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GROUP BUNDLE DUALITY

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ABSTRACT. This paper introduces a generalization of Pontryagin duality for locally compact Hausdorff Abelian groups to locally compact Hausdorff Abelian group bundles.

First, recall that a group bundle is just a groupoid where the range and source maps coincide. An Abelian group bundle is a bundle where each fibre is an Abelian group. When working with a group bundle G, we will use X to denote the unit space of G and $p: G \to X$ to denote the combined range and source maps. Furthermore, we will use G_x to denote the fibre over x. Group bundles, like general groupoids, may not have a Haar system but when they do the Haar system has a special form. If G is a locally compact Hausdorff group bundle with Haar system, denoted by $\{\beta^x\}$ throughout the paper, then β^x is Haar measure on the fibre G_x for all $x \in X$. At this point, it is convenient to make the standing assumption that all of the locally compact spaces in this paper are Hausdorff.

Now suppose G is an Abelian, second countable, locally compact group bundle with Haar system $\{\beta^x\}$. Then $C^*(G,\beta)$ is a separable Abelian C^*-algebra and in particular $\widehat{G} = C^*(G,\beta)^\wedge$ is a second countable locally compact Hausdorff space [1, Theorem 1.1.1]. We cite [2, Section 3] to see that each element of $\widehat{G}$ is of the form (ω, x) with $x \in X$ and ω a character in the Pontryagin dual of G_x, denoted $(G_x)^\wedge$. The action of (ω, x) on $C_c(G)$ is given by

$$(\omega, x)(f) = \int_G f(s)\omega(s)\, d\beta^x(s). \tag{1}$$

Since every element in $\widehat{G}$ is a character on a fibre of G, we are justified in thinking of $\widehat{G}$ as a bundle over X with fibres $\widehat{G}_x = (G_x)^\wedge$ and action on

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$C^*(G,\beta)$ given by (1). We will use $\hat{p}$ to denote the projection from $\widehat{G}$ to X and ω to denote the element $(\omega, \hat{p}(\omega))$ in $\widehat{G}$.

At this point, it is clear that $\widehat{G}$ is algebraically a group bundle. In order for it to be a topological groupoid, we must show that the groupoid operations are continuous with respect to the Gelfand topology on $\widehat{G}$. To this end, we reference the following characterization of the topology on $\widehat{G}$.

LEMMA 1 ([2, Proposition 3.3]). *Let G be a second countable locally compact abelian group bundle with Haar system. Then a sequence $\{\omega_n\}$ in $\widehat{G}$ converges to ω_0 in $\widehat{G}$ if and only if:*

(a) *$\hat{p}(\omega_n)$ converges to $\hat{p}(\omega_0)$ in X, and*
(b) *if $s_n \in G_{\hat{p}(\omega_n)}$ for all $n \geq 0$ and $s_n \to s_0$ in G, then $\omega_n(s_n) \to \omega_0(s_0)$.*

The first thing we can conclude from this lemma is that the restriction of the topology on $\widehat{G}$ to $\widehat{G}_x$ is the same as the topology on $\widehat{G}_x$ as the dual group of G_x. The second thing we conclude is that the topology on $\widehat{G}$ is independent of the Haar system β. Furthermore, recall that the groupoid operations on $\widehat{G}$ are those coming from the dual operations on $\widehat{G}_x$. In other words, the operations are pointwise multiplication and conjugation of characters, and it follows from Lemma 1 that these operations are continuous. Therefore, we have proven the lemma.

LEMMA 2 ([2, Corollary 3.4]). *Let G be a second countable locally compact Abelian group bundle with Haar system. Then $\widehat{G}$, equipped with the Gelfand topology, is a second countable locally compact Abelian group bundle with fibres $\widehat{G}_x = (G_x)^\wedge$.*

Now we can make our first definition.

DEFINITION 3. If G is a second countable locally compact Abelian group bundle with Haar system, then we define the dual bundle to be $\widehat{G} = C^*(G)^\wedge$ equipped with the groupoid operations coming from the identification of $\widehat{G}_x$ as the dual of G_x. We will use $\hat{p}$ to denote the projection on this bundle.

This definition gives rise to the notion of a duality theorem for group bundles. The main result of this paper is to prove the following theorem, stated without proof in [3, Proposition 1.3.7].

