ON A CLASS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Consider the system of linear differential equations

$$y'' + A(\lambda)y = \lambda\phi(t, \lambda)y + \lambda\psi(t, \lambda)y' \qquad (' = d/dt) \tag{1}$$

where λ is a real parameter, $y = (y_1, \cdots, y_n)$, $A(\lambda) = \operatorname{diag}(\sigma_1^2, \cdots, \sigma_n^2)$, ϕ and ψ are $n \times n$ matrices whose elements are real, periodic functions of t of period $T = 2\pi/\omega$, are L-integrable in $[0, T]$, are analytic in λ and have mean value zero. Further, suppose that each $\sigma_j^2(\lambda)$, $j = 1, 2, \cdots, n$ is a real positive analytic function of λ with

$$\sigma_j(0) \not\equiv \sigma_h(0), \ (\mathrm{mod}\ \omega i), \qquad j \neq h, \quad j, h = 1, 2, \cdots, n.$$

Systems of type (1) for $|\lambda|$ small have recently been extensively investigated by a method which has been successively developed and modified by L. Cesari, R. A. Gambill and J. K. Hale for both linear [1, 4, 5, 6, 9] and weakly nonlinear differential systems [7, 10]. The aim of the present paper is to prove a theorem, concerning the boundedness of the AC (absolutely continuous) solutions of (1), which contains as a particular case one of the various theorems proved in [1] and [4]. Applying the methods of [1], we prove the following:

THEOREM. *If*

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix},$$

where ϕ_{ij}, ψ_{ij} are matrices with ϕ_{11} and ψ_{11} of dimension $\mu \times \mu$, and if (α) *ϕ_{11}, ϕ_{22}, ψ_{21}, ψ_{12} are even in t,* (β) *ϕ_{21}, ϕ_{12}, ψ_{11}, ψ_{22} are odd in t, then, for $|\lambda|$ sufficiently small, all the* AC *solutions of* (1) *are bounded in* $(-\infty, +\infty)$.

For ψ identically zero, ϕ and A independent of λ, and each element of ϕ an even function of t having mean value zero and possessing absolutely convergent Fourier series, this theorem was first proved by L. Cesari [1] and then extended by the author [9] to L-integrable functions. Using the techniques in [1], R. A. Gambill [4] extended the theorem of Cesari to the case where ψ is odd in t.

We shall prove the above theorem by showing that there is a fundamental system of AC solutions of (1) which are bounded for all values of t. Furthermore, we shall see that each solution y of the fundamental system so obtained has the following property: if the first μ components of y are even (or odd), then the last $n - \mu$ components are odd (or even).

If we make the transformation of variables

$$y_j = \frac{1}{2i\sigma_j}(z_{2j-1} - z_{2j}), \qquad y_j' = \tfrac{1}{2}(z_{2j-1} - z_{2j}), \qquad j = 1, 2, \cdots, n, \tag{2}$$

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then system (1) is transformed into the first order system

$$(3)\quad \begin{cases} z'_{2j-1} = i\sigma_j z_{2j-1} + \dfrac{\lambda}{2}\sum_{k=1}^{n}\left[\left(\dfrac{1}{i\sigma_k}\phi_{jk} + \psi_{jk}\right)z_{2k-1} + \left(\dfrac{1}{i\sigma_k}\phi_{jk} - \psi_{jk}\right)z_{2k}\right], \\ z'_{2j} = -i\sigma_j z_{2j} - \dfrac{\lambda}{2}\sum_{k=1}^{n}\left[\left(\dfrac{1}{i\sigma_k}\phi_{jk} + \psi_{jk}\right)z_{2k-1} + \left(\dfrac{1}{i\sigma_k}\phi_{jk} - \psi_{jk}\right)z_{2k}\right], \\ \qquad\qquad j = 1, 2, \cdots, n. \end{cases}$$

Since we are looking for a fundamental system of solutions we shall write this in matrix form

$$(4)\qquad Z' = A^*Z + \lambda\Phi^*Z,$$

where $Z = ((z_{ij}))$ is an $n \times n$ matrix, $A^* = \text{diag}\,(\sigma_1^*, \cdots, \sigma_{2n}^*)$, $\sigma_{2j-1}^* = i\sigma_j$, $\sigma_{2j}^* = -i\sigma_j$ and Φ^* is a $2n \times 2n$ matrix.

