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UPPER POROUS MEASURES ON METRIC SPACES

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Abstract. We show how a standard method of geometric measure theory for providing density estimates may be used in general metric spaces to obtain information on the upper porosity of packing type measures. We also obtain a connection between lower densities and the upper porosity of measures on Euclidean spaces.

1. Introduction

In geometric measure theory several tools have been developed to study the local geometry of measures on Euclidean spaces. Among the most important are density estimates, in particular for conical densities. These were studied first by Besicovitch [3], [4] and later by Marstrand [13], Federer [8], Mattila [14], and many others. Another concept that is used for describing the local distribution of a given fractal (a set or a measure) is that of porosity. It is well known that these two concepts are related to each other. Indeed, upper conical density results lead to dimension estimates for lower porous sets and measures, see [14], [11], [12]. On the other hand, in this paper, we will show that lower conical densities are closely related to upper porosities of measures. The main purpose of this paper is, however, to show that a well-known technique in geometric measure theory, sometimes called "touching point arguments" or "space filling" used to provide lower conical density theorems in Euclidean spaces may be used in general metric spaces to get information on upper porosity of packing type measures.

Our main result implies that under rather general conditions on the metric space X and the gauge h, the packing measures $\mathcal{P}^h|A$ will be upper porous for all Borel sets $A \subset X$ for which $0 < \mathcal{P}^h(A) < \infty$ (see Section 2 for the definitions). We will show by an example that an analogous statement does not in general hold if packing measures are replaced by Hausdorff

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measures. In Euclidean spaces, we provide a characterization of upper porous measures in terms of their lower conical density properties. This connection between lower densities and upper porosities of measures suggests that upper porosity results for measures on general metric spaces may be viewed as analogies of the well-known lower conical density results on Euclidean spaces.

The paper is organized as follows: In Section 2, we will set up the necessary notation and recall some known properties of upper porous measures. In Section 3, we will work on general metric spaces by proving our main result and discussing its corollaries. Section 4 contains results on Euclidean spaces: a connection between conical densities and the upper porosity of measures is discussed in Section 4.1, and an example of Hausdorff-type measure on the real line which is not upper porous will be constructed in Section 4.2.

2. Basic concepts

In what follows, $X=(X,d)$ will always be a separable metric space. By a measure on X, we mean a finite Borel regular (outer) measure defined on all subsets of X. If μ is a measure on X and $A\subset X$, we let $\mu|A$ denote the restriction measure defined by setting $\mu|A(B)=\mu(A\cap B)$ for all $B\subset X$. For $x\in X$ and $r>0$, we let $B(x,r)$ denote the open ball centered at $x\in X$ with radius r. Closed balls will be denoted by $\overline{B}(x,r)$, respectively. If $A\subset X$, we denote by $\overline{A}$ the closure of A. We also let $\operatorname{spt}\mu$ denote the smallest closed set with full μ-measure.

Porosity of a set is a notion that concerns the size of holes of a given set on small scales. There are basically two kinds of porosity, the upper and the lower porosity. If the holes or "pores" are to be found in all small scales, one is concerned with the lower porosity. If, on the other hand, one is interested in the maximal relative size of "pores" that appear in arbitrarily small but not necessarily in all scales, then the upper porosity is the relevant notion. The surveys [22], [23] of Zajíček contain plenty of information about porosity concepts of sets and their applications.

Also, porosities of measures have attracted increasing attention during the past few years. As for sets, we have to distinguish between upper and lower porosity. The lower porosity of measures on $\mathbb{R}^n$ was first introduced by Eckmann, Järvenpää, and Järvenpää in [6]. Its relation to dimensions has been studied also in [10], [2], and [1]. Our interest will be focused on the upper porosity of measures, which has been previously studied by several authors in [16] and [17]. The definition is as follows: The upper porosity of a measure μ at a point $x\in X$ is

$$\overline{\operatorname{por}}(\mu,x)=\lim_{\varepsilon\downarrow 0}\limsup_{r\downarrow 0}\operatorname{por}(\mu,x,r,\varepsilon),$$

where

$$\operatorname{por}(\mu,x,r,\varepsilon)=\sup\{\varrho\geq 0 : \text{there is } y\in X \text{ such that} \\ d(y,x)+\varrho r\leq r \text{ and } \mu(B(y,\varrho r))\leq\varepsilon\mu(B(x,r))\}$$

for $\varepsilon, r>0$. We say that μ is upper porous provided $\overline{\operatorname{por}}(\mu,x)>0$ for μ-almost all $x\in X$. The upper porosity of a set $A\subset X$ at a point $x\in A$ is given by

$$\overline{\operatorname{por}}(A,x)=\limsup_{r\downarrow 0}\operatorname{por}(A,x,r),$$

where

$$\operatorname{por}(A,x,r)=\sup\{\varrho\geq 0 : \text{there is } y\in X \text{ such that} \\ d(y,x)+\varrho r\leq r \text{ and } B(y,\varrho r)\cap A=\emptyset\}.$$

We call A upper porous if there is $a>0$ such that $\overline{\operatorname{por}}(A,x)>a$ for all $x\in A$ and σ-upper porous if it is a countable union of upper porous sets.

