AMENABLE SEMIGROUPS[1]

BY MAHLON M. DAY

1. Introduction

We begin with the definitions needed to formulate the results of this paper and then survey the known results on existence and behaviour of invariant means on semigroups. Then follow new results, among which are some criteria for existence of invariant means; these are found in §4. In §5 it is proved that an amenable semigroup is strongly amenable; this settles a question that first arose in an earlier paper [10]. In §8 this result from §5 is applied to improve other results of the paper [10] on the relationships between means and ergodicity. In §5 the semigroup algebra $l_1(\Sigma)$ is discussed; it is used as the principal tool in the proof of the result on strong amenability. In §6 is discussed the specialization to the semigroup algebra of a semigroup of an idea of Arens [1]; Arens has given a construction which makes an algebra out of the second conjugate space of a Banach algebra, and has constructed an example of a commutative algebra whose second-conjugate algebra is not commutative. We show in §6 that the semigroup algebra of the additive semigroup of positive integers has this pathological property; the proof depends on showing that if an abelian semigroup has at least two invariant means, then they cannot commute in the second-conjugate algebra.

§7 discusses this necessary condition for commutativity in more detail. The best result there is that an abelian group G has a *unique* invariant mean if and only if G is a finite group. For general torsion groups the question of uniqueness and existence of invariant means is dependent on whether Burnside's conjecture, that every finitely generated torsion group is finite, is true or not.

§9 contains the proof that a theorem of G. G. Lorentz [17], about the set where all invariant means are uniquely determined, carries over to amenable semigroups.

§10 introduces the concepts of amenable and introverted subspaces of $m(\Sigma)$ and shows how many of the preceding results have depended only on these properties of $m(\Sigma)$. In §11 these results are applied to the space $C(\Sigma)$ of bounded continuous functions on a topological semigroup.

2. Preliminary definitions

All of the present study will start from the relationships between a set Σ, which shall usually be a semigroup or group, and certain function spaces de-

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termined by Σ. These spaces are defined as follows; see Banach [2], pp. 11–12 for the case where Σ is countable.

$l_1(\Sigma)$ is the set of all those real-valued functions θ defined on Σ for which

$$\| \theta \| = \sum_{\sigma \in \Sigma} | \theta(\sigma) |$$

is finite.

$m(\Sigma)$ is the set of all bounded, real-valued functions x on Σ with norm

$$\| x \| = \operatorname{lub}_{\sigma \in \Sigma} | x(\sigma) |.$$

$l_1(\Sigma)$ and $m(\Sigma)$ are Banach spaces.

As in Banach [2], p. 188, each Banach space B has a conjugate space B^* consisting of all the linear, real-valued functions β on B; B^* is a Banach space under the norm

$$\| \beta \| = \operatorname{lub}_{\| b \| \leqq 1} | \beta(b) |.$$

We shall be interested in certain elements of $m(\Sigma)^*$, but first we remark that the proof of isometry of $m(\Sigma)$ with $l_1(\Sigma)^*$, given in Banach [2], p. 67, for countable Σ, is valid in general; specifically:

For each x in $m(\Sigma)$ there is a $\xi = Tx$ defined for all θ in $l_1(\Sigma)$ by

$$\xi(\theta) = \sum_{\sigma \in \Sigma} x(\sigma)\theta(\sigma)$$

such that

(a) for each x in $m(\Sigma)$, Tx is in $l_1(\Sigma)^*$,

(b) T is linear; that is, additive, homogeneous, and continuous,

(c) for each x, $\| Tx \| = \| x \|$; that is,

$$\operatorname{lub}_{\| \theta \| \leqq 1} | Tx(\theta) | = \operatorname{lub}_{\sigma \in \Sigma} | x(\sigma) |,$$

(c) T carries all of $m(\Sigma)$ onto all of $l_1(\Sigma)^*$.

As in Banach [2], p. 100, each linear operator U from one Banach space B to another such space B' determines a conjugate or adjoint operator U^* from B'^* to B^* by means of the formula:

For each β' in B'^*, $U^*\beta'$ is that element of B^* for which

$$(U^*\beta')(b) = \beta'(Ub) \qquad \text{for all } b \text{ in } B.$$

Banach shows that U^* is also a linear operator and that $\| U^* \| = \| U \|$. In the special case of the isometric operator T from $m(\Sigma)$ onto $l_1(\Sigma)^*$, the adjoint operator T^* is also an isometry from $l_1(\Sigma)^{**}$ onto all of $m(\Sigma)^*$.

The weak topology of a Banach space B is defined, for example, in Hille [15], p. 23, by means of neighborhoods; for our purposes it will often be convenient to think of it in terms of convergence; for discussion of general, or Moore-Smith, convergence see G. Birkhoff [3], Tukey [22], and Kelley [16]; we shall use the terminology of Kelley [16].

Definition 1. If $\{b_n\}$, where n runs over a directed system $\mathfrak{N}$, is a net of elements in a Banach space B, then $\{b_n\}$ *converges weakly to* b (in symbols,

w-$\lim_n b_n = b$) means that $\lim_n \beta(b_n) = \beta(b)$ for every β in B^*. A dual notion, weak* convergence, can be defined for nets of elements of a conjugate space: If $\{\beta_n\}$ is a net of elements of B^*, then $\{\beta_n\}$ *converges weakly* to* β (in symbols, w^*-$\lim_n \beta_n = \beta$) means that $\lim_n \beta_n(b) = \beta(b)$ for every b in B.

The most important property of the w^*-topology is

(A) Spheres in B^* are w^*-compact.

Expressing compactness in terms of nets this says:

(A′) If $\{\beta_n\}$ is a net of elements in a sphere in B^*, then there exists a subnet $\{\beta'_m\}$ of $\{\beta_n\}$ such that $\{\beta'_m\}$ is w^*-convergent to some element of the sphere.

See Kelley [16], page 242, for a proof.

The following result is well-known (see Bourbaki [6], page 103) and is easy to prove directly from the definitions.

(B) If U is a linear operator from one Banach space B to another B', then from the norm continuity of U follows also the weak-to-weak continuity; that is, if w-$\lim_n b_n = b$, then w-$\lim Ub_n = Ub$; and also the w^*-w^* continuity of U^*; that is, if w^*-$\lim_n \beta'_n = \beta'$, then w^*-$\lim_n U^*\beta'_n = U^*\beta'$.

Banach [2], page 189, also shows that there is a natural way to embed a given B into its second conjugate space B^{**}. To apply this to the case that most interests us here, for each θ in $l_1(\Sigma)$, let $Q'\theta$ be that element of $l_1(\Sigma)^{**}$ defined by

$$Q'\theta(\beta) = \beta(\theta) \qquad \text{for all } \beta \text{ in } l_1(\Sigma)^*.$$

Banach observed that in general this operator Q' is a linear isometry of B into B^{**}; in this particular example where $B = l_1(\Sigma)$, the range of Q' *does not* fill up the space B^{**}. However we do have a general density theorem.

(C) $Q'(B)$ is dense in the w^*-topology in B^{**}; even better, if S is the unit sphere in B, then $Q'(S)$ is w^*-dense in the unit sphere of B^{**}.

For one proof, see Day [8].

In our particular spaces, define $Q = T^*Q'$; then Q is an isometry of $l_1(\Sigma)$ into $m(\Sigma)^*$. From the w^*-w^* continuity of T^* in both directions follows:

(D) The image $Q(S)$ is w^*-dense in the unit sphere of $m(\Sigma)^*$.

In most of the rest of this paper we shall identify each x in $m(\Sigma)$ with its image Tx in $l_1(\Sigma)^*$, and use the symbol x for either one. Similarly we shall identify an element μ in $l_1(\Sigma)^{**}$ with $T^*\mu$ in $m(\Sigma)^*$, and identify Q with Q'.

Following Banach's terminology [2], page 23, we shall call an operator U from one Banach space to another linear if U is additive and continuous. Then, see Banach [2], page 54, the number

$$\| U \| = \mathrm{lub}_{\|x\| \leqq 1} \| Ux \|$$

is finite. Under this norm the whole set $\mathfrak{L}$ of linear operators from B to B' becomes a Banach space. In $\mathfrak{L}$ there are two analogues of the weak* topology in B^*; these are defined in terms of neighborhoods in Hille [15], page 33; we describe them here in terms of convergence of nets.

$\{U_n\}$ has the strong limit U in $\mathfrak{L}$ (in symbols, $s\text{-}\lim_n U_n = U$) means that $\lim_n \| U_n x - U_x \| = 0$ for every choice of x in B.

$\{U_n\}$ has the weak limit U in $\mathfrak{L}$ (in symbols, $w\text{-}\lim_n U_n = U$) means that $\lim_n [\beta'(U_n b) - \beta'(Ub)] = 0$ for every choice of β' in B'^* and b in B.

In the special case in which $B' = B$, the set $\mathfrak{L}(B)$ of linear operators from B into B has still more structure; it becomes an algebra if we define multiplication in $\mathfrak{L}(B)$ as follows: For each S and T in $\mathfrak{L}(B)$, ST is that element of $\mathfrak{L}(B)$ for which

$$ST(b) = S(Tb) \qquad \text{for all } b \text{ in } B.$$

It is easily seen (Hille [15], page 33) that this multiplication is continuous in the norm topology; in fact,

$$\| ST \| \leqq \| S \| \| T \|,$$

so $\mathfrak{L}(B)$ is a Banach algebra.

Three other elementary processes will be useful in several later sections.

(1) Let Σ and Σ' be sets, and let f be a function carrying Σ onto all of Σ'. This determines a linear operator, which we call F, from $m(\Sigma')$ into $m(\Sigma)$: For each x' in $m(\Sigma')$, Fx' is that element of $m(\Sigma)$ such that

$$(Fx')(\sigma) = x'(f\sigma) \qquad \text{for every } \sigma \text{ in } \Sigma.$$

It can be checked that F is a linear operator carrying $m(\Sigma')$ isometrically into $m(\Sigma)$. Hence F^* is a linear operator of norm one carrying $m(\Sigma)^*$ onto $m(\Sigma')^*$.

(2) Let Σ' be a subset of Σ; then there is a natural mapping Π of $m(\Sigma)$ onto $m(\Sigma')$ in which for each x in $m(\Sigma)$, Πx is that function on Σ' which agrees with x on Σ';

$$(\Pi x)(\sigma') = x(\sigma') \qquad \text{for all } \sigma' \text{ in } \Sigma'.$$

Then it can be verified that Π is a linear operator of norm one and that Π^* is an isometry of $m(\Sigma')^*$ into $\mathrm{m}(\Sigma)^*$.

(3) If σ is an element of Σ, it determines an element $I\sigma$ of $l_1(\Sigma)$ by the formula

$$(I\sigma)(\sigma') = \begin{cases} 1 & \text{if} \quad \sigma' = \sigma, \\ 0 & \text{if} \quad \sigma' \neq \sigma. \end{cases}$$

We shall often inject Σ in this way into $l_1(\Sigma)$ and identify the image $I\sigma$ with σ and use the same label for both. This simplifies the notation much more than it adds to the confusion.

3. Means on $m(\Sigma)$

In the common usage of sophomore calculus, a mean value, or average value, of a function is a number chosen in some reasonable fashion between the least upper bound and greatest lower bound of the function. Here we ask that the choice be made simultaneously for all functions in $m(\Sigma)$ and made in a linear way.

DEFINITION 1. A mean μ on $m(\Sigma)$ is an element of $m(\Sigma)^*$ such that for each x in $m(\Sigma)$

$$\mathrm{glb}_{\sigma\in\Sigma}\, x(\sigma) \leqq \mu(x) \leqq \mathrm{lub}_{\sigma\in\Sigma}\, x(\sigma).$$

(A) Each mean μ on $m(\Sigma)$ has the following properties:

(a) μ is in the unit sphere in $m(\Sigma)^*$.

(b) If e is the function whose value is 1 at every point of Σ, then $\mu(e) = 1$.

(c) If $x(\sigma) \geqq 0$ for all σ in Σ, then $\mu(x) \geqq 0$.

(a′) $\| \mu \| = 1$.

(B) If an element μ of $m(\Sigma)^*$ satisfies (a) and (b), or if μ satisfies any two of the conditions (a′), (b), and (c) of (A), then μ is a mean on $m(\Sigma)$.

A useful corollary of this is

(C) The set of means on $m(\Sigma)$ is nonempty, convex, and w^*-compact.

DEFINITION 2. An element θ of $l_1(\Sigma)$ is called a *countable mean on* Σ if $\theta(\sigma) \geqq 0$ for all σ in Σ and if $\sum_{\sigma\in\Sigma} \theta(\sigma) = 1$. A countable mean φ is called a *finite mean on* Σ if, in addition, the set $\{\sigma \mid \varphi(\sigma) > 0\}$ is a finite set.

Clearly the set of finite means is norm-dense in the set of countable means. This nomenclature is a slight abuse of language, since the image, $Q\theta$ or $Q\varphi$, should, perhaps, more properly be called the countable or finite mean. See Day [10] for the next result.

(D) If Φ is the set of finite means on Σ, then $Q\Phi$ is w^*-dense in the set of means on $m(\Sigma)$.

Consider next the operations between sets which were introduced in §2 and their effect on means.

