ON MODULES OF TRIVIAL COHOMOLOGY OVER A FINITE GROUP

BY TADASI NAKAYAMA

We say that a module A over a finite group G is *of trivial cohomology* when we have $H^n(H, A) = 0$ for every integer n and every subgroup H of G. Recently the writer proved:

THEOREM. *Let A be a module over a finite group G. Assume that for every prime p dividing the order $[G]$ of G there is an integer r_p such that*

$$H^{r_p}(H_p, A) = H^{r_p+1}(H_p, A) = 0,$$

where H_p is a p-Sylow subgroup of G. Then the G-module A is of trivial cohomology.

Our proof of this theorem in [8] (or [7]) was a combination of representation-theoretical arguments and an argument by so-called fundamental exact sequences in group cohomology. In the present note we shall give two (partly) new proofs, one by means of fundamental exact sequences only, like former proofs of a weaker form of the theorem ([5], [1], [2]), and one quite representation- or module-theoretical. Indeed, in the course of our latter proof, which makes use of an idea of Gaschütz, we shall obtain a result which may be considered as a structural characterization of a module of trivial cohomology.

1. Proof by fundamental exact sequences

Let G be a group and H an invariant subgroup of G. The theorem of fundamental exact sequences in group cohomology states [6], [4]: If n is a natural number and if A is a G-module such that $H^i(H, A) = 0$ for $i = 1, 2, \cdots, n-1$, then the sequence

$$0 \to H^n(G/H, A^H) \xrightarrow{\lambda} H^n(G, A) \xrightarrow{\rho} H^n(H, A)^G \xrightarrow{\tau} H^{n+1}(G/H, A^H) \xrightarrow{\lambda} H^{n+1}(G, A)$$

is exact, where M^G with a G-module M denotes the submodule of M consisting of all G-invariant elements of M and where λ, ρ and τ denote lift, restriction and transgression maps respectively. Dually we have the theorem of fundamental exact sequences in homology, which in case of a finite group G may be formulated in terms of negative dimensional cohomology groups as follows: If $n \geqq 2$ and if $H^{-i}(H, A) = 0$ for $i = 2, 3, \cdots, n-1$, then we have an exact sequence

$$0 \leftarrow H^{-n}(G/H, A_H) \leftarrow H^{-n}(G, A) \leftarrow H^{-n}(H, A)_G \leftarrow H^{-(n+1)}(G/H, A_H) \leftarrow H^{-(n+1)}(G, A)$$

Received September 22, 1956.

where M_G with a G-module M denotes the residue module of M modulo the submodule generated by the elements of form $(1 - \sigma)u$ ($\sigma \in G$, $u \in M$).

As a bridge between these two series of exact sequences the writer proved recently [9]: If G is a finite group, H an invariant subgroup of G, and A a G-module, then we have an exact sequence

$$0 \leftarrow H^0(G/H, A^H) \leftarrow H^0(G, A) \leftarrow H^0(H, A)_G \leftarrow H^{-1}(G/H, A^H) \leftarrow H^{-1}(G, A);$$

if further $H^0(H, A) = 0$, then we have an exact sequence

$$0 \leftarrow H^{-1}(G/H, A^H) \leftarrow H^{-1}(G, A) \leftarrow H^{-1}(H, A)_G \leftarrow H^{-2}(G/H, A^H) \leftarrow H^{-2}(G, A).$$

(The first half of the former of these two exact sequences had been given by Artin-Tate [1]). We have also duals to these sequences.

Now we turn to a proof of our Theorem. By the Sylow group argument in cohomology it is sufficient to consider the case of a p-group. Thus, let G be a p-group and A a G-module such that $H^r(G, A) = H^{r+1}(G, A) = 0$ for some integer r. If here $r \geqq 0$ then we put $B = A \otimes J \otimes \cdots \otimes J$ (rJ's), while we put $B = A \otimes I \otimes \cdots \otimes I$ ($-r$ I's) if $r < 0$, where $\otimes$ denotes the tensor multiplication over the integer ring and the tensor product of G-modules is considered as a G-module in the usual way by component-wise operation and where I, J are the dimension shifters of Artin-Tate-Chevalley; thus I is the ideal of the group algebra $Z[G]$ over the integer ring Z generated by the elements of form $1 - \sigma$ ($\sigma \in G$) and J is the residue module of $Z[G]$ modulo the principal ideal generated by the element $\sum_{\sigma \in G} \sigma$. Then B satisfies $H^0(G, B) = H^1(G, B) = 0$. So the third terms in the exact sequences

