ be a maximal family of positive divisors on $V$, and $W$ the set of Chow points of the members in $\Sigma$. The family $\Sigma$ is called a *total* family if $\Sigma$ satisfies the following condition: Let $X$ be an arbitrary divisor on $V$ which is algebraically equivalent to zero. Then there exists a divisor $X_1$ in $\Sigma$ such that $X \sim X_1 - X_0$, where $X_0$ is a fixed member of $\Sigma$ and $\sim$ denotes the linear equivalence of divisors [4]. In this case the associated variety $W$ is birationally equivalent to the product of the Picard variety $\wp$ of $V$ and the linear space whose dimension is equal to the dimension of the complete linear system $| C |$, where $C$ is a generic member of $\Sigma$ [5]. We shall denote as before by $E_m$ the section of $V$ with the hypersurface of order $m$. Then we have the

LEMMA 1. *Let $\Sigma$ be a total family defined over $k$, and let $C$ be a generic member of $\Sigma$ over $k$. Let $\Sigma_m$ be the maximal family containing $C + E_m$. Then $\Sigma_m$ is again a total family, and a generic member $C_m$ of $\Sigma_m$ over $k$ is linearly equivalent to a divisor of the form $C + E_m$.*

*Proof.* It is easy to see that $\Sigma_m$ is total. Let $C_0$, $E_0$ be fixed members of $\Sigma$ and $| E_m |$ respectively, rational over $k$. Then since $C_m$ is algebraically equivalent to $C_0 + E_0$, we can find a divisor $C$ in $\Sigma$ such that $C_m - C_0 - E_0 \sim$

$C - C_0$. Since $C_m$ is a generic member of $\Sigma_m$ over $k$ and $\Sigma_m$ is total, the class of $C_m - C_0 - E_0$ is a generic point of $\wp$ over $k$. Hence we can find such $C$ among the generic members of $\Sigma$ over $k$.

Let $\Sigma$ be a total family, and $C$ a generic member of $\Sigma$. Let $\mathfrak{L}(C + E_m)$ be the sheaf[5] of the germs of the rational functions on $V$ such that $(f) \succ - C - E_m$. Then it is known that $\dim H^1(V, \mathfrak{L}(C + E_m)) = 0$ for sufficiently large $m$. Now as an immediate consequence of Lemma 1, together with Lemma 2 of [5], we get the following

LEMMA 2. *On a nonsingular variety, there exists a total maximal family $\Sigma$ such that* (1) *a generic member $C$ of $\Sigma$ is a nonsingular variety,* (2) $\dim H^1(V, \mathfrak{L}(C)) = 0$.

We shall call a total maximal algebraic family $\Sigma$ satisfying the conditions of Lemma 2 a *typical family*.

Let $\Sigma$ be a typical family, and $C$ a generic member of $\Sigma$. We shall consider the exact sequence of sheaves

$$0 \to \mathfrak{O} \to \mathfrak{L}(C) \to {}_C\mathfrak{L}(C\cdot C) \to 0,$$

where $\mathfrak{O}$ is the sheaf of local rings on $V$ and ${}_C\mathfrak{L}(C\cdot C)$ denotes the sheaf on $C$ defined by the trace of $|C|$ on $C$. Then we have the exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^0(V, \mathfrak{O}) \to H^0(V, \mathfrak{L}(C)) &\to H^0(C, {}_C\mathfrak{L}(C\cdot C)) \\ &\to H^1(V, \mathfrak{O}) \quad \to 0, \end{aligned}$$

since $H^1(V, \mathfrak{L}(C)) = 0$. From this we get

$$\dim |\operatorname{Tr}_C |C|| = \dim |C| + h^{0,1} - 1,$$

where $h^{0,1} = \dim H^1(V, \mathfrak{O})$.

Now we shall return to the characteristic linear system $\mathfrak{C}$ of $\Sigma$ on $C$. Since $\mathfrak{C}$ is contained in $|\operatorname{Tr}_C |C||$ (see corollary to Theorem 1), we have $\dim \mathfrak{C} \leqq \dim |C| + h^{0,1} - 1$. On the other hand, $\dim \mathfrak{C} = \dim \Sigma - 1 = \dim |C| + q - 1$ (where $q$ stands for the dimension of the Picard variety of $V$), since $\Sigma$ is a total family. Combining these two we get the

THEOREM 2. *On the nonsingular projective variety we have the inequality*[6] $q \leqq h^{0,1}$. *The equality holds if the characteristic linear system of some typical family is complete, and only if the characteristic system of every typical family is complete.*

*Remark.* Theorem 2 does not hold in general when $V$ has some singular points, even though $V$ is normal. For instance let $V$ be a surface in a projective 3-space which is obtained by projecting a plane nonsingular cubic curve from a point. Then it is readily seen that $q = 1$, but $h^{0,1} = 0$.