REMARKS 2.1. (a) It is clear that $0\leq\overline{\operatorname{por}}(A,x)\leq\frac{1}{2}$ for any $x\in\overline{A}$. For measures, the question is trickier. A measure μ is said to satisfy the doubling condition at $x\in X$ if

$$\limsup_{r\downarrow 0}\mu(B(x,2r))/\mu(B(x,r))<\infty. \tag{2.1}$$

It follows easily (see [17]) that $0\leq\overline{\operatorname{por}}(\mu,x)\leq\frac{1}{2}$ if μ satisfies the doubling condition at x and $\overline{\operatorname{por}}(\mu,x)=1$ otherwise. Moreover, it was shown in [16] and [17], that also $\mu(\{x\in X : 0<\overline{\operatorname{por}}(\mu,x)<\frac{1}{2}\})=0$. So μ is upper porous if and only if $\overline{\operatorname{por}}(\mu,x)\in\{\frac{1}{2},1\}$ almost everywhere.

(b) As shown in [17], upper porosity of measures may be defined in terms of upper porous sets: μ is upper porous if for all $\varepsilon>0$ there is an upper porous set $A\subset X$ with $\mu(X\setminus A)<\varepsilon$. Moreover, this upper porous set A may be chosen so that $\overline{\operatorname{por}}(A,x)=1/2$ for all $x\in A$.

Throughout the paper, we denote by $h:(0,\infty)\to(0,\infty)$ a nondecreasing gauge function that satisfies the doubling condition,

$$h(2r)\leq C_d h(r) \quad \text{for all } 0<r<\infty \tag{2.2}$$

with some constant $C_d<\infty$. The h-Hausdorff measure of $A\subset X$ is given by the definitions $(0<\delta\leq\infty)$

$$\mathcal{H}^h_\delta(A)=\inf\left\{\sum h(\operatorname{diam}(A_i)) : A\subset\bigcup_{i=1}^\infty A_i, \text{and } \operatorname{diam}(A_i)<\delta \text{ for all } i\right\},$$
$$\mathcal{H}^h(A)=\lim_{\delta\downarrow 0}\mathcal{H}^h_\delta(A).$$

When h is a power, $h(r)=r^s$ for some $0<s<\infty$, we use the familiar notation $\mathcal{H}^s$ to denote $\mathcal{H}^h$. For $\delta>0$ and $A\subset X$, we call any collection of pairwise disjoint balls with centers in A and radii $\le \delta$ a δ-packing of A. We define the radius-based packing premeasure P^h and packing measure $\mathcal{P}^h$ by

$$\begin{aligned} P^h_\delta(A) &= \sup\Big\{\sum h(2r_i) : \{B(x,r_i)\}_i \text{ is a } \delta\text{-packing of } A\Big\},\\ P^h(A) &= \lim_{\delta\downarrow 0} P^h_\delta(A),\\ \mathcal{P}^h(A) &= \inf \sum P^h(A_i), \end{aligned}$$

where the infimum is over all countable partitioning $\cup A_i = A$.

The following notation will be used on $\mathbb{R}^n$. If $x\in\mathbb{R}^n$ and $r>0$, we denote by $S(x,r)\subset\mathbb{R}^n$ the sphere $\{y\in\mathbb{R}^n : |x-y|=r\}$, where $|\cdot|$ refers to the Euclidean distance. For $\theta\in S^{n-1}=S(0,1)\subset\mathbb{R}^n$, $\eta>0$, $x\in\mathbb{R}^n$, and $r>0$ we put:

$$H(x,r,\theta,\eta)=\{y\in B(x,r) : (y-x)\cdot\theta>\eta|y-x|\}.$$

3. Upper porosity of packing type measures

In this section, we prove our main result concerning the upper porosity of packing type measures. As a technical tool, we first have to obtain a density point theorem for measures that are not upper porous. For that we need the following Vitali-type covering lemma. Related, more general results have been obtained by Shevchenko in [18].

Lemma 3.1. *Let $A\subset X$ and $\mathcal{B}$ be a collection of closed balls in X with $\inf\{r : \overline{B}(x,r)\in\mathcal{B}\}=0$ for all $x\in A$. Then there is a pairwise disjoint collection $\mathcal{B}'\subset\mathcal{B}$ such that $A\subset\cup\mathcal{B}'\cup P\cup N$, where P is σ-upper porous and $\mu(N)=0$.*

Proof. For any $\varepsilon>0$, it is enough to find a finite and pairwise disjoint sub-collection $\mathcal{B}_\varepsilon$ such that

$$A\subset\cup\mathcal{B}_\varepsilon\cup P_\varepsilon\cup N_\varepsilon, \tag{3.1}$$

where P_ε is upper porous and $\mu(N_\varepsilon)\le\varepsilon\mu(A)$. For if this holds, we first define $\mathcal{B}_{\frac12}$, $P_{\frac12}$, and $N_{\frac12}$ and then replace A by $A\setminus\cup\mathcal{B}_{\frac12}$ and $\mathcal{B}$ by $\{B\in\mathcal{B} : B\cap\cup\mathcal{B}_{\frac12}=\emptyset\}$ as well as define $\mathcal{B}_{\frac14}$, $P_{\frac14}$, and $N_{\frac14}$, and so on. Then the collection $\mathcal{B}'=\bigcup_{k\in\mathbb{N}}\mathcal{B}_{2^{-k}}$ is pairwise disjoint, $P=\bigcup_{k\in\mathbb{N}}P_{2^{-k}}$ is σ-upper porous and $\mu(A\setminus(\cup\mathcal{B}'\cup P))=0$.

