FINITE DIMENSIONALITY OF CERTAIN TRANSFORMATION GROUPS

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1. Introduction

The main result to be proved in this paper is as follows:

THEOREM A. *If G is a locally compact effective transformation group of a manifold M, then G is finite-dimensional.*

By a manifold M is meant a separable, metric, connected, locally euclidean space. For the proof of Theorem A it will be sufficient to consider the case where G is compact. This is because an infinite-dimensional locally compact group contains an infinite-dimensional compact subgroup [4, 5].

This theorem gives no information on whether or not G must be a Lie group. Information on this latter question depends on an analysis of the case where G is zero-dimensional.

The proof of Theorem A will be based on Theorem B which is known for compact Lie groups (see [7] and for extensions [6] and [8]).

THEOREM B. *Let G be a compact connected group which acts on an n-dimensional manifold M and let F be the set of points of M left fixed by every element of G. If* $\dim F \geqq n - 1$, *then* $F = M$.

2. Reduction of Theorem A to Theorem B

It has been shown in the introduction that for the proof of Theorem A, the group G may be assumed compact. It is therefore now assumed that G is a compact effective transformation group of a manifold M.

Let G^* be the identity component of G, and let k be the highest dimension of any orbit of G^*. The set of points of M which lie on k-dimensional orbits is an open set and a component of this open set will be denoted by Y. The group G^* acts as a transformation group of Y,

$$G^*(Y) = Y,$$

although it may, conceivably, not be effective. Let H be the subgroup of G^* which leaves every point of Y fixed. Then H is an invariant subgroup of G^*, and G^*/H acts in a natural way on Y and the action is effective. It is known [5, p. 243] that G^*/H is finite-dimensional.

Assuming that Theorem B is true, it follows that H^* leaves all of M fixed. But G was taken effective so H^* can contain only the identity, and H is zero-dimensional. Then

$$\dim G = \dim G^* = \dim G^*/H,$$

and G is finite-dimensional.

This completes the proof that Theorem B implies Theorem A.

Received September 8, 1956.

3. Reduction to a solenoid

A solenoid, by definition, is a compact connected one-dimensional group which is therefore an inverse limit of circle groups. (Note that for the purposes of this paper the definition of solenoid includes the circle as a special case.) In the statement of Theorem B, G is not required to be effective, but on the other hand the theorem for G not effective follows at once from the effective case. Hence it may be assumed in proving Theorem B that G is effective. Hence G is an inverse limit of a countable sequence of compact connected Lie groups [5, p. 63].

Lemma 1. *For the proof of Theorem* B *it is sufficient to let* G *be a solenoid.*

By the remarks above, the G of Theorem B may be assumed effective, and hence G is separable and the inverse limit of a sequence of compact connected Lie groups G_i,

$$G = \lim G_i, \qquad G_i = f_i(G_{i+1}).$$

In order to prove Lemma 1 it will be enough to show that elements of G which lie on solenoids are everywhere dense in G.

In order to do this let g be any element of G so that g, as a sequence of the inverse limit, has the form

$$g = (g_1, g_2, \cdots, g_i, g_{i+1}, \cdots), \qquad g_i \in G_i, \qquad g_i = f_i(g_{i+1}).$$

Let i be a fixed index, and let K_i be a circle subgroup of G_i which contains h_i as near as we please to g_i. There is a circle subgroup K_{i+1} of G_{i+1} such that

$$K_i = f_i(K_{i+1}).$$

Continuing, there is obtained the circle groups

$$K_i, K_{i+1}, K_{i+2}, \cdots$$

where each group is the image of the succeeding one. Projecting K_i to K_{i-1}, which is a circle group in G_{i-1}, and so on gives the sequence of circle groups

$$K_1, K_2, \cdots, K_i, K_{i+1}, \cdots.$$

This is an inverse sequence of circles having for limit a solenoid in G. This solenoid contains the element

$$g' = (h_1, h_2, \cdots, h_i, g'_{i+1}, \cdots).$$

The first i coordinates of g' are near the first i coordinates of g. Thus given g in G and an integer i, there is an element g' in G, where g' is on a solenoid in G, and g' and g differ in each of the first i coordinates by as little as we please. This proves that elements of G, which are in solenoids in G, are dense in G, and this proves Lemma 1.