LEMMA 1. *If f maps Σ onto Σ', then F^* maps M, the set of means on $m(\Sigma)$, onto M', the set of means on $m(\Sigma')$.*

If μ is in M and $\mu' = F^*\mu$, then $\| \mu' \| \leqq \| F^* \| \, \| \mu \| = 1$. Also $F(e') = e$, so $\mu'(e') = (F^*\mu)(e') = \mu(Fe') = \mu(e) = 1$. By (B), μ' is a mean, so

$$F^*M \subseteqq M'.$$

If μ' is a mean on $m(\Sigma')$, let $m_0 = \{Fx' \mid x' \in m(\Sigma')\}$, and let μ_0 be defined on m_0 by $\mu_0(x_0) = \mu'(F^{-1}x_0)$ for each x_0 in m_0. Then μ_0 is a linear functional on m_0 of norm one; by the Hahn-Banach theorem (Banach [2], page 27) μ_0 has at least one extension μ of norm one. Also

$$\mu(e) = \mu_0(e) = \mu'(e') = 1;$$

by (B), μ is a mean on $m(\Sigma)$. But

$$(F^*\mu)(x') = \mu(Fx') = \mu_0(Fx') = \mu'(F^{-1}Fx') = \mu'(x')$$

for all x' in $m(\Sigma')$; hence $F^*\mu = \mu'$, and F^* maps M *onto* M'.

LEMMA 2. *If Σ' is a subset of Σ, then $\Pi^*M' \subseteqq M$.*

Π^* is a linear operator from $m(\Sigma')^*$ into $m(\Sigma)^*$ which preserves norm, so for each μ' in $m(\Sigma')^*$, $\Pi^*\mu'$ is of norm one. But $\Pi^*\mu'(e) = \mu'(\Pi e) = \mu'(e') = 1$; by (B), $\Pi^*\mu'$ is a mean on $m(\Sigma)$.

4. Semigroups. Invariance of means

A *semigroup* Σ is a set in which an associative, binary operation is defined; we shall generally write it by putting the elements to be combined next to each other with no further symbols. Precisely,

(a) If σ and σ' are in Σ, then $\sigma\sigma'$ is an element of Σ.

(b) If σ, σ', and σ'' are elements of Σ, then $\sigma(\sigma'\sigma'') = (\sigma\sigma')\sigma''$.

In addition to groups some examples of semigroups are:

(1) The set of integers, or the subset of positive integers, under ordinary addition as the rule of operation.

(2) The set of n-by-n matrices, under matrix multiplication.

(3) The multiplicative semigroup in the operator algebra $\mathfrak{L}(B)$ over any Banach space B.

(4) Any set Σ in which the product of two elements is defined to be the second element of the pair; $\sigma\sigma' = \sigma'$ for all σ, σ' in Σ.

Example (4) seems a most artificial and uncommutative semigroup, but examples arise, as we shall show in §6, even when the only original intent is to study the semigroup of integers and its invariant means.

If Σ is a semigroup, then in $m(\Sigma)$ many new operations become possible; for example, it is possible to embed Σ homomorphically into $\mathfrak{L}(m(\Sigma))$ by the following device.

For each σ in Σ let r_σ be that element of $\mathfrak{L}(m(\Sigma))$ defined for each x in $m(\Sigma)$ by

$$(r_\sigma x)(\sigma') = x(\sigma'\sigma) \qquad \text{for all } \sigma' \text{ in } \Sigma.$$

Similarly we define

$$(l_\sigma x)(\sigma') = x(\sigma\sigma') \qquad \text{for all } \sigma' \text{ in } \Sigma.$$

It is easily verified that the correspondence of σ with r_σ is a homomorphism of Σ into $\mathfrak{L}(m(\Sigma))$, and that the correspondence of σ to l_σ is an antihomomorphism, that is, it reverses the order of factors:

$$r_{\sigma\sigma'} = r_\sigma r_{\sigma'} \qquad \text{and} \qquad l_{\sigma\sigma'} = l_{\sigma'} l_\sigma .$$

Also $\| r_\sigma x \| \leqq \| x \|$, and $r_\sigma e = e$, so $\| r_\sigma \| = 1$ for each σ; similarly, $\| l_\sigma \| = 1$ for each σ. It should be noted that these maps may not be isomorphic; for example, in the semigroup of example (4) every l_σ is the identity operator.

Definition 1. An element μ of $m(\Sigma)^*$ is called *left* [*right*] *invariant* if $\mu(l_\sigma x) = \mu(x)$ [$\mu(r_\sigma x) = \mu(x)$] for all x in $m(\Sigma)$ and σ in Σ.

This can easily be rephrased in terms of adjoint operations in the algebra $\mathfrak{L}(m(\Sigma)^*)$.

μ is left [right] invariant if and only if for every σ in Σ

$$l_\sigma^* \mu = \mu \qquad [r_\sigma^* \mu = \mu].$$

DEFINITION 2. A semigroup Σ is called *amenable* if there is a mean μ on $m(\Sigma)$ which is both left and right invariant. In case only a left [right] invariant mean exists, Σ is called l-[r-] amenable.

We give in this section the many properties of invariant means which had been announced with or without proofs before this paper and give references to at least one source for each. These are listed in order with capital letters to label them; the results called lemmas and theorems later in the section are new.

The first two properties simplify many calculations.

(A) *If Σ is both l- and r-amenable, then it is amenable.*

This was proved for groups by Day [10]. To prove it for semigroups is easiest after §6 of this paper; if λ and ρ are, respectively, left and right invariant, it will be shown in §6, Corollary 2, that $\lambda \odot \rho$ is both left and right invariant.

(B) *An l-[r-] amenable group is also r-[l-] amenable; and therefore is amenable.*

This also was proved in Day [10]; basically it depends on the fact that the operation $g \to g^{-1}$ transposes the order of products, and therefore interchanges left and right.

One of the earliest studies of invariant means is that of von Neumann [18]. The groups which he calls *measurable* can be seen to be those which are called l-amenable here; (A) and (B) show that this class coincides with the class of amenable groups, which shows that many of the results in Day [10] are consequences of results in von Neumann [18].

An example, (4) at the beginning of this section, shows that nothing like (B) is true for semigroups in general. In that semigroup,

$$(l_\sigma x)(\sigma') = x(\sigma\sigma') = x(\sigma')$$

so every l_σ is the identity and every mean is left invariant. (Means always exist.) But $(r_\sigma x)(\sigma') = x(\sigma'\sigma) = x(\sigma)$ for all σ', so $r_\sigma x = x(\sigma)e$, and if μ is right invariant, then $\mu(r_\sigma x) = \mu(x) = x(\sigma)\mu(e)$ for all σ and x. Therefore x is a constant function for all x in $m(\Sigma)$, or else $\mu(e) = 0$ and $\mu(x) = 0$ for all x in $m(\Sigma)$. But if Σ has more than one element in it, then $m(\Sigma)$ has non-constant functions in it, so a semigroup of the type in example (4) has no right invariant linear functionals on it unless it has but one element.

Next come techniques for creating new amenable semigroups from given ones.

(C) *If Σ is an (l-)[r-] amenable*[2] *semigroup and f a homomorphism of Σ onto Σ', then Σ' is (l-)[r-] amenable.*

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[2] In (B) two possibilities, left or right, are considered. In (C) and in §5, three choices left or right or both, are possible, the same choice to be used all the way through the sentence.

One proves that $\mu' = F^*\mu$ is left invariant on $m(\Sigma')$ if μ is left invariant over Σ, and similarly for right invariant means. See Day, [12] for groups.

(D) *If G is a (l-)[r-] amenable group, so is every subgroup.*

See Day [12]. Also this has been published recently by Følner [14].

The proof will be given in connection with a stronger result in §7, Theorem 2. It has not been published before.

This result too may fail for semigroups. As an example let Σ' be any non-amenable semigroup, and let Σ contain Σ' and one new element 0 such that $0\sigma' = \sigma'0 = 00 = 0$, and Σ' is a subsemigroup of Σ. Σ has an invariant mean: $\mu(x) = x(0)$. The subsemigroup Σ' has not.

(E) *Let H be a normal subgroup of a group G such that H and G/H are amenable; then G is amenable.*

See von Neumann [18] for left amenable; (B) and (A) complete the proof (see Day [10]).

(F) *Suppose that* $\{\Sigma_n\}$ *is a set of amenable subsemigroups of a semigroup* Σ *such that* (a) *for each m, n there exists p with* $\Sigma_p \supseteqq \Sigma_m \cup \Sigma_n$, *and* (b) $\Sigma = \bigcup_n \Sigma_n$. *Then Σ is amenable.*

von Neumann [18] has this for a well-ordered system of subgroups of a group. In the present generality it is in Day [10].

To be sure these methods of construction have some value, we need examples. We know already a non-amenable semigroup but we need also

(G) *A free group on 2 generators is not amenable.*

This can be gotten from von Neumann [18]; it is also in Day [10]. Used with (D) it asserts that

(G′) *A free group on 2 or more generators is not amenable. No amenable group has a free subgroup on more than one generator.*

We have two basic families of amenable semigroups.

(H) *Every abelian semigroup is amenable.*

For groups this is in von Neumann [18]; for semigroups in Day [9].

(I) *Every finite group is amenable.*

More precisely, for later use note that there is exactly one invariant mean (left or right) on a finite group; if G has n elements, then

$$\mu(x) = n^{-1}\textstyle\sum_{g \in G} x(g) \qquad \text{for all } x \text{ in } m(G)$$

is that mean.

A finite semigroup need not have any invariant mean. If Σ is a finite semigroup in which $\sigma\sigma' = \sigma'$, Σ is not amenable if it has more than one element.

These known results have many corollaries, some of them not printed before.

(J) *A solvable group is amenable.*

See von Neumann [18]. This follows from (H) and (E) by induction. The same technique yields

(J′) *If the chain of commutator subgroups of G ends at the identity in finitely many steps, then G is amenable.*

See von Neumann [18]. Day [10] has a generalization to semigroups. The example of the descending chain of commutator subgroups of the free group on two generators, which ends at the identity only after countably many steps, shows that some restriction on the chain is pertinent. In terms of the notions of direct limit and inverse limit of groups, (F) can be used to prove that

(F′) *A direct limit of amenable groups is amenable.*

Consider as an application an index set S, a family Σ_s, $s \in S$, of amenable semigroups, and the full and weak direct products: $\prod_{s \in S} \Sigma_s$ is the set of all functions f defined on S such that $f(s) \in \Sigma_s$, and the product operation is defined coordinatewise; in case each Σ_s has an identity i_s, the weak direct product $\prod_{s \in S}^{w} \Sigma_s$ is the subsemigroup of those elements f of $\prod_{s \in S} \Sigma_s$ such that the set $\{s : f(s) \neq i_s\}$ is a finite set. *For S finite* $\prod = \prod^{w}$ and, by (E), $\prod_{s \in S} \Sigma_s$ is amenable; hence (F) implies for every S

(F″) *If all Σ_s are amenable, then $\prod_{s \in S}^{w} \Sigma_s$ is amenable.*

The full direct product of amenable groups need not be amenable. For example, let G be the free group on two generators and let $\{G_n\}$ be the upper commutator chain for G; that is, $G_0 = G$ and $G_{i+1} = [G, G_i]$, the normal subgroup on those commutators $fgf^{-1}g^{-1}$ with f in G and g in G_i. Then G_i/G_{i+1} is abelian, so, by (J′), G/G_n is amenable for all n in N, the set of positive integers. Consider $H = \prod_{n \in N} (G/G_n)$; the inverse limiting system of groups G/G_n under the homomorphisms onto; $U_{mn}(gG_m) = gG_n$ if $m \geqq n$ and $g \in G$, is a subgroup of H; hence H contains a subgroup isomorphic to this limit. This particular inverse limit group is isomorphic to G, because $\bigcap_{n \in N} G_n = \{1\}$. By (G′) *inverse limits and full direct products of amenable groups need not be amenable.*

(K) *A group G is amenable if and only if every finitely generated subgroup of G is amenable.*

Sufficiency comes from (F), necessity from (D). For a semigroup we have only sufficiency. For groups we have another result.

(K′) *Every locally finite group is amenable.*

(*Locally finite* means that every finite subset of G generates a finite subgroup of G) (Day [10].)

R. Baer calls a group G "supersolvable" if every nontrivial homomorphic image of G has a nontrivial, abelian, normal subgroup; we shall use the term *Baer group* for a group such that every nontrivial homomorphic image of G has a nontrivial normal, amenable subgroup.

THEOREM 1. *Every Baer group is amenable.*

This depends on

LEMMA 1. *Every group G contains a normal, amenable subgroup G_1 which contains all other normal, amenable subgroups of G.*

Let $\{H\}$ be the family of normal, amenable subgroups of G. The family is closed under the process of taking unions of increasing simply ordered subsets, so Zorn's lemma (see Kelley [16], page 33) applies to give a normal, abelian G_1 not included in any other H in $\{H\}$. If, now, $H \in \{H\}$ and H is not a subset of G_1, let $G' =$ smallest subgroup of G spanned by G_1 and H; then G_1 is normal in G' and G'/G_1 is isomorphic to $H/G_1 \cap H$. Hence G_1 and G'/G_1 are amenable. By (E), G' is amenable.

But if g' is a word in G' and $g \in G$, $gg'g^{-1}$ is a word in G' too, since G_1 and H are both normal. Hence G' is a normal, amenable subgroup of G which contains G_1; this contradiction shows that $H \subseteqq G_1$.

To prove Theorem 1 we suppose that G is a Baer group and that $G_1 \subset G$. Then G/G_1 contains a normal, amenable subgroup $A \neq \{1\}$; also G', the inverse image of A, is an extension of G_1 by A. Because A and G_1 are amenable, (E) asserts that G' is amenable. G' is also normal. This contradicts Lemma 1.