By considering an auxiliary equation of (4),

$$(5)\qquad Z' = B^*Z + \lambda\Phi^*Z,$$

transforming it into an integral equation, and applying a convenient modification of successive approximations, we will obtain AC solutions of the equation

$$(6)\qquad Z' = (B^* - \lambda D)Z + \lambda\Phi^*Z,$$

where D is a constant matrix which depends on B^*, Φ^* and λ. Then, by determining B^* such that

$$(7)\qquad B^* - \lambda D = A^*,$$

the obtained solutions of (6) become solutions of (4).

In the following, we shall let C_ω denote the family of all functions which are finite sums of functions of the form $f(t) = e^{\alpha t}\phi(t)$, $-\infty < t < +\infty$, where α is any complex number and $\phi(t)$ is any complex-valued function of the real variable t, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$. If $\phi(t)$ has a Fourier series, $\phi(t) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\omega t}$, then we shall denote the series

$$(8)\qquad f(t) = e^{\alpha t}\phi(t) \approx \sum_{n=-\infty}^{+\infty} c_n e^{(in\omega+\alpha)t}$$

as the series associated with $f(t)$. Moreover, we shall denote by *mean value* $m[f]$ of $f(t)$ the number $m[f] = 0$ if $in\omega + \alpha \neq 0$ for all n, $m[f] = c_n$ if $in\omega + \alpha = 0$ for some n. It is known [8] that if $f(t) \in C_\omega$ and $m[f] = 0$, then there is one and only one primitive of $f(t)$, say $F(t)$, which belongs to C_ω and such that $m[F] = 0$.

Put $B^* = \text{diag}\,(\rho_1, \cdots, \rho_{2n})$, where

$$\rho_{2j-1} = i\tau_j, \qquad \rho_{2j} = -i\tau_j, \qquad j = 1, 2, \cdots, n,$$

each τ_j is a real number, and $\rho_i \not\equiv \rho_j \pmod{\omega i}$, $i \neq j$, $i, j = 1, 2, \cdots, 2n$. If we put

$$e^{B^*t} = \text{diag}\,(e^{\rho_1 t}, \cdots, e^{\rho_{2n} t}), \qquad X^{(0)} = \text{diag}\,(a_1 e^{\rho_1 t}, a_2 e^{\rho_2 t}, \cdots, a_{2n} e^{\rho_{2n} t}),$$

where each a_j is a nonzero complex number, then any solution of the equation

$$Z(t) = X^{(0)} + \lambda e^{B^*t} \int e^{-B^*\alpha}\Phi^*(\alpha, \lambda)Z(\alpha)\, d\alpha \tag{9}$$

is an AC solution of (5). Let

$$Z^{(m)} = X^{(0)} + \lambda X^{(1)} + \cdots + \lambda^m X^{(m)}, \tag{10}$$

where each $X^{(i)}$ is, in general, a function of λ, and define a matrix $D^{(r)}$ by

$$\mathfrak{A} D^{(r)} = m[e^{-B^*t}\Phi^* Z^{(r-1)}], \qquad r = 1, 2, 3, \cdots, \tag{11}$$

where $\mathfrak{A} = \operatorname{diag}(a_1, \cdots, a_{2n})$.

We now define the method of successive approximations as follows:

$$\begin{cases} Z^{(0)} = X^{(0)} = \operatorname{diag}(a_1 e^{\rho_1 t}, \cdots, a_{2n} e^{\rho_{2n} t}). \\ Z^{(m)} \equiv X^{(0)} + \lambda e^{B^*t} \int e^{-B^*\alpha}\, \Phi^{(m)} Z^{(m-1)}\, d\alpha \quad (\text{mod } \lambda^{m-1}), m = 1, 2, \cdots, \end{cases} \tag{12}$$