4. Invariant cycles

From this point on, it is assumed that G is a solenoid acting on the manifold M, and that F is the set of points left fixed by every element of G. Let $U = M - F$ so that every point of U has a one-dimensional orbit.

Let T be a compact zero-dimensional subgroup of G such that G/T is a circle group. The space M/T will be denoted by M^*, M/G by M^{**}, with a corresponding notation for subsets. Notice that (letting a homeomorphism be indicated by $=$)

$$M/T/G/T = M^*/G/T = M^{**}.$$

Since G/T is a circle, M^{**} is the orbit space of M^* as acted on by the circle group G/T.

The homology used in this paper will always have for coefficients the real numbers R, and this is to be understood whether R is indicated explicitly or not.

If $A \subset X$ are invariant compact subsets of M, let

$$I_i(X, A; R) = \{x \mid x \in H_i(X, A; R), g(x) = x, g \in T\},$$

that is let $I_i(X, A; R)$ denote all elements of $H_i(X, A; R)$ which are invariant under every element of T. As remarked above, the following abbreviations and similar ones are often used:

$$H_i(X, A) = H_i(X, A; R)$$

$$I_i(X, A) = I_i(X, A; R).$$

LEMMA 2. *If $A \subset X$ are invariant compact subsets of M, then*

$$I_i(X, A) = H_i(X^*, A^*).$$

When T is a finite group, this fact has been shown by Liao, Floyd, and Conner (see [1] and the references there given). The fact that the lemma is true for finite groups will be used to prove the general case. Let

$$T = T_1 \supset T_2 \supset \cdots$$

be a decreasing sequence of compact zero-dimensional subgroups with T_i/T_{i+1} finite for all i and $\bigcap T_i = e$. This determines the sequence of spaces

$$M/T = M/T_1 \leftarrow M/T_2 \leftarrow \cdots$$

where each is the image of the succeeding as indicated. Thus M/T_j is the image of M/T_{j+1} under a map determined by forming the orbit space of the finite group T_j/T_{j+1} in its action on M/T_{j+1}. With these maps, M is the inverse limit of the sequence of spaces M/T_j

$$M = \lim M/T_j,$$

and similarly the pair (X, A) is the limit:

$$(X, A) = \lim (X/T_j, A/T_j).$$

Since real Čech homology on compact pairs is continuous [2, p. 261], it follows that

$$H_i(X, A) = \lim H_i(X/T_j, A/T_j). \tag{1}$$

Now the known fact for finite groups is

$$H_i(X/T_j, A/T_j) = I_i(X/T_{j+k}, A/T_{j+k})$$

where in this latter equation invariance is with respect to the finite group T_j/T_{j+k} acting on $(X/T_{j+k}, A/T_{j+k})$. Hence

$$(2) \qquad H_i(X/T_1, A/T_1) = I_i(X/T_j ; A/T_j),$$

invariance being with respect to T_1/T_j, or, otherwise expressed, with respect to the action of T_1 since of course every element of T_j leaves every element of X/T_j fixed. For these groups of invariant cycles, there is the following map

$$(3) \qquad I_i(X/T_j, A/T_j) \leftarrow I_i(X/T_{j+1}, A/T_{j+1})$$

which is onto and is a part of the inverse sequence (1). Thus (3) determines an inverse sequence which is a subsequence of (1). We see that

$$I_i(X, A) = \lim I_i(X/T_j, A/T_j),$$

and because of (2)

$$I_i(X, A) = H_i(X/T_1, A/T_1),$$

which completes the proof.

5. Invariant cycles of dimension n

Lemma 3. *If (X, A) is an invariant compact pair, then for $i > 0$*

$$H_{n+i}(X^*, A^*) = I_{n+i}(X, A) = 0.$$

The proof is immediate.

The following is included as of some interest though it is not really needed.