Note again how the free group furnishes an example to prevent the assumption that a group must have a largest amenable subgroup. If a and b are the generators of a free group G, then the infinite cyclic subgroups on these generators are both amenable. But G is the only subgroup of G containing both a and b, and G is not amenable.

Since not every subsemigroup of an amenable semigroup is amenable, the following partial results add some information.

THEOREM 2. *Let Γ be a semigroup with a left invariant mean μ. Suppose that Σ is a subsemigroup of Γ such that $\mu(\chi) > 0$, where χ is the characteristic function of Σ. Then Σ is left amenable.*

Proof. Let λ_σ denote the left translation operator in $m(\Sigma)$, and for each γ in Γ define $\bar{l}_\gamma$ from $m(\Gamma)$ into $m(\Gamma)$ by

$$\bar{l}_\gamma x = (l_\gamma x)\chi \qquad \text{for all } x \text{ in } m(\Gamma).$$

Define T from $m(\Sigma)$ into $m(\Gamma)$ by

$$(Tx)(\gamma) = \begin{cases} x(\gamma) & \text{if} \quad \gamma \in \Sigma, \\ 0 & \text{if} \quad \gamma \notin \Sigma, \end{cases}$$

and let $\nu = T^*\mu/\mu(\chi)$. Then ν is a mean on $m(\Sigma)$. It can easily be checked that for each σ in Σ and x in $m(\Sigma)$

(a) $$T(\lambda_\sigma x) = \bar{l}_\sigma (Tx).$$

Now let us fix a σ in Σ. Let $v = l_\sigma \chi - \bar{l}_\sigma \chi$, so $v(\gamma) = \chi(\sigma\gamma) - \chi(\sigma\gamma)\chi(\gamma)$. This shows that $v(\gamma)$ can either be 0 or 1 and takes no other value, and, therefore, that v is the characteristic function of a set E. It is clear that

(b) $$E = \{\gamma \mid \gamma \in \Gamma, \quad \sigma\gamma \in \Sigma, \quad \gamma \notin \Sigma\}.$$

Let us take any γ in Γ and consider the sequence $\{\sigma^i\gamma \mid i \geqq 0\}$. If possible, suppose that there exists $0 \leqq k < j$ such that $\sigma^k\gamma$ and $\sigma^j\gamma$ both are in E. It follows from (b) and $\sigma^k\gamma \in E$ that $\sigma^j\gamma \in \Sigma$, which again with (b) shows that $\sigma^j\gamma \notin E$, and this is a contradiction. Thus, either no $\sigma^i\gamma$ belongs to E, or there is exactly one j such that $\sigma^j\gamma \in E$. Now let $n > 0$ be an integer, and let us consider

$$w_n = \sum\nolimits_{0 \leqq i \leqq n} l_{\sigma^i} v.$$

Then for each γ in Γ

$$w_n(\gamma) = \sum\nolimits_{0 \leqq i \leqq n} v(\sigma^i\gamma) = 0 \text{ or } 1$$

by our previous considerations; therefore,

$$\| w_n \| \leqq 1,$$

thus

$$(n + 1)\mu(v) = \mu(w_n) \leqq 1.$$

As this is true for every n, $\mu(v) = 0$. Now if we take any x in $m(\Sigma)$ such that $\| x \| \leqq 1$, then we can easily check that

$$-v \leqq l_\sigma (Tx) - \bar{l}_\sigma(Tx) \leqq v,$$

and therefore

$$\mu(l_\sigma Tx) - \mu(\bar{l}_\sigma Tx) = 0,$$

or

$$\mu(\bar{l}_\sigma Tx) = \mu(l_\sigma Tx) \qquad \text{for } \| x \| \leqq 1,$$

and by homogeneity it follows that

$$\mu(\bar{l}_\sigma Tx) = \mu(l_\sigma Tx) = \mu(Tx)$$

for all x in $m(\Sigma)$. From (a) this can be written as

$$\mu[T(\lambda_\sigma x)] = \mu[Tx],$$

and this is the same as

$$\nu(\lambda_\sigma x) = \nu(x).$$

This shows that ν is left invariant and we already know that it is a mean on $m(\Sigma)$, so Σ is left amenable.

The amenability, or lack of it, of finite semigroups was settled in the thesis of Rosen [19]. (See also [20].)

THEOREM 3. *A finite semigroup Σ has an invariant mean if and only if it has just one minimal left ideal, and just one minimal right ideal; then these ideals coincide, and the resulting two-sided ideal, the kernel of Σ, is a finite group G. The unique invariant mean on $m(\Sigma)$ is that of G; if n = number of elements of G, then*

$$\mu(x) = n^{-1}\textstyle\sum_{g\epsilon G} x(g).$$

Rosen [20] also discusses left and right amenability of finite semigroups and the corresponding results in compact semigroups.

The lemmas above show that the family of amenable *groups* is closed under four standard processes of constructing groups from given groups: (a) subgroup, (b) factor group, (c) group extension, (d) expanding unions (or direct limits).

Also finite groups and abelian groups are amenable. Let EG, for elementary groups, be the smallest family of groups containing all finite groups and all abelian groups, and closed under the processes (a)–(d). Then $EG \subseteqq AG$, the family of amenable groups. Also define NF to be the family of groups with no free subgroup on two generators. It is easily seen that NF is also closed under the processes (a)–(d). It follows from (a) and (G) that $AG \subseteqq NF$.

It is not known whether $EG = AG$ or $AG = NF$ or both.

5. Finite means and the semigroup algebra of a semigroup

In §3 we noted that the set of finite means is w^*-dense in the set of means. It is also useful to recall from Day [10] how this combines with invariance.

DEFINITION 1. Say that a net $\{\mu_n\}$ of means is w^*- [*norm-*] *convergent to right invariance* if

$$w^*\text{-}\lim_n [r_\sigma^* \mu_n - \mu_n] = 0 \quad [\lim_n \| r_\sigma^* \mu_n - \mu_n \| = 0].$$

The dual definitions can be made for left invariance.

(A) If $\{\mu_n\}$ is any net of means which is w^*-convergent to μ, then $\{\mu_n\}$ is w^*-convergent to right [left] invariance if and only if μ is right [left] invariant.

By w^*-continuity of r_σ^*

$$\lim_n (r_\sigma^* \mu_n(x) - \mu_n(x)) = (r_\sigma^* \mu(x) - \mu(x)).$$

(B) If $\{\mu_n\}$ is a net of means w^*-convergent to right [left] invariance, then every w^*-cluster point of $\{\mu_n\}$ is a right [left] invariant mean on $m(\Sigma)$.

Every w^*-cluster point μ of $\{\mu_n\}$ is the limit of a subnet $\{\nu_m\}$ or $\{\mu_n\}$.

For every σ in Σ and x in $m(\Sigma)$, $r_\sigma^* \nu_m(x) - \nu_m(x)$ is a subnet of $r_\sigma^* \mu_n(x) - \mu_n(x)$; hence it also tends to zero. By (A), μ is right invariant.

In Day [10] it was proved that

(C) A semigroup Σ is (l-)[r-] amenable if and only if there exists a net $\{\varphi_n\}$ of finite means such that the net $\{Q\varphi_n\}$ is w^*-convergent to (l-)[r-] invariance.

This follows from (A) and (B) of the present section and (C) and (D) of §3.

Observe that $\{Q\theta_n\}$ is w^*-convergent to zero in $m(\Sigma)^*$ if and only if $\{\theta_n\}$ is w-convergent to 0 in $l_1(\Sigma)$; hence we can convert this to

(C′) A semigroup is (l-)[r-] amenable if and only if there exists a net $\{\varphi_n\}$ of finite means such that $\{\varphi_n\}$ converges weakly to (l-)[r-] invariance.

A condition formally stronger than amenability was used in Day [10] and was named, for groups, in an abstract of that period, Day [11].

Definition 2. Σ is called (r-)[l-] *strongly amenable* if there exists a net $\Phi = \{\varphi_n\}$ of finite means convergent in norm to (right) [left] invariance; that is such that for each σ in Σ,

$$(\lim_n \| r_\sigma^* Q\varphi_n - Q\varphi_n \| = 0) \quad [\lim_n \| l_\sigma^* Q\varphi_n - Q\varphi_n \| = 0].$$

The notation was so unwieldy that while many properties of amenable groups could be shown to have analogues for strongly amenable groups, it was not then possible to decide whether every amenable group is strongly amenable, nor was it convenient to discuss strong amenability of semigroups. This can be handled by changing the problem to one stated in $l_1(\Sigma)$, and this in turn requires a discussion of a multiplication operation which makes a Banach algebra out of $l_1(\Sigma)$. This definition of multiplication is a familiar one in the classical case where Σ is a finite semigroup; see, for example, van der Waerden [23], page 49.

Definition 3. For each choice of θ_1 and θ_2 in $l_1(\Sigma)$, define $\theta_1 \theta_2$ by the formula

$$(\theta_1 \theta_2)(\sigma) = \sum_{\sigma_1\sigma_2=\sigma} \theta_1(\sigma_1)\theta_2(\sigma_2).$$

If each element σ in Σ is identified with the vector $I\sigma$ in $l_1(\Sigma)$ (see definition in §2, (3)), then it is easy to check that for all σ_i in Σ

$$I(\sigma_1 \sigma_2) = (I\sigma_1)(I\sigma_2).$$

Hence if we drop the I, it will cause no confusion in the multiplication in Σ, since I is an isomorphism of Σ into $l_1(\Sigma)$. Hereafter we shall use the symbol σ both for σ in Σ and $I\sigma$ in $l_1(\Sigma)$. This gives the formula

$$\theta = \sum_{\sigma\in\Sigma} \theta(\sigma)\sigma$$

for every θ in $l_1(\Sigma)$.

Then this multiplication in $l_1(\Sigma)$ also determines right and left translation operations $\theta\sigma$ and $\sigma\theta$ in $l_1(\Sigma)$;

$$(\sigma\theta)(\sigma') = \sum_{\sigma\sigma_2=\sigma'} \theta(\sigma_2) \quad \text{and} \quad (\theta\sigma)(\sigma') = \sum_{\sigma_1\sigma=\sigma'} \theta(\sigma_1).$$

Direct calculations with the definitions prove

(D) $r_\sigma^*(Q\theta) = Q(\theta\sigma)$ and $l_\sigma^*(Q\theta) = Q(\sigma\theta)$ for all σ in Σ and θ in $l_1(\Sigma)$.

Therefore the elements $r_\sigma^* Q\varphi_n - Q\varphi_n$ and $l_\sigma^* Q\varphi_n - Q\varphi_n$ which were discussed in the definitions of weak and strong amenability are images under Q of the elements $\varphi_n \sigma - \varphi_n$ and $\sigma\varphi_n - \varphi_n$. Under the mapping Q, norms are preserved and weak convergence to zero in $l_1(\Sigma)$ is equivalent to weak* convergence to zero of the images in $m(\Sigma)^*$. This proves the following reformulation of the preceding amenability conditions.

LEMMA 1. *A semigroup Σ is amenable (strongly amenable) if and only if there exists a net $\Phi = \{\varphi_n\}$ of finite means such that for every σ in Σ*

$$\lim_n (\varphi_n \sigma - \varphi_n) = 0 = \lim_n (\sigma\varphi_n - \varphi_n)$$

in the weak (norm) topology of $l_1(\Sigma)$.

This displays clearly that strong amenability is not less of a restriction on a semigroup than is amenability. The purpose of this section is to prove these two conditions equivalent, but we now turn aside from the main stream of that proof to give some information about the semigroup algebra which will be needed.

LEMMA 2. *Suppose that θ_1 and θ_2 are in $l_1(\Sigma)$, or are countable means, or are finite means; then the same property is possessed by $\theta_1 \theta_2$. Hence $\sigma\theta$ and $\theta\sigma$ have for each σ in Σ the same of these properties as has θ. Also multiplication in $l_1(\Sigma)$ is associative, so $(\sigma\theta)\sigma' = \sigma(\theta\sigma')$. Finally, $\| \theta_1 \theta_2 \| \leqq \| \theta_1 \| \| \theta_2 \|$ and*

$$\theta_1 \theta_2 = \sum_{\sigma \epsilon \Sigma} \theta_1(\sigma)\sigma\theta_2 = \sum_{\sigma\epsilon\Sigma} \theta_2(\sigma)\theta_1 \sigma.$$

Proof.

$$\begin{aligned} \| \theta_1 \theta_2 \| &= \sum_{\sigma\epsilon\Sigma} | \theta_1 \theta_2(\sigma)| = \sum_\sigma \Big|\sum_{\sigma_1\sigma_2=\sigma} \theta_1(\sigma_1)\theta_2(\sigma_2)\Big| \\ &\leqq \sum_\sigma \sum_{\sigma_1\sigma_2=\sigma} | \theta_1(\sigma_1)| \, | \theta_2(\sigma_2)| \\ &= \sum_{\sigma_1\epsilon\Sigma} | \theta_1(\sigma_1)| \sum_{\sigma_2\epsilon\Sigma} | \theta_2(\sigma_2)| \\ &= \| \theta_1 \| \| \theta_2 \| . \end{aligned}$$

Hence $\theta_1 \theta_2$ is an element of $l_1(\Sigma)$ if the θ_i are, and $l_1(\Sigma)$ is a Banach algebra, possibly without unit. When the numbers $\theta_i(\sigma)$ are all nonnegative, then the only possible proper inequality in the above chain is prevented from occurring, and then $\| \theta_1 \theta_2 \| = \| \theta_1 \| \| \theta_2 \|$; in particular, $\theta_1 \theta_2$ is a countable mean if the θ_i are countable or finite means. When the θ_i are finite means, $\theta_1 \theta_2(\sigma) = 0$ except in the finite set

$$\{\sigma_1 \sigma_2 \mid \theta_1(\sigma_1) > 0 \quad\text{and}\quad \theta_2(\sigma_2) > 0\}.$$

COROLLARY 1. *The set of countable means and the set of finite means are subsemigroups in the multiplicative semigroup of the Banach algebra $l_1(\Sigma)$.*

This technical matter settled, we turn now to averages and nets of averages of elements of a linear space.