where

$$\Phi^{(m)} = \Phi^* - \textstyle\sum_{r=1}^m \lambda^{r-1} D^{(r)}, \qquad m = 1, 2, \cdots, \tag{13}$$

and the integrations are performed so as to obtain the unique primitive of mean value zero. The symbol (mod λ^{m-1}) denotes that the function $Z^{(m)}$ contains only the terms in λ_j, $j = 0, 1, \cdots, m$, of the expression on the right. If we replace $Z^{(m)}$ in (12) by (10) and equate coefficients of powers of λ^m, we obtain

$$\begin{cases} X^{(0)} = \operatorname{diag}(a_1 e^{\rho_1 t}, \cdots, a_{2n} e^{\rho_{2n} t}), \\ X^{(m)} = e^{B^*t} \int e^{-B^*\alpha}[\Phi^* X^{(m-1)} - (D^{(1)}X^{(m-1)} + \cdots + D^{(m)}X^{(0)})]\, d\alpha, \\ \qquad\qquad m = 1, 2, \cdots. \end{cases} \tag{14}$$

By defining the method in this manner, it is clear, since the ρ_j are two by two incongruent modulo ωi, that $D^{(r)} = \operatorname{diag}(d_{11}^{(r)}, d_{22}^{(r)}, \cdots, d_{2n,2n}^{(r)})$, the integrand belongs to the class C_ω of functions and is of mean value zero; consequently, there is a unique primitive of mean value zero. Furthermore, if $X^{(m)} = ((x_{jh}^{(m)}))$ it is clear that

$$x_{jh}^{(m)} = e^{\rho_j t} p_{jh}^{(m)}(t), \qquad j = 1, 2, \cdots, 2n, \quad h = 1, 2, \cdots, 2n, \tag{15}$$

where each $p_{jh}^{(m)}$ is periodic of period $2\pi/\omega$. This method of successive approximations is exactly the same as the one defined by L. Cesari [1] except for the congruence modulo λ^{m+1} (see also [4]). The proof of the convergence of the method to a solution of an equation of the form (6) may be supplied in the same way as described in [1] or [9].

We shall first show that for a proper choice of the constants $a_1, \cdots, a_{2n}$

the numbers $d_{jj}^{(r)}$ are such that $d_{2h-1,2h-1}^{(r)} = \bar{d}_{2h,2h}^{(r)}$ ($-$ is the complex conjugate), $h = 1, 2, \cdots, n$, $r = 1, 2, \cdots$, for every system of equations of the form (9). Under the conditions of the theorem and some additional restrictions on $a_1, \cdots, a_{2n}$, we shall show that $d_{2h-1,2h-1}^{(r)} = -d_{2h,2h}^{(r)}$, $h = 1, 2, \cdots, n$, $r = 1, 2, \cdots$. Consequently, the system of $2n$ equations (7) reduces to the n equations

$$i\tau_h - \lambda\, d_{2h-1,2h-1} = i\sigma_h, \qquad h = 1, 2, \cdots, n,$$

where each $d_{2h-1,2h-1} = \sum_{r=1}^{\infty} \lambda^{r-1} d_{2h-1,2h-1}^{(r)}$, $h = 1, 2, \cdots, n$, is purely imaginary. From the implicit function theorem, there exist real numbers $\tau_1, \cdots, \tau_n$ analytic in λ for $|\lambda|$ sufficiently small satisfying the above system of equations. Consequently, there will be a solution of (4) with components z_{jh} of the form $\sum_{m=1}^{\infty} \lambda^m x_{jh}^{(m)}(t)$, where $x_{jh}^{(m)}(t)$ is given by (15). We shall then show that this solution leads to a fundamental system of bounded real AC solutions of (1) and, thus, the theorem will be proved.

