HOMOLOGY OF NOETHERIAN RINGS AND LOCAL RINGS

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Introduction

This paper contains a collection of results on the homology of a residue class ring R/M of a commutative Noetherian ring R, as R-module. More important than the individual results is the general method by which they are obtained, namely, the systematic use of skew-commutative graded differential algebras (called R-algebras in this paper, cf. §1). The functor

$$\mathrm{Tor}^R(R/M,\ R/N)$$

has naturally the structure of an R-algebra (cf. §5), so why not exploit this fact? We show in §2 that it is always possible to construct a free resolution of R/M which is an R-algebra, and in §3 and §4, we show that in some important cases our abstract method of construction yields a concrete efficient resolution (Theorem 4). Our "adjunction of variables" is a naive approach to the exterior algebras and twisted polynomial rings familiar to topologists, and the ideas involved were clarified in my mind by conversations with John Moore. In the long §6 we apply our methods to a local ring R and obtain generalizations of results of Serre and Eilenberg. In particular, Theorem 8 gives the correct lower bound for the Betti numbers of a nonregular local ring. I wish to thank Zariski and Artin for several stimulating general discussions in connection with these problems.

1. R-algebras

Let R be a commutative Noetherian ring with unit element. In this note we shall use the brief term *R-algebra* to denote an associative algebra X over R in which there is defined an R-linear mapping $d: X \to X$, such that the following axioms are satisfied:

(1) X is graded, i.e. $X = \sum_{\lambda=-\infty}^{\infty} X_\lambda$ is the direct sum of R-modules X_λ such that $X_\lambda X_\mu \subset X_{\lambda+\mu}$.

(2) $X_\lambda = 0$ for $\lambda < 0$; X has a unit element $1 \in X_0$ such that $X_0 = R1$; and X_λ is a finitely generated R-module for $\lambda > 0$.

(3) X is strictly skew-commutative, that is:

$$xy = (-1)^{\lambda\mu} yx, \qquad \text{for } x \in X_\lambda,\ y \in X_\mu$$

and

$$x^2 = 0, \qquad \text{for } x \in X_\lambda,\ \lambda \text{ odd}.$$

(4) The map d is a skew derivation of degree -1, that is, $dX_\lambda \subset X_{\lambda-1}$ for all λ, $d^2 = 0$, and

$$(*) \qquad d(xy) = (dx)y + (-1)^\lambda x(dy), \qquad \text{for } x \in X_\lambda,\ y \in X_\mu.$$

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A *homomorphism*, f, of an R-algebra X into an R-algebra Y is an R-linear ring homomorphism $f: X \to Y$ such that $f(X_\lambda) \subset Y_\lambda$ for all λ, $f(1) = 1$, and $fdx = dfx$ for all $x \in X$. An R-algebra X is an *R-subalgebra* of an R-algebra Y, if X is a subset of Y, and the inclusion map $X \to Y$ is a homomorphism.

An R-algebra X can be viewed as a complex of R-modules with boundary operator d:

$$\cdots \to X_n \xrightarrow{d} X_{n-1} \to \cdots\cdots \to X_1 \xrightarrow{d} X_0 \to 0 \to \cdots,$$

Let $Z = Z(X)$ be the kernel of d (group of cycles) and let $B = B(X)$ be the image of d (group of boundaries). Then $Z = \sum_\lambda Z_\lambda$ (direct sum), where $Z_\lambda = Z \cap X_\lambda$, and $B = \sum_\lambda B_\lambda$ (direct sum), where $B_\lambda = B \cap X_\lambda = dX_{\lambda+1}$. Since $d^2 = 0$ we have $B \subset Z$. The formula (*) for the derivative of a product shows

$$Z_\lambda Z_\mu \subset Z_{\lambda+\mu}, \qquad B_{\lambda-1} Z_\mu \subset B_{\lambda+\mu-1}, \quad \text{and} \quad Z_\lambda B_{\mu-1} \subset B_{\lambda+\mu-1}.$$

Hence Z is a graded subalgebra of X, and B is a homogeneous two-sided ideal in Z. The residue class algebra Z/B is called the *homology algebra* of X and is denoted by $H = H(X)$. Obviously H is graded; $H = \sum_\lambda H_\lambda$, where $H_\lambda = Z_\lambda/B_\lambda$. We say that X is *acyclic* if $H = H_0$, i.e. if $H_\lambda = 0$ for all $\lambda > 0$. We shall call X *free* if X_λ is a free R-module for each λ. If X is free, we have $X_0 = R1 \approx R$ and $B_0 \approx M$, a certain ideal of R, hence $H_0 \approx R/M$ is a residue class ring of R. If X is free and acyclic it furnishes us with a free resolution of the R-module R/M, that is, an exact sequence

$$\cdots \to X_2 \xrightarrow{d} X_1 \xrightarrow{d} R \xrightarrow{\varepsilon} R/M \to 0$$

in which the modules X_λ are R-free. It is our purpose to construct resolutions of this type and to show their usefulness by a few applications.