To prove (3.1), we first take numbers $\varepsilon_j>0$ such that $\sum_{j=1}^\infty\varepsilon_j<\varepsilon$ and use the $5R$-covering theorem, see [9, Theorem 1.2], to find a disjoint subcollection $B_1^1,B_1^2,\ldots$ of $\mathcal{B}$ with radii at most 1 such that $A\subset\bigcup_i 5B_1^i$. Now use the Borel regularity of μ to find a large $k_1\in\mathbb{N}$ so that $\mu(A\setminus\bigcup_{i=1}^{k_1}5B_1^i)<\varepsilon_1\mu(A)$. Next, define $A_1=A\cap\bigcup_{i=1}^{k_1}5B_1^i\setminus\bigcup_{i=1}^{k_1}B_1^i$ and $\mathcal{B}_1=\{B=\overline{B}(x,r)\in\mathcal{B} : r\le\frac12 \text{ and } B\cap\bigcup_1^{k_1}B_1^i=\emptyset\}$. Again, use the $5R$-covering theorem to find a disjoint subcollection $B_2^1,B_2^2,\ldots$ of $\mathcal{B}_1$ such that $A_1\subset\bigcup_i 5B_2^i$. Let k_2 be so large that $\mu(A_1\setminus\bigcup_1^{k_2}5B_2^i)<\varepsilon_2\mu(A)$ and define $A_2=A_1\cap\bigcup_1^{k_2}5B_2^i\setminus\bigcup_1^{k_2}B_2^i$. Proceeding in this way, we define A_j and $B_j^1, B_j^2, \dots, B_j^{k_j}$ for all $j \in \mathbb{N}$. We require that the radius of B_j^i be at most 2^{-j} for all i and j.

We now define $\widetilde{\mathcal{B}}_\varepsilon = \{B_j^i : 1 \le i \le k_j, j \in \mathbb{N}\}$, $P_\varepsilon = \bigcap_{k\in\mathbb{N}} A_k$, and $\widetilde{N}_\varepsilon = A \setminus (\cup\widetilde{\mathcal{B}}_\varepsilon \cup P_\varepsilon)$. It is clear that the collection $\widetilde{\mathcal{B}}_\varepsilon$ is pairwise disjoint. Also,

$$\mu(\widetilde{N}_\varepsilon) = \mu\big(A \setminus (\cup\widetilde{\mathcal{B}}_\varepsilon \cup P_\varepsilon)\big) \le \mu\left(\bigcup_{j=0}^{\infty}\left(A_j \Big\backslash \bigcup_{i=1}^{k_{j+1}} 5B_{j+1}^i\right)\right) \le \sum_{j=1}^{\infty} \varepsilon_j \mu(A) < \varepsilon\mu(A),$$

denoting $A_0 = A$. Since $\widetilde{B}_\varepsilon$ is countable, we must have $\mu(N_\varepsilon) < \varepsilon\mu(A)$ also for some finite subcollection $\mathcal{B}_\varepsilon$ of $\widetilde{\mathcal{B}}_\varepsilon$, where $N_\varepsilon = A \setminus (\cup\mathcal{B}_\varepsilon \cup P_\varepsilon))$ (recall that μ is Borel regular). Finally, if $x \in P_\varepsilon$, then $x \in A_j$ for all j. Thus for all j there is $B = \overline{B}(x,r) \in \{B_j^1, \dots, B_j^{k_j}\}$ with $x \in \overline{B}(x,5r) \setminus \overline{B}(x,r)$. Since $B \cap P_\varepsilon = \emptyset$, this implies $\mathrm{por}(P_\varepsilon, x, 6r) \ge \frac{1}{6}$ giving $\overline{\mathrm{por}}(P_\varepsilon, x) \ge \frac{1}{6}$ as $r \downarrow 0$. □

Below, we have a density point theorem suitable for our purposes.

LEMMA 3.2. *If $\overline{\mathrm{por}}(\mu, x) = 0$ for μ-almost every $x \in A$ for a Borel set $A \subset X$, then $\lim_{r\downarrow 0} \mu(B(x,r) \setminus A)/\mu(B(x,r)) = 0$ for μ-almost all $x \in A$.*

Proof. By [17, Proposition 3.5], $\mu(A \cap P) = 0$ for any σ-upper porous set $P \subset X$. By the previous lemma, we are able to use Vitali's covering theorem. The claim then follows, since it is well known that Vitali's covering theorem implies the density point theorem. See [9, Remark 1.13], for example. □

We are now ready to prove our main result, Theorem 3.3 below.

THEOREM 3.3. *Let*

$$0 < \liminf_{r\downarrow 0} \mu(B(x,r))/h(2r) < \infty \tag{3.2}$$

for μ-almost all $x \in X$ and suppose that for all $\delta > 0$ there is a Borel set D with $\mu(X \setminus D) < \delta$ such that

$$\liminf_{r\downarrow 0} \mathcal{H}^h_\infty\big(B(x,r) \setminus D\big)/h(2r) > 0 \tag{3.3}$$

for μ-almost every $x \in X$. Then μ is upper porous.

Proof. It is enough to show that

$$\lim_{\varepsilon\downarrow 0} \limsup_{r\downarrow 0} \mathrm{por}(\mu, x, r, \varepsilon) \ge \frac{1}{3} \tag{3.4}$$

for μ-almost all points $x \in X$. We argue by contradiction assuming that (3.4) fails in a set of positive μ-measure. Then there is a Borel set $B \subset X$ with

$\mu(B) > 0$, and $\varepsilon > 0$ with

$$\limsup_{r\downarrow 0} \operatorname{por}(\mu, x, r, \varepsilon) < \frac{1}{3} \tag{3.5}$$

for all $x \in B$. We refer to [17, Proposition 3.2] for measurability arguments.

