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# BURKHOLDER'S SUBMARTINGALES FROM A STOCHASTIC CALCULUS PERSPECTIVE

GIOVANNI PECCATI AND MARC YOR

ABSTRACT. We provide a simple proof, as well as several generalizations, of a recent result by Davis and Suh, characterizing a class of continuous submartingales and supermartingales that can be expressed in terms of a squared Brownian motion and of some appropriate powers of its maximum. Our techniques involve elementary stochastic calculus, as well as the Doob–Meyer decomposition of continuous submartingales. These results can be used to obtain an explicit expression of the constants appearing in the Burkholder–Davis–Gundy inequalities. A connection with some *balayage formulae* is also established.

## 1. Introduction

Let $W = \{W_t : t \geq 0\}$ be a standard Brownian motion initialized at zero, set $W_t^* = \max_{s\leq t} |W_s|$, and write $\mathcal{F}_t^W = \sigma\{W_u : u \leq t\}$, $t \geq 0$. In [3], Davis and Suh proved the following result.

THEOREM 1 ([3] Theorem 1.1). *For every $p > 0$ and every $c \in \mathbb{R}$, set*

$$(1)\qquad \begin{aligned} Y_t &= Y_t(c,p) = (W_t^*)^{p-2}[W_t^2 - t] + c(W_t^*)^p, \quad t > 0,\\ Y_0 &= Y_0(c,p) = 0. \end{aligned}$$

(i) *For every $p \in (0,2]$, the process $Y_t$ is a $\mathcal{F}_t^W$-submartingale, if and only if, $c \geq \frac{2-p}{p}$.*
(ii) *For every $p \in [2,+\infty)$, the process $Y_t$ is a $\mathcal{F}_t^W$-supermartingale, if and only if, $c \leq \frac{2-p}{p}$.*

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As the title of [3] clearly indicates, the results of Theorem 1 were discovered mainly by Burkholder in [2], apart from the precise constant $(2-p)/p$. However, the emphasis in [2] is to obtain certain best constants for *all* martingales, whereas in [3] and in the present paper, the authors focus on continuous local martingales, hence, due to the Dubins–Schwarz theorem, the emphasis is on Brownian motion. Furthermore, as pointed out in [3, p. 314] and in Section 4 below, very simple derivations of explicit expressions of the best constants appearing in the Burkholder–Davis–Gundy (BDG) inequalities (see [1], or [5, Chapter IV, Section 4]) derived from Part (i) of Theorem 1. The proof of Theorem 1 given in [3] uses several delicate estimates related to a class of Brownian hitting times. Such an approach can be seen as a ramification of the discrete time techniques developed in [2]. In particular, in [3], it is observed that the submartingale (or supermartingale) characterization of $Y_t(c,p)$ basically relies on the properties of the random subset of $[0,+\infty)$ consisting of the instants $t$ when $|W_t| = W_t^*$. The aim of this note is to bring this last connection into further light, by providing an elementary proof of Theorem 1, based on a direct application of Itô formula and on an appropriate version of the Doob–Meyer decomposition of submartingales. See also Theorem 4 below for some generalizations.

The rest of the paper is organized as follows. In Section 2, we state and prove a general result involving a class of stochastic processes that are functions of a positive submartingale and of a monotone transformation of its maximum. In Section 3, we focus once again on the Brownian setting, and establish a generalization of Theorem 1. Section 4 deals with an application of the previous results to (strong) BDG inequalities. Finally, in Section 5, we provide an explicit connection with some classic *balayage formulae* for continuous-time semimartingales (see e.g., [6]).

All the objects appearing in the subsequent sections are defined on a common probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

## 2. A general result

Throughout this section, $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$ stands for a filtration satisfying the usual conditions. We will write $X = \{X_t : t \geq 0\}$ to indicate a *continuous $\mathcal{F}_t$-submartingale* issued from zero and such that $\mathbb{P}\{X_t \geq 0, \forall t\} = 1$. We will suppose that the Doob–Meyer decomposition of $X$ (see for instance [4, Theorem 1.4.14]) is of the type $X_t = M_t + A_t$, $t \geq 0$, where $M$ is a *square-integrable* continuous $\mathcal{F}_t$-martingale issued from zero, and $A$ is an increasing (integrable) natural process. We assume that $A_0 = M_0 = 0$; the symbol $\langle M \rangle = \{\langle M \rangle_t : t \geq 0\}$ stands for the quadratic variation of $M$. We note $X_t^* = \max_{s \leq t} X_s$, and we also suppose that $\mathbb{P}\{X_t^* > 0\} = 1$ for every $t > 0$. The following result is an extension of Theorem 1.

THEOREM 2. *Fix* $\varepsilon > 0$.