2. The process of adjoining a variable of degree ρ in order to kill a cycle of degree $\rho - 1$

Let X be an R-algebra. Let $\rho > 0$ be a positive integer. Let $t \in Z_{\rho-1}(X)$ be a cycle of degree $\rho - 1$. We shall now describe a canonical procedure for constructing an extension R-algebra $Y \supset X$ such that

(a) $$Y_\lambda = X_\lambda, \text{ for } \lambda < \rho, \text{ and}$$

(b) $$B_{\rho-1}(Y) = B_{\rho-1}(X) + Rt.$$

The procedure is quite different for the case of even ρ and the case of odd ρ, so we discuss the two cases separately:

ρ odd. Let XT be the free X-module with one basis element, T, and put $Y = X + XT$, direct sum. Grade Y by giving T the degree ρ; that is, put $Y_\lambda = X_\lambda + X_{\lambda-\rho} T$. This defines Y as a graded R-module, and it is now a completely straightforward matter to check that there is a unique way to make Y into an extension R-algebra of X, such that $dT = t$, and that the conditions (a) and (b) are then achieved. Since T is of odd degree, we must have $T^2 = 0$, and $Tx = (-1)^\lambda xT$ for $x \in X_\lambda$. These rules determine a mul-

tiplication in Y which turns out to be associative and skew-commutative. Similarly, we must have $d(xT) = (dx)T + (-1)^{\lambda}x(dT)$ for $x \in X_\lambda$, and this rule determines an extension of d to Y which turns out to be a skew derivation. We leave the details to the reader.

ρ even. In this case we let Y be the free X-module on a countable basis $\{1, T, T^{(2)}, T^{(3)}, \cdots\}$:

$$Y = X + XT + XT^{(2)} + \cdots .$$

For convenience in writing formulas, we sometimes put $1 = T^{(0)}$ and $T = T^{(1)}$. We grade Y by giving $T^{(i)}$ the degree ρi; that is, we put

$$Y_\lambda = X_\lambda + X_{\lambda-\rho}\,T + X_{\lambda-2\rho}\,T^{(2)} + \cdots .$$

This is a finite sum because $X_{\lambda - i\rho} = 0$ for $i\rho > \lambda$. We define the multiplication in Y by the rules

$$T^{(i)}T^{(j)} = \frac{(i+j)!}{i!j!}\,T^{(i+j)}, \quad \text{and} \quad T^{(i)}x = xT^{(i)}, \, x \in X.$$

The derivation in X is extended to Y in the unique way such that

$$dT^{(i)} = tT^{(i-1)}, \qquad \text{for } i > 0.$$

It is a straightforward matter, which we leave to the reader, to check that all requirements are met by these definitions. Notice that if R contains a subfield of characteristic 0, then we have $T^{(i)} = T^i/(i!)$. In this case Y is just the ring of polynomials $X[T]$ in one commuting variable T with coefficients in X, and the derivation is then uniquely determined by the requirement $dT = t$.

In both cases, ρ even and ρ odd, we shall denote the R-algebra Y which we have constructed by the combination of symbols:

$$Y = X\langle T\rangle; \qquad dT = t,$$

and we shall call Y the R-algebra obtained from X *by the adjunction of a variable T which kills t.* Suppose that we are given a finite sequence of homology classes $\tau_1, \tau_2, \cdots, \tau_n \in H_{\rho-1}(X)$. Select cycles $t_1, t_2, \cdots, t_n \in Z_{\rho-1}(X)$ representing these classes. Then by adjoining successively variables T_1, $T_2, \cdots, T_n$ of degree ρ which kill the cycles t_j we obtain an R-algebra

$$Y = X\langle T_1, T_2, \cdots, T_n\rangle; \qquad dT_j = t_j,$$

satisfying the following conditions:

(a) $$Y \supset X, \text{ and } Y_\lambda = X_\lambda \text{ for } \lambda < \rho.$$

(b) $$H_{\rho-1}(Y) = H_{\rho-1}(X)/(R\tau_0 + R\tau_1 + \cdots + R\tau_n).$$

Furthermore, it is clear from our construction that Y is free if X is free. Now it is almost obvious how to prove

THEOREM 1. *Let M be any ideal in R. Then there exists a free acyclic R-algebra X such that $H_0(X) = R/M$. In other words, there exists a free resolution of R/M which is an R-algebra.*

Proof. We shall obtain X as the union of an ascending chain of R-algebras $X^0 \subset X^1 \subset X^2 \subset \cdots$ which we shall now define inductively. We define X^0 to be the R-algebra R itself ($X_0^0 = R$; $X_\lambda^0 = 0$, $\lambda \neq 0$; $d = 0$). To construct X^1 we take generators $t_1, \cdots, t_n$ for the ideal M. Then, viewing the t_j as 0-cycles in the algebra R, we adjoin variables $T_1, \cdots, T_n$ of degree 1 to R which kill the t_j and put

$$X^1 = R\langle T_1, \cdots, T_n\rangle; \qquad dT_j = t_j.$$

Clearly, $H_0(X^1) = R/M$. Next we choose 1-cycles $s_1, \cdots, s_m \in Z_1(X^1)$, whose homology classes σ_j generate $H_1(X^1)$, and adjoin variables S_j of degree 2 to X^1 which kill the cycles s_j, obtaining an R-algebra

$$X^2 = X^1\langle S_1, \cdots, S_m\rangle; \qquad dS_j = s_j$$

such that $H_1(X^2) = 0$, and $H_0(X^2) = R/M$. Continuing in this way we define inductively for $k > 0$

$$X^{k+1} = X^k\langle U_1, \cdots, U_{n_k}\rangle; \qquad dU_j = u_j,$$

where $u_1, \cdots, u_{n_k}$ are generators for the k-cycles (mod boundaries) in X^k. Since $(X^k)_\lambda$ is constant as function of k for $k \geqq \lambda$, and since X^k is by construction acyclic in degrees $0 < \lambda < k$, it is obvious that the algebra $X = \bigcup_{k=0}^{\infty} X^k$ furnishes a free resolution of R/M.