$$(1) \qquad 0 \leftarrow H^0(G/H, B^H) \leftarrow H^0(G, B) \leftarrow H^0(H, B)_G \leftarrow H^{-1}(G/H, B^H),$$

$$(2) \qquad 0 \rightarrow H^1(G/H, B^H) \rightarrow H^1(G, B) \rightarrow H^1(H, B)^G \rightarrow H^2(G/H, B^H)$$

are 0, where H is any invariant subgroup of G. Hence the second terms, i.e. $H^0(G/H, B^H)$, $H^1(G/H, B^H)$ are 0 too. Now, assume that G/H is cyclic. Then we have $H^n(G/H, B^H) = 0$ for every n, and in particular $H^{-1}(G/H, B^H) = H^2(G/H, B^H) = 0$. From the same exact sequences (1), (2) we have $H^0(H, B)_G = H^1(H, B)^G = 0$. But G is a p-group and the order of any element of $H^0(H, B)$, $H^1(H, B)$ divides the order $g = [G]$ of G. It follows that $H^0(H, B)$, $H^1(H, B)$ themselves are 0. For, as the ideal I modulo $gZ[G]$ of the group algebra $(Z/gZ)[G]$ is nilpotent, $H^0(H, B) \neq 0$ would entail $H^0(H, B) \neq IH^0(H, B)$, i.e. $H^0(H, B)_G \neq 0$. Further, if $H^1(H, B) \neq 0$, then it would contain a minimal (nonzero) G-submodule, say M, as we readily see by considering a G-submodule of $H^1(H, B)$ generated by a single element. As I modulo $gZ[G]$ is nilpotent, we should have $IM = 0$, i.e. $M \subset H^1(H, B)^G$. These prove $H^0(H, B) = H^1(H, B) = 0$. Hence we have the exact sequences

$$(3) \qquad 0 \leftarrow H^{-1}(G/H, B^H) \leftarrow H^{-1}(G, B) \leftarrow H^{-1}(H, B)_G,$$

$$(4) \qquad 0 \rightarrow H^2(G/H, B^H) \rightarrow H^2(G, B) \rightarrow H^2(H, B)^G.$$

Now, assume that the theorem is true for proper subgroups of G. (On assuming $G \neq 1$) take a proper invariant subgroup (for instance a maximal invariant subgroup) H of G such that G/H is cyclic. Then, as we have seen, $H^0(H, B) = H^1(H, B) = 0$. Hence, by our assumption, the H-module B is of trivial cohomology. We have in particular $H^{-1}(H, B) = H^2(H, B) = 0$ in (3), (4). As we have seen $H^{-1}(G/H, B^H) = H^2(G/H, B^H) = 0$ too, we have thus $H^{-1}(G, B) = H^2(G, B) = 0$. Hence B satisfies $H^1(G, B) = H^2(G, B) = 0$, and, therefore, $B \otimes J$ satisfies $H^0(G, B \otimes J) = H^1(G, B \otimes J) = 0$. Taking $B \otimes J$ instead of B, we have then $H^2(G, B \otimes J) = 0$, i.e. $H^3(G, B) = 0$. In this way we obtain $H^n(G, B) = 0$ for any $n > 0$. Similarly we have $H^n(G, B) = 0$ for $n \leqq 0$. Thus $H^n(G, A) = H^{n-r}(G, B) = 0$ for every n.

Further, for any proper subgroup K of G, there is an H as above which contains K; for instance a maximal invariant subgroup of G containing K may be taken as H. Since the H-module B is of trivial cohomology, we have $H^n(K, B) = 0$ for every n. Thus $H^n(K, A) = 0$ for every n. Hence the G-module A is of trivial cohomology. Now our Theorem for p-groups G follows readily by induction with respect to composition lengths.