Now (3.5) implies that for each $x \in B$ there is $r_0 = r_0(x) > 0$, such that

$$\mu(B(y, r)) > \varepsilon\mu(B(x, 3r)) \tag{3.6}$$

for all $y \in B(x, 2r) \setminus B(x, r)$ and $0 < r < r_0$. By the Borel regularity of μ, we may take a closed set $F \subset B$ with $\mu(F) > 0$ such that (3.6) holds for all $x \in F$ with a fixed constant $r_0 > 0$. Using the left-hand side estimate of (3.2) and (3.3), we may assume, making F and r_0 smaller if necessary, that also

$$h(2r) < c_1\mu(B(x, r)), \quad \text{and} \tag{3.7}$$

$$h(2r) < c_2\mathcal{H}^h_\infty\big(B(x, r) \setminus F\big) \tag{3.8}$$

for all $x \in F$ and $0 < r < r_0$ with some constants $1 < c_1, c_2 < \infty$. To obtain (3.8), take $D \subset X$ that satisfies (3.3) with $\delta < \mu(F)$ and replace F by a suitable compact subset of $D \cap F$. Furthermore, by combining (3.6) and (3.7) it follows that

$$\mu(B(y, r)) > c_3 h(6r) \tag{3.9}$$

for all $x \in F$, $y \in B(x, 2r) \setminus B(x, r)$, and $0 < r < \frac{r_0}{2}$ where $c_3 = \frac{\varepsilon}{c_1}$.

Now, we use Lemma 3.2 and the right-hand side estimate of (3.2) to find $0 < c_4 < \infty$ and $x \in F$, such that $\lim_{r\downarrow 0} \mu(B(x, r) \setminus F)/\mu(B(x, r)) = 0$ and $\liminf_{r\downarrow 0} \mu(B(x, r))/h(2r) < c_4$. The use of Lemma 3.2 is justified since by (3.5) and Remarks 2.1(a), we have $\overline{\operatorname{por}}(\mu, x) = 0$ for μ-almost all $x \in F$. Let $c_5 = c_3/(c_2 C_d^2)$, where C_d is the doubling constant of (2.2). We may then find a radius $0 < r < r_0$ for which

$$\mu\big(B(x, r) \setminus F\big) < \frac{c_5}{c_4}\mu(B(x, r)) < c_5 h(2r). \tag{3.10}$$

For each $y \in B(x, \frac{r}{2}) \setminus F$, there is a unique $0 < r_y \leq \frac{r}{2}$ such that $B(y, r_y) \cap F = \emptyset$, while $B(y, r) \cap F \neq \emptyset$ for all $r > r_y$. Applying the $5R$-covering theorem ([9, Theorem 1.2]) to the collection $\{B(y, r_y) : y \in B(x, \frac{r}{2}) \setminus F\}$, we get pairwise disjoint balls $B(y_1, r_1), B(y_2, r_2), \ldots \subset B(x, r) \setminus F$ such that $B(x, \frac{r}{2}) \setminus F \subset \bigcup_i \overline{B}(y_i, 5r_i)$. Choosing points $x_i \in B(y_i, 2r_i) \cap F \subset B(y_i, 2r_i) \setminus B(y_i, r_i)$, we get, using (3.9), the doubling condition (2.2), and (3.8)

$$\begin{aligned}
\mu\big(B(x, r) \setminus F\big) &\geq \sum_i \mu(B(y_i, r_i)) > c_3 \sum_i h(6r_i) > \frac{c_3}{C_d} \sum_i h(12r_i) \\
&\geq \frac{c_3}{C_d}\mathcal{H}^h_\infty\big(B\big(x, \tfrac{r}{2}\big) \setminus F\big) > \frac{c_3}{c_2 C_d} h(r) \geq \frac{c_3}{c_2 C_d^2} h(2r) \\
&= c_5 h(2r)
\end{aligned}$$

contrary to (3.10). This completes the proof. □

As an immediate corollary to Theorem 3.3 we get the following corollary.

Corollary 3.4. *Suppose that* (3.3) *holds for all closed sets* $D \subset X$ *such that* $\mathcal{P}^h(D) < \infty$. *If* $A \subset X$ *is a Borel set with* $0 < \mathcal{P}^h(A) < \infty$, *then* $\mathcal{P}^h|A$ *is upper porous.*

Proof. By [5, Theorem 3.11] $\mathcal{P}^h|A$ is Borel regular, and thus for all $\delta > 0$ there is a closed set $D \subset A$ such that $\mathcal{P}^h(A \setminus D) < \delta$. The claim now follows directly from Theorem 3.3 together with Cutler [5, Theorem 3.16], which implies (3.2) for $\mu = \mathcal{P}^h|A$. □

As will be shown by an example in Section 4.2, Corollary 3.4 is not in general true if one replaces packing measures by Hausdorff measures. On the other hand, if $\mathcal{H}^h|A$ has positive lower density almost everywhere, then we may use Theorem 3.3 to obtain the following result.

Corollary 3.5. *Suppose that* (3.3) *holds for all closed sets* $D \subset X$ *such that* $\mathcal{H}^h(D) < \infty$. *If* $A \subset X$ *is a Borel set with* $0 < \mathcal{H}^h(A) < \infty$ *and if* $\liminf_{r\downarrow 0} \mathcal{H}^h(A \cap B(x,r))/h(2r) > 0$ *for* $\mathcal{H}^h$*-almost all* $x \in A$, *then the measure* $\mathcal{H}^h|A$ *is upper porous.*

Proof. This follows directly from Theorem 3.3. It is well known that $\mathcal{H}^h|A$ is Borel regular, see [15, Theorem 4.2]. Moreover, it is a simple consequence of the $5R$-covering theorem that even $\limsup_{r\downarrow 0} \mathcal{H}^h(A \cap B(x,r))/h(2r) < \infty$ for $\mathcal{H}^h$-almost all $x \in A$. Recall that h is doubling and use the argument of [15, Theorem 6.6], for example. □

Remark 3.6. In order for μ to be upper porous, it is necessary, by Remark 2.1(b), to find upper porous sets with measure arbitrarily close to $\mu(X)$. Thus, we need to impose conditions on μ that ensure the existence of such sets. To obtain this, the condition (3.3) is used. It may look a bit awkward but it is easily seen to hold in many natural cases. Below, we give one concrete example.