DEFINITION 4. If L is a linear topological space and E is a subset of L, then an element x is called *an average of* E if it is in the closure of the convex hull of E; x is a *finite average of* E if it is in the convex hull of E; that is, if there exists a real-valued function φ on E such that $\varphi(y) \geqq 0$ if $y \in E$, $\varphi(y) = 0$ except for a finite set of y in E, $\sum_{y \in E} \varphi(y) = 1$, and $x = \sum_{y \in E} \varphi(y)y$.

For example, the finite means in $l_1(\Sigma)$ are the finite averages of the set Σ of basic means, and the countable means are the averages of Σ. Using the w^*-topology of the space $m(\Sigma)^*$ the set of all means is the set of averages of $Q(\Sigma)$.

DEFINITION 5. If $D = \{d_n\}$ is a net of elements of a locally convex linear topological space, say that $C = \{c_i\}$ is *a net of finite averages far out in* D if (a) each c_i is a finite average, $c_i = \sum_{n \in N} \varphi_i(n)\, d_n$, of values of the function D, and (b) for each n_0 in N, there is an i_0 in I such that, for each $i \geqq i_0$ and each n such that $\varphi_i(n) > 0$, it follows that $n \geqq n_0$.

LEMMA 3. *Let L be a locally convex linear topological space, and let $D = \{d_n\}$ be a net of elements of L converging to an element z of L. If C is a net of finite averages of elements far out in D, then C also converges to the limit z.*

Take a convex neighborhood U of z, and take n_U so that $d_n \in U$ if $n \geqq n_U$. Choose i_U by (b) of Definition 5 so that $i \geqq i_U$ and $\varphi_i(n) > 0$ imply $n \geqq n_U$. Then all d_n for which $\varphi_i(n) > 0$ are in U, so, by convexity of U, $c_i = \sum_{n \in N} \varphi_i(n)\, d_n \in U$ when $i \geqq i_U$; that is, $\lim_i c_i = z$.

We need also a result well-known for sequences to be a consequence of Mazur's theorem. (See Mazur [24]; see Bourgin [7] for the general case.)

LEMMA 4. *Let L be a locally convex linear topological space, and let $D = \{d_n\}$ be a net of elements weakly convergent to an element z. Then there is a net C of finite averages of elements far out in D such that C converges to z in the topology originally given in L.*

Let K_n = closed (in L) convex hull of $\{d_n \mid m \geqq n\}$. Then by the Ascoli-Mazur-Bourgin theorem (Bourbaki's "geometric form of the Hahn-Banach theorem", [5], page 69), each K_n is weakly closed; hence z is in each K_n. Let $\mathfrak{I}$ be the Cartesian product of the directed system $\mathfrak{N}$ with the directed system of neighborhoods of z, ordered by $\subseteqq$; that is, $(U, n) \geqq (V, m)$ means that $U \subseteqq V$ and $n \geqq m$. Then for each $i = (U, n)$ there is a c_i in U which is a finite average of the d_m, $m \geqq n$, because z is in K_n, the closure in L of this set of finite averages. Then $C = \{c_i\}$ has the desired properties.

With this machinery we are prepared to prove the main theorem of this section.

THEOREM 1. *A semigroup Σ is amenable if and only if it is strongly amenable.*

The characterizations of Lemma 1 show that a strongly amenable semigroup is amenable. If, on the other hand, Σ is amenable, Lemma 1 asserts the existence of a net $\Phi = \{\varphi_n\}$ of finite means such that in the weak topology of $l_1(\Sigma)$ we have for each σ of Σ

$$\lim_n (\varphi_n \sigma - \varphi_n) = 0 = \lim_n (\sigma\varphi_n - \varphi_n).$$

Let δ be any finite subset of Σ, and enumerate the elements of δ in some order as $\sigma_1, \sigma_2, \cdots, \sigma_k$. Then $\sigma_1 \varphi_n - \varphi_n$ tends to zero weakly in $l_1(\Sigma)$; by Lemma 4 there is a net Φ_1 of finite averages of elements far out in Φ such that $\lim_m \| \sigma_1 \varphi_{1m} - \varphi_{1m} \| = 0$. By Lemma 3 the weak limit of $\sigma_j \varphi_{1m} - \varphi_{1m}$ is zero for $j = 2, \cdots, k$; hence there is a subnet $\Phi_2 = \{\varphi_{2p}\}$ of Φ_1 such that $\lim_p \| \sigma_j \varphi_{2p} - \varphi_{2p} \| = 0$ for $j = 1, 2$, while $\sigma_j \varphi_{2p} - \varphi_{2p}$ still tends weakly to zero for $j = 3, \cdots, k$. Continuing by induction there exists a subnet $\Phi_k = \{\varphi_{kq}\}$ such that

$$\lim_q \| \sigma_j \varphi_{kq} - \varphi_{kq} \| = 0 \qquad \text{for } 1 \leqq j \leqq k.$$

If Σ is finite, this net will do to show one side of strong amenability if $\delta = \Sigma$. If Σ is infinite, let $\mathfrak{I}$ be the cartesian product of $\mathfrak{N}$, the directed system of integers, with Δ, where Δ is the net of all finite subsets of Σ ordered by $\supseteqq$, so $(n, \delta) \geqq (n', \delta')$ means $n \geqq n'$ and $\delta \supseteqq \delta'$. Then for each $i = (n, \delta)$ let $\psi(i) = \psi(n, \delta)$ be so chosen that

(1) $\psi(n, \delta)$ is a finite average of elements φ_m, $m \geqq n$, and

(2) for each element σ of δ

$$\| \sigma\psi(n, \delta) - \psi(n, \delta)\| < 1/(\text{number of elements in } \delta).$$

Such an element $\psi(n, \delta)$ can be chosen from the net Φ_k associated to δ by the construction of the preceding paragraph, for each φ_{kq} is a finite average of finite means φ_m, and is therefore a finite mean itself, and, once δ is chosen and n given as well, φ_{kq} for q large enough uses only elements φ_m with $m \geqq n$ and can be taken as close to zero in norm as may be desired.

This net $\Psi = \{\psi_i\}$ is a net of finite averages of elements far out in Φ, and $\lim_i \| \sigma\psi_i - \psi_i \| = 0$ for each σ in Σ. By Lemma 3 the weak limit of $\psi_i \sigma - \psi_i$ still is zero for each σ in Σ. Hence the argument just used will yield a net Ψ' which is norm convergent to right invariance as well as to left invariance. This proves the theorem by displaying a net with the characteristic property which Lemma 1 says is equivalent to strong amenability.

It is worthy of note that there is truly something that needed proof in this theorem. It is well-known (Banach, page 137, gives the case where Σ is countable, but the proof does not depend on that property of Σ) that *for sequences* in $l_1(\Sigma)$ weak convergence to an element is equivalent to strong convergence to the same element. But this is a theorem for sequences; for nets in general the facts that (a) weak and norm topologies are distinct in

$l_1(\Sigma)$ if Σ is not finite, and (b) these topologies can both be determined by convergence of nets, show that a net $\{\theta_n\}$ might converge weakly to zero while at the same time it need not converge to zero in norm.

A recent theorem of Følner [14] gives two new characteristic properties of amenable *groups*.

THEOREM OF FØLNER. *Amenability of a group G is equivalent to each of the following conditions:*

(a) *For each number k such that $0 \leqq k < 1$ and each finite subset γ of G, there is a finite subset E of G such that for each g in γ.*

$$(\textit{no. of elements common to } E \textit{ and to } gE)/(\textit{no. of elements in } E) > k.$$

(b) *There is a number k_0, $0 < k_0 < 1$, such that for each choice of finitely many, not necessarily distinct, elements $g_1, g_2, \cdots, g_n \in G$ there is a finite set $E \leqq G$ such that*

$$n^{-1}\textstyle\sum_{i \leqq n} (\textit{no. of elements common to } E \textit{ and } g_i E) \geqq k_0(\textit{no. of elements in } E).$$

For groups this yields another proof that left amenability is equivalent to strong amenability. For a given finite subset γ of G and a given $\varepsilon > 0$, take E by Følner's condition (a) with $k = 1 - \varepsilon$; then set $\varphi_{\gamma,\varepsilon}(g) = 1/|E|$ if $g \in E$, $= 0$ if $g \notin E$. This net converges in norm to left invariance.

It is not now clear whether Følner's condition can be derived from strong amenability in general. A related question is: How much tampering can a net of means strongly convergent to invariance take before it loses its desirable property. In this vein we have two results

LEMMA 5. *If $\{\varphi_n\}$ is a net of finite means which is weak [norm] convergent to right $\{left\}$ invariance, then for each θ in $l_1(\Sigma)$ such that $e(\theta) = 1$,*

$$\lim_n (\varphi_n \theta - \varphi_n) = 0 \quad \{\lim_n (\theta\varphi_n - \varphi_n) = 0\}$$

in the weak [norm] topology in $l_1(\Sigma)$.

For one typical case of the proof assume that $w\text{-}\lim_n (\varphi_n \sigma - \varphi_n) = 0$ for each σ in Σ. Then for each finite mean ψ we have

$$\varphi_n \psi - \varphi_n = \textstyle\sum_\sigma \psi(\sigma)\varphi_n \sigma - \varphi_n = \sum_\sigma \psi(\sigma)(\varphi_n \sigma - \varphi_n);$$

therefore $\varphi_n \psi - \varphi_n$ tends weakly to zero. But each mean θ in $l_1(\Sigma)$ can be approximated arbitrarily closely in norm by a finite mean ψ, and for all φ_n in Φ we have $\| \varphi_n \psi - \varphi_n \theta \| \leqq \| \psi - \theta \|$. For each x in $m(\Sigma)$ and each $\varepsilon > 0$, take $\| \psi - \theta \| < \varepsilon$, and then take φ_n so that $| x(\varphi_n \psi - \varphi_n)| < \varepsilon$. Then

$$| x(\varphi_n \theta - \varphi_n)| \leqq | x(\varphi_n \theta - \varphi_n \psi)| + | x(\varphi_n \psi - \varphi_n)| < (1 + \| x \|)\varepsilon;$$

hence $\{\varphi_n \theta - \varphi_n\}$ tends weakly to zero. Similar proofs yield the corresponding results for $\{\theta\varphi_n - \varphi_n\}$ and for norm convergence.

LEMMA 6. *If Σ is a semigroup and $\Phi = \{\varphi_n\}$ is a net of finite means con-*

verging in norm to right invariance, then each of the following nets converges in norm to right invariance:

(1) *The net* $\{\psi_n \varphi_n\}$, *where* $\{\psi_n\}$ *is any other net of finite means defined on the same directed system* $\mathfrak{N}$.

(2) *Any right multiple of* $\{\varphi_n\}$ *by a single countable mean,* $\{\varphi_n \theta\}$.

(3) *Any net of finite (or countable) averages of elements far out in* $\{\varphi_n\}$.

Dual results hold for left invariance.

Proof of (1).

$$\| \psi_n \varphi_n \sigma - \psi_n \varphi_n \| \leqq \| \psi_n \| \| \varphi_n \sigma - \varphi_n \| \to 0.$$

Proof of (2).

$$\| \varphi_n \theta\sigma - \varphi_n \theta \| \leqq \| \varphi_n \theta\sigma - \varphi_n \| + \| \varphi_n \theta - \varphi_n \| .$$

Both terms tend to zero by Lemma 5.

Proof of (3). If ψ_m is an average, finite or countable, of elements $\{\varphi_n\}$, then $\psi_m \sigma - \psi_m$ is the same average of $\{\varphi_n \sigma - \varphi_n\}$. Hence $\{\psi_m \sigma - \psi_m\}$ is a net of averages of elements far out in $\{\varphi_n \sigma - \varphi_n\}$ and, by Lemma 3, has the same limit, zero, as the latter net.

A related result is

LEMMA 7. *Under the hypotheses of Lemma* 6, *let* $K(n, \theta)$ *be the closure of* $\{\gamma\varphi_n \theta - \gamma\varphi_n \mid \gamma$ *a mean*$\}$; *then for each* θ *the diameter of* $K(n, \theta)$ *tends to zero.*

Proof.

$$\| \gamma\varphi_n \theta - \gamma\varphi_n \| \leqq \| \varphi_n \theta - \varphi_n \|$$

which tends to zero by (2) of the preceding lemma.

6. The second-conjugate algebra of a semigroup algebra

This section contains an application of an idea of Arens to semigroup algebras, and in turn applies what we now know about invariant means to construct examples of interest for Arens's own work.

