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ERGODIC COMPONENTS OF AN EXTENSION BY A NILMANIFOLD

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ABSTRACT. We prove that all ergodic components of an extension of an ergodic system by translations on a nilmanifold X are isomorphic to extensions of this system by translations on subnilmanifolds of X.

If G is a compact group and V a subgroup of G, then under the (left) action of V, G splits into a disjoint union of isomorphic "orbits": if H is the closure of V in G, then the right cosets Ha, $a \in G$, are minimal closed V-invariant subsets of G, and the action of V on each of these sets is ergodic (with respect to the Haar measure). If X is a compact homogeneous space of a locally compact group G and V is a subgroup of G, then the structure of orbits of the action of V on X may be much more complicated. However, if G is a nilpotent Lie group, and X is, respectively, a compact *nilmanifold*, then the orbit structure on X is almost as simple as in the case of a compact G:

THEOREM 1. *Let X be a compact nilmanifold and let V be a group of translations of X. Then X is a disjoint union of closed V-invariant (not necessarily isomorphic) subnilmanifolds, on each of which the action of V is minimal and ergodic with respect to the Haar measure.*

(See [Le], [L1], and [L2]; this is also a corollary of the general theory of Ratner and Shah on unipotent flows, see [Sh].)

Let us now turn to the "relative" situation. We say that a measure space Y is *an extension* of Y', and that Y' is *a factor* of Y, if a measure preserving mapping $p: Y \longrightarrow Y'$ is fixed. If P and P' are measure preserving actions of a group V on Y and Y', respectively, such that $P'_v \circ p = p \circ P_v$, $v \in V$, we say that *P is an extension of P' on Y*, and that Y' is *a factor of Y under the action P*.

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Throughout the paper, (Ω,ν) will be a probability measure space, and S will be an ergodic measure preserving action of a group V on Ω. We will assume that V is countable. (This assumption is not crucial for our argument, but saves us from measure theoretical troubles: under this assumption, if some statement is true a.e. for every $v\in V$, then it is true a.e. for all $v\in V$ simultaneously.) Let G be a compact group; we say that an extension T of S on the space $\Omega\times G$ is *a group extension* if T is defined by the formula $T_v(\omega,x)=(S_v\omega,a_{v,\omega}x)$, $x\in G$, where $a_{v,\omega}\in G$, $\omega\in\Omega$, $v\in V$, and for every $v\in V$, the mapping $\omega\mapsto a_{v,\omega}$ is assumed to be measurable. The family $(a_{v,\omega})_{v\in V,\omega\in\Omega}$ of elements of G defining T is called *a cocycle*; we will say that T *is given by* the cocycle $(a_{v,\omega})$. If H is a subgroup of G and $a_{v,\omega}\in H$ for all $v\in V$ and $\omega\in\Omega$, we will say that $(a_{v,\omega})_{v\in V,\omega\in\Omega}$ is *an H-cocycle*. Clearly, if T is given by an H-cocycle, the sets $\Omega\times(Hx)$, $x\in G$, are T-invariant.

We will call a self-mapping of $\Omega\times G$ defined by the formula $(\omega,x)\mapsto(\omega,b_\omega x)$, $x\in G$, where $b_\omega\in G$, $\omega\in\Omega$, and measurably depend on ω, *a reparametrization of* $\Omega\times G$ *over* Ω. When reparametrizing $\Omega\times G$ we allow ourself to ignore a null set of Ω, so that the reparametrization function b_ω can only be defined on a subset Ω' of full measure in Ω, and we substitute Ω by Ω'. After a reparametrization given by b_ω, the cocycle $(a_{v,\omega})$, defining a group extension T of S on $\Omega\times G$, changes to the cocycle $(b_{S_v\omega}a_{v,\omega}b_\omega^{-1})$ (which is said to be *cohomologous* to $(a_{v,\omega})$).

Let G be a compact metric group and let T be a group extension of S on $\Omega\times G$. Then in complete analogy with the absolute case, a simple decomposition of $\Omega\times G$ takes place.