In order to prove the following two lemmas, we need to rewrite (14) in terms of the components of the matrix $X^{(m)}$ as follows:

$$(16)\quad \begin{cases} x_{2j-1,q}^{(m)} = e^{i\tau_j t} \displaystyle\int e^{-i\tau_j \alpha} \Big\{ \sum_{l=1}^{n} \Big[\frac{1}{2i\sigma_l} \phi_{jl}(x_{2l-1,q}^{(m)} + x_{2l,q}^{(m)}) + \tfrac{1}{2}\psi_{jl}(x_{2l-1,q}^{(m)} \\ \qquad - x_{2l,q}^{(m)}) \Big] - (d_{2j-1,2j-1}^{(m)} x_{2j-1,q}^{(0)} + d_{2j-1,2j-1}^{(1)} x_{2j-1,q}^{(m-1)} \\ \qquad + \cdots + d_{2j-1,2j-1}^{(m-1)} x_{2j-1,q}^{(1)}) \Big\}\, d\alpha, \\ x_{2j,q}^{(m)} = e^{-i\tau_j t} \displaystyle\int e^{i\tau_j \alpha} \Big\{ -\sum_{l=1}^{n} \Big[\frac{1}{2i\sigma_l} \phi_{jl}(x_{2l-1,q}^{(m)} + x_{2l,q}^{(m)}) + \tfrac{1}{2}\psi_{jl}(x_{2l-1,q}^{(m)} \\ \qquad - x_{2l,q}^{(m)}) \Big] - (d_{2j,2j}^{(m)} x_{2j,q}^{(0)} + d_{2j,2j}^{(1)} x_{2j,q}^{(m-1)} \\ \qquad + \cdots + d_{2j,2j}^{(m-1)} x_{2j,q}^{(1)}) \Big\}\, d\alpha, \\ \qquad j = 1, 2, \cdots, n, \quad q = 1, 2, \cdots, 2n. \end{cases}$$

LEMMA 1. *If we apply the preceding algorithm to system* (9) *with* $a_{2j-1} = b_j$, $a_{2j} = -\bar{b}_j$, $j = 1, 2, \cdots, n$, *then*

$$x_{2j-1,2h-1} = -\bar{x}_{2j,2h}^{(r)}, \qquad x_{2j,2h-1}^{(r)} = -\bar{x}_{2j-1,2h}, \qquad d_{2h-1,2h-1}^{(r)} = \bar{d}_{2h,2h}^{(r)}$$

for $r = 0, 1, 2, \cdots, h = 1, 2, \cdots, n, j = 1, 2, \cdots, n$.

Proof. We shall first prove by induction that $x_{2j-1,2h-1}^{(r)} = -\bar{x}_{2j,2h}^{(r)}$ and $x_{2j,2h-1}^{(r)} = -\bar{x}_{2j-1,2h}^{(r)}$ for all r, j, h. The assertion is clearly true for $r = 0$ by our choice of $X^{(0)}$. Assume the assertion true for $r = 0, 1, \cdots, m - 1$ and all j, h. From the definition of Φ^* in (4) and relation (11), we have

$$(17)\quad \begin{cases} b_h d^{(r)}_{2h-1,2h-1} = \sum_{p=1}^{r} \lambda^{p-1} m\left[e^{-i\tau_h t}\sum_{l=1}^{n}\left\{\frac{1}{2i\sigma_l}\phi_{hl}(x^{(p)}_{2l-1,2h-1}+x^{(p)}_{2l,2h-1})\right.\right. \\ \qquad\qquad\left.\left. +\tfrac{1}{2}\psi_{hl}(x^{(p)}_{2l-1,2h-1}-x^{(p)}_{2l,2h-1})\right\}\right], \\ \bar{b}_h d^{(r)}_{2h,2h} = \sum_{p=1}^{r} \lambda^{p-1} m\left[e^{i\tau_h t}\sum_{l=1}^{n}\left\{\frac{1}{2i\sigma_l}\phi_{hl}(x^{(p)}_{2l-1,2h}+x^{(p)}_{2l,2h})\right.\right. \\ \qquad\qquad\left.\left. +\tfrac{1}{2}\psi_{hl}(x^{(p)}_{2l-1,2h}-x^{(p)}_{2l,2h})\right\}\right], \quad r = 1, 2, \cdots. \end{cases}$$

