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AN OPTIMAL LOWER CURVATURE BOUND FOR CONVEX HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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Abstract. It is proved that a convex hypersurface in a Riemannian manifold of sectional curvature $\geq \kappa$ is an Alexandrov's space of curvature $\geq \kappa$. This theorem provides an optimal lower curvature bound for an older theorem of Buyalo.

The purpose of this paper is to provide a reference for the following theorem.

Theorem 1. *Let M be a Riemannian manifold with sectional curvature $\geq \kappa$. Then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is an Alexandrov's space with curvature $\geq \kappa$.*

The following is a slightly weaker statement.

Theorem 2 ([Buyalo]). *If M is a Riemannian manifold, then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is locally an Alexandrov's space.*

In the proof of Theorem 2 in [Buyalo], the (local) lower curvature bound depends on (local) upper as well as lower curvature bounds of M. We show that the approach in [Buyalo] can be modified to give Theorem 1.

Definition 3. A locally Lipschitz function f on an open subset of a Riemannian manifold is called λ-*concave* ($\lambda \in \mathbb{R}$) if for any unit-speed geodesic γ, the function

$$t \mapsto f \circ \gamma(t) - \frac{\lambda}{2} t^2$$

is concave.

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LEMMA 4. *Let $f:\Omega\to\mathbb{R}$ be a λ-concave function on an open subset Ω of a Riemannian manifold. Then there is a sequence of nested open domains Ω_i, with $\Omega_i\subset\Omega_j$ for $i<j$ and $\bigcup_i\Omega_i=\Omega$, and a sequence of smooth λ_i-concave functions $f_i:\Omega_i\to\mathbb{R}$ such that:*

(i) *on any compact subset $K\subset\Omega$, f_i converges uniformly to f;*
(ii) *$\lambda_i\to\lambda$ as $i\to\infty$.*

This lemma is a slight generalization of [Greene–Wu, Theorem 2] and can be proved exactly the same way.

Proof of Theorem 1. Without loss of generality one can assume that:

(a) $\kappa\ge-1$,
(b) F bounds a compact convex set C in M,
(c) there is a (-2)-concave function μ defined in a neighborhood of C and $|\mu(x)|<1/10$ for any $x\in C$,
(d) there is unique minimal geodesic between any two points in C.

(If not, rescale and pass to the boundary of the convex piece cut by F from a small convex ball centered at $x\in F$, taking $\mu=-10\,\mathrm{dist}_x^2$.)

Consider the function $f=\mathrm{dist}_F$. By Rauch comparison (as in [Petersen, 11.4.8]), for any unit-speed geodesic γ in the interior of C, $(f\circ\gamma)''$ is bounded in the barrier sense by the corresponding value in the model case—when M is Lobachevsky plane and F is a geodesic. In particular,

$$(f\circ\gamma)''\le f\circ\gamma.$$

Therefore, $f+\varepsilon\mu$ is $(-\varepsilon)$-concave in $\Omega_\varepsilon=f^{-1}((0,\varepsilon))\cap C$. Take $K_\varepsilon=f^{-1}([\frac{1}{3}\varepsilon,\frac{2}{3}\varepsilon])\cap C$. Applying Lemma 4, we can find a smooth $(-\frac{\varepsilon}{2})$-concave function f_ε which is arbitrarily close to $f+\varepsilon\mu$ on K_ε and which is defined on a neighborhood of K_ε. Take a regular value $\vartheta_\varepsilon\approx\frac{1}{2}\varepsilon$ of f_ε. (In fact, one can take $\vartheta_\varepsilon=\frac{1}{2}\varepsilon$, but it requires a little work.) Since $|\mu|_C|<1/10$, the level set $F_\varepsilon=f_\varepsilon^{-1}(\vartheta_\varepsilon)$ will lie entirely in K_ε. Therefore, F_ε forms a smooth closed convex hypersurface.

Let us denote by ρ and ρ_ε the induced intrinsic metrics on correspondingly F and F_ε. By the Gauss formula, $(F_\varepsilon,\rho_\varepsilon)$ has curvature $\ge\kappa$. Further, F_ε bounds a compact convex set C_ε and $F_\varepsilon\to F$, $C_\varepsilon\to C$ in Hausdorff sense as $\varepsilon\to0$. By property (d), the restricted metrics from M to C and to C_ε are intrinsic. Thus, C_ε is an Alexandrov space with F_ε as boundary, that converges in Gromov–Hausdorff sense to C. It follows from [Petrunin, Theorem 1.2] (compare [Buyalo, Theorem 1]) that $(F_\varepsilon,\rho_\varepsilon)$ converges in Gromov–Hausdorff sense to (F,ρ). Therefore, (F,ρ) is an Alexandrov space with curvature $\ge\kappa$. □

REMARK 5. We are not aware of any proof of Theorem 1 which is not based on the Gauss formula. (Although if M is Euclidean space, there is a beautiful purely synthetic proof in [Milka].) Finding such a proof would be interesting on its own, and also could lead to the generalization of Theorem 1 to the case when M is an Alexandrov space.

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