3. The change in the homology ring produced by killing a cycle

Let X be an R-algebra, let t be a cycle of degree $\rho - 1$ in X, and let $Y = X\langle T\rangle$; $dT = t$ be the result of killing t. Then the inclusion map $i: X \to Y$ induces a homomorphism $i_*: H(X) \to H(Y)$ of the homology algebra of X into that of Y. We wish now to examine this homomorphism more closely and in doing so to prove the following

THEOREM 2. *In the situation just described, suppose that the homology class τ of t is a skew non-zerodivisor; that is, assume for $\xi \in H(X)$*

$$\tau\xi = 0 \Rightarrow \xi = 0, \qquad \text{if } \tau \text{ is of even degree } (\rho \text{ odd}),$$

$$\tau\xi = 0 \Rightarrow \xi \in \tau H(X), \qquad \text{if } \tau \text{ is of odd degree } (\rho \text{ even}).$$

Then i_* is a surjection with kernel $\tau H(X)$; and hence $H(Y) \approx H(X)/\tau H(X)$.

Proof. We treat the cases of odd and even ρ separately, giving first a general discussion with no assumptions on τ and then proving the theorem.

ρ *odd*: In this case $Y = X + XT$. Consider the map $j: Y \to X$ defined by $j(x_1 + x_2T) = x_2$. Obviously the sequence

$$0 \to X \xrightarrow{i} Y \xrightarrow{j} X \to 0 \tag{1}$$

is exact. Furthermore, j commutes with d because

$$jd(x_1 + x_2T) = j(dx_1 + (dx_2)T \pm x_2t) = dx_2 = dj(x_1 + x_2T).$$

Hence our exact sequence yields an exact homology triangle

$$(1_*)\qquad \begin{tikzcd} & H(Y) \arrow[dr, "j_*"] & \\ H(X) \arrow[ur, "i_*"] & & H(X) \arrow[ll, "d_*"'] \end{tikzcd}$$

Here j_* is of degree $-\rho$ and d_* is of degree $\rho - 1$. I contend that, except for a sign, the connecting homomorphism d_* is just multiplication by τ, the homology class of t. Indeed, let $\xi \in H_\lambda(X)$ be represented by a cycle $x \in Z_\lambda(X)$. Then $j(xT) = x$, hence $d_*(\xi)$ is the homology class of the cycle $d(xT) = (-1)^\lambda xt$, that is, $d_*\xi = (-1)^\lambda \xi\tau$, as contended. From the exactness of 1_* we now obtain the following information about the homomorphism i_* :

$$\text{Kernel } i_* = \text{Image } d_* = \tau H(X)$$
$$\text{Cokernel } i_* \approx \text{Kernel } d_* = \{\xi \in H(X) \mid \tau\xi = 0\}.$$

In particular, if τ is not a zerodivisor in $H(X)$, then i_* is an onto mapping and $H(Y) \approx H(X)/\tau H(X)$.

ρ even: In this case, $Y = X + XT + XT^{(2)} + \cdots$. Consider the map $j: Y \to Y$ defined by

$$j(x_0 + x_1 T + x_2 T^{(2)} + \cdots) = x_1 + x_2 T + x_3 T^{(2)} + \cdots.$$

Obviously the sequence

$$(2)\qquad 0 \to X \xrightarrow{i} Y \xrightarrow{j} Y \to 0$$

is exact, and j commutes with d. Therefore (2) gives rise to an exact homology triangle

$$(2_*)\qquad \begin{tikzcd} & H(Y) \arrow[dr, "j_*"] & \\ H(X) \arrow[ur, "i_*"] & & H(Y) \arrow[ll, "d_*"'] \end{tikzcd}$$

in which j_* is of degree $-\rho$ and d_* of degree $\rho - 1$. I contend that the map $d_* i_*: H(X) \to H(X)$ obtained by skipping j_* in the triangle is none other than (left) multiplication by τ, the homology class of t. Indeed, let $\xi \in H(X)$ and let $x \in Z(X)$ be a cycle representing ξ. Then $j(xT) = x = ix$, hence $d_* i_* \xi$ is the homology class (in X) of the cycle $d(xT) = d(Tx) = tx$, as contended. The complete analysis of the information contained in our triangle (2_*) leads to a spectral sequence (Cf. W. S. Massey, *Exact couples in algebraic topology*, Ann. of Math., vol. 56 (1952), pp. 363–396). Here we treat only the extremely simple case in which τ satisfies the hypothesis of Theorem 2. In order to establish the conclusion of Theorem 2 it is enough, in view of the exactness of (2_*), to prove that $j_* = 0$. We first prove that Image j_* and Image i_* have 0 intersection. Indeed, suppose $i_*\xi = j_*\eta$. Then

$\tau\xi = d_* i_* \xi = d_* j_* \eta = 0$, so there exists ξ_1 such that $\xi = \tau\xi_1$, and hence $i_*\xi = i_*\tau\xi_1 = i_* d_* i_* \xi_1 = 0$. Since Image $i_* =$ Kernel j_* we now know that $j_*^2\eta = 0$ implies $j_*\eta = 0$. By induction it follows that $j_*^n\eta = 0$ implies $j_*\eta = 0$. However, for any $\eta \in H(Y)$ there exists an n such that $j_*^n\eta = 0$, because j_* is of negative degree $-\rho$ (simply take n so large that $n\rho >$ degree η). Thus we have shown $j_*\eta = 0$ for all η.

In Theorem 2, adjunction of a variable to X divides $H(X)$ by a non-zero-divisor. In the next theorem, division of X by a non-zerodivisor adjoins a variable to $H(X)$.