THEOREM 4. *If G is a second countable locally compact (Hausdorff) Abelian group bundle with Haar system then the dual $\widehat{G}$ has a dual group bundle, denoted $\widehat{\widehat{G}}$. Furthermore, the map $\Phi : G \to \widehat{\widehat{G}}$ such that*

$$\Phi(s)(\omega) = \hat{s}(\omega) := \omega(s)$$

is a (topological) group bundle isomorphism between G and $\widehat{\widehat{G}}$.

Before we continue, it will be useful to see that the group bundle notion of duality is a natural extension of the usual Pontryagin dual, as illustrated by the following proposition.

PROPOSITION 5. *Let G be a second countable locally compact Abelian group bundle with Haar system. Then $C^*(G) \cong C_0(\widehat{G})$ via the Gelfand transform. Furthermore, if $f \in C_c(G)$ then the Gelfand transform of f restricted to $\widehat{G}_x$ is the Fourier transform of $f|_{G_x}$.*

Proof. The first statement follows from the fact that we defined $\widehat{G}$ to be the spectrum of the Abelian C^*-algebra $C^*(G)$. Next, let $\hat{f}$ be the Gelfand transform of f. Then for $\omega \in \widehat{G}$ we see from (1) that

$$\hat{f}(\omega) = \omega(f) = \int_{G_{\hat{p}(\omega)}} f(s)\omega(s)\, d\beta^{\hat{p}(\omega)}(s).$$

This of course implies that $\hat{f}$ is the usual Fourier transform on $\widehat{G}_x$. □

We can now begin the process of proving Theorem 4. The first step is to show that $\widehat{G}$ has a dual bundle. We have already verified that $\widehat{G}$ is a second countable locally compact Abelian group bundle. The only remaining requirement is that $\widehat{G}$ has a Haar system. Recall that given a locally compact Abelian group H and Haar measure λ the Plancharel theorem guarantees the existence of a dual Haar measure $\hat{\lambda}$ such that $L^2(H,\lambda) \cong L^2(\widehat{H},\hat{\lambda})$. The existence of a dual Haar system is then taken care of by the following lemma.

LEMMA 6 ([2, Proposition 3.6]). *If G is an Abelian second countable locally compact group bundle with Haar system $\{\beta^x\}$, then the collection of dual Haar measures $\{\hat{\beta}^x\}$ is a Haar system for $\widehat{G}$.*

Now that $\widehat{\widehat{G}}$ is well defined, we must show that Φ is a group bundle isomorphism. In some sense, the following proposition gets us most of the way there.

PROPOSITION 7. *The map $\Phi : G \to \widehat{\widehat{G}} : s \mapsto \hat{s}$ is a continuous bijective groupoid homomorphism.*

Proof. It follows from Lemma 2 that $\widehat{\widehat{G}}_x$ is the double dual of G_x. Furthermore, classical Pontryagin duality says that $s \to \hat{s}$ is an isomorphism from G_x onto $\widehat{\widehat{G}}_x$ [4, Theorem 1.7.2]. Since Φ is formed by gluing all of these fibre isomorphisms together it is clear that Φ is a bijective groupoid homomorphism. Next, we need to see that it is continuous. Suppose $s_i \to s_0$ in G. We know from Lemma 1 that it will suffice to show that

(a) $\hat{\hat{p}}(\Phi(s_i)) \to \hat{\hat{p}}(\Phi(s_0))$, and
(b) given $\omega_i \in \widehat{G}_{\hat{\hat{p}}(\Phi(s_i))}$ such that $\omega_i \to \omega_0$ in $\widehat{G}$ then $\Phi(s_i)(\omega_i) \to \Phi(s_0)(\omega_0)$.

First, let $x_i = p(s_i) = \hat{p}(\Phi(s_i))$. Since p is continuous, it is clear that $x_i \to x_0$ and that the first condition is satisfied. Now suppose $\omega_i \in \widehat{G}_{x_i}$ for all $i \geq 0$ such that $\omega_i \to \omega_0$. All we have to do is cite Lemma 1 again to see that

$$\Phi(s_i)(\omega_i) = \omega_i(s_i) \to \omega_0(s_0) = \Phi(s_0)(\omega_0). \qquad \square$$

If we were working with groups, we would be done since continuous bijections between second countable locally compact groups are automatically homeomorphisms [5, Theorem D.3], [1, Corollary 2, p. 72]. However, there currently no automatic continuity results for the inverse of a continuous bijective group bundle homomorphism. Regardless, we can still show that in this case Φ is a homeomorphism.