Lemma 3′. *Let X be a compact invariant set in M with boundary A and with $X - A$ a connected open set. Let (Y, B) be a compact pair, satisfying*

(1) *$(X, A) \subset (Y, B)$,*
(2) *Y is a closed n-cell,*
(3) *B is a spherical shell and $Y - B$ is an open n-cell in $X - A$,*
(4) *for any $g \in T$, and $x \in A$, the segment from x to $g(x)$ is in B.*

Then for dimension n, we have

$$H_n(X, A) = H_n(X^*, A^*).$$

Proof. By Lemma 2

$$(*) \qquad I_n(X, A) = H_n(X^*, A^*).$$

It will next be proved that

$$(**) \qquad I_n(X, A) = H_n(X, A).$$

Let i be the identity map of (X, A) into (Y, B), so that there is the induced homology map

$$i_*: H_n(X, A) \rightarrow H_n(Y, B),$$

and take

$$y \in H_n(X, A).$$

Then

$$i_* y = z_1 \in H_n(Y, B),$$

$$i_* g_* y = z_2 \in H_n(Y, B).$$

By hypothesis the map g is homotopic to the identity so that

$$i_* y = i_* g_* y.$$

The map

$$i_*: H_n(X, A) \to H_n(Y, B)$$

is one–one, and therefore

$$y = g_* y.$$

This proves (**) and therefore completes the proof of the lemma.

Notice that if an X is given which is invariant under G and in an n-cell neighborhood, then T can always be chosen so that B exists, and it will always be assumed that this has been done.

6. Formulation of cases

The proof of Theorem B for G a solenoid will be made by means of a contradiction. Thus from this point on, it is assumed that Theorem B is false. There must then be a point p,

$$p \in B = \text{Boundary } U,$$

such that B separates any sufficiently small neighborhood of p. Let V be an open neighborhood of p such that

(a) $G(V) = V$,
(b) $\bar{V}$ is compact,
(c) any neighborhood of p which is in V is separated by B,
(d) $\bar{V}$ is contained in an n-cell neighborhood of p.

Let Y be a component of $V - B$ which is of course in U,

$$Y \subset U.$$

Let

$$X = \bar{Y}, \qquad A = \text{Boundary } X.$$

Then, letting $=$ denote isomorphism,

$$H_n(X, A) = R = \text{reals},$$

so that

$$H_n(X^*, A^*) = R.$$

Let z be a nonzero element of $H_n(X, A)$,

$$0 \neq z \in H_n(X, A),$$

and let y be the image of z under the map ∂, that is

$$\partial z = y \in H_{n-1}(A).$$

Now let

$$A_1 = \text{closure } (A \cap U), \qquad A_2 = (A \cap B),$$

so that

$$A = A_1 \cup A_2 ,$$

and define

$$D = A_1 \cap A_2 .$$

It will be necessary to use the Mayer-Vietoris sequence [2, p. 39] which is exact. This sequence is as follows:

$$\text{(I)} \qquad \begin{aligned} \cdots \leftarrow H_{n-2}(A_1) + H_{n-2}(A_2) \xleftarrow{\psi} H_{n-2}(D) \xleftarrow{\Delta} H_{n-1}(A) \\ \xleftarrow{\phi} H_{n-1}(A_1) + H_{n-1}(A_2) \xleftarrow{\psi} H_{n-1}(D) \leftarrow \cdots \end{aligned}$$

Two cases will be considered.

Case I. For $y \in H_{n-1}(A)$ as defined above, it is true that y is carried to zero in (I), that is

$$\Delta(y) = 0.$$

In this case the Mayer-Vietoris sequence shows that [2, p. 39]

$$y = m_{1*} v_1 + m_{2*} v_2$$

where $v_1 \in H_{n-1}(A_1)$, $v_2 \in H_{n-1}(A_2)$ and m_1, m_2 are inclusion maps of A_1, A_2 into A.

Case II. The element y is not carried to zero:

$$\Delta(y) = u \neq 0, \qquad u \in H_{n-2}(D).$$

In this case

$$u = \partial y_1 , \qquad y_1 \in H_{n-1}(A_1 , D),$$

$$u = \partial y_2 , \qquad y_2 \in H_{n-1}(A_2 , D).$$

Let f' be the map from X to X^{**}.