Proposition 3.7. *Suppose that* h *and* g *are gauge functions which satisfy the doubling condition* (2.2) *and that* $r \mapsto h(r)/g(r)$ *is decreasing on* $(0,\infty)$. *Suppose also that* $h(r)/g(r) \to \infty$ *as* $r \downarrow 0$ *and that there is a measure* μ *on* X *and constants* $r_0 > 0$, $1 \le C < \infty$ *such that* $\frac{1}{C} g(2r) \le \mu(B(x,r)) \le C g(2r)$ *for every* $x \in X$ *and all* $0 < r < r_0$. *Then the assumption* (3.3) *is satisfied for the measures* $\nu = \mathcal{P}^h|A$ *and* $\lambda = \mathcal{H}^h|B$ *provided* A *and* B *are Borel sets with* $\mathcal{P}^h(A) < \infty$ *and* $\mathcal{H}^h(B) < \infty$.

Proof. Let $D \subset X$ be either of the sets A or B. Since $\mathcal{H}^h(A) \le \mathcal{P}^h(A)$, see [5, Theorem 3.11], we conclude that $M = \mathcal{H}^h(D) < \infty$. We first check that

$$\mu(D) = 0. \tag{3.11}$$

Let $\delta > 0$. Choose $0 < r_1 < r_0$ such that $g(r) < \delta h(r)$ for all $0 < r < r_1$ and cover D with sets $A_1, A_2, A_3, \ldots$ such that $d_i = \operatorname{diam}(A_i) < r_1/2$ for all i and $\sum_i h(d_i) < 2M$. Choosing $x_i \in A_i$ for each i, we get

$$\begin{aligned}\mu(D) &\le \sum_i \mu(B(x_i, d_i)) \le C \sum_i g(2d_i) \le \delta C \sum_i h(2d_i)\\ &\le \delta C C_d \sum_i h(d_i) < \delta C C_d 2M,\end{aligned}$$

where C_d is the doubling constant of h, see (2.2). Letting $\delta \downarrow 0$ gives (3.11).

It remains to show that

$$\liminf_{r\downarrow 0} \mathcal{H}^h_\infty\big(B(x,r)\setminus D\big)/h(2r) > 0 \tag{3.12}$$

for all $x \in X$. Let $0 < r < r_0/2$ and suppose $B(x,r)\setminus D \subset \bigcup_{i=1}^\infty A_i$. We may assume that $d_i = \operatorname{diam}(A_i) \le 2r$ for each i. Choosing $x_i \in A_i$ and using the assumption that $h(r)/g(r)$ is decreasing in r and (3.11), we get

$$\begin{aligned}\sum_i h(d_i) &= \sum_i \frac{h(d_i)}{g(d_i)} g(d_i) \ge \frac{h(2r)}{g(2r)} \sum_i g(d_i) \ge \frac{h(2r)}{C' g(2r)} \sum_i g(2d_i)\\ &\ge \frac{h(2r)}{CC' g(2r)} \sum_i \mu(B(x_i, d_i)) \ge \frac{h(2r)}{CC' g(2r)} \mu(B(x,r)) \ge \frac{h(2r)}{C^2 C'},\end{aligned}$$

where C' is the doubling constant of g, see (2.2). Hence it follows that $\mathcal{H}^h_\infty(B(x,r)\setminus D)/h(2r) \ge 1/(C^2C')$ giving (3.12). □

Remark 3.8. In Theorem 3.3, we have no assumptions on the space X besides separability and (3.3). For example, X may be infinite dimensional. This is not surprising given the nature of our statement: We only have to care about the points where μ satisfies the doubling condition, so our measures are essentially finite dimensional. I do not know whether all finite dimensional measures on an infinite dimensional space are upper porous but, for instance, in any infinite dimensional Banach space all Radon measures are even lower porous [22, p. 518].

4. Results on $\mathbb{R}^n$

Throughout this section, we shall work on $\mathbb{R}^n$ with the Euclidean metric $d(x,y) = |x-y|$.

4.1. Upper porosity and conical densities. In this subsection, we show that in Euclidean spaces, upper porosity is equivalent to a lower conical density property by proving the following theorem.

Theorem 4.1. *A measure μ on $\mathbb{R}^n$ is upper porous if and only if for μ-almost all $x \in \mathbb{R}^n$ there is $\theta = \theta(x) \in S^{n-1}$ such that*

$$\liminf_{r\downarrow 0} \mu(H(x,r,\theta,\eta))/\mu(B(x,r)) = 0 \tag{4.1}$$

for all $\eta > 0$.