Arens [1] showed how to define a multiplication in the second-conjugate space of a Banach algebra B. The process works in three steps:

For each β in B^* and b in B, define $\beta \odot b$ in B^* by

$$(\beta \odot b)(b') = \beta(bb') \qquad \text{for all } b' \text{ in } B.$$

For each ν in B^{**} and β in B^*, define $\nu \odot \beta$ in B^* by

$$(\nu \odot \beta)(b) = \nu(\beta \odot b) \qquad \text{for all } b \text{ in } B.$$

For each μ in B^{**} and ν in B^{**}, define $\mu \odot \nu$ in B^{**} by

$$(\mu \odot \nu)(\beta) = \mu(\nu \odot \beta) \qquad \text{for all } \beta \text{ in } B^*.$$

If for B we choose $l_1(\Sigma)$, where Σ is a semigroup, if we make the identifications of $l_1(\Sigma)^*$ with $m(\Sigma)$ and of $l_1(\Sigma)^{**}$ with $m(\Sigma)^*$, then for x in $m(\Sigma)$

and θ in $l_1(\Sigma)$, the first definition gives for all θ' in $l_1(\Sigma)$,

$$(x \odot \theta)(\theta') = x(\theta\theta') = x(\textstyle\sum_{\sigma\epsilon\Sigma} \theta(\sigma)\sigma\theta') = \sum_{\sigma\epsilon\Sigma} \theta(\sigma)x(\sigma\theta')$$
$$= \textstyle\sum_{\sigma\epsilon\Sigma} \theta(\sigma)[(x \odot \sigma)(\theta')] = [\sum_{\sigma\epsilon\Sigma} \theta(\sigma)x \odot \sigma](\theta').$$

This shows that for each x and θ,

$$x \odot \theta = \textstyle\sum_{\sigma\epsilon\Sigma} \theta(\sigma)x \odot \sigma.$$

But

$$(x \odot \sigma)(\theta') = x(\sigma\theta') = \textstyle\sum_\tau x(\tau)\sigma\theta'(\tau)$$
$$= \textstyle\sum_\tau x(\tau)\sum_{\sigma\sigma'=\tau} \theta'(\sigma')$$
$$= \textstyle\sum_{\sigma'} x(\sigma\sigma')\theta'(\sigma') = (l_\sigma x)(\theta').$$

so

$$x \odot \sigma = l_\sigma x.$$

Hence

$$(\nu \odot x)(\sigma) = \nu(x \odot \sigma) = \nu(l_\sigma x) = l_\sigma^* \nu(x).$$

We take this as our basic definition, now that we have checked that it agrees with Arens's definition; that is, we rewrite the definitions for our case as:

$$x \odot \sigma = l_\sigma x.$$
$$(\nu \odot x)(\sigma) = \nu(x \odot \sigma) = \nu(l_\sigma x) = (l_\sigma^* \nu)(x).$$
$$(\mu \odot \nu)(x) = \mu(\nu \odot x).$$

We add two new definitions

$$l_\theta = \textstyle\sum_{\sigma\epsilon\Sigma} \theta(\sigma)l_\sigma ; \qquad r_\theta = \sum_{\sigma\epsilon\Sigma} \theta(\sigma)r_\sigma .$$

We are now ready to describe the properties of this multiplication in $m(\Sigma)^*$ and to show how invariant means appear.

LEMMA 1. (Arens) $\odot$ *is associative and distributive*; *also, the norm of the product is not greater than the product of the norms.*

For each x

$$[\lambda \odot (\mu \odot \nu)](x) = \lambda[(\mu \odot \nu) \odot x],$$

and

$$[(\mu \odot \nu) \odot x](\sigma) = (\mu \odot \nu)(x \odot \sigma) = \mu[\nu \odot (x \odot \sigma)] = \mu[\nu \odot l_\sigma x]$$

for all x and σ. Also

$$[(\lambda \odot \mu) \odot \nu](x) = (\lambda \odot \mu)(\nu \odot x) = \lambda[\mu \odot (\nu \odot x)],$$

and

$$[\mu \odot (\nu \odot x)](\sigma) = \mu[l_\sigma(\nu \odot x)]$$

for all x and σ. But for all σ', x, and σ

$$[l_\sigma(\nu \odot x)](\sigma') = (\nu \odot x)(\sigma\sigma') = \nu(l_{\sigma\sigma'} x) = \nu(l_{\sigma'} l_\sigma x) = (\nu \odot l_\sigma x)(\sigma').$$

Hence for all σ and x

$$l_\sigma(\nu \odot x) = \nu \odot (l_\sigma x).$$

Hence the last expressions in the second and fourth equations of this proof are equal for all x and σ; hence the last expressions in the first and third equations are equal for all x. Hence $\odot$ is associative.

For distributivity we check first that $(\mu + \nu) \odot x = \mu \odot x + \nu \odot x$ for all μ, ν in $m(\Sigma)^*$ and x in $m(\Sigma)$; this is true because for all σ in Σ

$$\begin{aligned}[(\mu + \nu) \odot x](\sigma) &= (\mu + \nu)(x \odot \sigma) = \mu(x \odot \sigma) + \nu(x \odot \sigma) \\ &= (\mu \odot x)(\sigma) + (\nu \odot x)(\sigma) = [\mu \odot x + \nu \odot x](\sigma).\end{aligned}$$

Then for all x

$$\begin{aligned}[\lambda \odot (\mu + \nu)](x) &= \lambda[(\mu + \nu) \odot x] = \lambda[\mu \odot x + \nu \odot x] \\ &= \lambda(\mu \odot x) + \lambda(\nu \odot x) \\ &= (\lambda \odot \mu)(x) + (\lambda \odot \nu)(x) \\ &= [\lambda \odot \mu + \lambda \odot \nu](x).\end{aligned}$$

To prove the boundedness, if μ, $\nu \in m(\Sigma)^*$ and $x \in m(\Sigma)$, then

$$|(\mu \odot \nu)(x)| = | \mu(\nu \odot x)| \leqq \| \mu \| \| \nu \odot x \| ,$$

and for each σ

$$| (\nu \odot x)(\sigma) | = | \nu(l_\sigma x) | \leqq \| \nu \| \| l_\sigma x \| \leqq \| \nu \| \| x \|.$$

Hence if $\| x \| \leqq 1$, then $\| \nu \odot x \| \leqq \| \nu \|$, so

$$\| \mu \odot \nu \| \leqq \| \mu \| \| \nu \|.$$

LEMMA 2. *If $\theta \in l_1(\Sigma)$ and $\nu \in m(\Sigma)^*$, then $Q\theta \odot \nu = l_\theta^* \nu$, and $\nu \odot Q\theta = r_\theta^* \nu$.*

For each x in $m(\Sigma)$

$$\begin{aligned}(l_\theta^* \nu)(x) &= \nu(l_\theta x) = \nu\left(\textstyle\sum_\sigma \theta(\sigma) l_\sigma x\right) = \textstyle\sum_\sigma \theta(\sigma)\nu(l_\sigma x) \\ &= \textstyle\sum_\sigma \theta(\sigma)\nu(x \odot \sigma) = \textstyle\sum_\sigma \theta(\sigma)[(\nu \odot x)(\sigma)] \\ &= (Q\theta)(\nu \odot x) = (Q\theta \odot \nu)(x).\end{aligned}$$

For the other conclusion, start with ν in $m(\Sigma)^*$ and θ in $l_1(\Sigma)$; then

$$\nu \odot Q\theta = \textstyle\sum_\sigma \theta(\sigma)\nu \odot Q\sigma.$$

Then for each σ in Σ and x in $m(\Sigma)$

$$(\nu \odot Q\sigma)(x) = \nu(Q\sigma \odot x),$$

and for each τ in Σ

$$\begin{aligned}(Q\sigma \odot x)(\tau) &= (Q\sigma)(x \odot \tau) = (Q\sigma)(l_\tau x) = (l_\tau x)(\sigma) \\ &= x(\tau\sigma) = (r_\sigma x)(\tau).\end{aligned}$$

Hence for each x

$$Q\sigma \odot x = r_\sigma x,$$

and

$$(\nu \odot Q\sigma)(x) = \nu(Q\sigma \odot x) = \nu(r_\sigma x) = (r_\sigma^* \nu)(x);$$

hence

$$\nu \odot Q\sigma = r_\sigma^* \nu,$$

and

$$\begin{aligned}(\nu \odot Q\theta) &= \textstyle\sum_\sigma \theta(\sigma) r_\sigma^* \nu = \left(\sum_\sigma \theta(\sigma) r_\sigma^*\right)(\nu) \\ &= \left(\textstyle\sum_\sigma \theta(\sigma) r_\sigma\right)^*(\nu) = r_\theta^* \nu.\end{aligned}$$

COROLLARY 1. *If θ is fixed, then $Q\theta \odot \nu$ is w^*-w^* continuous in the second variable, and $\mu \odot Q\theta$ is w^*-w^* continuous in the first variable.*

Proof. Every adjoint operation is w^*-w^* continuous by §2, (B).

COROLLARY 2. *If ν and μ are means, then $\nu \odot \mu$ is left invariant if ν is left invariant, and $\nu \odot \mu$ is right invariant if μ is right invariant.*

For one of these proofs we have from Lemmas 1 and 2

$$\begin{aligned}l_\sigma^*(\nu \odot \mu) &= Q\sigma \odot (\nu \odot \mu) = (Q\sigma \odot \nu) \odot \mu \\ &= (l_\sigma^* \nu) \odot \mu = \nu \odot \mu.\end{aligned}$$

COROLLARY 3. *Q is an isomorphism of the algebra $l_1(\Sigma)$ into $m(\Sigma)^*$; that is, $Q\theta_1 \odot Q\theta_2 = Q(\theta_1 \theta_2)$.*

We already know that Q is isometric into $m(\Sigma)^*$ and is linear. For all x in $m(\Sigma)$

$$\begin{aligned}(Q\theta_1 \odot Q\theta_2)(x) &= (l_{\theta_1}^* Q\theta_2)(x) = (Q\theta_2)(l_{\theta_1} x) = \textstyle\sum_{\sigma_2 \in \Sigma} \theta_2(\sigma_2) l_{\theta_1} x(\sigma_2) \\ &= \textstyle\sum_{\sigma_2 \in \Sigma} \theta_2(\sigma_2)[\sum_{\sigma_1 \in \Sigma} \theta_1(\sigma_1) l_{\sigma_1} x(\sigma_2)] \\ &= \textstyle\sum_{\sigma_1} \sum_{\sigma_2} \theta_2(\sigma_2)\theta_1(\sigma_1) x(\sigma_1 \sigma_2) = \sum_\sigma \sum_{\sigma_1\sigma_2 = \sigma} \theta_1(\sigma_1)\theta_2(\sigma_2) x(\sigma) \\ &= \textstyle\sum_\sigma \theta_1 \theta_2(\sigma) x(\sigma) = [Q(\theta_1 \theta_2)](x).\end{aligned}$$

LEMMA 3. *The operation $\odot$ is w^*-w^* continuous in the first variable if the second variable is fixed.*

Let $\{\mu_n\}$ be a net of elements of $m(\Sigma)^*$ such that $\lim_n \mu_n(y) = \mu(y)$ for all y in $m(\Sigma)$; then

$$(\mu_n \odot \nu)(x) = \mu_n(\nu \odot x) \to \mu(\nu \odot x) = (\mu \odot \nu)(x)$$

for all x in $m(\Sigma)$; that is,

$$w^*\text{-}\lim_n (\mu_n \odot \nu) = \mu \odot \nu \qquad \text{if } w^*\text{-}\lim_n \mu_n = \mu.$$

Continuity in the other variable may not be present; see Corollary 5 of Theorem 1 of this section or Arens [1]. This limited weak*-continuity in the second variable found in Corollary 1, is due to the asymmetry of our definition of multiplication; r_σ and l_σ do not enter into it together.

We have now gathered together the elementary properties of the algebra $m(\Sigma)^*$, and we know that Q is an isometric isomorphism of the algebra $l_1(\Sigma)$ into the algebra $m(\Sigma)^*$. To connect what we know about amenable semigroups with Arens's results, we prove

THEOREM 1. *If ν is an element of $m(\Sigma)^*$ which is fixed under all the operators l_σ^*, then for every μ in $m(\Sigma)^*$*

$$\mu \odot \nu = \mu(e)\nu,$$

where e is the function constantly one on Σ.

For all $\theta \in l_1(\Sigma)$

$$\begin{aligned} Q\theta \odot \nu &= l_\theta^* \nu = \textstyle\sum_\sigma \theta(\sigma) l_\sigma^* \nu \\ &= \left(\textstyle\sum_\sigma \theta(\sigma)\right)\nu = (Q\theta(e))\nu. \end{aligned}$$

But for each μ in $m(\Sigma)^*$, there is a net $\{\theta_n\}$ in $l_1(\Sigma)$ such that $w^*\text{-}\lim_n Q\theta_n = \mu$. By Lemma 3

$$\begin{aligned} \mu \odot \nu &= w^*\text{-}\lim_n (Q\theta_n \odot \nu) = w^*\text{-}\lim_n (Q\theta_n(e)\nu) \\ &= [\lim_n Q\theta_n(e)]\nu = \mu(e)\nu. \end{aligned}$$

COROLLARY 4. *If ν is a left invariant mean, then $\mu \odot \nu = \nu$ for every mean μ.*

$\mu(e) = 1$ if μ is a mean.

COROLLARY 5. *If Σ' is the set of all left invariant means on $m(\Sigma)$, then Σ' is a semigroup in which the product of two elements is always the second element of the pair.*

COROLLARY 6. *If Σ is a left amenable semigroup and if $m(\Sigma)^*$ is a commutative Banach algebra when $\odot$ is used as the multiplication, then there is only one left invariant mean on $m(\Sigma)$.*

For Σ' can be both commutative and nonempty if and only if it has just one element; if there were two, Corollary 5 would assert that $\mu \odot \nu = \nu \neq \mu = \nu \odot \mu$, which would prevent commutativity.

