ON A PROBLEM OF PICARD CONCERNING SYMMETRIC COMPOSITUMS OF FUNCTION-FIELDS[1]

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1. Introduction

Let K be a fixed algebraically closed ground-field of arbitrary characteristic. Function-fields and varieties will be considered over K. Function-fields of Abelian varieties will be called Abelian function-fields. Let Σ be a function-field of dimension r, and let $\Sigma(m)$ be its m-fold symmetric compositum, i.e., the invariant subfield of the m-fold direct compositum of Σ under the symmetric group of permutations of factors. Obviously, $\Sigma(1)$ is an Abelian function-field if and only if Σ is so. Moreover, $\Sigma(m)$ is an Abelian function-field if and only if m is the genus of Σ in case $r = 1$. In this paper, *we shall show that $\Sigma(m)$ can never become an Abelian function-field for r, $m > 1$.* This fact was already remarked by Picard in the case $r = 2$.[2] His reasoning applies to the case of even r, but, as he himself observed, not directly to the case of odd r. Thus, our result includes the case which Picard failed to discuss.

2. Reduction of the problem

Let V be a projective model of Σ, and let U and $V(m)$ be the m-fold direct and symmetric products of V. Then, there is a canonical rational map from U to $V(m)$, and $\Sigma(m)$ is the function-field of $V(m)$. Moreover, V and $V(m)$ have the same Albanese variety, say A. In fact, let p_i be the projection of U to its i^{th} factor for $i = 1, \cdots, m$; let f be a canonical map of V to its Albanese variety A. Then, $F = \sum_{i=1}^{m} f \circ p_i$ is the product of the canonical rational map from U to $V(m)$ and a canonical map of $V(m)$ to A. The converse is also true.[3] On the other hand, if we replace V by the graph of f, we can assume that f is regular on V. Furthermore, if we replace V by its derived normal model, we can assume, in addition, that V has negligible singularities, i.e., that the singular locus of V is of co-dimension at least equal to 2.[4] In this case, U has also negligible singularities.

Now, assume that $\Sigma(m)$ is an Abelian function-field for some r, $m > 1$. Let A be an Abelian variety such that $\Sigma(m)$ is the corresponding function-field. Then, A is the Albanese variety of $V(m)$, and a canonical map of $V(m)$

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[2] Cf. E. PICARD AND G. SIMART, *Théorie des fonctions algébriques de deux variables indépendentes*, 2, Paris, 1906, pp. 469–474. The problem is raised on p. 474.

[3] These are immediate consequences of the definition of Albanese varieties and of the Corollary on p. 32 of A. WEIL, *Variétés abéliennes et courbes algébriques*, Paris, 1948.

[4] The passage from V to the derived normal model of the graph of f is a standard process introduced by Zariski. Cf., *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc., vol. 53 (1943), pp. 490–542.

to A is birational. In general, we shall denote by $\Omega = \sum_{q=0}^{\infty} \Omega_q$ the graded algebra over K of differential forms of the first kind on a variety. Also, we shall use the symbol δ for contravariant map of differential forms. Let $\theta_1, \cdots, \theta_m$ be differential forms on V. Then, the exterior product

$$\delta p_1 \cdot \theta_1 \wedge \cdots \wedge \delta p_m \cdot \theta_m$$

is a differential form on U, which we shall denote by $p^* \cdot (\theta_1, \cdots, \theta_m)$. We note that p^* is Σ-multilinear. The following lemma is a consequence of a general theorem due to Koizumi:[5]

LEMMA 1. *The map p^* induces a bijective K-linear isomorphism of the m-fold tensor product of $\Omega(V)$ over K to $\Omega(U)$.*

On the other hand, the symmetric group of degree m, say G, operates on U as group of permutations of m factors. Hence, G operates canonically on the space of differential forms on U. We shall denote by $\Omega^G(U) = \sum_{q=0}^{\infty} \Omega_q^G(U)$ the graded algebra over K of G-invariant differential forms of the first kind on U. Since, in the notation and the convention we introduced before, F is a separable map, δF is injective.

LEMMA 2. *The map δF gives a bijective K-linear isomorphism of $\Omega(A)$ to $\Omega^G(U)$.*

Proof. Let θ be a differential form of the first kind on A. Then, $\delta F \cdot \theta$ is a differential form of the first kind on U.[6] Moreover, $\delta F \cdot \theta$ is G-invariant, hence it is an element of $\Omega^G(U)$. Conversely, let θ^* be a G-invariant differential form on U. Then, we can find a differential form θ on A such that $\theta^* = \delta F \cdot \theta$. If θ is not of the first kind on A, it has a polar variety, say W, because A is nonsingular. Let $\tau_1, \tau_2, \cdots$ be a system of local coordinates of A along W. Then, in the expression of θ as a linear combination of exterior products of $d\tau_1, d\tau_2, \cdots$, at least one coefficient, say ϕ, is not contained in the local ring of A along W. Let w be a generic point of W over K, and let

$$x = (x_1, \cdots, x_m)$$

be a point of U such that $F(x) = w$. There exists such a point, because U is complete and F is regular on U. Since w is of co-dimension 1 over K, so is x. In particular, $x_1, \cdots, x_m$ are distinct simple points of V. Therefore, $\delta F \cdot \tau_1, \delta F \cdot \tau_2, \cdots$ form a system of local coordinates of U along the locus, say W^*, of x over K, i.e., W^* is unramified over W. Moreover, $\delta F \cdot \phi$ is a coefficient of $\delta F \cdot \theta = \theta^*$ as a linear combination of the exterior products of $d(\delta F \cdot \tau_1), d(\delta F \cdot \tau_2), \cdots$. Since the local ring of A along W is the intersection of the local ring of U along W^* and the function-field of A, we see that $\delta F \cdot \phi$

[5] S. KOIZUMI, *On the differential forms of the first kind on algebraic varieties. II*, J. Math. Soc. Japan, vol. 2 (1951), pp. 267–269.