(i) *Suppose that the function* $\phi : (0, +\infty) \mapsto \mathbb{R}$ *is of class* $C^1$*, nonincreasing, and such that*

$$\mathbb{E}\left[\int_\varepsilon^T \phi(X_s^*)^2 \, d\langle M\rangle_s\right] < +\infty, \tag{2}$$

*for every* $T > \varepsilon$*. For every* $x \geq z > 0$*, we set*

$$\Phi(x, z) = -\int_z^x y\phi'(y)\, dy; \tag{3}$$

*then for every* $\alpha \geq 1$*, the process*

$$Z_\varepsilon(\phi, \alpha; t) = \phi(X_t^*)(X_t - A_t) + \alpha\Phi(X_t^*, X_\varepsilon^*), \quad t \geq \varepsilon, \tag{4}$$

*is a* $\mathcal{F}_t$*-submartingale on* $[\varepsilon, +\infty)$.

(ii) *Suppose that the function* $\phi : (0, +\infty) \mapsto \mathbb{R}$ *is of class* $C^1$*, nondecreasing and such that* (2) *holds for every* $T > \varepsilon$*. Define* $\Phi(\cdot, \cdot)$ *according to* (3)*, and* $Z_\varepsilon(\phi, \alpha; t)$ *according to* (4)*. Then for every* $\alpha \geq 1$*, the process* $Z_\varepsilon(\phi, \alpha; t)$ *is a* $\mathcal{F}_t$*-supermartingale on* $[\varepsilon, +\infty)$.

REMARK 1. Note that the function $\phi(y)$ (and $\phi'(y)$) need not be defined at $y = 0$.

REMARK 2. In Section 3, where we will focus on the Brownian setting, we will exhibit specific examples where the condition $\alpha \geq 1$ is necessary and sufficient to have that the process $Z_\varepsilon(\alpha, \phi; t)$ is a submartingale (when $\phi$ is nonincreasing) or a supermartingale (when $\phi$ is nondecreasing).

*Proof.* (*Part* (i)) Observe first that since $M_t = X_t - A_t$ is a continuous martingale, $X^*$ is nondecreasing and $\phi$ is differentiable, then a standard application of Itô formula gives that

$$\begin{aligned} &\phi(X_t^*)(X_t - A_t) - \phi(X_\varepsilon^*)(X_\varepsilon - A_\varepsilon) \\ &\quad = \phi(X_t^*)M_t - \phi(X_\varepsilon^*)M_\varepsilon \\ &\quad = \int_\varepsilon^t \phi(X_s^*)\, dM_s + \int_\varepsilon^t (X_s - A_s)\phi'(X_s^*)\, dX_s^*. \end{aligned} \tag{5}$$

The assumptions in the statement imply that the application

$$t \mapsto \widetilde{M}_{\varepsilon,t} := \int_\varepsilon^t \phi(X_s^*)\, dM_s$$

is a continuous square integrable $\mathcal{F}_t$-martingale on $[\varepsilon, +\infty)$. Moreover, the continuity of $X$ implies that the support of the random measure $dX_t^*$ (on $[0, +\infty)$) is contained in the (random) set $\{t \geq 0 : X_t = X_t^*\}$, thus yielding that

$$\int_\varepsilon^t (X_s - A_s)\phi'(X_s^*)\,dX_s^* = \int_\varepsilon^t (X_s^* - A_s)\phi'(X_s^*)\,dX_s^*$$
$$= -\int_\varepsilon^t A_s\phi'(X_s^*)\,dX_s^* - \Phi(X_t^*, X_\varepsilon^*),$$

where $\Phi$ is defined in (3). As a consequence,

$$Z_\varepsilon(\phi,\alpha;t) = \widetilde{M}_{\varepsilon,t} + \int_\varepsilon^t (-A_s\phi'(X_s^*))\,dX_s^* + (\alpha-1)\Phi(X_t^*, X_\varepsilon^*). \tag{6}$$

Now, observe that the application $t \mapsto \Phi(X_t^*, X_\varepsilon^*)$ is nondecreasing (a.s.-$\mathbb{P}$), and also that by assumption, $-A_s\phi'(X_s^*) \geq 0$ for every $s > 0$. This entails immediately that $Z_\varepsilon(\phi,\alpha;t)$ is a $\mathcal{F}_t$-submartingale for every $\alpha \geq 1$.

(*Part* (ii)) By using exactly the same line of reasoning as in the proof of Point (i), we obtain that

$$Z_\varepsilon(\phi,\alpha;t) = \int_\varepsilon^t \phi(X_s^*)\,dM_s + \int_\varepsilon^t (-A_s\phi'(X_s^*))\,dX_s^* + (\alpha-1)\Phi(X_t^*, X_\varepsilon^*). \tag{7}$$

Since (2) is in order, we deduce that $t \mapsto \int_\varepsilon^t \phi(X_s^*)dM_s$ is a continuous (square-integrable) $\mathcal{F}_t$-martingale on $[\varepsilon, +\infty)$. Moreover, $-A_s\phi'(X_s^*) \leq 0$ for every $s > 0$, and we also have that $t \mapsto \Phi(X_t^*, X_\varepsilon^*)$ is a.s. decreasing. This implies that $Z_\varepsilon(\phi,\alpha;t)$ is a $\mathcal{F}_t$-supermartingale for every $\alpha \geq 1$. □

The next result allows to characterize the nature of the process $Z$ appearing in (4) on the whole positive axis. Its proof can be immediately deduced from formulae (6) (for Part (i)) and (7) (for Part (ii)).