An example pertinent to Arens's work is

COROLLARY 7. *Let Σ be the semigroup of nonnegative integers; then $l_1(\Sigma)$ is, but its second-conjugate algebra is not, commutative.*

The sequence of finite means $\{\varphi_n\}$ defined by

$$\begin{aligned} \varphi_n(i) &= 1/n \quad \text{if} \quad 1 \leqq i \leqq n, \\ \varphi_n(i) &= 0 \qquad \text{if} \quad i > n, \end{aligned}$$

converges in norm to invariance, for

$$\| \varphi_n k - \varphi_n \| = 2k/n.$$

By taking a function x in $m(\Sigma)$ so that there are longer and longer blocks of consecutive zeros and ones, it is easy to construct a sequence such that

$$\limsup_n Q\varphi_n(x) = 1 > 0 = \liminf_n Q\varphi_n(x).$$

By §5, (B), every w^*-cluster point of this sequence $\{Q\varphi_n\}$ is an invariant mean. By w^*-compactness of the set of means, there is at least one w^*-convergent subnet $\{\phi'_m\}$ of $\{\varphi_n\}$ with $\lim_m \varphi'_m(x) = 1$, and at least one w^*-convergent subnet $\{\varphi''_m\}$ with $\lim_m \varphi''_m(x) = 0$. Then $\mu = w^*\text{-}\lim_m \varphi'_m$ and $\mu' = w^*\text{-}\lim_m \varphi''_m$ are distinct invariant means. By Corollary 6, $m(\Sigma)^*$ is not commutative.

Due to the asymmetry of our definition of multiplication in $m(\Sigma)^*$, the dual conditions for right invariant means may fail, and indeed do fail for most abelian groups. For example, in the situation of Corollary 7, where Σ is the semigroup of nonnegative integers, $l_\sigma = r_\sigma$ for all σ, but the product of two right invariant means is not necessarily the first one, while it must be the second one.

7. Uniqueness of invariant means on $m(\Sigma)$

From the preceding section we saw that a necessary condition for commutativity of $m(\Sigma)^*$ is that there exist only one invariant mean on $m(\Sigma)$. We devote this section to a proof that the behaviour of the semigroup of integers is typical of the behaviour of abelian groups; on every infinite abelian group there are many invariant means.

We begin with the observation that by §5, (B) we want a net of means on $m(\Sigma)$ which is w^*-convergent to invariance but is not actually w^*-convergent.

LEMMA 1. *If $\{\mu_n\}$ is a net of means on $m(\Sigma)$ which is not w^*-convergent but is w^*-convergent to right [left] invariance, then there is more than one right [left] invariant mean on $m(\Sigma)$.*

By w^*-compactness of the set of means (§3, (C)) $\{\mu_n\}$ must have at least one cluster point; since it is not w^*-convergent it has more than one cluster point, so it has subnets converging to at least two different points; these subnets both converge to the same-sided invariance as that possessed by $\{\mu_n\}$, so the limits, by §5, (B) are invariant on that side.

To reduce the class of semigroups requiring special investigation we give two simplifying results.

THEOREM 1. (I. S. Luthar) *Let f be a homomorphism of a left [right] amenable semigroup Σ onto a semigroup Σ', and let F be defined from $m(\Sigma')$ into $m(\Sigma)$ as in §2, (1); $Fx'(\sigma) = x'(f\sigma)$ for all σ in Σ. Then F^* carries the set*

M_l [M_r] of left [right] invariant means on $m(\Sigma)$ onto the set M_l' [M_r'] of left [right] invariant means on $m(\Sigma')$.

We already know from Lemma 1 of §3 that F^* carries the set of all means on $m(\Sigma)$ onto the set of all means on $m(\Sigma')$. That it carries left invariant means to left invariant means is easily verified; it is, indeed, the proof of (C) of §4.

To refine that proof slightly, as in the proof of §3, Lemma 1, set $m_0 = \{Fx' \mid x' \in m(\Sigma')\}$, and define μ_0 on m_0 by $\mu_0(x_0) = \mu'(F^{-1}x_0)$, so that $\mu_0(Fx') = \mu'(x')$ *for all* x' *in* $m(\Sigma')$.

Lemma 2. *m_0 is carried into itself by all l_σ and all r_σ.*

For each x' in $m(\Sigma')$ and each σ, τ in Σ,

$$\begin{aligned}(l_\sigma Fx')(\tau) &= (Fx')(\sigma\tau) = x'(f(\sigma\tau)) = x'((f\sigma)(f\tau)) \\ &= (l'_{f\sigma} x')(f\tau) = [F(l'_{f\sigma} x')](\tau);\end{aligned}$$

that is, for each σ and x'

$$l_\sigma(Fx') = F(l'_{f\sigma} x').$$

Hence it is an element of m_0. A similar calculation gives

$$r_\sigma(Fx') = F(r'_{f\sigma} x').$$

Lemma 3. *μ_0 is left invariant on m_0.*

For all σ in Σ and x' in $m(\Sigma')$

$$\begin{aligned}\mu_0(l_\sigma Fx') &= \mu_0(F(l'_{f\sigma} x')) = \mu'(l'_{f\sigma} x') \\ &= \mu'(x') = \mu_0(Fx').\end{aligned}$$

We could now use a theorem of Hahn-Banach type for extension of functions invariant under groups of transformations; see the thesis of R. J. Silverman [21]. However, a check of the proof there shows that the second-conjugate algebra gives us a technique for proving this in the framework in use here.

By the classical Hahn-Banach theorem, Banach [2], page 27, there is an extension μ_1 of μ_0 defined on all $m(\Sigma)$ with

$$\| \mu_1 \| = \| \mu_0 \| = \| \mu' \| = 1.$$

Also $\mu_1(e) = \mu_0(e) = \mu'(e') = 1$, so μ_1 is a mean. Σ is left amenable, so choose any left invariant mean ν on $m(\Sigma)$ and let $\mu = \nu \odot \mu_1$.

Then by Corollary 2 of §6, μ is a left invariant mean on $m(\Sigma)$. To see that μ is also an extension of μ_0, take x_0 in m_0; then

$$\mu(x_0) = (\nu \odot \mu_1)(x_0) = \nu(\mu_1 \odot x_0),$$

and for every σ

$$(\mu_1 \odot x_0)(\sigma) = \mu_1(x_0 \odot \sigma) = \mu_1(l_\sigma x_0) = \mu_0(l_\sigma x_0) = \mu_0(x_0),$$

so

$$(\mu_1 \odot x_0) = \mu_0(x_0)e,$$

and

$$\mu(x_0) = \nu[\mu_0(x_0)e] = \mu_0(x_0)\nu(e) = \mu_0(x_0).$$

Hence μ is a left invariant extension of μ_0. Then for every x' in $m(\Sigma')$

$$F^*\mu(x') = \mu(Fx') = \mu_0(Fx') = \mu'(x'),$$

so $F^*\mu = \mu'$. Therefore F^* carries the set of left invariant means on $m(\Sigma)$ *onto* the set of left invariant means on $m(\Sigma')$.

For right invariance we pass to the transposed semigroup Σ^t in which products are taken in the other order from that used in Σ. This interchanges l_σ and r_σ, and a left invariant mean on Σ^t can be constructed as above, and the result transposed back to get a right invariant mean on Σ. As Arens has emphasized, and as these examples continue to emphasize, this repeated transposition need not return one to the original product operation in the second-conjugate space.

Using the operation (2) of §2, we are able to give a reduction in the case of groups.

THEOREM 2. *Let G be an amenable group, and let H be a (not necessarily normal) subgroup of G. Suppose that μ is a mean [left invariant] on $m(G)$ and that γ' is a left invariant element of $m(H)^*$. Define Π, as in §2, (2), from $m(G)$ onto $m(H)$ by $\Pi x(h) = x(h)$ for all h in H, and set $\gamma = \Pi^*\gamma'$. Then $\| \mu \odot \gamma \| = \| \gamma' \|$, [and $\mu \odot \gamma$ is left invariant].*

We have already seen (§3, Lemma 2) that $\gamma = \Pi^*\gamma'$ is in $m(G)^*$, and, by Corollary 2 of §6, $\mu \odot \gamma$ is left invariant when μ is. We need now to use the construction which proves that H is amenable (§4, (D)). Let K be a set of representatives for left cosets of H in G so that every g in G has a unique representation as a product, $g = kh$, with k in K and h in H. Define U from $m(H)$ into $m(G)$ by

$$(Uy')(kh) = y'(h) \qquad \text{for all } k, h.$$

Then U is an isometry of $m(H)$ into $m(G)$, and $\Pi Uy' = y'$ for all y' in $m(H)$.

Now choose $\varepsilon > 0$, and then choose x' in $m(H)$ with $\| x' \| = 1$ and $\gamma'(x') > \| \gamma \| - \varepsilon$. Let $x = Ux'$. Then

$$(\mu \odot \gamma)(x) = \mu(\gamma \odot x),$$

and

$$(\gamma \odot x)(g) = \gamma(x \odot g) = \gamma(l_g x) = (\Pi^*\gamma')(l_g x)$$
$$= \gamma'(\Pi l_g x).$$

But, setting $g = kh$,

$$(\Pi l_g x)(h') = (l_g x)(h') = x(gh') = x(khh')$$
$$= (Ux')(khh') = x'(hh') = (l'_h x')(h'),$$

so

$$\Pi(l_{kh}\, Ux') = l'_h\, x'.$$

Hence

$$(\gamma \odot x)(g) = \gamma'(l'_h\, x') = \gamma'(x') \qquad \text{for all } g \text{ in } G.$$

Therefore

$$\gamma \odot x = \gamma'(x')e,$$

so

$$(\mu \odot \gamma)(x) = \mu(\gamma'(x')e) = \gamma'(x')\mu(e) = \gamma'(x') > \| \gamma' \| - \varepsilon.$$

Hence $\| \mu \odot \gamma \| > \| \gamma' \| - \varepsilon$ for every $\varepsilon > 0$; it follows that $\| \mu \odot \gamma \| = \| \gamma' \|$, since, by §6, Lemma 1, $\| \mu \odot \gamma \| \leqq \| \gamma \|$.

COROLLARY 1. *For each mean ζ the operator Z defined by $Z\gamma' = \zeta \odot (\Pi^*\gamma')$ from $m(H)^*$ to $m(G)^*$ is an isometry of the set of left invariant elements of $m(H)^*$ into $m(G)^*$. If also ζ is left invariant, then Z carries the set of left invariant means on $m(H)$ isometrically into the set of left invariant means on $m(G)$.*

These results imply

THEOREM 3. *If a left amenable group G has either a subgroup or a factor group with more than one left invariant mean, then G has more than one left invariant mean.*

If f is a homomorphism of G onto G', and if $\mu'_1 \neq \mu'_2$ are left invariant elements of $m(G')$, then there exist, by Theorem 1, μ_1 and μ_2 such that $F^*\mu_i = \mu'_i$; each μ_i is left invariant, and $\mu_1 \neq \mu_2$.

If H is a subgroup of G and ζ is left invariant on $m(G)$, and if $\mu'_1 \neq \mu'_2$ are left invariant means on $m(H)$, then by Theorem 2 each $Z\mu'_i$ is left invariant, and $Z\mu'_1 - Z\mu'_2 = Z(\mu'_1 - \mu'_2) \neq 0$, since Z is an isometry on left invariant elements.

This is our first main result, and with the known structure theorems for abelian groups it enables us to prove

THEOREM 4. *An abelian group G has only one invariant mean if and only if G is a finite group.*

If an abelian group has finite order, then its mean is unique. If the group is of infinite order, then either there is or there is not an element of infinite order in G. If there is an element of infinite order, it generates a cyclic subgroup which, by Corollary 7 of §6, has more than one invariant mean; by Theorem 3, G also has more than one invariant mean.

If G has only elements of finite order, we observe that this implies that there exists in G an expanding sequence of finite subgroups $H_1 \subset H_2 \subset \cdots \subset H_n \cdots$.

LEMMA 4. *Let $H = \bigcup_n H_n$, where $H_{n+1} \supset H_n$, each H_n is a finite subgroup of H_{n+1}, and where the number of elements in H_{n+1} is more than ten times the* *number of elements in H_n. Then there is more than one invariant mean on $m(H)$.*

Define x in $m(H)$ by $x(h) = 1$ if $h \in H_{2n+1} - H_{2n}$ and $x(h) = 0$ if $h \in H_{2n} - H_{2n-1}$ for any n. Let k_n be the number of elements in H_n, and define φ_n in $l_1(H)$ by

$$\varphi_n(h) = 1/k_n \quad \text{if} \quad h \in H_n,$$

$$\varphi_n(h) = 0 \quad \text{if} \quad h \notin H_n.$$

Then $\{\varphi_n\}$ converges in norm to invariance, for as soon as $h \in H_n$ then $\varphi_n h - \varphi_n = 0$. But $\lim \inf_n \varphi_n(x) \leqq .1$, and $\lim \sup_n \varphi_n(x) \geqq .9$, so $\{\varphi_n\}$ is not weakly convergent. By Lemma 1, $m(H)$ has at least two invariant means.

Now return to the proof of Theorem 4. By taking a subsequence if necessary, the finite subgroups there can be chosen to have all the properties of Lemma 4, so $H = \bigcup_n H_n$ has more than one invariant mean. Hence G has more than one invariant mean.

COROLLARY 2. *If G is an abelian group, then the second-conjugate algebra $m(G)^*$ is commutative if and only if G is finite.*

If G is finite, then $l_1(G)$ and $l_1(G)^{**}$ are isomorphic, and are, therefore, both commutative with G. If G is infinite, Theorem 4 asserts that there are many invariant means; Corollary 6 of §6 asserts that $m(G)^*$ is not commutative.