THEOREM 2. (See, for example, [Z1].) *There exists a closed subgroup H of G (called* the Mackey group *of T) such that after a certain reparametrization of $\Omega\times G$ over Ω, T is given by an H-cocycle and T is ergodic on the right cosets Ha, $a\in G$, with respect to the measures $\nu\times(\mu_H a)$, where μ_H is the left Haar measure on H. Moreover, any T-ergodic measure on $\Omega\times G$ whose projection to Ω is ν has the form $\nu\times(\mu_H a)$ for some $a\in G$.*

Now let G be a locally compact group and let X be a compact homogeneous space of G. The notion of a group extension of S on $\Omega\times X$ given by a G-cocycle is transferred without changes to this case; we will only call it *a homogeneous space extension*, not a group extension. A reparametrization of $\Omega\times X$ over Ω with the help of a function $b_\omega\in G^\Omega$ is also defined similarly. Our goal is to show that in the framework of relative actions, compact nilmanifolds, again, behave as well as compact groups.

THEOREM 3. *Let X be a compact nilmanifold and let T be a homogeneous space extension of S on $\Omega\times X$. There exists a closed subgroup H of G such that after a certain reparametrization of $\Omega\times X$ over Ω, T is given by an H-cocycle, and if $\bigcup_{\theta\in\Theta}X_\theta$ is the partition of X into the minimal subnilmanifolds with respect to the action of H, then the measures $\nu\times\mu_{X_\theta}$, $\theta\in\Theta$, where*

μ_{X_θ} is the Haar measure on X_θ, are T-ergodic, and are the only T-ergodic measures on $\Omega\times X$ whose projection to Ω is ν.

We will use the following notation and terminology. If a is a transformation of a (measure) space Y and f is a function on Y, then a acts on f from the right by the rule $(fa)(y)=f(ay)$. If a space Y' is a factor of Y, then any function h' on Y' lifts to a function h on Y; we identify h' with h, and say that h *comes from* Y' in this case.

If Y' is a factor of a measure space Y, P' is an action of a group V on Y', and P is an extension of P' on Y, we will say that a function $f\in L^\infty(Y)$ is *an eigenfunction of P over Y* if $fP_v=\alpha_v f$, where $\alpha_v\in L^\infty(Y')$, for every $v\in V$. (Our definition of an eigenfunction over Y is more restricted than the standard definition of a *generalized eigenfunction of P over Y*, which assumes that the module spanned by the functions fT_v, $v\in V$, has finite rank over $L^\infty(\Omega)$.)

G will stand for a nilpotent Lie group of nilpotency class r, Γ for a cocompact subgroup of G, and X for the compact nilmanifold G/Γ. By μ_X we will denote the Haar measure on X, and will always mean this measure on X if the opposite is not stated.

T will stand for a homogeneous space extension of S on $\Omega\times X$ by a cocycle $(a_{v,\omega})_{v\in V,\omega\in\Omega}$.

If Z is a factor of X under the action of G, then T induces an action of V on $\Omega\times Z$, which is defined by the same cocycle $(a_{v,\omega})_{v\in V,\omega\in\Omega}$. We will identify this action with T and denote it by the same symbol.

A subnilmanifold X' of X is a closed subset of X of the form Kx, where K is a closed subgroup of G and $x\in X$. (Note that the notion of a subnilmanifold depends on the group acting of X; what is a subnilmanifold of X with respect to the action of G may not be a subnilmanifold with respect to the action of, say, the identity component of G.) For a subnilmanifold $X'=Kx$ of X, we will denote by $\mu_{X'}$ the Haar measure on X' with respect to the action of K, and will always mean this measure on X' if the opposite is not stated.

Let G^o be identity component of G. If X is connected, then X is a homogeneous space of G^o, $X=G^o/(\Gamma\cap G^o)$. If X is disconnected, then X is a finite union of connected subnilmanifolds; this subnilmanifolds are all isomorphic, are homogeneous spaces of G^o, and are permuted by elements of G.

We define $G_{(1)}=G^o$, $G_{(k)}=[G_{(k-1)},G]$, $k=2,3,\dots,r$, and $X_{(k)}=G_{(k+1)}\backslash X$, $k=0,1,\dots,r-1$. When X is connected, we also define $X_2=[G^o,G^o]\backslash X$; then X_2 is a torus, *the maximal factor-torus* of X. We will denote by p the canonical projection $\Omega\times X\longrightarrow\Omega$.