Therefore, from our assumption on the $x^{(r)}_{jh}$ and the fact that $\bar{m}[f] = m[\bar{f}]$ and the τ_j are real numbers, we have

$$(18)\qquad \bar{d}^{(r)}_{2h-1,2h-1} = d^{(r)}_{2h,2h}, \qquad r = 1, 2, \cdots, m, \quad h = 1, 2, \cdots, n.$$

Now, if we use (18) and our assumption on the $x^{(r)}_{jh}$, we see easily from (16) that $\bar{x}^{(m)}_{2j-1,2h-1} = -x^{(m)}_{2j,2h}$, $\bar{x}^{(m)}_{2j,2h-1} = -x^{(m)}_{2j-1,2h}$ and the induction on the $x^{(r)}_{jh}$ is completed. Finally, (18) holds for all r and the lemma is proved.

LEMMA 2. *If Φ and Ψ satisfy conditions (α) and (β) of the theorem and*

$$a_{2s-1} = b_s = ic_s, \qquad a_{2s} = -\bar{b}_s = ic_s, \qquad s = 1, 2, \cdots, \mu,$$
$$a_{2t-1} = b_t = c_t, \qquad a_{2t} = -\bar{b}_t = -c_t,$$

where $c_1, \cdots, c_n$ are nonzero real numbers, then

$$x^{(r)}_{2u-1,2j-1}(-t) = x^{(r)}_{2u,2j}(t), \qquad x^{(r)}_{2u-1,2j}(-t) = x^{(r)}_{2u,2j-1}(t), \qquad u = 1, 2, \cdots, \mu,$$
$$x^{(r)}_{2v-1,2j-1}(-t) = -x^{(r)}_{2v,2j}(t), \qquad x^{(r)}_{2v-1,2j}(-t) = -x^{(r)}_{2v,2j-1}(t),$$
$$v = \mu+1, \cdots, n, \quad j = 1, 2, \cdots, n,$$

and

$$d^{(r)}_{2h-1,2h-1} = -d^{(r)}_{2h,2h}, \qquad r = 0, 1, 2, \cdots, \quad h = 1, 2, \cdots, n.$$

Proof. We shall first prove by induction that the $x^{(r)}_{jh}$ satisfy the above property. By our choice of the numbers a_j, we see that the assertion is true for $r = 0$. Assume the assertion true for $r = 0, 1, \cdots, m-1$. Let

$$Q^{(p)}_{jk}(t) = \phi_{jk}(t)[x^{(p)}_{2k-1,2j-1}(t) + x^{(p)}_{2k,2j-1}(t)],$$
$$R^{(p)}_{jk}(t) = \phi_{jk}(t)[x^{(p)}_{2k-1,2j}(t) + x^{(p)}_{2k,2j}(t)],$$
$$S^{(p)}_{jk}(t) = \psi_{jk}(t)[x^{(p)}_{2k-1,2j-1}(t) - x^{(p)}_{2k,2j-1}(t)],$$
$$T^{(p)}_{jk}(t) = \psi_{jk}(t)[x^{(p)}_{2k-1,2j}(t) - x^{(p)}_{2k,2j}(t)].$$

Then, from our assumption regarding the $x^{(r)}_{jk}$, we have

$$Q^{(p)}_{h,l}(-t) = R^{(p)}_{h,l}(t), \qquad S^{(p)}_{h,l}(-t) = T^{(p)}_{h,l}(t), \qquad \text{for } h \leqq \mu,\ l \leqq n,$$
$$Q^{(p)}_{h,l}(-t) = -R^{(p)}_{h,l}(t), \qquad S^{(p)}_{h,l}(-t) = -T^{(p)}_{h,l}(t), \qquad \text{for } \mu < h \leqq n,\ l \leqq n.$$

Therefore, since $m[f(-t)] = m[f(t)]$, we may replace t by $-t$ in the first part of (17) to obtain

$$b_s\, d^{(r)}_{2s-1,2s-1} = \bar{b}_s\, d_{2s,2s}\,, \qquad s = 1, 2, \cdots, \mu,$$
$$b_t\, d^{(r)}_{2t-1,2t-1} = -\bar{b}_s\, d^{(r)}_{2t,2t}\,, \qquad t = \mu + 1, \cdots, n.$$