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[5] Concerning the sheaf theory, the reader is referred to papers [8] and [13].

[6] I heard from J. Igusa that he already knew this inequality.

## 3. A property of the ample linear systems

Let $V^r$ be a nonsingular variety defined over $k$, belonging to the projective space $L^n$, i.e. $V$ is not contained in any hyperplane of $L$. Let $E$ be the linear system on $V$ which is composed of hyperplane sections of $V$; then the generic member $C$ of $E$ is also a nonsingular variety [6]. Let $L'$ be the dual space of $L$; we shall denote by $x, x', \cdots$ the points of $L'$. Then a point $x$ of $L'$ defines uniquely a hyperplane (which will be denoted by the same letter) $x$ in $L$, and we get a member $C_x$ of $E$ (the intersection cycle of $V$ and the hyperplane $x$). We shall denote by $\mathfrak{M}_d$ the set of points of $L'$ such that the hyperplane $x$ either has at least $d$ contacts[7] with $V$ (in this case the divisor $C_x$ has at least $d$ multiple points), or else is a specialization of such a hyperplane over $k$.

Let $P$ be a generic point of $V$ over $k$, and $T_P$ the tangent linear variety to $V$ at $P$. Then the Plücker coordinates $c(T_P)$ are rational over $k(P)$, and the point $c(T_P)$ has a locus $V^*$ over $k$. The variety $V^*$ is a subvariety of the Grassmann variety $\mathfrak{G}(r, n)$ which is the set of $r$-dimensional linear varieties in $L^n$. We shall call this variety $V^*$ the *dual variety* of $V$. If $\varphi$ is the rational map from $V$ onto $V^*$ defined over $k$ by $\varphi(P) = c(T_P)$, then $\varphi$ is defined everywhere on $V$, since $V$ is nonsingular.

We shall recall here that $\mathfrak{G}(r, n)$ is an irreducible variety, defined over the prime field $k_0$, and that it is a nonsingular variety[8] of dimension $(n - r)(r + 1)$.

LEMMA 3. *The point set $\mathfrak{M}_d$ is a bunch of subvarieties of $L'$, normally algebraic over $k$. Moreover if the dual variety $V^*$ of $V$ has the same dimension $r$ as $V$, then a component of $\mathfrak{M}_d$ has a dimension $\geqq n - d$ provided a generic member of that component has only a finite number of points of contact with $V$.*

*Proof.* Let $T$ be the correspondence between $L'$ and $\mathfrak{G}(r, n)$ such that for any point $x$ in $L'$, $T(x)$ is the set of $r$-dimensional linear varieties contained in the hyperplane $x$. Then $T(x)$ is also a Grassmann variety of dimension $(r + 1)(n - r - 1)$, and $T$ is an irreducible correspondence of dimension $n + (r + 1)(n - r - 1)$ defined over $k_0$. Let $\mathfrak{G}^{(d)} = \mathfrak{G}(r, n) \times \cdots \times \mathfrak{G}(r, n)$ and $T^{(d)}$ be the correspondence between $L'$ and $\mathfrak{G}^{(d)}$ such that

$$T^{(d)}(x) = T(x) \times \cdots \times T(x).$$

Then $T^{(d)}$ is an irreducible correspondence of dimension

$$n + (r + 1)(n - r - 1)\, d.$$

We shall now consider the intersection $(L' \times V^* \times \cdots \times V^*) \cap T^{(d)}$. Since $L' \times \mathfrak{G}^{(d)}$ is a nonsingular variety, the components $\mathfrak{L}_i$ $(i = 1, \cdots, s)$ of this

[7] We say that a hyperplane $x$ has $d$ contacts with $V$ if $x$ contains $d$ different tangent linear varieties to $V$. If the correspondence between $V$ and the dual variety $V^*$ is 1 to 1 everywhere, then $x$ has $d$ contacts with $V$ if and only if the divisor $C_x$ has $d$ multiple points.