THEOREM 3. *Let X be an R-algebra. For odd ρ, let a be an element of $B_{\rho-1}(X)$ which is not a zerodivisor in X, and select $s \in X_\rho$ such that $ds = a$. Then the residue class algebra $\bar{X} = X/aX$, with the derivation $\bar{d}$ induced by d, is an R-algebra, and the residue class $\bar{s}$ of s* (mod aX) *is a ρ-cycle in $\bar{X}$, whose homology class we denote by $\sigma \in H_\rho(\bar{X})$. The canonical map $j: X \to \bar{X}$ induces an isomorphism j_* of $H(X)$ into $H(\bar{X})$, and we have $H(\bar{X}) = (j_* H(X))\langle\sigma\rangle; d\sigma = 0$.*

Proof. Since a is not a zerodivisor in X, the sequence

$$(3) \qquad 0 \to X \xrightarrow{a} X \xrightarrow{j} \bar{X} \to 0$$

is exact. Since a is a boundary, the induced map $a_*: H(X) \to H(X)$ is zero, and consequently the homology triangle associated wth (3) reduces to an exact sequence

$$(3_*) \qquad 0 \to H(X) \xrightarrow{j_*} H(\bar{X}) \xrightarrow{d_*} H(X) \to 0.$$

I contend that for any $\xi \in H(\bar{X})$ we have

$$(*) \qquad \xi = j_* d_*(\sigma\xi) + \sigma j_* d_*(\xi).$$

To prove this, choose $x \in X$ such that $\bar{x} \in \bar{X}$ is a cycle representing ξ. Then $dx = ay$ for some $y \in X$, and y is a cycle whose homology class $\eta \in H(X)$ is the image of ξ under d_*. Thus $j_* d_* \xi = \bar{\eta}$, the homology class of $\bar{y} \in \bar{X}$. On the other hand, to compute $j_* d_* \sigma\xi$ we write $d(sx) = (ds)x - sdx = ax - say = a(x - sy)$, which shows that $j_* d_* \sigma\xi = \xi - \sigma\bar{\eta} = \xi - \sigma j_* d_* \xi$, as contended. Now (*) shows

$$H(\bar{X}) = j_* H(X) + \sigma j_* H(X)$$

and using (*), together with $\sigma^2 = 0$, and the exactness of (3_*), one easily checks that 1 and σ are in fact a $j_* H(X)$-basis for $H(\bar{X})$.

4. A special free resolution

A sequence of elements $a_1, a_2, \cdots, a_r \in R$ is said to be an R-sequence if a_1 is not a zerodivisor in R, and if, for each i, $1 \leqq i < r$, the residue class of a_{i+1} is not a zerodivisor in the residue class ring $R/(a_1, \cdots, a_i)$.

THEOREM 4. *Let $t_1, \cdots, t_n$ and $a_1, \cdots, a_r$ be R-sequences such that the ideal $A = (a_1, \cdots, a_r)$ generated by the a_j is contained in the ideal $M = (t_1, \cdots, t_n)$ generated by the t_i. Write $a_j = \sum_{i=1}^{n} c_{ji} t_i$, $1 \leqq j \leqq r$, with $c_{ji} \in R$. Let $\bar{R} = R/A$ and $\bar{M} = M/A$, and let $\bar{c}_{ji}$ and $\bar{t}_i$ denote the A-residues of c_{ji} and t_i. Then the algebra*

$$Y = \bar{R}\langle T_1, \cdots, T_n; S_1, \cdots, S_r\rangle$$

with T_i of degree 1, S_j of degree 2, and with

$$dT_i = \bar{t}_i, \qquad dS_j = \sum_{i=1}^{n} \bar{c}_{ji} T_i,$$

is acyclic, and therefore yields a free resolution of the $\bar{R}$-module $\bar{R}/\bar{M}$.

Proof. The R-algebra Y can be reached in three successive steps as follows. We start with the R-algebra R itself, and adjoin variables T_i to kill the t_i, obtaining an R-algebra

$$X = R\langle T_1, \cdots, Tn\rangle; \qquad dT_i = t_i.$$

By induction on n, using the fact that $t_1, \cdots, t_n$ is an R-sequence, together with the case $\rho = 1$ of Theorem 2, we see that

$$H(X) = R/(t_1, \cdots, t_n) = R/M.$$

In X_1, we have elements $s_j = \sum c_{ji} T_i$ such that $ds_j = \sum c_{ji} t_i = a_j$ for each j. Next we take everything mod A, obtaining the algebra

$$\bar{X} = \bar{R}\langle T_1, \cdots, T_n\rangle; \qquad dT_i = \bar{t}_i.$$

By induction on r, using the fact that $a_1, \cdots, a_r$ is an R-sequence, together with the case $\rho = 1$ of Theorem 3, we find that

$$H(\bar{X}) = (\bar{R}/\bar{M})\langle \sigma_1, \cdots, \sigma_r\rangle,$$

where σ_j is the homology class of the 1-cycle $\bar{s}_j = \sum \bar{c}_{ji} T_i \in \bar{X}_1$. Finally to obtain the algebra Y we adjoin variables S_j which kill these cycles $\bar{s}_j$, and we prove by induction on r, using the case $\rho = 2$ of Theorem 2, that $H(Y) = \bar{R}/\bar{M}$. (Theorem 2 is applicable because for any ring P, σ is evidently a skew non-zerodivisor in $P\langle\sigma\rangle$.)

Application 1. Let F be the free abelian group on generators $u_1, \cdots, u_n$, and let $R = Z(F) = Z[u_1, u_1^{-1}, \cdots, u_n, u_n^{-1}]$ be the group ring of F with integer coefficients. Let $t_i = u_i - 1$, $1 \leqq i \leqq n$, and let $M = (t_1, \cdots, t_n)$. Let $a_i = u_i^{e_i} - 1$, $1 \leqq i \leqq r$, with positive integers $e_1 | e_2 | \cdots | e_r$, and let $A = (a_1, \cdots, a_r)$. Then $\bar{R} = R/A$ is the group ring of the abelian group $\bar{F}$ generated by elements $\bar{u}_i$ with the relations $\bar{u}_i^{e_i} = 1$, $1 \leqq i \leqq r$, that is, of the direct product of cyclic groups of order e_i, $1 \leqq i \leqq r$, and $n - r$ infinite cyclic groups. Theorem 4 yields then a free resolution of the $\bar{F}$-module $Z = R/M = \bar{R}/\bar{M}$, a resolution which can be used efficiently to compute the cohomology and homology groups of the finitely generated abelian group $\bar{F}$.