Before the proof, let us consider the following example. Suppose that $A \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(A) < \infty$ for some $0 < s < n$. Then the basic lower density results originating from the works of Besicovitch [3], [4], and Marstrand [13] imply that for $\mathcal{H}^s$-almost every $x \in A$, there is $\theta \in S^{n-1}$ so that

$$\liminf_{r \downarrow 0} \mathcal{H}^s\big(A \cap H(x,r,\theta,\eta)\big)/(2r)^s = 0 \tag{4.2}$$

for all $\eta > 0$. If $\liminf_{r\downarrow 0} \mathcal{H}^s(A \cap B(x,r))/(2r)^s > 0$ for $\mathcal{H}^s$-almost all $x \in A$, we see from Theorem 4.1 that (4.2) is equivalent to saying that the measure $\mathcal{H}^s|A$ is upper porous. Of course, this follows also from Corollary 3.5, but the connection given by Theorem 4.1 between upper porosity and lower conical densities enables us to consider upper density results on metric spaces as analogies of (4.2) and the other known lower conical density results for many Hausdorff and packing type measures, see [19].

Proof of Theorem 4.1. If $\varrho(\eta) = \sqrt{1-\eta^2}/(1+\sqrt{1-\eta^2})$, it follows by elementary geometry that $B(x + (1-\varrho(\eta))r\theta, \varrho(\eta)r) \subset H(x,r,\theta,\eta)$, for all $x \in \mathbb{R}^n$, $0 < \eta < \frac{1}{2}$ and $\theta \in S^{n-1}$. Thus, (4.1) implies that $\overline{\mathrm{por}}(\mu,x) \geq \varrho(\eta)$. Observe that $\varrho(\eta) \to \frac{1}{2}$ as $\eta \downarrow 0$.

To prove that an upper porous measure μ satisfies (4.1), it is actually enough to show that for all $\eta > 0$ and almost all x,

$$\liminf_{r \downarrow 0} \inf_{\theta \in S^{n-1}} \mu(H(x,r,\theta,\eta))/\mu(B(x,r)) = 0. \tag{4.3}$$

That (4.3) implies (4.1) follows by a simple compactness argument, see [20, p. 504].

Let $D \subset \mathbb{R}^n$ be the set of points where μ satisfies the doubling condition (2.1). We first prove that (4.3) is satisfied almost everywhere on D. If this is not the case, we use the Borel regularity of μ and the Borel measurability of the mapping $x \mapsto \mu(H(x,r,\theta,\eta))/\mu(B(x,r))$ to find numbers $0 < a, r_0, \varrho < \frac{1}{2}$ and a compact set $F \subset D$ with $\mu(F) > 0$ such that $\overline{\mathrm{por}}(\mu,x) > \varrho$ and

$$\mu(H(x,r,\theta,\eta)) > a\mu(B(x,r)) \tag{4.4}$$

for all $x \in F$, $\theta \in S^{n-1}$, and $0 < r < r_0$. Since $F \subset D$, we may also assume by making F and r_0 smaller if necessary that

$$\mu(B(x,2r)) < c_1\mu(B(x,r)) \quad \text{for all } 0 < r < r_0 \tag{4.5}$$

for some constant $1 < c_1 < \infty$. Now, we fix $x \in F$ such that $\lim_{r\downarrow 0} \mu(B(x,r) \setminus F)/\mu(B(x,r)) = 0$ (use [15, 2.14], for example) and choose $0 < r_1 < r_0$, for which

$$\mu\big(B(x,2r) \setminus F\big) < \frac{1}{2}ac_1^{-k}\mu(B(x,2r)) \quad \text{for all } 0 < r < r_1, \tag{4.6}$$

where k is an integer to be defined later (depending only on η and ϱ). Since $\overline{\mathrm{por}}(\mu,x)>\varrho$, we may find $0<r<r_1/8$ and y such that $d(x,y)<(1-\varrho)r$ and

$$\mu(B(y,\varrho r))\le\frac{1}{2}ac_1^{-k}\mu(B(x,r)). \tag{4.7}$$

Let $t=\min\{t\ge\varrho r: S(y,t)\cap F\neq\emptyset\}$ and pick $z\in F\cap S(y,t)$. Putting $\theta=(y-z)/|y-z|$, we have $H(z,c_2r,\theta,\eta)\subset B(y,t)$ for a constant $c_2=c_2(\eta,\varrho)>0$. We now let k be the smallest positive integer for which $2^k>\frac{4}{c_2}$. Then k depends only on η and ϱ. It is easy to see that $|z-x|<2r$, giving $B(x,2r)\subset B(z,4r)\subset B(z,2^kc_2r)$ and $H(z,c_2r,\theta,\eta)\subset B(y,t)\subset B(y,\varrho r)\cup(B(x,2r)\setminus F)$. We now get a contradiction, since

$$\mu(H(z,c_2r,\theta,\eta))\ge a\mu(B(z,c_2r))\ge ac_1^{-k}\mu(B(x,2r))$$

by (4.4) and the repeated use of (4.5), and, on the other hand,

$$\mu(H(z,c_2r,\theta,\eta))\le\mu(B(y,\varrho r))+\mu\big(B(x,2r)\setminus F\big)<ac_1^{-k}\mu(B(x,2r)),$$

using (4.6) and (4.7).

To finish the proof, it is enough to show that (4.3) holds also for μ-almost every $x\in\mathbb{R}^n\setminus D$ when $\eta>0$. We argue by contradiction by assuming that there is a compact set $F\subset\operatorname{spt}\mu\setminus D$ with positive μ-measure and numbers $0<a,r_0<1$ so that (4.4) holds true for all $x\in F$ and $0<r<r_0$.