[8] The nonsingularity of the Grassman variety follows from the fact that $\mathfrak{G}(r, n)$ is an algebraic homogeneous space.

intersection have dimensions $\geqq n - d$. Let $\mathrm{proj}_{L'}\, \mathfrak{L}_i = B_i$,[9] and select among the $B_i$'s those for which the hyperplane $x$, for the generic point $x$ of $B_i$, has at least $d$ contacts with $V$. Let $B_i$ $(i = 1, \cdots t)$ $(t < s)$ be such components. We shall show that $\bigcup_{i=1}^{t} B_i = \mathfrak{M}_d$. It is clear, by definition, that $\bigcup_{i=1}^{t} B_i$ is contained in $\mathfrak{M}_d$. Let $x$ be a point of $\mathfrak{M}_d$, and assume that $x$ has $d$ contacts $P_1^*, \cdots, P_d^*$ with $V$. Then the point $x \times P_1^* \times \cdots \times P_d^*$ is contained in $(L' \times V^* \times \cdots \times V^*) \cap T^{(d)}$, where $P_i^*$ is the Plücker coordinates of the tangent linear variety to $V$. Let $\mathfrak{L}$ be the component of this intersection containing the point $x \times P_1^* \times \cdots \times P_d^*$, and let

$$\bar{x} \times \bar{P}_1^* \times \cdots \times \bar{P}_d^*$$

be a generic point of $\mathfrak{L}$ over $\bar{k}$. Then

$$\bar{x} \times \bar{P}_1^* \times \cdots \times \bar{P}_d^* \rightarrow x \times P_1^* \times \cdots \times P_d^*$$

is a specialization over $\bar{k}$, and we see that $\bar{P}_i^* \neq \bar{P}_j^*$ for $i \neq j$, since $P_i^* \neq P_j^*$ for $i \neq j$. Hence the hyperplane $\bar{x}$ has at least $d$ contacts with $V$ and $\mathrm{proj}_{L'}\, \mathfrak{L} = B$ must be one of $B_i\,(i = 1, \cdots, t)$. This proves that $x \in \bigcup_{i=1}^{t} B_i$, or equivalently $\mathfrak{M}_d \subset \bigcup_{i=1}^{t} B_i$.

Let $x$ be a generic point of one of $B_i\,(1 \leqq i \leqq t)$ over $\bar{k}$. Since $\mathrm{proj}_{L'}\, \mathfrak{L}_i = B_i$, we can find the points $P_1^*, \cdots, P_d^*$ such that

$$x \times P_1^* \times \cdots \times P_d^*$$

is a generic point of $\mathfrak{L}_i$ over $\bar{k}$. Now assume that the hyperplane $x$ has only a finite number of points of contact with $V$; then $P_i^*$ are all algebraic over $k(x)$. Hence we have $\dim B_i = \dim \mathfrak{L}_i \geqq n - d$.

We see immediately that $\mathfrak{M}_d$ is a bunch of varieties normally algebraic over $k$. Thus the lemma is proved completely.

Let $E$ be an ample linear system on a nonsingular variety $V$; then the linear system $E$ defines a biregular birational transformation of $V$ into a variety $\bar{V}$, in which the linear system $E$ is transformed into the linear system of hyperplane sections of $\bar{V}$. Let $n = \dim E$, and let $\bar{L}$ be the ambient space of $\bar{V}$. Then $V$ is not contained in any hyperplane of $\bar{L}$. Let $\bar{V}^*$ be the dual variety of $\bar{V}$; we call this variety the dual variety of $V$ with respect to the linear system $E$. Then we have the

LEMMA 4. *The dual variety $V_E^*$ of $V$ with respect to the linear system $E$ has the same dimension as $V$ if $E$ contains a member $C$ such that $C$ has at least one but at most a finite number of multiple points.*

*Proof.* Without loss of generality we can assume that $E$ itself is the linear system of hyperplane sections of $V$. Assume that $\dim V^* < \dim V$. Let $C$ be a member of $E$, and $P$ a multiple point of $C$. Let $P^* = \varphi(P)$; then $\varphi^{-1}(P^*)$ has a component $\Pi$ of dimension $\geqq 1$. This means that any point of $\Pi$ is a multiple point of $C$. This proves the lemma.