Application 2. Let R be a regular local ring of dimension n, and let A be an ideal of dimension $n - r$ in R such that A can be generated by r elements $a_1, \cdots, a_r$. Then it is known, in connection with the theorem of Cohen-Macaulay, that the a_j form an R-sequence. In particular, the maximal ideal, M, of R is generated by an R-sequence $t_1, \cdots, t_n$. Thus, Theorem 4 yields a free resolution of the residue field $\bar{R}/\bar{M}$ as $\bar{R}$-module, for any local ring $\bar{R}$ which can be obtained from a regular local ring by factoring by an ideal of type A. Geometrically, a ring of type $\bar{R}$ would arise for example as the local ring of a point P on a variety V, such that P is simple on the ambient variety, and such that V is locally a complete intersection at P. In this case A is the prime ideal of functions regular at P on the ambient variety which vanish along the subvariety V. Consideration of this special case in conversations with Zariski gave the first impetus to this work. Zariski has independently obtained the resolution of Theorem 4 in the case of local complete intersections.

5. Applications to the torsion functor

Let X and Y be R-algebras. It is easy to check that their tensor product $X \otimes Y$ over R can be made into an R-algebra in a unique way such that $x \otimes y = (x \otimes 1)(1 \otimes y)$ and such that the maps $x \to x \otimes 1$ and $y \to 1 \otimes y$ are homomorphisms of X and Y into $X \otimes Y$. Suppose that X and Y are both *free* and *acyclic*, with $H(X) = R/M$ and $H(Y) = R/N$. Denote by $j: X \to R/M$ and $k: Y \to R/N$ the canonical homomorphisms. Then it is well known (elementary theory of the torsion functor; cf. [1]) that the homomorphisms

$$(R/M) \otimes Y \xleftarrow{j \otimes 1} X \otimes Y \xrightarrow{1 \otimes k} X \otimes (R/N)$$

induce *isomorphisms*

$$H((R/M) \otimes Y) \approx H(X \otimes Y) \approx H(X \otimes (R/N)).$$

Thus, the homology algebra $H(X \otimes Y)$, *ring structure included*, is independent of the resolutions X and Y, and, up to canonical isomorphisms, depends only on R/M and R/N. It is denoted of course by $\mathrm{Tor}^R(R/M, R/N)$. The multiplication is the $\pitchfork$ product; see [1], p. 215 bottom.

THEOREM 5. *Let M and N be ideals of R. Let $a \in MN$ be a non-zerodivisor in R. Let $\bar{R} = R/aR$, and put $K = R/M$, $L = R/N$. Then*

$$\mathrm{Tor}^{\bar{R}}(K, L) = \mathrm{Tor}^R(K, L)\langle U\rangle,$$

where U is a variable of degree 2.

Proof. Let X be a free acyclic R-algebra such that $H(X) = K = R/M$.

Then $dNX_1 = NdX_1 = NM$, so we can choose $s \in NX_1$ such that $ds = a$. Let $\bar{X} = X/aX$ and let $\bar{s} \in \bar{X}_1$ be the residue of s. From Theorem 3 we know that $H(\bar{X}) = K\langle\sigma\rangle$, where σ is the class of s; and from Theorem 2 it follows that

$$\bar{X}\langle S\rangle; \qquad dS = \bar{s}$$

is a free resolution of the $\bar{R}$-module K. To compute $\mathrm{Tor}^{\bar{R}}(K, L)$ we must now tensor with $L = R/N$ and pass to homology. Tensoring with L commutes with adjunction of S, and subsumes the passage from X to $\bar{X}$ because $a \in N$. Hence

$$\mathrm{Tor}^{\bar{R}}(K, L) = H((X \otimes L)\langle S\rangle) = H\left(\sum_{i=0}^{\infty} (X \otimes L)S^{(i)}\right).$$

Since $s \in NX$, we have $dS^{(i)} = sS^{(i-1)} \equiv 0 \pmod{NX\langle S\rangle}$, hence the direct sum decomposition is stable with respect to d and we can continue:

$$= \sum_{i=1}^{\infty} H(X \otimes L)U^{(i)} = H(X \otimes L)\langle U\rangle,$$

where $U^{(i)}$ is the homology class of $S^{(i)}$ (mod $NX\langle S\rangle$). Since $H(X \otimes L) = \mathrm{Tor}^{R}(K, L)$, our theorem is proved.