We next define constants $0<c,\varepsilon<1$ and $k\in\mathbb{N}$. By simple geometric inspections, it is easy to see that the choice of these constants only depends on the dimension n and the fixed constant $\eta>0$, although one might find it tedious to calculate the exact values. First, we choose $c>0$ so that $H(y,2c,-y,\eta)\subset B(0,1)$ for all $y\in S^{n-1}$. Given $0<\varepsilon<\frac{c}{2}$, we denote

$$C_{x,r,\theta}=H\left(x-(1+2\varepsilon)r\theta,\frac{c}{2}r,\theta,\eta\right)\cap B\big(x,(1+\varepsilon)r\big)\setminus B(x,r)$$

and we fix ε so small that

$$H(y,cr,\theta,\eta)\subset B(x,r)\cup C_{x,r,\theta} \tag{4.8}$$

for all $x\in\mathbb{R}^n$, $\theta\in S^{n-1}$, $r>0$, and $y\in\overline{C}_{x,r,\theta}$. Figure 1 might help visualize the situation. We finally choose a large integer k so that any annulus $B(x,(1+\varepsilon)r)\setminus B(x,r)\subset\mathbb{R}^n$ may be covered by at most k different sets of the form $C_{x,r,\theta}$.

Since $F\subset\operatorname{spt}\mu\setminus D$, we have $\limsup_{r\downarrow0}\mu(B(x,(1+\varepsilon)r))/\mu(B(x,r))=\infty$ for all $x\in F$. By [15, 2.14], also $\lim_{r\downarrow0}\mu(B(x,r)\setminus F)/\mu(B(x,r))=0$ for μ-almost all $x\in F$. Thus, we may fix $x\in F$ and $0<r<r_0$, for which

$$\mu\big(B\big(x,(1+\varepsilon)r\big)\setminus F\big)<\frac{a}{4k}\mu\big(B\big(x,(1+\varepsilon)r\big)\big), \tag{4.9}$$

$$\mu(B(x,r))<\frac{a}{4k}\mu\big(B\big(x,(1+\varepsilon)r\big)\cap F\big). \tag{4.10}$$

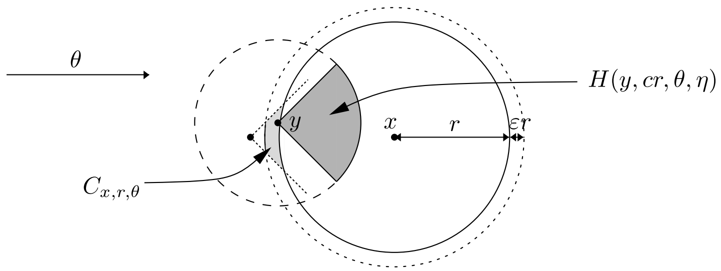

FIGURE 1. Illustration for the second part of the proof of Theorem 4.1.

Now the annulus $B(x,(1+\varepsilon)r)\setminus B(x,r))$ may be covered by at most k cones $C_{x,r,\theta}$. Hence, we may fix a $\theta\in S^{n-1}$ so that for $C=C_{x,r,\theta}$ we have

$$\mu(C\cap F)\geq\frac{1}{k}\mu\big(B\big(x,(1+\varepsilon)r\big)\cap F\setminus B(x,r)\big)>\frac{1}{2k}\mu\big(B\big(x,(1+\varepsilon)r\big)\big); \tag{4.11}$$

use (4.9)–(4.10) to obtain the last estimate. We now choose a point $y\in\overline{C}\cap F$ which maximizes the inner product $y\cdot\theta$ in $\overline{C}\cap F$. From (4.8), it follows that $H(y,cr,\theta,\eta)\subset B(x,r)\cup(C\setminus F)\subset B(x,r)\cup(B(x,(1+\varepsilon)r)\setminus F)$, see Figure 1. Also $C\subset B(y,cr)$ since $\operatorname{diam}(C)<cr$, and using (4.9)–(4.11), we get

$$\begin{aligned}\mu(H(y,cr,\theta,\eta))&\leq\mu(B(x,r))+\mu\big(B\big(x,(1+\varepsilon)r\big)\setminus F\big)\\&<\frac{a}{2k}\mu\big(B\big(x,(1+\varepsilon)r\big)\big)<a\mu(C)\leq a\mu(B(y,cr))\end{aligned}$$

contrary to (4.4). This completes the proof. □

REMARK 4.2. The above theorem gives also an alternative proof for the fact that $\overline{\text{por}}(\mu,x)\in\{\frac{1}{2},1\}$ for μ-almost every $x\in\mathbb{R}^n$, if μ is an upper porous measure on $\mathbb{R}^n$. The original proof given in [16] makes use of tangent measures, and the one given in [17] that works in metric spaces is based on a general fact, according to which any upper porous set $A\subset X$ is a countable union of sets with upper porosity arbitrarily close to $\frac{1}{2}$, see [17, Proposition 1.3] and [21, Proposition 4.1].

4.2. Hausdorff measures need not be upper porous. It is natural to ask if μ is upper porous in Theorem 3.3 also if lim inf is replaced by lim sup in (3.2). In particular, if Corollary 3.5 remains true if $\liminf_{r\downarrow 0}\mathcal{H}^h(A\cap B(x,r))/h(2r)=0$ in a set of a positive μ-measure. The following example shows that this need not be the case even for $\mu=\mathcal{H}^s|A$, where $A\subset\mathbb{R}$ is a Borel set and $0<\mathcal{H}^s(A)<\infty$ for some $0<s<1$.