COROLLARY 3. *Let G be an infinite group; then each of the following conditions is sufficient that G have more than one invariant mean:*

(i) *The commutator chain $G \supset G_1 \supset \cdots \supset G_n = 1$ ends at the identity in a finite number of steps.*

(ii) *G is amenable and contains an element of infinite order.*

(iii) *G is locally finite; that is, every finite subset of G generates a finite subgroup of G.*

(i) and (iii) are already known to be sufficient conditions for amenability of G. (ii) and (iii) yield many means by Theorem 2 and Lemma 4. If (i) holds, we have first that every G_i/G_{i+1} is abelian, by the definition of commutator groups. At least one of these groups must be infinite, since an extension of a finite group by a finite group is always finite. By Theorem 4 that group G_k/G_{k+1} has many means. But G_k/G_{k+1} is a factor group of G_k, so, by Theorem 1, G_k has many means. By Theorem 3, G itself has many means.

It should be observed that the case not covered by the theorem brings us directly up against one of the outstanding important problems of group theory. A group is called a *torsion group* if every element is of finite order.

Burnside's conjecture. Every finitely generated torsion group is a finite group.

If Burnside's conjecture is true, then all infinite torsion groups are amenable, and indeed come under (iii) of the last theorem, so they have many means. It should be remarked that the set of left [right] invariant means over any semigroup is *convex*; hence a semigroup with more than one invariant mean must have at least a continuum of invariant means.

8. Means and ergodicity

We take advantage of the results of §4 on strong amenability of amenable groups to improve some of the results of the paper [10] on the relationships between mean values and ergodicity. First we need some definitions:

If B is any Banach space, we let $\mathcal{L}(B)$ be the Banach algebra of all linear operators from B into B. $\mathcal{S}$ is called an *operator semigroup* over B if $\mathcal{S}$ is a subsemigroup of the multiplicative semigroup of $\mathcal{L}(B)$. A linear operator A is an *average* of the subset $\mathcal{S}$ of $\mathcal{L}(B)$ if and only if for each b in B the point Ab is in the closed convex hull of the set of Sb, $s \in \mathcal{S}$. A is a *finite average* of $\mathcal{S}$ if A is in the convex hull of $\mathcal{S}$; that is, if there exists a finite mean φ on $\mathcal{S}$ such that $A = \sum_{\mathcal{S}} \varphi(S)S$.

A bounded operator semigroup $\mathcal{S}$ is called *weakly, strongly,* or *uniformly ergodic* under a net $\{A_n\}$ of averages of $\mathcal{S}$ when for each S in $\mathcal{S}$

$$\lim_n A_n(S - I) = 0 = \lim_n (S - I)A_n$$

in the appropriate topology of $\mathcal{L}(B)$; that is,

$$(\text{weak}) \quad \beta[A_n(S - I)b] \to 0 \quad \text{and} \quad \beta[(S - I)A_n b] \to 0$$

for each b in B and β in B^*,

$$(\text{strong}) \quad \| A_n(S - I)b \| \to 0 \quad \text{and} \quad \| (S - I)A_n b \| \to 0$$

for each b in B, or

$$(\text{uniform}) \quad \| A_n(S - I) \| \to 0 \quad \text{and} \quad \| (S - I)A_n \| \to 0;$$

respectively.

For an ergodic operator semigroup $\mathcal{S}$ over B, we define two closed linear subspaces of B.

$\mathfrak{F} = \mathfrak{F}(\mathcal{S})$ is the set of common fixed points of all the elements of $\mathcal{S}$; that is, $b \in \mathfrak{F}$ if and only if $Sb = b$ for all S in $\mathcal{S}$.

$\mathcal{V} = \mathcal{V}(\mathcal{S})$ is the smallest closed linear subspace of B containing

$$\{Sb - b \mid b \in B \text{ and } S \in \mathcal{S}\}.$$

For convenience, for each b in B we also set $K(b)$ for the closed convex hull of $\{Sb \mid S \in \mathcal{S}\}$.

Eberlein [13] has shown that if a semigroup $\mathcal{S}$ is strongly ergodic under a net of averages $\{A_n\}$, then the following conditions on an element b in B are equivalent:

(a) The net $\{A_n b\}$ has a weak cluster point b_0 in B.

(b) There is a b_0 in $\mathfrak{F}$ such that $b - b_0 \in \mathfrak{V}$.

(c) $K(b) \cap \mathfrak{F}$ has in it exactly one point b_0.

(d) $\lim_n A_n b = b_0$.

From these and the rest of the ergodic theorem come other results (see Day [9] and [10] for references); let $\mathcal{E} = \mathcal{E}(\mathcal{S})$ be the vector sum of $\mathfrak{F}$ and $\mathfrak{V}$; $\mathcal{E} = \{b_0 + b_1 \mid b_0 \in \mathfrak{F},\ b_1 \in \mathfrak{V}\}$. Then

(e) $\mathfrak{V} \cap \mathfrak{F}$ contains only the point 0.

(f) $Pb = \text{norm-lim}_n A_n b$ exists if and only if $b \in \mathcal{E}$.

(g) $\mathcal{E}$ is a closed linear subspace of B.

(h) P is the projection of $\mathcal{E}$ on $\mathfrak{F}$ along $\mathfrak{V}$; that is, $PPb = Pb$ if $b \in \mathcal{E}$, $Pb = b$ if and only if $b \in \mathfrak{F}$, $Pb = 0$ if and only if $b \in \mathfrak{V}$.

(i) $PS = SP = P$ for all S in $\mathcal{S}$.

(j) P is a linear operator whose norm is not greater than $\text{lub}_{S \in \mathcal{S}} \| S \|$.

We recall more definitions from Day [10]. A *right* [*left*] *representation* of a semigroup Σ over a Banach space B is a homomorphism ρ [antihomomorphism λ] of Σ onto an operator semigroup over B; that is, for each σ in Σ, ρ_σ [λ_σ] is in $\mathfrak{L}(B)$, and $\rho_{\sigma\sigma'} = \rho_\sigma \rho_{\sigma'}$ [$\lambda_{\sigma\sigma'} = \lambda_{\sigma'}\lambda_\sigma$] for each σ, σ' in Σ. The *right* [*left*] *regular representation* of Σ is the representation r [l] already defined over $m(\Sigma)$ by $(r_\sigma x)(\sigma') = x(\sigma'\sigma)$ [$(l_\sigma x)(\sigma') = x(\sigma\sigma')$] for all σ, σ' in Σ and all x in $m(\Sigma)$.

Hereafter we shall use ρ for a right representation, λ for a left representation, and π for a representation which may be either right or left.

Attached to each bounded representation π of the semigroup, is a representation of the algebra $l_1(\Sigma)$ over B defined for each θ in $l_1(\Sigma)$ by

$$\pi_\theta = \Sigma_{\sigma' \in \Sigma}\, \theta(\sigma')\pi_{\sigma'}.$$

LEMMA 1. *Each representation π of the semigroup Σ determines a representation of the same kind of the semigroup algebra $l_1(\Sigma)$ by the definition above; then*

(i) $\| \pi_\varphi \| \leqq (\text{lub}_\sigma \| \pi_\sigma \|)\| \varphi \|$,

(ii) $\pi_{\theta+\varphi} = \pi_\theta + \pi_\varphi$ *if θ and $\varphi \in l_1(\Sigma)$*,

(iii) $\pi_{c\varphi} = c\pi_\varphi$ *if c is a scalar*,

(iv) $\rho_{\theta\varphi} = \rho_\theta\, \rho_\varphi$, *so* $\rho_{\sigma\varphi} = \rho_\sigma\, \rho_\varphi$ *and* $\rho_{\varphi\sigma} = \rho_\varphi\, \rho_\sigma$,

(v) $\lambda_{\theta\varphi} = \lambda_\varphi\, \lambda_\theta$, *so* $\lambda_{\sigma\varphi} = \lambda_\varphi\, \lambda_\sigma$ *and* $\lambda_{\varphi\sigma} = \lambda_\sigma\, \lambda_\varphi$.

For an example we give a proof of one such relation.

$$\begin{aligned}\rho_{\theta\varphi} &= \textstyle\sum_{\sigma \in \Sigma} \theta\varphi(\sigma)\rho_\sigma = \sum_\sigma \sum_{\sigma_1\sigma_2=\sigma} \theta(\sigma_1)\varphi(\sigma_2)\rho_{\sigma_1\sigma_2} \\ &= \textstyle\sum_{\sigma_1, \sigma_2 \in \Sigma} \theta(\sigma_1)\varphi(\sigma_2)\rho_{\sigma_1}\rho_{\sigma_2} \\ &= \left(\textstyle\sum_{\sigma_1 \in \Sigma} \theta(\sigma_1)\rho_{\sigma_1}\right)\left(\textstyle\sum_{\sigma_2 \in \Sigma} \varphi(\sigma_2)\rho_{\sigma_2}\right) = \rho_\theta\, \rho_\varphi .\end{aligned}$$

In the paper [10] (essentially in Theorem 2 and Corollary 4) *the following properties of an abstract semigroup were proved equivalent*:

(1) *Σ is an amenable semigroup.*

(2) *There exists a net* $\{\varphi_n\}$ *of finite means which converges weakly to invariance.*

(3) *Every bounded (right or left) representation* π *of* Σ *over a Banach space* B *is weakly ergodic under a net* $\{\pi_{\varphi_n}\}$ *of finite averages of the* π_σ.

(4) *The right and left regular representations,* r *and* l *of* Σ *over* $m(\Sigma)$ *are weakly ergodic under nets of finite averages,* $\{r_{\varphi_n}\}$ *and* $\{l_{\varphi_n}\}$.

In this section we show that these conditions are also equivalent to the (formally stronger) conditions obtained from the last three by replacing weak by norm convergence. The only cost is in the extra care required in constructing a new system of finite means from the original system of means which is assured by (2).

THEOREM 1. *The following statements are equivalent to the conditions* (1) *to* (4), *above*:

(2s) Σ *is strongly amenable, under a net* $\{\varphi_n\}$ *of finite means.*

(3s) *Every bounded right or left representation* π *of* Σ *over a space* B *is uniformly ergodic under a net* $\{\pi_{\varphi_n}\}$ *of finite averages of the* π_σ.

(4s) *The right and left regular representations,* r *and* l, *of* Σ *over* $m(\Sigma)$ *are uniformly ergodic under nets of finite averages,* $\{r_{\varphi_n}\}$ *and* $\{l_{\varphi_n}\}$.

From §5, Theorem 1, we know that (2) implies (2s). For the next step, if $\Phi = \{\varphi_n\}$ is a net of finite means converging in norm to invariance, then by Lemma 1, $\{\pi_{\varphi_n}\}$ is a net of finite averages of the π_σ such that

$$\| \pi_{\varphi_n} \pi_\sigma - \pi_{\varphi_n} \| = \| \pi_{\varphi_n \sigma - \varphi_n} \| \leqq \| \varphi_n \sigma - \varphi_n \| M,$$

where M is a bound for $\| \pi_\sigma \|$. The other relations follow in the same way, so $\pi(\Sigma)$ is uniformly ergodic under the net $\{\pi_{\varphi_n}\}$. (3s) clearly implies (4s), and (4s) implies (4); this completes the proof of equivalence of these new conditions with the earlier ones.

9. A theorem of G. G. Lorentz on almost convergence

We observed in Day [10] that Lorentz[3] [17] had proved that when Σ is the semigroup of integers, certain conditions on an element x in $m(\Sigma)$ are equivalent; in this section we state these conditions for amenable semigroups and prove that they are still equivalent.

DEFINITION 1. Let Σ be an amenable semigroup. An element x of $m(\Sigma)$ is called *almost convergent* if all invariant means on $m(\Sigma)$ coincide at x.

As in Day [10] let $\mathfrak{R}$ and $\mathfrak{L}$ be the operator semigroups $r(\Sigma)$ and $l(\Sigma)$ in $\mathfrak{L}(m(\Sigma))$, and let $\mathfrak{P} = \mathfrak{R}\mathfrak{L} = \{RL \mid R \in \mathfrak{R},\ L \in \mathfrak{L}\}$. It is observed in [10] that $\mathfrak{R}$ and $\mathfrak{L}$ commute, so $\mathfrak{P}$ is also a semigroup. If Σ^t is the transposed semigroup of Σ, then $\mathfrak{L}$ is a homomorphic image of Σ^t, so $\mathfrak{P}$ is a homomorphic

[3] The referee has remarked that some of Lorentz's results can be found in the lectures of von Neumann on invariant measures, The Institute for Advanced Study, 1940–41.

image of $\Sigma \times \Sigma^t$. But Σ^t is amenable along with Σ, and the direct product of two amenable semigroups is amenable, so $\mathcal{P}$ is a representation of an amenable semigroup, and all of the ergodic theorem of §8 applies. First, $\mathcal{P}$ is uniformly ergodic under some net of finite averages $\{\pi_n\}$ of the elements of $\mathcal{P}$, and second, an element x of $m(\Sigma)$ is in the ergodic subspace $\mathcal{E}$ of $\mathcal{P}$ if and only if $\pi x = \text{norm-lim}_n \pi_n x$ exists.