A base tool in studying orbits in nilmanifolds is a lemma by W. Parry ([P1] and [P2]), that says that a shift-transformation of a compact connected nilmanifold X is ergodic iff it is ergodic on the maximal factor-torus of X. Here is a "relative" analogue of Parry's lemma; another proof of it can be found in [Z2].

PROPOSITION 4. (Cf. [Z2], Corollary 3.4.) *Assume that X is connected. If T is ergodic on $\Omega\times X_2$, then T is ergodic on $\Omega\times X$, and any eigenfunction f of T over Ω comes from $\Omega\times X_2$ and is such that $f(\omega,\cdot)$ is a character on X_2, times a constant, for a.e. $\omega\in\Omega$.*

Proof. We will assume by induction on r that T is ergodic on $\Omega\times X_{(r-1)}$, and that if g is an eigenfunction of T on $\Omega\times X_{(r-1)}$ over Ω, then g comes from $\Omega\times X_2$ and $g(\omega,\cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega\in\Omega$.

Let $f\in L^\infty(\Omega\times X)$ be an eigenfunction of T over Ω, $fT_v=\alpha_v(\omega)f$, $\alpha_v:\Omega\longrightarrow\mathbb{C}$, $v\in V$. The action of the group $G_{(r)}$ on $\Omega\times X$ factors through an action of the compact commutative group (the torus) $G_{(r)}/(G_{(r)}\cap\Gamma)$, thus $L^2(\Omega\times X)$ is a direct sum of eigenspaces of $G_{(r)}$. Let f' be a nonzero projection of f to one of these eigenspaces, then $f'c=\lambda_c f'$, $\lambda_c\in\mathbb{C}$, for every $c\in G_{(r)}$. Since the eigenspaces of $G_{(r)}$ are T-invariant and invariant under multiplication by functions from $L^\infty(\Omega)$, we have $f'T_v=\alpha_v(\omega)f'$, $v\in V$.

For every $b\in G$ and $c\in G_{(r)}$, $(f'b)c=f'cb=\lambda_c f'b$, so the function $f'_b=(f'b)/f'$ is $G_{(r)}$ invariant, and thus comes from $\Omega\times X_{(r-1)}$.

Assume, by induction on decreasing k, that for some $k\in\{2,\dots,r\}$ we have $f'c=\lambda_c f'$, $\lambda_c\in\mathbb{C}^\Omega$, for any $c\in G_{(k)}$. Then $(f'\mathbf{c})(\omega,x)=\lambda_{c(\omega)}(\omega)f'(\omega,x)$, $\omega\in\Omega$, $x\in X$, for any $\mathbf{c}=c(\omega)\in G_{(k)}^\Omega$. Now, for any $b\in G_{(k-1)}$ and $v\in V$,

$$\begin{aligned}(f'bT_v)(\omega,x)&=f'(S_v\omega,ba_{v,\omega}x)=f'(S_v\omega,a_{v,\omega}[a_{v,\omega},b^{-1}]bx)\\&=(f'T_v)(\omega,[a_{v,\omega},b^{-1}]bx)=\alpha_v(\omega)f'(\omega,[a_{v,\omega},b^{-1}]bx)\\&=\alpha_v(\omega)\lambda_{c_{v,b}(\omega)}(\omega)f'(\omega,bx)=\alpha_v(\omega)\lambda_{c_{v,b}(\omega)}(\omega)(f'b)(\omega,x),\end{aligned}$$

where $c_{v,b}(\omega)=[a_{v,\omega},b^{-1}]\in G_{(k)}$, $\omega\in\Omega$. So, for any $b\in G_{(k-1)}$ and $v\in V$, $f'_bT_v=\lambda_{c_{v,b}(\omega)}(\omega)f'_b$, and since f'_b comes from $X_{(r-1)}$, by our first induction assumption, $f'_b(\omega,\cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega\in\Omega$. Thus, for a.e. $\omega\in\Omega$, we have a continuous mapping from $G_{(k-1)}$ to the set of characters on X_2, and since this set is discrete and $G_{(k-1)}$ is connected, this mapping is constant. (For a.e. ω, the considered mapping may not be a priori defined on a null subset of $G_{(k-1)}$, but since it is locally uniformly continuous, it extends to a continuous mapping on $G_{(k-1)}$.) Hence, $f'_b(\omega,\cdot)=\lambda_b(\omega)$, $\lambda_b\in\mathbb{C}$, for all $b\in G_{(k-1)}$ and a.e. $\omega\in\Omega$, that is, $f'b=\lambda_b f'$ with $\lambda_b\in\mathbb{C}^\Omega$, for all $b\in G_{(k-1)}$, which gives us the induction step.