Proof of Theorem 4. Given Proposition 7, all we need to do to prove that Φ is a homeomorphism is show that if $\hat{s_i} \to \hat{s_0}$ in $\widehat{\widehat{G}}$ then $s_i \to s_0$ in G. First, we let $x_i = p(s_i)$ for all i. Recall that $\widehat{\widehat{G}}$ has the Gelfand topology as the spectrum of $C^*(\widehat{G}, \hat{\beta})$. Therefore, for all $\phi \in C_c(\widehat{G})$ we have $\hat{s_i}(\phi) \to \hat{s_0}(\phi)$. When we remember that characters in $\widehat{\widehat{G}}$ act on functions in $C_c(\widehat{G})$ via equation (1) we see that this says, for all $\phi \in C_c(\widehat{G})$,

$$\int_{\widehat{G}} \phi(\omega)\omega(s_i)\, d\hat{\beta}^{x_i}(\omega) \to \int_{\widehat{G}} \phi(\omega)\omega(s_0)\, d\hat{\beta}^{x_0}(\omega). \tag{2}$$

Now suppose we have a relatively compact open neighborhood V of x_0 in G. Then using the continuity of multiplication, there exists a relatively compact open neighborhood U of x_0 in G such that $U^2 \subseteq V$. Choose $h \in C_c(G)$ such that $h(x_0) = 1$ and $\operatorname{supp}(h) \subseteq U$. Let $f = h^* * h$. Then $f \in C_c(G)$ and a simple calculation shows that $\operatorname{supp}(f) \subseteq V$. From now on, let f^x denote the restriction of f to G_x. It is clear from the definition of f and [4, Section 1.4.2] that it is a positive definite function on each fibre and therefore satisfies the conditions of Bochner's theorem and the inversion theorem on each fibre. In particular, it can be shown using [4, Section 1.4.3] that for each x there exists a finite positive measure μ^x on $\widehat{G}_x$ (extended to $\widehat{G}$ by giving everything else measure zero) such that

$$f(s) = \int_{\widehat{G}} \overline{\omega(s)} \mu^{p(s)}(\omega).$$

Furthermore, it is easy to prove using [4, Section 1.4.1] that $\mu^x(\widehat{G}) = \mu^x(\widehat{G}_x) = \|f^x\|_\infty \leq \|f\|_\infty$ for all $x \in X$ so that $\{\mu^x\}$ is a bounded collection of finite measures. Additionally, it is shown in the proof of [4, Section 1.5.1] that, as measures on $\widehat{G}_x$,

$$\widehat{f^x}\, d\hat{\beta}^x = d\mu^x.$$

Proposition 5 states that given $f \in C_c(G)$ the Gelfand transform of f restricts to the usual Fourier transform fibrewise. Therefore, since everything outside

$\widehat{G}_x$ has measure zero, we may as well write

$$\hat{f}\,d\hat{\beta}^x = d\mu^x. \tag{3}$$

Now, if $\phi \in C_c(\widehat{G})$ then $\phi\hat{f}$ is compactly supported. It follows from (2) that

$$\int_{\widehat{G}} \phi(\omega)\hat{f}(\omega)\omega(s_i)\,d\hat{\beta}^{x_i}(\omega) \to \int_{\widehat{G}} \phi(\omega)\hat{f}(\omega)\omega(s_0)\,d\hat{\beta}^{x_0}(\omega). \tag{4}$$

Using (3), we can rewrite (4) as

$$\int_{\widehat{G}} \phi(\omega)\omega(s_i)\,d\mu^{x_i}(\omega) \to \int_{\widehat{G}} \phi(\omega)\omega(s_0)\,d\mu^{x_0}(\omega). \tag{5}$$

We can extend (5) to functions $\phi \in C_0(\widehat{G})$ by noting that $C_c(\widehat{G})$ is uniformly dense in $C_0(\widehat{G})$ and doing a straightforward approximation argument using the fact that the $\{\mu^{x_i}\}$ are uniformly bounded.