EXAMPLE 4.3. Let $0 < s < 1$. We construct a probability measure μ on $[0,1]$ such that with some constants $0 < c_1 < c_2 < \infty$,

$$c_1 < \limsup_{r \downarrow 0} \mu(x-r, x+r)/(2r)^s < c_2 \tag{4.12}$$

for μ-almost every $x \in [0,1]$, and, moreover,

$$c^{-1}\mu(I) < \mu(I') < c\mu(I) \tag{4.13}$$

for some constant $c < \infty$ whenever I and I' are adjacent triadic sub-intervals of $[0,1]$ with equal length. Condition (4.12) implies that μ is comparable with a Hausdorff measure $\widetilde{\mu} = \mathcal{H}^s|A$, where $A \subset [0,1]$ is a Borel set with $0 < \mathcal{H}^s(A) < \infty$, see [15, Theorem 6.9]. On the other hand, (4.13) implies that

$$\overline{\text{por}}(\mu, x) = \overline{\text{por}}(\widetilde{\mu}, x) = 0 \quad \text{for } \mathcal{H}^s\text{-almost all } x \in A. \tag{4.14}$$

Indeed, let $\alpha > 0$, $r > 0$ and $x, y \in (0,1)$ so that $(y - \alpha r, y + \alpha r) \subset (x - r, x + r)$. Then there is a triadic interval $I_1 \subset (y - \alpha r, y + \alpha r)$ with length at least $\alpha r/3$ and also triadic intervals I_2 and I_3 with length at most $3r$ so that $(x - r, x + r) \subset I_2 \cup I_3$. Let $i = i(\alpha)$ be the smallest natural number for which $\alpha \geq 3^{2-i}$. Then a repeated application of (4.13) gives $\mu(y - \alpha r, y + \alpha r) \geq \mu(I_1) \geq (1 + c)^{-i} \min\{\mu(I_2), \mu(I_3)\} \geq (1+c)^{-i-1}\mu(I_2 \cup I_3) \geq c(\alpha)\mu(x - r, x + r)$. This implies $\overline{\text{por}}(\mu, x) \leq \alpha$ for all $x \in (0,1)$ and letting $\alpha \downarrow 0$ gives $\overline{\text{por}}(\mu, x) = 0$ for $x \in (0,1)$. Moreover, the equality $\overline{\text{por}}(\mu, x) = \overline{\text{por}}(\widetilde{\mu}, x)$ holds almost everywhere when we compare (4.12) with [15, Theorem 6.9].

To construct μ, we fix $\frac{1}{3} < p < 1$ such that

$$s > -p \log_3 p - 2p' \log_3 p', \tag{4.15}$$

where $p' = (1-p)/2$. We start by defining $\mu[0, \frac{1}{3}] = \mu[\frac{2}{3}, 1] = p'$ and $\mu[\frac{1}{3}, \frac{2}{3}] = p$. Suppose that $\mu(I)$ was already defined for a triadic interval $I = [j3^{-k}, (j+1)3^{-k}]$ ($k \in \mathbb{N}$ and $j = 0, \ldots, 3^k - 1$). Denote $I_1 = [j3^{-k}, j3^{-k} + 3^{-k-1}]$, $I_2 = [j3^{-k} + 3^{-k-1}, j3^{-k} + 2 \cdot 3^{-k-1}]$, and $I_3 = [j3^{-k} + 2 \cdot 3^{-k-1}, (j+1)3^{-k}]$. If $\mu(I) \leq \ell(I)^s = 3^{-ks}$, we let

$$\mu(I_1) = \mu(I_3) = p'\mu(I) \quad \text{and} \quad \mu(I_2) = p\mu(I). \tag{4.16}$$

Otherwise, we put $\mu(I_i) = \mu(I)/3$ for each $i = 1, 2, 3$. Repeating this procedure defines μ on all triadic sub-intervals of $[0,1]$ and μ then easily extends to a probability measure on $[0,1]$.

It is clear from the construction that $\limsup_{r \downarrow 0} \mu(x - r, x + r)/(2r)^s < c_2 < \infty$ for all $x \in [0,1]$. The estimate (4.13) is also easily obtained. Proving that $\limsup_{r \downarrow 0} \mu(x - r, x + r)/(2r)^s > c_1 > 0$ almost everywhere reduces to showing that

$$\mu(B) = 0 \tag{4.17}$$

for the set

$$B = \{x \in [0,1] : \mu(I) \geq \ell(I)^s \text{ for only finitely many triadic intervals } I \ni x\}.$$

The claim (4.17) follows by the observation that on the set B the measure μ is absolutely continuous with respect to the self-similar probability measure μ_p that satisfies (4.16) for all triadic intervals $I \subset [0,1]$. Notice that $\mu_p(B) = 0$, since $\limsup_{j\to\infty} B_x^j/j \leq \frac{\log_3 p' + s}{\log_3 p' - \log_3 p} < p$ for all $x \in B$ by (4.15), where B_x^j is the number of indices $1 \leq i \leq j$ such that $x_i = 1$ in the ternary decomposition $x = \sum_{i=1}^{\infty} x_i 3^{-i}$, $x_i \in \{0,1,2\}$. See [7, Proposition 10.4].

Remarks 4.4. By Theorem 4.1, it follows that the above constructed μ (or equivalently $\widetilde{\mu} = \mathcal{H}^s|A$) satisfies

$$\liminf_{r\downarrow 0} \frac{\mu(x, x+r)}{\mu(x-r, x+r)} > 0 \quad \text{and} \quad \liminf_{r\downarrow 0} \frac{\mu(x-r, x)}{\mu(x-r, x+r)} > 0$$

for μ-almost all $x \in \mathbb{R}$. This gives a negative answers to a question posed in [19, p. 12]. The results of [16] on upper porous measures also imply that $\mu(P) = 0$ for all σ-upper porous sets $P \subset \mathbb{R}$ and that for μ-almost all $x \in \mathbb{R}$ all tangent measures ν of μ at x satisfy $\operatorname{spt} \nu = \mathbb{R}$. See [16] for details.

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