2. Torsion free modules of trivial cohomology over a p-group

We now turn to our representation-theoretical proof of the Theorem. In [8] we proved

LEMMA 1. *Let G be a p-group and A a G-module such that $H^r(G, A/pA) = H^{r+1}(G, A^{(p)}) = 0$ for some integer r, where $A^{(p)} = \{a \in A \mid pA = 0\}$. For any p-torsion free G-module M we have*

$$H^r(G, A \otimes M) = 0.$$

We shall in the sequel make use of the following special case:

LEMMA 1′. *Let G be a p-group and A a p-torsion free G-module satisfying $H^r(G, A/pA) = 0$ for some integer r. For any representation module M of G over Z (i.e. finitely generated torsion free G-module) we have $H^r(G, A \otimes M) = 0$.*

As in the situation of Lemma 1′ we may consider $A \otimes M_1$ as a submodule of $A \otimes M_2$, where M_1 and M_2 are two representation modules of G over Z satisfying $pM_2 \subset M_1 \subset M_2$, the wording is made a little simpler in the proof of Lemma 1′ than in the proof of Lemma 1. Though we do not repeat these proofs here, they may easily be reconstructed from our proof of Lemma 7 and the second approach to Proposition 3 below.

LEMMA 2. *Let G be a finite group and A a q-torsion group, where q is a natural number. If $H^r(G, A) = H^{r+1}(G, A) = 0$ for some integer r, then $H^r(G, A/qA) = 0$.*

Proof. This is clear from the exact sequence

$$H^r(G, A) \to H^r(G, A/qA) \to H^{r+1}(G, qA)$$

and the observation that qA is G-isomorphic to A.

Now, with a finite group G, let us say that a G-module A is *of quasi-trivial cohomology* when we have $H^n(G, A) = 0$ for all integers n.

PROPOSITION 3. *Let G be a p-group and A a p-torsion free G-module. If $H^r(G, A/pA) = 0$ for some r, then $H^n(G, A) = 0$ for all n, i.e. the G-module A is of quasi-trivial cohomology.*

Proof. This follows readily from Lemma 1′ by taking as M the dimension shifters I, J and their products.

PROPOSITION 4. *Let G be a p-group and A a p-torsion free G-module. If $H^r(G, A) = H^{r+1}(G, A) = 0$ for some r, then $H^n(G, A/pA) = 0$ for all n.*

Proof. We have $H^r(G, A/pA) = 0$ by Lemma 2. Then by Proposition 3 $H^n(G, A) = 0$ for all n. Again by Lemma 2 we obtain $H^n(G, A/pA) = 0$ for all n.

From Propositions 3, 4 we obtain

PROPOSITION 5. *Let G be a p-group and r an integer. A p-torsion free module G-module A is of quasi-trivial cohomology if and only if*

$$H^r(G, A/pA) = 0.$$

The same propositions suggest, if do not prove:

PROPOSITION 6. *Let G be a p-group and A a G-module satisfying $pA = 0$, $H^r(G, A) = 0$ for some r. Then $H^n(G, A) = 0$ for all n, i.e. the G-module A is of quasi-trivial cohomology.*

Proof. This follows, if we take I/pI, J/pJ and their products as M, from

LEMMA 7. *Let G, A be the same as in Proposition 6. Then $H^r(G, A \otimes \mathrm{M}) = 0$ for any representation module M of G over the field Z/pZ.*

Proof. The proof is similar to, and simpler than, a part of the proof of Lemma 1 (or Lemma 1′). Thus, let N be a maximal G-submodule of M. Then M/N is G-isomorphic to Z/pZ, G operating on Z trivially. As N is a direct summand of M as Z/pZ-module (or as Z-module) $A \otimes N$ may be looked upon as a submodule of $A \otimes M$ and we have readily the G-isomorphism $(A \otimes M)/(A \otimes N) \cong A/pA = A$. Hence we have the exact sequence

$$H^r(G, A \otimes N) \rightarrow H^r(G, A \otimes M) \rightarrow H^r(G, (A \otimes M)/(A \otimes N)) = H^r(G, A) = 0.$$

It follows that $H^r(G, A \otimes M) = 0$ whenever we have $H^r(G, A \otimes N) = 0$. Now the lemma follows readily by induction with respect to the composition length of M.