[9] We mean by "proj" the geometric projection.

Combining these two lemmas we get the following

THEOREM 3.[10] *Let $E$ be an ample linear system of dimension $n$ on a nonsingular variety $V$, and let $k$ be a common field of definition for $V$ and $E$. Assume that the correspondence between $V$ and the dual variety of $V$ with respect to the linear system is everywhere* 1 *to* 1. *Let $L'$ be the parameter space of $E$, and let $\mathfrak{M}_d$ be the set of points of $L'$ corresponding to the members $C$ of $E$ such that either* (1) *$C$ has at least $d$ multiple points,*[11] *or else* (2) *$C$ is a specialization, over $k$, of a divisor having the property described in* (1). *Then $\mathfrak{M}_d$ is a bunch of varieities, normally algebraic over $k$. Moreover, if $B$ is a component of $\mathfrak{M}_d$ such that the generic member of $B$ has at most a finite number of multiple points, then* dim $B$ *is not less than* $n - d$.

## 4. The theory on the nonsingular surfaces

In this section we shall restrict ourselves to a nonsingular surface $V^2$ in a projective space. Let $\Sigma$ be a maximal algebraic family on $V$, and let $C$ be a generic member of $\Sigma$ over a field $k$ which is a common field of definition for $V$ and $\Sigma$. We shall assume that $C$ is a nonsingular variety. Let $|E|$ be the complete linear system on $V$ which is composed of the sections of $V$ with the hypersurfaces of such an order $m$ that the linear system $|E| - C$ is also ample. Let $D$ be a generic member of $|E| - C$ over $k(C)$; then $D$ is also a nonsingular curve. Let $\Delta$ be the set of all divisors which are the specializations of the divisor $C + D$ over $k$, and put $(C \cdot D) = d$. Let $W$ be the parameter variety of $\Delta$; then $W$ is an irreducible variety defined over $k$. When $m \geqq 2$, $|E|$ satisfies the assumption of Theorem 3, hence $W$ is contained in the bunch $\mathfrak{M}_d$, where $\mathfrak{M}_d$ is a bunch of varieties in the parameter space of the linear system $E$ defined as in Theorem 3. Since the divisor $C + D$ is a member of $E$ and since the number of multiple points in $C + D$ is $d$, we can apply Theorem 3 to this linear system $|E|$, and we see that the component of $\mathfrak{M}_d$ has dimension $\geqq n - d$ if a generic member of this component contains no multiple components.

LEMMA 5. *The parameter variety $W$ of $\Delta$ is a component of $\mathfrak{M}_d$, and* dim $W \geqq n - d$.

*Proof.* Let $B$ be a component of $\mathfrak{M}_d$, and assume that $B \supset W$, $B \neq W$. Let $x$ be a generic point of $B$ over $\bar{k}$, and $X$ the divisor corresponding to the point $x$. Then $X$ must be an irreducible curve. In fact if $X$ is reducible, it must be of the form $\bar{C} + \bar{D}$. Since $C + D$ is a specialization of $\bar{C} + \bar{D}$, one of them, say $\bar{C}$, must have $C$ as a specialization. Since $\Sigma$ is maximal, it follows that $\bar{C}$ is a member of $\Sigma$. Hence $\bar{D}$ is a member of $|E| - \bar{C}$, and $\bar{C} + \bar{D}$ is contained in $\Delta$. Now assume that $X$ is an irreducible curve; we shall de-

[10] The corresponding results to Theorem 3 were obtained elementarily by the joint effort of Akizuki and Matsumura just before I succeeded in proving this theorem (see [15]).

[11] Cf. footnote 7.

note by $p(X)$ and $\pi(X)$ the arithmetic genus and effective genus[12] of $X$, respectively. Since $X$ has at least $d$ multiple points, we must have $\pi(X) \leqq p(X) - d$ [2]. On the other hand, $p(X)$ is invariant by algebraic equivalence, whence

$$p(X) = p(C + D) = p(C) + p(D) + (C \cdot D) - 1 = p(C) + p(D) + d - 1.$$

Since $C$ and $D$ are nonsingular curves, we have $p(C) = \pi(C)$, $p(D) = \pi(D)$, and we have $\pi(X) \leqq \pi(C) + \pi(D) - 1$. On the other hand, $C + D$ is a specialization of $X$, hence we must have $\pi(X) \geqq \pi(C) + \pi(D)$ [2]. This is a contradiction, and the first half of the lemma is proved. Since the generic member of $W$ is $C + D$ and since it has no multiple component, the assertion on the dimension follows from Theorem 3, Q.E.D.