As the result of our induction on k, we obtain that for every $b\in G_{(1)}=G^o$ there exists a function $\lambda_b\in\mathbb{C}^\Omega$ such that $f'b=\lambda_b f'$. Thus, for any $b_1,b_2\in G^o$ we have $f'[b_1,b_2]=f'$. Hence, f' is $[G^o,G^o]$-invariant, and so, comes from $\Omega\times X_2$. The equality $f'b=\lambda_b f'$, $b\in G^o$, now implies that $f'(\omega,\cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega\in\Omega$.

It follows that f also comes from $\Omega\times X_2$. In particular, there are no T-invariant functions on $\Omega\times X$ since there are no T-invariant functions on $\Omega\times X_2$, so T is ergodic.

Now assume that for at least two distinct eigenspaces of $G_{(r)}$ the projections f', f'' of f to these eigenspaces are nonzero. Then both $f'T_v=\alpha_v(\omega)f'$ and $f''T_v=\alpha_v(\omega)f''$, $v\in V$, and so, f'/f'' is T-invariant, which contradicts the ergodicity of T. Hence, f belongs to one of the eigenspaces of $G_{(r)}$, and so, as this has been proven for f', $f(\omega,\cdot)$ is a character-times-a-constant on X_2 for a.e. $\omega\in\Omega$. □

REMARK. In contrast with the absolute case (the case $\Omega=\{\cdot\}$), the stronger statement "T is ergodic if it is ergodic on $\Omega\times({}_{[G,G]}\backslash X)$" (where it is assumed that G is generated by G^o and $\{T_v,v\in V\}$) is no longer true in the relative case. Here is an example: let $\Omega=\mathbb{Z}_2$, let $X=\mathbb{T}^2_{x_1,x_2}$ where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, let G be the group of transformations of X of the form $(x_1,x_2)\mapsto(x_1+\alpha,x_2+lx_1+\beta)$, $\alpha,\beta\in\mathbb{T}$, $l\in\mathbb{Z}$, and let V be the group generated by the transformation $T(\omega,x_1,x_2)=(\omega+1,x_1+\omega\alpha,x_2+(-1)^\omega x_1)$ of $\Omega\times X$, where α is an irrational element of $\mathbb{T}$. Then $[G,G]=\{(0,x_2),\ x_2\in\mathbb{T}\}$, and ${}_{[G,G]}\backslash X\simeq\mathbb{T}_{x_1}$. One checks that T is ergodic on $\Omega\times({}_{[G,G]}\backslash X)$, whereas the function

$$f(\omega,x_1,x_2)=\begin{cases}x_2, & \omega=0,\\ x_2-x_1, & \omega=1,\end{cases}\quad\text{on } \Omega\times X$$

is T-invariant. The reason of this effect is clear, it is a "bad parametrization" of $\Omega\times X$; after a proper reparametrization, T acts as a rotation on X, G can be reduced to the group of rotations of X, and then ${}_{[G,G]}\backslash X=X$.

REMARK. We do not know whether Proposition 4 can be extended to the (more general) class of generalized eigenfunctions of T over Ω.

Let X be connected. Having Proposition 4, we may deal with the maximal factor-torus X_2 of X instead of X; indeed, if T is not ergodic on $\Omega\times X$, then T is not ergodic on $T\times X_2$ as well. The problem is that G, if disconnected, may act on X_2 not only by conventional rotations, but also by affine unipotent transformation. Thus, we will still have to treat X_2 as a nilmanifold, not as a conventional torus. Since this does not change our argument, we will not assume that X is a torus; we will, however, call "characters" on X those on X_2.