(In case G is cyclic, Proposition 6 can easily be proved structurally. Indeed, all the cohomology groups $H^n(G, A)$ $(n = 0, \pm 1, \cdots)$ are isomorphic then. For the group algebra $(Z/pZ)[G]$ is then uni-serial and A is a direct sum of G-submodules (perhaps infinite in number) each of which is isomorphic to an ideal of $(Z/pZ)[G]$. If m is the (finite or infinite) number of those summands which are not isomorphic to $(Z/pZ)[G]$ itself, then we see easily that both $H^{-1}(G, A)$ and $H^0(G, A)$ are a direct sum of m cyclic modules of order p.

Having thus proved the cyclic case of Proposition 6 structurally, we may derive the general case of Proposition 6 from it by means of the exact sequences (1), (2) (only)).

Proposition 6 being thus proved directly, it is perhaps of some interest to derive Proposition 3, and thence Propositions 4, 5, from Proposition 6. Thus:

Second approach to Propositions 3, 4, 5. To prove Proposition 3, assume $H^r(G, A/pA) = 0$ where A is a p-torsion free module over a p-group G. By Proposition 6 we have $H^n(G, A/pA) = 0$ for every n. What we have to do is to derive $H^n(G, A) = 0$ from $H^n(G, A/pA) = 0$, for each n. As a special case $M = Z$ of Lemma 1′, this may be seen as follows by specializing the proof of Lemma 1′. Thus, observe that the sum $Z(\sum_{\sigma\in G}\sigma) + I$ in $Z[G]$ is direct and that $gZ[G]$ is contained in this sum, where $g = [G]$ denotes the order of G and is a power of p. There exists therefore an increasing finite series of G-modules $N_i\,(i = 0, 1, \cdots, t)$ such that $N_0 = gZ[G]$, $N_t = Z(\sum_G \sigma) + I$, and for each $i = 0, \cdots, t-1$ the residue module N_{i+1}/N_i is G-isomorphic to Z/pZ. Now, as A is p-torsion free, $A \otimes N_i$ may be looked upon as a submodule of $A \otimes N_{i+1}$ and the residue module $(A \otimes N_{i+1})/(A \otimes N_i)$ is G-isomorphic to A/pA. From the exact sequence $H^n(G, A \otimes N_i) \to H^n(G, A \otimes N_{i+1}) \to H^n(G, A/pA) = 0$ we see that $H^n(G, A \otimes N_{i+1})$ is a homomorphic image of $H^n(G, A \otimes N_i)$. However, since $A \otimes N_0 = A \otimes gZ[G]$ is, together with $gZ[G]$, a regular G-module, we have $H^n(G, A \otimes N_0) = 0$. It follows that $H^n(G, A \otimes N_t) = 0$. Here $A \otimes N_t$ is the direct sum $A \otimes Z(\sum_G \sigma) + A \otimes I$ and we have $H^n(G, A \otimes Z(\sum_G \sigma)) = 0$. So $H^n(G, A) = 0$ since $Z(\sum_G \sigma)$ is G-isomorphic to Z. This proves Proposition 3.

However, the above derivation of $H^n(G, A) = 0$ from $H^n(G, A/pA) = 0$ is, by a specialization of Lemma 1′, a lemma which is so designed as to be applied also to "dimension shifting", and is somewhat cumbersome. Perhaps the following argument, which makes use of a remark of G. Rayna, is more natural. Thus, assume $H^n(G, A/pA) = 0$ with a p-torsion free G-module A and consider the exact sequence