We shall denote, as usual, by $K$ the canonical divisor on $V$, and we shall introduce a numerical character for the divisor $Z$ on $V$ by $\sigma(Z) = -(K \cdot Z)$. Let $Z$ be a divisor on $V$ such that $\dim H^1(V, \mathfrak{L}(Z)) = 0$; then the theorem of Riemann-Roch on the surface yields

$$\dim | Z | = \tfrac{1}{2}[(Z^2) + \sigma(Z)] + p_a(V) - i(Z),$$

where $i(Z)$ is the speciality index and $p_a$ is the arithmetic genus of $V$ [12], [13].

THEOREM 4. *Let $C$ be a nonsingular curve on $V$, and $\Sigma$ the maximal algebraic family containing $C$. Then we have* $\dim \Sigma \geqq (C^2) - p(C) + 1$.

*Proof.* Let $E$, $D$, and $\Delta$ have the same meaning as before, and assume that $D$ is nonspecial and $\dim H^1(V, \mathfrak{L}(D)) = 0$. Then we have

$$\dim \Delta = \dim \Sigma + \dim | D | = \dim \Sigma + \tfrac{1}{2}[(D^2) + \sigma(D)] + p_a .$$

By Lemma 5 we see that this number is not less than

$$\dim | E | - (C \cdot D) = \tfrac{1}{2}[(C^2) + (D^2) + \sigma(C) + \sigma(D)] + p_a$$

since $E \sim C + D$. Hence we get the inequality

$$\dim \Sigma \geqq \tfrac{1}{2}[(C^2) + \sigma(C)].$$

Since $(K + C) \cdot C$ is the canonical divisor on the nonsingular curve $C$, we have $\sigma(C) = (C^2) - 2p(C) + 2$. Substituting this relation in the above inequality we get the relation $\dim \Sigma \geqq (C^2) - p(C) + 1$.

THEOREM 5. *Let $V$ be a nonsingular surface, and assume that the geometric genus $p_g$ of $V$ is 0. Let $\Sigma$ be a maximal algebraic family (not necessarily total) such that the generic curve $C$ of $\Sigma$ satisfies the conditions*: (1) *$C$ is a nonsingular curve*, (2) $\dim H^1(V, \mathfrak{L}(K - C)) = 0$. *Then the characteristic linear system $\mathfrak{C}$ of $\Sigma$ is complete. Hence we have* $q = h^{0,1}$.

*Proof.* Since $\dim H^1(V, \mathfrak{L}(K - C)) = 0$, we have $\dim | K \cdot C | =$

[12] The definition of the arithmetic genus of divisors on a normal variety will be found in [12]. The effective genus of an irreducible curve $X$ is the genus of the nonsingular $\bar{X}$ which is birationally equivalent to $X$.

$\dim | K | = -1$. This means that the characteristic linear system $| C \cdot C |$ on $C$ is nonspecial. Hence $\dim | C \cdot C | = C^2 - p(C)$. By Theorems 1 and 4, the dimension of the characteristic linear system $\mathfrak{C}$ attains this number, and $\mathfrak{C}$ is contained in $| C \cdot C |$. Hence $\mathfrak{C}$ must be complete, Q.E.D.

As an application of Theorem 4 we have a slightly more precise result than Theorem 5.

THEOREM 6. *On a nonsingular surface we have the inequality* $q + p_a \geqq 0$, *or* $p_g \geqq h^{0,1} - q \geqq 0$.

*Proof.* Since $V$ is nonsingular, there exists a typical family $\Sigma$. Let $C$ be a generic member of $\Sigma$; then we have

$$\dim \Sigma = q + \dim | C | = q + \tfrac{1}{2}[(C^2) + \sigma(C)] + p_a .$$

By the Theorem 4, $\dim \Sigma \geqq \frac{1}{2}[(C^2) + \sigma(C)]$, whence we get at once the relation $q + p_a \geqq 0$. The second inequality follows from the well known relation $h^{0,1} = p_g - p_a$ [12], [13] and the first one by applying Theorem 2.

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