Note that for any character χ on X and any $a\in G$, $\chi a=\lambda\chi'$, where χ' is a character on X and $\lambda\in\mathbb{C}$, $|\lambda|=1$. On the other hand, if $\lambda\in\mathbb{C}$, $|\lambda|=1$, and χ is a character on X, then clearly, there exists a translation a of X such that $\chi a=\lambda\chi$.

Rather than Proposition 4, we will actually need the following, more technical fact.

LEMMA 5. *Let X be connected. Assume that T is ergodic on $X_{(r-1)}$ and that $f\in L^\infty(\Omega\times X)$ is T-invariant and is an eigenfunction of $G_{(r)}$. Then $f(\omega,\cdot)$ is a character-times-a-constant on X for a.e. $\omega\in\Omega$.*

Of course, if X_2 is a factor of $X_{(r-1)}$, this lemma follows from Proposition 4; otherwise it has to be proven separately, though its proof is very similar to that of Proposition 4.

Proof of Lemma 5. Let $fc=\lambda_c f$, $\lambda_c\in\mathbb{C}$, $c\in G_{(r)}$. For every $b\in G$ and $c\in G_{(r)}$, $(fb)c=fcb=\lambda_c fb$, so the function $f_b=(fb)/f$ is $G_{(r)}$ invariant, and thus comes from $\Omega\times X_{(r-1)}$. Assume, by induction on decreasing k, that for some $k\in\{2,\dots,r\}$ we have $fc=\lambda_c f$, $\lambda_c\in\mathbb{C}^\Omega$, for any $c\in G_{(k)}$. Then $(f\mathbf{c})(\omega,x)=\lambda_{c(\omega)}(\omega)f(\omega,x)$, $\omega\in\Omega$, $x\in X$, for any $\mathbf{c}=c(\omega)\in G_{(k)}^\Omega$. Now, for any $b\in G_{(k-1)}$ and $v\in V$,

$$\begin{aligned}(fbT_v)(\omega,x)&=f(S_v\omega,ba_{v,\omega}x)=f(S_v\omega,a_{v,\omega}[a_{v,\omega},b^{-1}]bx)\\&=(fT_v)(\omega,[a_{v,\omega},b^{-1}]bx)=f(\omega,[a_{v,\omega},b^{-1}]bx)\\&=\lambda_{c_{v,b}(\omega)}(\omega)f(\omega,bx)=\lambda_{c_{v,b}(\omega)}(\omega)(fb)(\omega,x),\end{aligned}$$

where $c_{v,b}(\omega)=[a_{v,\omega},b^{-1}]\in G_{(k)}$, $\omega\in\Omega$. So, for any $b\in G_{(k-1)}$ and $v\in V$, $f_bT_v=\lambda_{c_{v,b}(\omega)}(\omega)f_b$, and since f_b comes from $X_{(r-1)}$ where T is ergodic, by Proposition 4, $f_b(\omega,\cdot)$ is a character-times-a-constant on X for a.e. $\omega\in\Omega$. Thus, for a.e. $\omega\in\Omega$, we have a continuous mapping from $G_{(k-1)}$ to the set of characters on X, and since this set is discrete and $G_{(k-1)}$ is connected, this mapping is constant. Hence, $f_b(\omega,\cdot)=\lambda_b(\omega)$, $\lambda_b\in\mathbb{C}$, for all $b\in G_{(k-1)}$ and a.e. $\omega\in\Omega$, that is, $fb=\lambda_b f$ with $\lambda_b\in\mathbb{C}^\Omega$ for all $b\in G_{(k-1)}$, which gives us the induction step.

As the result of induction on k, we obtain that for every $b\in G_{(1)}=G^o$ there exists a function $\lambda_b\in\mathbb{C}^\Omega$ such that $fb=\lambda_b f$. Hence, $f(\omega,\cdot)$ is a character-times-a-constant on X for a.e. $\omega\in\Omega$. □

We will also need the following corollary of Theorem 2.