$$H^n(G, pA/p^2A) \to H^n(G, A/p^2A) \to H^n(G, A/pA) = 0.$$

Since pA/p^2A is G-isomorphic to A/pA, the first term of this exact sequence is 0 too. Hence $H^n(G, A/p^2A) = 0$. By recursion we obtain $H^n(G, A/p^hA) = 0$ for any $h \geqq 1$, and in particular $H^n(G, A/gA) = 0$ with $g = [G]$. However, by a remark of Rayna, communicated to the writer, $H^n(G, A)$ is monomorphically mapped into $H^n(G, A/gA) = 0$. For, we have the exact sequence $H^n(G, gA) \to H^n(G, A) \to H^n(G, A/gA)$ where the first map is induced by the natural embedding. Since A is g-torsion free, this embedding map is the trace of the map $ga \to a$ ($a \in A$). So the first arrow of the above exact sequence is a zero map. Hence $H^n(G, A)$ is mapped monomorphically intn $H^n(G, A/gA)$, and $H^n(G, A/gA) = 0$ entails $H^n(G, A)$. Thus Proposition 3 is proved.

As before, Proposition 4 follows from Proposition 3 and Lemma 2 while Proposition 5 follows from Propositions 3 and 4.

On returning to our main trend of study we prove

LEMMA 8. *Let G be a p-group. A G-module A satisfying $pA = 0$ has an independent basis over $(Z/pZ)[G]$ if, and only if, $H^{-1}(G, A) = 0$.*

Proof. It suffices to prove the "if" part. The radical R of the group algebra $(Z/pZ)[G]$ is the ideal generated by the elements $1 - \sigma$ ($\sigma \in G$) and is thus nothing but I modulo $pZ[G]$. Let a_γ, γ running over an index set Γ, be an independent Z/pZ-basis of A modulo RA. As clearly a_γ G-generate A modulo RA, they G-generate A. Now we introduce a set of elements b_γ in 1–1 correspondence with Γ and construct a $(Z/pZ)[G]$-module B having b_γ as a free (i.e. independent) $(Z/pZ)[G]$-basis. By $b_\gamma \to a_\gamma$ we obtain a G-epimorphism of B onto A. If S_G denotes the trace map with respect to G, $S_G((Z/pZ)[G])$ is the ideal of $(Z/pZ)[G]$ generated by the element $\sum_G \sigma$ and is the totality of elements $u \in (Z/pZ)[G]$ with $Ru = 0$. So $S_G(B)$ is the totality of elements $y \in B$ with $Ry = 0$, or, what is the same, $S_G(B)$ is the maximal fully reducible submodule of the G-module B.

Let C be the kernel of our epimorphism $B \to A$ and suppose that $C \neq 0$. Then $C \cap S_G(B) \neq 0$ and there is an element $\sum_{\gamma\in\Gamma} \xi_\gamma \sum_{\sigma\in G} \sigma b_\gamma \neq 0$ ($\xi_\gamma \in Z/pZ$) in $C \cap S_G(B)$. Consider the element $\sum_\gamma \xi_\gamma a_\gamma$ of A. Since $S_G(a)$ is the image, by the epimorphism $B \to A$, of the element $S_G(\sum_\gamma \xi_\gamma b_\gamma) = \sum_\gamma \xi_\gamma \sum_\sigma \sigma b_\gamma \in C$, we have $S_G(a) = 0$. However, as a_γ are (Z/pZ)-independent modulo RA, we have $a \notin RA$. Here RA is nothing but the submodule of A generated by the elements of form $(1 - \sigma)x$ ($\sigma \in G$, $x \in A$). Thus we have $H^{-1}(G, A) \neq 0$, contrary to our assumption. This shows that $C = 0$ and our G-epimorphism $B \to A$ is an isomorphism. So A has, together with B, an independent basis over $(Z/pZ)[G]$.

In combination with Proposition 6 we have

PROPOSITION 9. *Let G be a p-group, and r an integer. A G-module A satisfying $pA = 0$ has an independent basis over $(Z/pZ)[G]$ if, and only if, $H^r(G, A) = 0$.*

For a representation module over Z/pZ (and for $r, n \geqq 1$) Propositions 6 and 9 have been obtained by Gaschütz [3]. However, his argument, as it stands, does not mean to be applied to infinitely generated A.