6. Local rings

In this section we assume that R is a local ring with maximal ideal M and residue field K. Let $t_1, t_2, \cdots, t_n$ be a minimal system of generators for M. Then the M^2-residues of the elements t_i are a K-base for the vector space M/M^2, and we have $n = \dim_K(M/M^2)$. Consider the R-algebra $E = R\langle T_1, \cdots, T_n\rangle$; $dT_i = t_i$. Though we shall not make use of the fact, it is perhaps well to sketch here a proof that E is uniquely determined by R up to a (noncanonical) isomorphism. Indeed, suppose $t_1', \cdots, t_n'$ is another choice of generators for M, and let $E' = R\langle T_1', \cdots, T_n'\rangle$; $dT_j' = t_j'$. Let $t_i = \sum a_{ij} t_j'$ with $a_{ij} \in R$. Reading this last equation mod M^2, we see that the determinant of the matrix (a_{ij}) does not belong to M, and consequently the matrix is invertible in R. Since E_1 and E_1' are free R-modules with bases $\{T_i\}$ and $\{T_j'\}$ it follows that the R-linear map $\varphi_1: E_1 \to E_1'$ defined by $\varphi_1(T_i) = \sum a_{ij} T_j'$ is bijective. Now φ_1 extends to a ring isomorphism $\varphi: E \approx E'$ because E and E' are just the exterior algebras $\wedge E_1$ and $\wedge E_1'$ over the R-modules E_1 and E_1'. Furthermore, φ commutes with d because for each generator T_i we have $d\varphi T_i = d\sum a_{ij} T_j' = \sum a_{ij} t_j' = t_i = \varphi\, dT_i$, and a skew derivation is determined by its effect on generators. Thus the homology algebra $H(E)$ is an invariant of the local ring R. It might be of interest to investigate the relationship between $H(E)$ and the more conventional homological invariants of R such as the algebra $\mathrm{Tor}^{R}(K, K)$ and the "Betti numbers" $B_q = \dim_K \mathrm{Tor}_q^{\bar{R}}(K, K)$.[1] In the case of "complete intersections" we have the whole story:

THEOREM 6. *Suppose that $R = R'/(a_1, \cdots, a_r)$, where R' is a regular local ring and $a_1, \cdots, a_r$ is an R'-sequence which is contained in the square of the maximal ideal M' of R'. Then*

$$H(E) = K\langle \sigma_1, \cdots, \sigma_r \rangle; \qquad \deg \sigma_j = 1,$$

and

$$\mathrm{Tor}^R(K, K) = K\langle T_1, \cdots, T_n ; S_1, \cdots, S_r \rangle;$$

$$\deg T_i = 1, \qquad \deg S_j = 2.$$

In particular, the Betti numbers of R are given by the power series identity

$$\sum_{q=0}^{\infty} B_q Z^q = \frac{(1 + Z)^n}{(1 - Z^2)^r}.$$

Proof. Let $t_i' \in M'$ be a pre-image of t_i, $1 \leqq i \leqq n$. Since the ideal $A = (a_1, \cdots, a_r)$ is contained in $(M')^2$, the t_i' constitute a minimal system of generators for M'. Since R' is regular, the t_i' form an R'-sequence, and we can now apply Theorem 4. Our present objects R', t_i', R, E, t_i are, respectively, denoted in Theorem 4 by the symbols R, t_i, $\bar{R}$, $\bar{X}$, $\bar{t}_i$. From the second step of the proof of Theorem 4 one finds $H(E) = K\langle \sigma_i, \cdots, \sigma_r \rangle$, as contended. Concerning Tor, we have

$$\mathrm{Tor}^R(K, K) = H(Y \otimes K),$$

where

$$Y = R\langle T_1, \cdots, T_n ; S_1, \cdots, S_r \rangle$$

is the free resolution of the R-module K constructed in Theorem 4. To complete our proof we must show $H(Y \otimes K) = Y \otimes K$, that is, $dY \subset MY$. Clearly $dT_i = t_i \in MY$. To show the same for dS_j we first write $a_j = \sum c_{ji}' t_i'$, $c_{ji}' \in R'$, and notice that $c_{ji}' \in M'$ because $a_j \in (M')^2$. Letting c_{ji} denote the image in R of c_{ji}', we have then $dS_j = \sum c_{ji} T_i \in MY$. More generally, $dS_j^{(k)} = (dS_j)S^{(k-1)} \in MY$ for all k, and it follows now that $dY \subset MY$, because d is a derivation and every element in Y is a linear combination of products of T_i's and $S_j^{(k)}$'s with coefficients in R.

Having had a look at a good case where we know the full story, let us return to the consideration of our arbitrary local ring R. In constructing a free resolution of the R-module K in the manner of Theorem 1, we would

[1] Using his technique of minimal resolutions, Eilenberg has proved $B_2 = \binom{n}{2} + \varepsilon$ and $B_3 \geqq \binom{n}{3} + \varepsilon n$, where $\varepsilon = \dim_K H_1(E)$. A resolution X is *minimal* if $dX \subset MX$; for example, the resolution Y which we construct in Theorem 6 has this property. One difficulty is that while *minimal* resolutions of K always exist, and while *R-algebra* resolutions always exist, it is doubtful whether *minimal R-algebra* resolutions exist in all cases.

begin with our R-algebra E, then adjoin variables of degree 2 to annihilate $H_1(E)$, then adjoin variables of degree 3, etc. At any given stage of this process we would have before us a free R-algebra X, containing E as subalgebra. Let us consider such an X.

LEMMA 1. *Let X be a free R-algebra. Then from a congruence*

$$\sum_{i=1}^{n} t_i x_i \equiv 0 \pmod{M^2 X}, \qquad x_i \in X,$$

we can conclude $x_i \in MX$, $1 \leqq i \leqq n$.

Proof. Let $\{y_\alpha\}$ be an R-basis for X, and write $x_i = \sum b_{i\alpha} y_\alpha$, $b_{i\alpha} \in R$. Our congruence implies $\sum_i t_i b_{i\alpha} \equiv 0 \pmod{M^2}$ for each α, hence $b_{i\alpha} \in M$ for all i, α, hence $x_i \in MX$.