[6] Basic properties of differential forms are summarized with references in our earlier paper, *A fundamental inequality in the theory of Picard varieties*, Proc. Nat. Acad. Sci. U.S.A., vol. 41 (1955), pp. 317–320.

is not contained in the local ring of U along W^*. Hence, θ^* is not of the first kind on U, i.e., the image of $\Omega(A)$ by δF is $\Omega^G(U)$, Q.E.D.

The following lemma is well known:

LEMMA 3. *The graded algebra $\Omega(A)$ is a Grassmann algebra over K; the dimension of $\Omega_1(A)$ over K is equal to the dimension rm of A.*

The above three lemmas permit us to derive a contradiction by formal arguments.

3. Derivation of a contradiction

The vector space $\Omega_0(V)$ is of dimension 1. We shall first determine the dimension of $\Omega_1(V)$. Let $\omega_1, \omega_2, \cdots$ be a base of $\Omega_1(V)$ over K. Then, by Lemma 1 an element θ of $\Omega_1(U)$ can be written uniquely in the form $\theta = \sum a_{ij}\, \delta p_i \cdot \omega_j$ with a_{ij} in K. Moreover, θ is G-invariant if and only if $a_{ij} = a_{1j}$ for $i = 1, \cdots, m$. Therefore, $\Omega_1(V)$ and $\Omega_1^G(U)$ are of the same dimension. However, by Lemma 2 we know that $\Omega_1^G(U)$ is of the same dimension as $\Omega_1(A)$, and by Lemma 3 this vector space is of dimension rm. Next, we shall show that $\Omega_2(V) = 0$. Let $\theta_1, \theta_2, \cdots$ be a base of $\Omega_2(V)$ over K. Then, by Lemma 1 an element θ of $\Omega_2(U)$ can be written uniquely in the form

$$\theta = \textstyle\sum_{i<i'} a_{ii'jj'}\, \delta p_i \cdot \omega_j \wedge \delta p_{i'} \cdot \omega_{j'} + \sum b_{ij}\, \delta p_i \cdot \theta_j$$

with $a_{ii'jj'}$ and b_{ij} in K. Moreover, θ is G-invariant if and only if $a_{ii'jj'} = a_{12jj'}$ and $b_{ij} = b_{1j}$ for $i < i'$ and for $i = 1, \cdots, m$, and, in addition, if $a_{12jj'}$ are skew-symmetric in the last two suffixes. Therefore, if p denotes the characteristic of K, we have

$$\dim \Omega_2^G(U) = \begin{cases} C_2^{rm} + rm + \dim \Omega_2(V), & p = 2 \\ C_2^{rm} + \dim \Omega_2(V), & p \neq 2. \end{cases}$$

However, by Lemmas 2 and 3 we know that $\dim \Omega_2^G(U)$ is equal to $\dim \Omega_2(A) = C_2^{rm}$. Hence, we have $p \neq 2$, and also $\Omega_2(V) = 0$. Once we know that $\Omega_2(V) = 0$, by the same argument we get $\Omega_3(V) = 0$, and, in this way, we finally get $\Omega_2(V) = \cdots = \Omega_m(V) = 0$. Naturally, in case $m > r$, the part $\Omega_{r+1}(V) = \cdots = \Omega_m(V) = 0$ is trivial. Anyway, in the case $m \geqq r$, we have $\Omega_r(V) = 0$, and then, by Lemma 1 we get $\Omega_{rm}(U) = 0$, hence a fortiori $\Omega_{rm}^G(U) = 0$. However, by Lemmas 2 and 3 this vector space is of dimension 1. This is a contradiction. Therefore, the case $m \geqq r$ is not possible, and we have $r > m$. In this case, by an argument similar to that used before, we get $\dim \Omega_{m+1}^G(U) = \dim \Omega_{m+1}(V)$, and this is equal to $\dim \Omega_{m+1}(A) = C_{m+1}^{rm}$. Here, even if $r = m + 1$, we can go one step further. We get $\dim \Omega_{m+2}^G(U) = \dim \Omega_1(V) \cdot \dim \Omega_{m+1}(V) + \dim \Omega_{m+2}(V)$, and this is equal to $\dim \Omega_{m+2}(A) = C_{m+2}^{rm}$. However, $\dim \Omega_1(V) \cdot \dim \Omega_{m+1}(V) = rm \cdot C_{m+1}^{rm}$ is strictly larger than C_{m+2}^{rm}. This is a contradiction, and the contradiction is derived from the assumption that $\Sigma(m)$ is an Abelian function-field for some $r, m > 1$.

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