Since $b_s = ic_s$, $s = 1, 2, \cdots, \mu$, and $b_t = c_t$, $t = \mu + 1, \cdots, n$, where $c_1, \cdots, c_n$ are nonzero real numbers, we have

$$d^{(r)}_{2h-1,2h-1} = -d^{(r)}_{2h,2h}\,, \qquad h = 1, 2, \cdots, n, \quad r = 1, 2, \cdots, m. \tag{19}$$

If $f(t) \in C_\omega$, $m[f] = 0$ and $F(t) \in C_\omega$ is the unique primitive of f of mean value zero, then it is clear that $F(-t)$ is equal to the negative of the primitive of $f(-t)$ of mean value zero. Consequently, if we use (19) and the above properties of the functions $Q^{(r)}_{jk}$, $R^{(r)}_{jk}$, $S^{(r)}_{jk}$, $T^{(r)}_{jk}$, then it is easily shown that

$$x^{(m)}_{2u-1,2j-1}(-t) = x^{(m)}_{2u,2j}(t), \qquad x^{(m)}_{2u-1,2j}(-t) = x^{(m)}_{2u,2j-1}(t),$$
$$u = 1, 2, \cdots, \mu, \quad j = 1, 2, \cdots, n,$$
$$x^{(m)}_{2v-1,2j-1}(-t) = -x^{(m)}_{2v,2j}(t), \qquad x^{(m)}_{2v-1,2j}(-t) = -x^{(m)}_{2v,2j-1}(t),$$
$$v = \mu + 1, \cdots, n, \quad j = 1, 2, \cdots, n,$$

and the induction on the $x^{(r)}_{jh}$ is completed. Consequently, formula (19) holds for all r and the lemma is proved.

Thus, from the remarks preceding Lemma 1, it remains only to show that the above solution of (4) leads to a fundamental system of bounded real AC solutions of (1). This is clearly the case since the solution of (4) constructed in the above manner is a fundamental system of AC solutions and it is obviously bounded since the numbers τ_j are real. However, we wish to go further and actually show how to obtain, from the solution of (4), a fundamental system of real solutions of (1) and discuss the nature of this solution.

Let the solution of (4) be denoted by $((z_{jh}))$ where

$$z_{jh} = \textstyle\sum_{m=1}^{\infty} \lambda^m x^{(m)}_{jh}(t), \qquad j, h = 1, 2, \cdots, 2n, \tag{20}$$

and $x^{(m)}_{jh}(t)$ is defined by (16). Then applying Lemma 1 and the transformation formulas (2), we see that the columns of the $n \times 2n$ matrix $Y = ((y_{jk}))$ where

$$\begin{cases} y_{j,2h-1} = \dfrac{1}{2i\sigma_j}\,(z_{2j-1,2h-1} + z_{2j,2h}) + \dfrac{1}{2i\sigma_j}\,(z_{2j,2h-1} + z_{2j-1,2h}), \\ y_{j,2h} = \tfrac{1}{2}(z_{2j-1,2h-1} - z_{2j,2h}) + \tfrac{1}{2}(z_{2j,2h-1} - z_{2j-1,2h}), \\ \qquad\qquad h = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, n \end{cases} \tag{21}$$

form a fundamental system of real AC solutions of (1). Furthermore, using Lemma 2, we see that

$$
(22) \quad \begin{cases} y_{u,2h-1}(-t) = y_{u,2h-1}(t), \quad y_{u,2h}(-t) = -y_{u,2h}(t), \quad u = 1, 2, \cdots, \mu, \\ y_{v,2h-1}(-t) = -y_{v,2h-1}(t), \quad y_{v,2h}(-t) = y_{v,2h}(t), \\ \qquad\qquad v = \mu + 1, \cdots, n, \quad h = 1, 2, \cdots, n \end{cases}
$$

or each solution of the fundamental system of solutions of (1) is such that if the first μ components of $y_k = (y_{1k}, \cdots, y_{nk})$ are even (or odd), then the last $n - \mu$ components of y_k are odd (or even). R. A. Gambill [6] has given more detailed expressions for solutions of (1) represented by (21).

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