As a G-module with an independent $(Z/pZ)[G]$-basis has naturally an independent $(Z/pZ)[H]$-basis for every subgroup H of G, we see in particular that a module A over a p-group G satisfying $pA = 0$ is of trivial cohomology whenever it is of quasi-trivial cohomology. By Propositions 3, 4 the same is true for a p-torsion free G-module. In fact, the same remains the case for any module over a p-group as we shall see in the next section.

3. General modules of trivial cohomology

PROPOSITION 10. *Let G be a p-group and A a G-module. If $H^r(G, A) = H^{r+1}(G, A) = 0$ for some integer r, then A is of trivial cohomology. In particular, A is of trivial cohomology when it is of quasi-trivial cohomology.*

Proof. Take a p-torsion free G-module A_0 of trivial cohomology of which A is a G-homomorphic image; for instance we may take as A_0 a free G-module over a G-generating system of A. Let A_1 be the kernel of the G-homomorphism $A_0 \to A$, thus $A \cong A_0/A_1$. We have the exact sequence

$$H^n(G, A) \to H^{n+1}(G, A_1) \to H^{n+1}(G, A_0)$$

for every n. Here the last term is always 0, while by our assumption $H^r(G, A) = H^{r+1}(G, A) = 0$ the first term is 0 for $n = r, r + 1$. It follows that $H^{r+1}(G, A_1) = H^{r+2}(G, A_1) = 0$. Moreover, A_1 is p-torsion free. By Propositions 3, 4 the G-module A_1 is of quasi-trivial cohomology, and hence of trivial cohomology too, as has been observed at the end of the preceding section. From the exact sequence

$$0 = H^n(H, A_0) \to H^n(H, A) \to H^{n+1}(H, A_1) = 0,$$

where H is any subgroup of G, we have $H^n(H, A) = 0$, for every n, and A is of trivial cohomology.

By the Sylow group argument we obtain our Theorem from Proposition 10. Note that in our proof Proposition 9 is used for $A = A_1/pA_1$ with a p-torsion free A_1 and thus Proposition 6 and Lemma 7 are unnecessary.

The above considerations give in a sense *a structural characterization of modules of trivial cohomology*. Namely, let G be a finite group and A a G-module. Let A_0 be a free G-module of which A is a G-homomorphic image, and let A_1 be the kernel of the homomorphism. Then, our *G-module A is of trivial cohomology if and only if for every prime p dividing $[G]$ the module A_1/pA_1 has an independent basis over $(Z/pZ)[H_p]$, where H_p is a p-Sylow subgroup of G.* (If A itself is $[G]$-torsion free, then the transition to A_1 is unnecessary and we may assert that it is of trivial cohomology if and only if A/pA has an independent $(Z/pZ)[H_p]$-basis for every $p \mid [G]$).

Added in proof. G. Rayna kindly communicates to the writer a further different proof. He points out namely that the "passage to subgroups" is effected readily also by Lemma 1 and Shapiro's relation ([5]; cf. CARTAN-EILENBERG, *Homological algebra*, Princeton, 1956, X, 7.4).

REFERENCES

1. E. ARTIN AND J. TATE, *Algebraic numbers and algebraic functions*, to appear.
2. C. CHEVALLEY, *Class field theory*, Lecture notes, Nagoya University, 1953–54.

3. W. GASCHÜTZ, *Habilitationsschrift*, University of Kiel.
4. A. HATTORI, *On exact sequences of Hochschild and Serre*, J. Math. Soc. Japan, vol. 7 (1955), pp. 312–321.
5. G. HOCHSCHILD AND T. NAKAYAMA, *Cohomology in class field theory*, Ann. of Math., vol. 55 (1952), pp. 348–372.
6. G. HOCHSCHILD AND J-P. SERRE, *Cohomology of group extensions*, Trans. Amer. Math. Soc., vol. 74 (1953), pp. 110–134.
7. T. NAKAYAMA, *A theorem on modules of trivial cohomology over a finite group*, Proc. Japan Acad., vol. 32 (1956), pp. 373–376.
8. ———, *Cohomology of class field theory and tensor product modules*, to appear in Ann. of Math.
9. ———, *A remark on fundamental exact sequences in cohomology of finite groups*, to appear in Proc. Japan Acad.

NAGOYA UNIVERSITY
NAGOYA, JAPAN