LEMMA 6. *Let K be a compact metric group, let Z be a homogeneous space of K, and let R be a homogeneous space extension of S on $\Omega\times Z$. If R is not ergodic, then K has a proper closed subgroup H such that after a reparametrization of $\Omega\times Z$ over Ω, R is given by an H-cocycle.*

Proof. The cocycle defining the action R defines a group action $\widetilde{R}$ of V on $\Omega\times K$, for which R is a factor. If R is not ergodic, then $\widetilde{R}$ is not ergodic as well, and the assertion of the lemma follows from Theorem 2. □

PROPOSITION 7. *Assume that T is not ergodic on $\Omega\times X$. Then there exists a proper closed subgroup H of G such that after a certain reparametrization of $\Omega\times X$ over Ω, T is given by an H-cocycle.*

Proof. We will use induction on r, the nilpotency class of X. First, for simplicity, consider the case where X is connected. If T is not ergodic on $\Omega\times X_{(r-1)}$, then we are done by induction on r. Thus, we assume that T is ergodic on $\Omega\times X_{(r-1)}$. Let f be a nonzero measurable T-invariant function on $\Omega \times X$. We replace f by its nonzero projection to one of the eigenspaces of $G_{(r)}$, which is also a T-invariant function. By Lemma 5, $f(\omega,\cdot) = \lambda(\omega)\chi_\omega$, where χ_ω is a character on X and $\lambda(\omega) \in \mathbb{C}$, for a.e. $\omega \in \Omega$. Since S is ergodic, $|\lambda(\omega)| = \text{const}$ on a subset Ω' of Ω of full measure, and we may assume that $|\lambda| \equiv 1$. There are only countably many characters on X, therefore a subset Ω'' of full measure in Ω' is partitioned into the union of sets of positive measure where χ_ω is constant. Since S is ergodic, we can choose a character χ on X and elements $b(\omega)$, $\omega \in \Omega''$, measurably depending on ω, such that for every $\omega \in \Omega'$ one has $\lambda_\omega \chi_\omega = \chi b_\omega$, so that $f(\omega, x) = \lambda(\omega)\chi_\omega(x) = \chi(b_\omega x)$, $x \in X$. After the reparametrization of $\Omega \times X$ defined by the function b_ω (and replacing Ω by Ω''), f takes the form $f(\omega, x) = \chi(x)$, $\omega \in \Omega$, $x \in X$. Let H be the stabilizer of χ in G, $H = \{c \in G : \chi c = \chi\}$; then H is a proper closed subgroup of G and the cocycle defining T takes values in H.

Now let X be disconnected. G acts on the finite set $\mathcal{X}$ of connected components of X; let $\widetilde{G}$ be the subgroup (of finite index) of G that acts trivially on $\mathcal{X}$. Then the action of G on $\mathcal{X}$ factorizes through the action of the finite group $G/\widetilde{G}$, and if T is not ergodic on $\Omega \times \mathcal{X}$, we are done by Lemma 6. Thus, we may assume that T is ergodic $\Omega \times \mathcal{X}$.