LEMMA 2. *Let X be a free R-algebra containing E as subalgebra. Then there exist K-linear maps*

$$D_i : H(X \otimes K) \to H(X \otimes K), \qquad 1 \leqq i \leqq n,$$

of degree -1, such that for $\xi \in H_\lambda(X \otimes K)$, $\eta \in H_\mu(X \otimes K)$,

$$(*) \qquad D_j(\xi\eta) = (D_i \xi)\eta + (-1)^\lambda \xi(D_i \eta),$$

and such that

$$D_i \tau_j = \delta_{ij} \qquad \text{(Kronecker delta)},$$

where $\tau_j \in H_1(X \otimes K)$ denotes the homology class of the 1-cycle $T_j \otimes 1$.

Proof. Let $\xi \in H_\lambda(X \otimes K)$ be represented by $x \in X_\lambda$. Then x is a cycle (mod $MX = t_1 X + \cdots + t_n X$), and we can write $dx = \sum t_i y_i$ with $y_i \in X_{\lambda-1}$. Differentiation yields $0 = \sum t_i \, dy_i$, and Lemma 1 shows now that $dy_i \in MX$. Let $\eta_i \in H_{\lambda-1}(X \otimes K)$ be the class represented by y_i. I contend that η_i is uniquely determined by ξ independently of the choices of x and the y_i's. Indeed, according to Lemma 1, the choice of x determines the y_i's uniquely (mod MX); changing x by a boundary doesn't affect dx nor the y_i's, and changing x by an element $\sum t_i u_i \in MX$ changes each y_i by a boundary du_i. We can therefore define $D_i \xi = \eta_i$ for each i. Bearing in mind Lemma 1 which allows one to compare coefficients of the t_i's, it is a straightforward matter to check that each map D_i inherits from d the property (*), and writing $dT_j = t_j = \sum_i \delta_{ij} t_i$ shows $D_i \tau_j = \delta_{ij}$.

Let $\Lambda = E \otimes K = K\langle T_1, \cdots, T_n \rangle$ be the exterior algebra of the n-dimensional vector space $KT_1 + \cdots + KT_n$. Obviously we can view $H(X \otimes K)$ as a left Λ-module in a canonical way such that $T_i \xi = \tau_i \xi$ for $\xi \in H(X \otimes K)$. Viewing the T_i's as operators on $H(X \otimes K)$ we have the operator identity

$$(**) \qquad D_i T_j + T_j D_i = \delta_{ij} \qquad \text{(Kronecker delta)},$$

because

$D_i T_j \xi = D_i(\tau_j \xi) = (D_i \tau_j)\xi - \tau_j(D_i \xi) = \delta_{ij} \xi - T_j D_i \xi$ for $\xi \in H(X \otimes K)$.

LEMMA 3. *Let V be any left Λ-module in which there exist K-linear maps $D_i: V \to V$ satisfying the identity (**). Then V is Λ-free.*

Proof. Let $N = \Lambda_1 + \cdots + \Lambda_n$ be the radical of Λ. Then $\Lambda/N = K$. Select a family of elements $v_\alpha \in V$ whose residues (mod NV) form a K-base for the vector space V/NV. I contend that the v_α constitute a Λ-base for V. Obviously since the v_α generate V(mod NV) the elements $T_{i_1} T_{i_2} \cdots T_{i_r} v_\alpha$ generate $N^r V$ (mod $N^{r+1}V$), and since $N^{n+1} = 0$ it follows that the elements $T_{i_1} \cdots T_{i_r} v_\alpha$, $0 \leqq r \leqq n$, $1 \leqq i_1 < i_2 < \cdots < i_r \leqq n$, K-generate V. We have to show that they are K-linearly independent. Suppose therefore we have a nontrivial K-linear relation between these elements and let

$$c_{i_1, i_2, \cdots, i_r, \alpha}$$

be a nonvanishing coefficient with a minimal value of r. Applying the operator $D_{i_1} D_{i_2} \cdots D_{i_r}$ to our relation we find

$$\sum_\alpha c_{i_1, i_2, \cdots, i_r, \alpha} v_\alpha \equiv 0 \text{ (mod } NV),$$

contradicting the K-linear independence of the v_α(mod NV).

LEMMA 4. *Let X be a free R-algebra containing E, such that $X_1 = E_1$. Then $\dim_K H_1(X \otimes K) = n$, and the subalgebra, L, of $H(X \otimes K)$ which is generated by $H_1(X \otimes K)$ is just the exterior algebra of the vector space $H_1(X \otimes K)$. Furthermore, $H(X \otimes K)$ is L-free, possessing a homogeneous L-basis.*

Proof. We have $dX_1 = dE_1 = M$, and on the other hand $dX_2 \subset MX_1$ because $dX_2 = B_1(X) \subset Z_1(X) = Z_1(E) \subset ME_1$; namely, if $\sum a_i T_i \in Z_1(E)$, $a_i \in R$, then $\sum a_i t_i = 0$ implies $a_i \in M$, for all i. It follows now that

$$H_1(X \otimes K) = E_1/ME_1 = K\tau_1 + \cdots + K\tau_n$$

is of dimension n. Viewing $H(X \otimes K)$ as a Λ-module in the manner described before Lemma 3, we see now that L is the image of Λ under the canonical homomorphism $T_i \to \tau_i$, $1 \leqq i \leqq n$. On the other hand, applying Lemma 3 we see that $H(X \otimes K)$ is a free Λ-module and that a Λ-base $\{\xi_\alpha\}$ for it can be obtained by taking a K-base for $H(X \otimes K)/NH(X \otimes K)$. Since the latter is graded, we can select the ξ_α's homogeneous, and we can obviously choose 1 as the first ξ_α. Hence $L = \Lambda \cdot 1$ is *isomorphic* to, and can be identified with, Λ, and all is proven.