Let $g \in C_c(G)$. Observe that

$$\begin{aligned}
\overline{\widehat{g^{x_i}}(\omega)}\omega(s_i) &= \int_{G_{x_i}} \overline{g^{x_i}(s)\omega(s)}\omega(s_i)\,d\beta^{x_i}(s) \\
&= \int_{G_{x_i}} \overline{g^{x_i}(s)}\omega(s^{-1}s_i)\,d\beta^{x_i}(s) \\
&= \int_{G_{x_i}} \overline{g^{x_i}(s_i s)}\omega(s^{-1})\,d\beta^{x_i}(s) \\
&= \overline{(\mathrm{lt}_{s_i^{-1}}\, g^{x_i})^\wedge}(\omega).
\end{aligned}$$

Therefore, for all i, we have

$$\begin{aligned}
\int_{\widehat{G}} \overline{\hat{g}(\omega)}\omega(s_i)\,d\mu^{x_i}(\omega) &= \int_{\widehat{G}} \overline{\hat{g}(\omega)}\hat{f}(\omega)\omega(s_i)\,d\hat{\beta}^{x_i}(\omega) \\
&= \int_{\widehat{G}_{x_i}} \overline{\widehat{g^{x_i}}(\omega)}\widehat{f^{x_i}}(\omega)\omega(s_i)\,d\hat{\beta}^{x_i}(\omega) \\
&= \int_{\widehat{G}_{x_i}} \overline{(\mathrm{lt}_{s_i^{-1}}\, g^{x_i})^\wedge}\widehat{f^{x_i}}\,d\hat{\beta}^{x_i} \\
&= \int_{G_{x_i}} \overline{\mathrm{lt}_{s_i^{-1}}\, g^{x_i}} f^{x_i}\,d\beta^{x_i},
\end{aligned}$$

where the last equality follows from the Plancharel theorem [4, Theorem 1.6.1]. Since $\bar{\hat{g}} \in C_0(\widehat{G})$, it follows from (5) that

$$\int_{G_{x_i}} \overline{\mathrm{lt}_{s_i^{-1}}\, g^{x_i}} f^{x_i}\,d\beta^{x_i} \to \int_{G_{x_0}} \overline{\mathrm{lt}_{s_0^{-1}}\, g^{x_0}} f^{x_0}\,d\beta^{x_0}. \tag{6}$$

We are now ready to attack the convergence of the s_i. Choose an open neighborhood O of s_0. Using the continuity of multiplication, we can find relatively compact open neighborhoods V and W in G such that $x_0 \in V$,

$s_0 \in W$ and $VW \subseteq O$. Furthermore, by intersecting V and V^{-1} we can assume that $V^{-1} = V$. Construct f for V as in the beginning of the proof. Now choose $g \in C(G)$ so that $0 \le g \le 1$, $g(s_0) = 1$, and g is zero off W. Then $g \in C_c(G)$ and $\overline{g} = g$ so that by equation (6) we have

$$\int_{G_{x_i}} g(s_i t) f(t)\, d\beta^{x_i}(t) \to \int_{G_{x_0}} g(s_0 t) f(t)\, d\beta^{x_0}(t). \tag{7}$$

It turns out that $\int g(s_i t) f(t)\, d\beta^{x_i}(t) = 0$ unless $s_i \in WV^{-1} = WV \subseteq O$. Furthermore, both $g(s_0 x_0)$ and $f(x_0)$ are nonzero by construction, and since both functions are continuous, this implies

$$\int_{G_{x_0}} g(s_0 t) f(t)\, d\beta^{x_0}(t) \neq 0.$$

It follows from (7) that eventually $s_i \in O$. This of course implies that $s_i \to s_0$ and we are done. □

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References

[1] W. Arveson, *An invitation to C^*-algebras*, Graduate Texts in Mathematics, vol. 39, Springer, New York, 1976. MR 0512360

[2] J. N. Renault, P. S. Muhly and D. P. Williams, *Continuous trace groupoid C^*-agebras, III*, Trans. Amer. Math. Soc. **348** (1996), 3621–3641. MR 1348867

[3] B. Ramazan, *Quantification par déformation des variétés de Lie-Poisson*, Ph.D. thesis, Université d'Orléans, 1998.

[4] W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, vol. 12, Interscience Publishers, New York, 1962. MR 0152834

[5] D. P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, Amer. Math. Soc., Providence, RI, 2007. MR 2288954

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