Let X^o be a connected component of X; then X, under the action of $\widetilde{G}$, is isomorphic to $\{1,\dots,n\} \times X^o$, where n is the number of components in X. Consider $\Omega \times X = \Omega \times \{1,\dots,n\} \times X^o$ as $\widetilde{\Omega} \times X^o$ where $\widetilde{\Omega} = \Omega \times \{1,\dots,n\}$; by our assumption, T acts ergodically on $\widetilde{\Omega}$. Since X^o is connected and has nilpotency class $\le r$, we may, as in the first part of the proof, find a subset Ω' of full measure in Ω and a measurable T-invariant function f on $\widetilde{\Omega}' \times X^o = \Omega' \times X$ such that $f(\omega, i, \cdot) = \lambda(\omega, i)\chi_{\omega,i}$, where $\chi_{\omega,i}$ is a character on X^o and $\lambda(\omega, i) \in \mathbb{C}$ for all $\omega \in \Omega'$ and all $i \in \{1,\dots,n\}$. For all $\omega \in \Omega'$ we, therefore, have the (nonordered) set $C_\omega = \{\chi_{\omega,1},\dots,\chi_{\omega,n}\}$ of characters on X^o such that $T_v C_\omega = C_{S_v\omega}$, $v \in V$, for all $\omega \in \Omega'$, and since only countably many possibilities for C_ω exist, a certain reparametrization of $\Omega \times X$ over Ω (with replacing Ω by Ω') makes C_ω to be constant, $C_\omega = C = \{\chi_1,\dots,\chi_n\}$ for all $\omega \in \Omega$. Moreover, since T acts ergodically on $\Omega \times \mathcal{X}$, G acts transitively on C; thus, after some change of coordinates in distinct connected components of X, we may make $\chi_1,\dots,\chi_n$ to be all equal to the same character χ. After this, we obtain that $\chi T_v = \frac{\lambda(\omega,i)}{\lambda(S_v\omega,j)}\chi$, $j = j(v,\omega,i)$, for all $v \in V$, $\omega \in \Omega$, and $i \in \{1,\dots,n\}$, that is, T maps the fibers of χ to fibers. Let us assume, as we may, that G is generated by G^o and the entries of the cocycle defining T; then G maps the fibers of χ to fibers, and we may factorize X by these fibers. Let Z be the factor; then Z is a finite union of circles, $Z = \{1,\dots,n\} \times \mathbb{T}$, and G acts by rotations on $\mathbb{T}$, that is, for any $a \in G$, $a(i, x) = (ai, x + \alpha_{a,i})$, $x \in \mathbb{T}$, $i \in \{1,\dots,n\}$, with $\alpha_{a,i} \in \mathbb{T}$ (and ai is defined by $X_{ai} = aX_i$). We obtain that the action of G on Z factorizes through the action of a compact group (the group of rotations of components of Z and of permutations of these components). Since T is not ergodic on $\Omega\times Z$, we are done by Lemma 6. □

LEMMA 8. *If T is ergodic on $\Omega\times X$ (with respect to $\nu\times\mu_X$), then $\nu\times\mu_X$ is the only T-ergodic probability measure whose projection on Ω is ν.*

Proof. Let $G_1=G$ and $G_k=[G_{k-1},G]$ for $k=2,3,\dots,r$, let $X_{r-1}=G_r\backslash X$, and let $\pi_r: X\longrightarrow X_{r-1}$ be the canonical projection. If T is ergodic on $\Omega\times X$ with respect to $\nu\times\mu_X$, by induction on r, $\nu\times\mu_{X_{r-1}}$ is the only T-ergodic probability measure on $\Omega\times X_{r-1}$ whose projection on Ω is ν. Thus, if τ is a T-ergodic probability measure on $\Omega\times X$ with $p(\tau)=\nu$, then $(\mathrm{Id}_\Omega\times\pi_r)(\tau)=\nu\times\mu_{X_{r-1}}$. $\Omega\times X$ is a group extension of $\Omega\times X_{r-1}$ with the fiber $F_r=G_r/(\Gamma\cap G_r)$, which is a compact commutative Lie group. Hence, by Theorem 2, $\tau=\nu\times\mu_{X_{r-1}}\times\mu_{F_r}=\nu\times\mu_X$. □

Proof of Theorem 3. Let H be a minimal closed subgroup of G such that there exists a reparametrization of $X\times\Omega$ over Ω after which T is given by an H-cocycle. (Such a subgroup exists since any chain of decreasing subgroups of G is finite.) Let $X=\bigcup_{\theta\in\Theta}X_\theta$ be the partition of X into the union of subnilmanifolds minimal under the action of H, as in Theorem 1. After the reparametrization corresponding to H, $\Omega\times X$ splits into the disjoint union $\bigcup_{\theta\in\Theta}\Omega\times X_\theta$ of T-invariant subsets on each of which T is given by an H-cocycle. If T is not ergodic on one of these subsets, then by Proposition 7, H contains a proper closed subgroup H' such that after a reparametrization of $\Omega\times X$ over Ω, T is given by an H'-cocycle; this contradicts the choice of H. Thus, T is ergodic on each of $\Omega\times X_\theta$, $\theta\in\Theta$. Moreover, if τ is an ergodic measure on $\Omega\times X$ with $p(\tau)=\nu$, then τ must be supported by $\Omega\times X_\theta$ for some $\theta\in\Theta$, and thus $\tau=\nu\times\mu_{\Omega_\theta}$ by Lemma 8. □

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