Now $\mathcal{E} = \mathcal{F} + \mathcal{V}$ and π projects $\mathcal{E}$ on $\mathcal{F}$ along $\mathcal{V}$. If μ is an invariant mean on $m(\Sigma)$, then

$$\mu(\pi x) = \lim_n \mu(\pi_n x) = \lim_n \mu(x) = \mu(x).$$

But $\mathcal{F}(\mathcal{P}) = \{te \mid t \text{ real}\}$ (see [10]), so for each x in $\mathcal{E}$, there is a t_x such that $\mu(\pi x) = \mu(t_x e) = t_x$; hence for each x in $\mathcal{E}$, $\mu(x)$ is independent of the invariant mean μ. Let μ_0 be the linear functional defined in $\mathcal{E}$ by $\mu_0(x) = t_x$ for each x in $\mathcal{E}$.

For y in $m(\Sigma)$ define $p_1(y)$ by

$$p_1(y) = \text{glb}\{\mu_0(x) \mid x \in \mathcal{E}(\mathcal{P}) \text{ and } x \geqq y\}.$$

(The notation $x \geqq y$ means that $x(\sigma) \geqq y(\sigma)$ for all σ in Σ.) Set

$$p_2(y) = -p_1(-y); \quad \text{then} \quad p_2(y) = \text{lub}\ \{\mu_0(z) \mid z \in \mathcal{E}(\mathcal{P}) \text{ and } y \geqq z\}.$$

For the special case in which Σ is the set of positive integers, Lorentz [17] proved that (1) and (3) below are equivalent; he also uses a function p defined in a somewhat different way than our p_1 as an aid in his proof.

THEOREM 1. *Let Σ be an amenable semigroup; then the following conditions on an element x of $m(\Sigma)$ are equivalent:*

(1) *x is almost convergent.*

(2) $p_1(x) = p_2(x)$.

(3) *There exist finite averages of transforms of x under $\mathcal{P}$ which are arbitrarily near some constant function.*

(4) *x is in the ergodic subspace $\mathcal{E}$.*

That (4) implies (1) is proved in defining μ_0; that (4) implies (2) is trivial, for $p_1(x) = \mu_0(x) = p_2(x)$ if $x \in \mathcal{E}$. Hence, if x does not satisfy (2), then $x \notin \mathcal{E}$. By the Hahn-Banach theorem, Banach [2] page 27, there exist at least two extensions μ_i of μ_0 such that $p_1(y) \geqq \mu(y)$ for all y, because for each value of r with $p_1(x) \geqq r \geqq p_2(x)$ there is an extension μ_r of μ_0 such that μ_r is dominated by p_1 and $\mu_r(x) = r$.

But if μ is any one such extension, $\mu(x - Px) = \mu_0(x - Px) = 0$ because $x - Px \in \mathcal{V}$. Hence μ is invariant under $\mathcal{P}$; that is, μ is both right and left invariant.

Since $x \leqq \| x \| e$ which is an element of $\mathcal{E}$, $p_1(x) \leqq \| x \|$, so $\| \mu \| \leqq 1$. Also $\mu(e) = \mu_0(e) = 1$, so μ is a mean. Hence (1) fails for x if (2) does; that is, (1) implies (2).

If (2) holds for x, take $\varepsilon > 0$, and take y, z in $\mathcal{E}$ such that $y \geqq x \geqq z$ and $\mu_0(y - z) < \varepsilon$. Then by the ergodic theorem (§8) there is a finite average

π_φ of elements of $\mathcal{P}$ such that

$$\| \pi_\varphi y - \mu_0(y)e \| < \varepsilon,$$

so

$$\pi_\varphi z \leqq \pi_\varphi x \leqq \pi_\varphi y \leqq (\mu_0(y) + \varepsilon)e.$$

Then there is a π_ψ such that

$$\| \pi_\psi \pi_\varphi z - \mu_0(\pi_\varphi z)e \| < \varepsilon,$$

but

$$\mu_0(\pi_\varphi z) = \mu_0(z),$$

so

$$(\mu_0(y) - 2\varepsilon)e \leqq (\mu_0(z) - \varepsilon)e \leqq \pi_\psi \pi_\varphi z \leqq \pi_\psi \pi_\varphi x$$
$$\leqq (\mu_0(y) + \varepsilon)\pi_\psi \pi_\varphi e = (\mu_0(y) + \varepsilon)e.$$

Hence

$$\| \pi_\psi \pi_\varphi x - \mu_0(y)e \| < 2\varepsilon.$$

This proves that (2) implies (3). (3) implies (4) is part of the ergodic theorem quoted in §8.

10. Amenable subspaces of $m(\Sigma)$

A subspace of $m(\Sigma)$ may behave better under translations than does the full space; an extreme example is the subspace of constant functions on Σ. *We shall assume throughout this section that X is an invariant linear closed subspace of $m(\Sigma)$ such that $e \in X$.* Occasionally it will be useful to have X a sublattice, and for many results we also wish to have X *introverted*; this means that for every x in X and ξ in X^*, the function $\xi_l x$, defined by $\xi_l x(\sigma) = \xi(l_\sigma x)$ for every σ in Σ (the function on Σ denoted by $\xi \odot x$ in §6), and the dual function $\xi_r x$, defined by $\xi_r x(\sigma) = \xi(r_\sigma x)$ for every σ in Σ, are in X.

In §3 the definition of means can be copied in X, and 3(A), (B) proved as before. Writing J for the injection map of X into $m(\Sigma)$, it is easy to see that *J^* carries the set of means in $m(\Sigma)^*$ onto the set of means in X^*.* 3(C), (D) follow at once with J^*Q replacing Q. Lemma 3.1 must be restated as: *If f maps Σ onto Σ', if X and X' are in $m(\Sigma)$ and $m(\Sigma')$ respectively and such that $F(X') \subseteqq X$*, and if M and M' are the sets of means in X and X', then $F^*(M) = M'$. To get isometry in Lemma 3.2 requires X to be a vector lattice.

Paralleling §4 we define: X is *amenable* if there is a mean μ in X^* such that $\mu(r_\sigma x) = \mu(x) = \mu(l_\sigma x)$ for all x in X and σ in Σ. 4(A) need not hold unless X is introverted (see discussion of §6). 4(B) requires X to be inverse-invariant. 4(C) becomes: *f a homomorphism of Σ onto Σ' and X amenable in $m(\Sigma)$ imply $F^{-1}(X)$ amenable in $m(\Sigma')$.* 4(D) suggests the true result: *If X is amenable, so is every invariant subspace of X.* We skip 4(E) through (K) except to remark that (F) has a parallel:

LEMMA 1. *If* $\{X_n\}$ *is an increasing net of amenable subspaces of* $m(\Sigma)$, *then* X, *the closure of the union of the* $\{X_n\}$, *is amenable.*

This and the existence of amenable subspaces allow us to use Zorn's Lemma to prove that *every* $m(\Sigma)$ *has maximal amenable subspaces.* It does not tell whether there can be more than one such subspace. The formulas at the end of the proof of Lemma 7.2 also allow us to prove

LEMMA 2. *If* f *maps* Σ *homomorphically onto* Σ', *if* X' *is contained in* $m(\Sigma')$, *and if* $X = F(X')$, *then* X *has in* $m(\Sigma)$ *the same ones of the following properties as has* X' *in* $m(\Sigma')$: (a) *Contains constant functions.* (b) *Right* (*left*) *invariant.* (c) *Right* (*left*) *introverted.* (d) *Amenable.* (e) *Sublattice of* $m(\cdot\cdot)$.

As an application of these lemmas, let G be a group, and let G_s, where s runs over S, be the family of all amenable homomorphic images of G. For each finite subset σ of S, let $G'_\sigma = \prod_{s\epsilon\sigma} G_s$, and let f_σ be defined from G into G'_σ coordinatewise using the original homomorphisms f_s mapping G onto G_s. Then each G'_σ is amenable by 4(F), so the subgroup $G_\sigma = f_\sigma(G)$ is also amenable. Let $X_\sigma = F_\sigma\, m(G_\sigma)$; then each X_σ has all of the properties of Lemma 2. If $\sigma \supseteqq \sigma'$, then $X_\sigma \supseteqq X_{\sigma'}$, so the sets X_σ are expanding with σ. By Lemmas 1 and 2, X, the closure of the union of the X_σ, is amenable; indeed, it can be shown to have the other properties discussed in Lemma 2. Since the full direct product $\prod_{s\epsilon S} G_s$ need not be amenable, it is by no means sure that X is itself determined by a homomorphism of G. If G is the free group on two generators, the calculations at the end of §4 show that X is at least large enough to separate points of G.

In §5 we make no use of introversion. If we modify Definition 5.1 to consider means in X^*, then 5(A), (B), (C) hold, and (C′) must be modified by using *X-convergence* of finite means in place of weak convergence: that is, we use in $l_1(\Sigma)$ the topology that Bourbaki ([6], p. 50) calls $\sigma(l_1(\Sigma), X)$. The corresponding relatively strong or Mackey topology, $\tau(l_1(\Sigma), X)$ (see [6], p. 70), must then be used in the definition of strong amenability of X and in the later lemmas. J^*Q must replace Q in 5(D) and elsewhere. Then everything goes swimmingly through the main theorem of §5.

In §6 we must assume that X is left introverted as this returns $\xi \odot x$ to X where $\eta(\xi \odot x)$ can be computed for η in X^*. J^*Q replaces Q in Lemma 2 and Corollary 3, and the latter requires that X be a lattice; the rest down to Corollary 6.6 holds as before, with $m(\Sigma)$ replaced by an introverted X.

We have already had use for the extension of Lemma 7.2. The rest of Theorem 7.1 extends if X is an amenable subspace containing $F(X')$, $X' \subseteqq m(\Sigma')$. Theorem 7.2 is yet unadapted to this situation.

For §8 we concentrate our attention on *X-representations*; that is, representations π of Σ over B such that for each β in B^* and b in B the function $\beta_\pi b$, defined by $\beta_\pi b(\sigma) = \beta(\pi_\sigma b)$ for all σ in Σ, is in X. Clearly *the right and left regular representations of* Σ *over* X *are X-representations if and only if* X *is introverted.* The conditions (1)–(4) of §8 can be rephrased as:

(X_1) *X is an amenable subspace of $m(\Sigma)$.*

(X_2) *There exists a net $\{\varphi_n\}$ of finite means X-convergent to invariance in $l_1(\Sigma)$.*

(X_3) *Every bounded (right or left) X-representation π is weakly ergodic under a net $\{\pi_{\varphi_n}\}$ of finite averages of the π_σ.*

(X_4) *The right and left regular representations of Σ over X are weakly ergodic under the nets $\{r_{\varphi_n}\}$ and $\{l_{\varphi_n}\}$ respectively.*

The theorem quoted from Day [10] becomes now: (X_4) *implies* (X_1) *implies* (X_2) *implies* (X_3); *if X is introverted,* (X_3) *implies* (X_4). The proofs go as in [10]: see also Rosen [19] for the case where X is the space of continuous functions on a topological semigroup.

Theorem 8.1 goes with the strong topology on $\mathfrak{L}(B)$ replacing the uniform topology and with the Mackey topology in $l_1(\Sigma)$. This is a consequence of two facts for each b in B: (a) If a net $\{\beta_n\}$ in the unit sphere U of B^* is w^*-convergent to β, then in $m(\Sigma)$, $\beta_{n\pi}\, b$ is w^*-convergent to $\beta_\pi\, b$. (b) U is w^*-compact, so its image in X is w^*-compact and therefore determines a τ-neighborhood of 0 in $l_1(\Sigma)$.

§9 goes through, with $m(\Sigma)$ replaced by an amenable, introverted subspace X, though the proof looks back a little farther than does §9 into the proof of the ergodic theorem in [10] and [13] to get the appropriate information about $\mathcal{P}$, the product of $r(\Sigma)$ and $l(\Sigma)$. Applying this to the discussion of of §7 gives: *If X is an amenable introverted subspace of $m(\Sigma)$, then there is just one invariant mean in X^* if and only if the ergodic subspace of the corresponding $\mathcal{P}$ is all of X.*

Theorem 8 and Corollary 11 of Day [10] and the similar theorem of Dixmier [4] for topological semigroups have the following common generalization: *Let G be a group and let X be an amenable subspace of $m(G)$; then every bounded X-representation of G over a Hilbert space H is equivalent to a unitary representation.*

11. Topological semigroups

In case there is a topology in Σ in which multiplication is continuous, a natural choice for X is the space $C(\Sigma)$ of continuous real-valued functions on Σ. $C(\Sigma)$ is a lattice, is invariant, and contains e, but it is not always introverted or amenable.

DEFINITION 1. A topological semigroup is called WCR (for weakly continuously representable) if the regular representations over $C(\Sigma)$ are $C(\Sigma)$-representations.

This happens if and only if $C(\Sigma)$ is introverted, and if and only if $\xi_r\, x$ and $\xi_l\, x$ are continuous for every x in $C(\Sigma)$ and ξ in $C(\Sigma)^*$. As Rosen [19] points out, *discrete semigroups and compact semigroups are* WCR; it is easily seen that the additive group of real numbers is not WCR. Looking back now through §10, we find Rosen's result that in a WCR semigroup, $C(\Sigma)$ is amenable if and only if all of the bounded weakly continuous representations of Σ are weakly ergodic, and if and only if the regular representations over $C(\Sigma)$ are weakly ergodic. Also Lorentz's theorem carries over to an amenable, introverted $C(\Sigma)$. Rosen [19] also observes that if there is an invariant mean only on the space of uniformly continuous functions on a group G, then Eberlein's results [13] on weak almost periodicity carry over almost intact to the continuous wap functions on G.

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University of Illinois
Urbana, Illinois
University of Washington
Seattle, Washington