THEOREM 7. *We have $B_1 = \dim_K \operatorname{Tor}_1^R(K, K) = n$. The subalgebra, L, of $\operatorname{Tor}^R(K, K)$ which is generated by $\operatorname{Tor}_1^R(K, K)$ is just the exterior algebra of the vector space $\operatorname{Tor}_1^R(K, K)$, and $\operatorname{Tor}^R(K, K)$ is a free L-module with a homogeneous base.*

Proof. Construct a free R-algebra resolution, X, of the R-module K as in the proof of Theorem 1, starting with E and adjoining elements successively.

Now apply Lemma 4 to X.

Since $\dim_K L_r = \binom{n}{r}$, we have as a corollary a result of Serre [2], namely

$$B_r = \dim_K \operatorname{Tor}_r^R(K, K) \geqq \binom{n}{r}.$$

Of course, if R is regular, we have equality; and if R is not regular, we can in fact prove a much stronger inequality. We begin with

LEMMA 5. (Eilenberg, M.I.T. Lecture, Spring 1956) *If* $H_1(E) = 0$, *then* R *is regular.*

Outline of proof shown me by Zariski. Since $H_1(E) = 0$, the sequence

$$E_2 \xrightarrow{d} E_1 \xrightarrow{d} M \to 0$$

is exact. From elementary properties of tensor products it follows that for each natural number ρ the sequence

$$(*) \qquad E_1^{(\rho-1)} \otimes E_2 \xrightarrow{1^{(\rho-1)} \otimes d} E_1^{(\rho)} \xrightarrow{d^{(\rho)}} M^{(\rho)} = M^\rho$$

is exact, where $E_1^{(\rho)}$ denotes the ρ^{th} *symmetric* tensor power of E_1, i.e., the result of dividing $\otimes_1^\rho E_1$ by the relations

$$(\cdots \otimes x \otimes \cdots \otimes y \otimes \cdots) - (\cdots \otimes y \otimes \cdots \otimes x \otimes \cdots).$$

Since E_1 has an R-base consisting of the elements $T_1, \cdots, T_n$, we can identify $E_1^{(\rho)}$ with the space of all homogeneous polynomials $f(T_1, \cdots, T_n)$ of degree ρ in n variables T_i with coefficients in R, and the map $d^{(\rho)}: E^{(\rho)} \to M^\rho$ becomes now the substitution $T_i \to t_i$. Translating the exactness of (*) into these terms we find: If a form $f(T_1, \cdots, T_n)$ of degree ρ is such that

$$f(t_1, \cdots, t_n) = 0,$$

then there exist forms g_{ij} of degree $(\rho - 1)$ such that

$$f(T_1, \cdots, T_n) = \sum_{i<j} g_{ij}(T_1, \cdots, T_n)(t_i T_j - t_j T_i),$$

and in particular, all coefficients of f are in M. Thus, the graded ring obtained by filtering R with the powers of the maximal ideal $M = (t_1, \cdots, t_n)$ is just the polynomial ring $K[T_1, \cdots, T_n]$, and R is regular.

THEOREM 8. *If R is not regular, then* $\operatorname{Tor}^R(K, K)$ *contains a subalgebra of the form* $L\langle S\rangle$, *where S is a variable of degree* 2, *and L is the exterior algebra on* $\operatorname{Tor}_1^R(K, K)$ *discussed in Theorem* 7. *In particular, we have*

$$B_r \geqq \binom{n}{r} + \binom{n}{r-2} + \binom{n}{r-4} + \cdots,$$

and therefore $B_r \geqq 2^{n-1}$ *for* $r \geqq n$.

Proof. Consider the R-algebra E. Since R is not regular, we have

$$H_1(E) \neq 0$$

by the previous lemma. Let $\dim_K H_1(E) = \varepsilon$, and adjoin $\varepsilon - 1$ variables of degree 2 to E to obtain an algebra $X^{(2)} \supset E$ such that $H_1(X^{(2)})$ is *one-dimensional*. Now adjoin variables of degree 3 to $X^{(2)}$ to kill *all* 2-cycles, then variables of degree 4 to kill *all* 3-cycles, etc. We end up then with a free R-algebra X such that $X_1 = E_1$, and such that $H_0(X) = K$, $H_1(X)$ is one-dimensional, and $H_r(X) = 0$ for $r \geqq 2$. Let σ be a basis element for $H_1(X)$, let $s \in E_1$ be a cycle representing σ, and let $Y = X\langle S\rangle$; $dS = s$: I contend Y is acyclic. This follows from the case $\rho = 2$ of Theorem 2, because σ is obviously a skew non-zerodivisor in $H(X) = K + K\sigma$. Now $S^{(k)}$ is a cycle (mod MY) for every k, because $dS^{(k)} = sS^{(k-1)}$, and $s \in Z_1(E) \subset ME_1$. Therefore

$$\operatorname{Tor}^R(K, K) = H(X\langle S\rangle \otimes K) \approx (H(X \otimes K))\langle S\rangle.$$

Applying Lemma 4 to X, we see that $L \subset H(X \otimes K)$, and consequently $\operatorname{Tor}^R(K, K)$ contains $L\langle S\rangle$ as was to be shown.

Our lower bound for the Betti numbers of nonregular local rings is obviously the best possible because, as Theorem 4 shows, our inequalities become equalities whenever R can be obtained from a regular local ring R' by dividing R' by a nonzero principal ideal. Theorem 8 affords a new quantitative proof of the characterization of regular local rings as those with finite homological dimension (Serre [2]). More generally, our Theorem 8 gives regularity criteria of the following sort: *If* $B_r = \binom{n}{r}$ *for one single dimension* $r \geqq 2$, *then* R *is regular*. For $r = 2$ and 3 this criterion has been proved by Eilenberg, using his result (Lemma 5) on which our general proof is based (cf. footnote 1).

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