LEMMA 4. *In both Case* I *and Case* II, *for* $f'_*(y) \in H_{n-1}(A^{**})$,

$$y^{**} = f'_*(y) \neq 0,$$

and also in both cases y^{**} bounds in X^{**}, that is under the map $H_{n-1}(A^{**}) \to H_{n-1}(X^{**})$, y^{**} is carried to zero.

Case I will be considered first and to begin with *it will be shown in this case that $v_2 \neq 0$. Let x be an inner point of X and assume $v_2 = 0$*, so that $y = m_{1*} v_1$. Then x may be deformed outside of X (going through A_2) without touching A_1, and hence the point has index zero with respect to $y = m_{1*} v_1$. This is a contradiction which proves $v_2 \neq 0$. Now f' is a homeomorphism on A_2

so $f'_* m_{2*} v_2 \neq 0$. Then $y^{**} = f'_* y \neq 0$. Since y bounds in X, y^{**} bounds in X^{**} in Case I and Case II both. This completes the proof in Case I.

Case II will be considered next. In this case (for u defined above)

$$u^{**} = f'_*(u) \neq 0$$

because f' is a homeomorphism on D. Hence $y^{**} \neq 0$, $y^{**} \in H_{n-1}(A^{**})$. This completes the proof. It follows that $H_n(X^{**}, A^{**}) \neq 0$.

7. Product spaces and carriers

Let (W, E) be a compact pair, and let C be the circle so that

$$(W \times C, E \times C)$$

is also a compact pair.

LEMMA 5. *If $H_n(W, E) \neq 0$, then (real coefficients)*

$$H_{n+1}(W \times C, E \times C) \neq 0.$$

We use Čech homology with real coefficients as always. Let y be a nonzero element in $H_n(W, E)$ and let x be a nonzero element of $H_1(C)$. Then $y \times x$ can be defined in a natural way as a cycle in $H_n(W \times C, E \times C)$ and is not zero. This completes the proof.

Let (W, P, Q) be a compact triple

$$W \supset P \supset Q.$$

Then the following sequence is exact [2, p. 25]:

$$\cdots \leftarrow H_{n-1}(P, Q) \leftarrow H_n(W, P) \leftarrow H_n(W, Q) \leftarrow H_n(P, Q) \leftarrow \cdots .$$

From exactness it follows that if $H_n(W, P) = 0$, then the map

$$H_n(W, Q) \leftarrow H_n(P, Q)$$

is onto. This fact will be used below.

8. Proof of Theorem B

Now let P be a compact invariant subset of X so chosen that

(1) $Cl(X^* - P^*)$ is homeomorphic to a direct product of a circle and a subset of X^{**},

(2) $Cl(X - P) \subset U$,

(3) $A \subset P \subset X$.

By the strengthened excision property [2, p. 266],

$$H_n(X^{**}, P^{**}) = H_n[Cl(X^{**} - P^{**}), P^{**} \cap Cl(X^{**} - P^{**})].$$

Then by the lemma on a product by a circle we have

$$H_n(X^{**}, P^{**}) = 0;$$

for if this were not true, we would have $H_{n+1}(X^*, P^*) \neq 0$ which is false by Lemma 3. Hence the map

$$H_n(X^{**}, A^{**}) \leftarrow H_n(P^{**}, A^{**})$$

is onto. There is a minimal set $N^{**} \supset A^{**}$ such that the map

$$H_n(X^{**}, A^{**}) \leftarrow H_n(N^{**}, A^{**})$$

is onto (to see this, use continuity and the sequence for triples). We see from the result mentioned just above that

$$N^{**} = A^{**};$$

for if $N^{**} \neq A^{**}$, it can always be reduced, because in any closed, invariant set in U, there always exist invariant, relatively open sets where the circle group acts without singularity. But

$$H_n(A^{**}, A^{**}) = 0,$$

and hence

$$H_n(X^{**}, A^{**}) = 0.$$

This contradiction (Lemma 4 shows $H_n(X^{**}, A^{**}) \neq 0$) proves Theorem B which has been seen to prove Theorem A.

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