PROPOSITION 3. *Let the assumptions and notation of this section prevail.*

(i) *Consider a decreasing function* $\phi : (0, +\infty) \mapsto \mathbb{R}$ *verifying the assumptions of part* (i) *of Theorem* 2, *and such that*

$$\Phi(x, 0) := -\int_0^x y\phi'(y)\,dy \quad \textit{is finite } \forall x > 0. \tag{8}$$

*Assume moreover that*

$$\mathbb{E}\left[\int_0^T \phi(X_s^*)^2\,d\langle M\rangle_s\right] < +\infty, \tag{9}$$

*and also*

(10) $\phi(X_\varepsilon^*)M_\varepsilon = \phi(X_\varepsilon^*)(X_\varepsilon - A_\varepsilon)$ *converges to zero in* $L^1(\mathbb{P})$, *as* $\varepsilon \downarrow 0$,

(11) $\Phi(X_t^*, 0) \in L^1(\mathbb{P})$.

*Then for every* $\alpha \geq 1$ *the process*

$$Z(\phi,\alpha;t) = \begin{cases} 0 & \text{for } t=0, \\ \phi(X_t^*)(X_t - A_t) + \alpha\Phi(X_t^*,0) & \text{for } t>0, \end{cases} \tag{12}$$

*is a* $\mathcal{F}_t$*-submartingale.*

(ii) *Consider an increasing function* $\phi : (0,+\infty) \mapsto \mathbb{R}$ *as in part* (ii) *of Theorem* 2 *and such that assumptions* (8)–(11) *are satisfied. Then for every* $\alpha \geq 1$ *the process* $Z(\phi,\alpha;t)$ *appearing in* (12) *is a* $\mathcal{F}_t$*-supermartingale.*

REMARK 3. A direct application of the Cauchy–Schwarz inequality shows that a sufficient condition to have (10) is the following:

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}[\phi(X_\varepsilon^*)^2] \times \mathbb{E}[M_\varepsilon^2] = \lim_{\varepsilon\downarrow 0} \mathbb{E}[\phi(X_\varepsilon^*)^2] \times \mathbb{E}[\langle M\rangle_\varepsilon] = 0 \tag{13}$$

(observe that $\lim_{\varepsilon\downarrow 0} \mathbb{E}[M_\varepsilon^2] = 0$, since $M_0 = 0$ by assumption). In other words, when (13) is verified the quantity $\mathbb{E}[M_\varepsilon^2]$ "takes care" of the possible explosion of $\varepsilon \mapsto \mathbb{E}[\phi(X_\varepsilon^*)^2]$ near zero.

REMARK 4. Let $\phi$ be nonincreasing or nondecreasing on $(0,+\infty)$, and suppose that $\phi$ satisfies the assumptions of Theorem 2 and Proposition 3. Then the process $t \mapsto \int_0^t \phi(X_s^*)\,dM_s$ is a continuous square-integrable $\mathcal{F}_t^W$-martingale. Moreover, for any choice of $\alpha \in \mathbb{R}$, the process $Z(\phi,\alpha;t)$, $t \geq 0$, defined in (12) is a semimartingale, with canonical decomposition given by

$$Z(\phi,\alpha;t) = \int_0^t \phi(X_s^*)\,dM_s + \int_0^t \big((\alpha-1)X_s^* - A_s\big)\phi'(X_s^*)\,dX_s^*.$$

## 3. A generalization of Theorem 1

The forthcoming Theorem 4 is a generalization of Theorem 1. Recall the notation: $W$ is a standard Brownian motion issued from zero, $W_t^* = \max_{s\leq t}|W_s|$ and $\mathcal{F}_t^W = \sigma\{W_u : u \leq t\}$. We also set for every $m \geq 1$, every $p > 0$, and every $c \in \mathbb{R}$:

$$\begin{aligned} J_t &= J_t(m,c,p) = (W_t^*)^{p-m}[|W_t|^m - A_{m,t}] + c(W_t^*)^p, \quad t>0, \\ J_0 &= J_0(m,c,p) = 0, \end{aligned} \tag{14}$$

where $t \mapsto A_{m,t}$ is the increasing natural process in the Doob–Meyer decomposition of the $\mathcal{F}_t^W$-submartingale $t \mapsto |W_t|^m$. Of course, $J_t(2,c,p) = Y_t(c,p)$, as defined in (1).

THEOREM 4. *Under the above notation:*

(i) *For every* $p \in (0,m]$*, the process* $J_t$ *is a* $\mathcal{F}_t^W$*-submartingale, if and only if,* $c \geq \frac{m-p}{p}$.

(ii) *For every* $p \in [m,+\infty)$*, the process* $J_t$ *is a* $\mathcal{F}_t^W$*-supermartingale, if and only if,* $c \leq \frac{m-p}{p}$.

*Proof.* Recall first the following two facts: (A) $W_t^* =_{\text{law}} \sqrt{t} W_1^*$ (by scaling), and (B) there exists $\eta > 0$ such that $\mathbb{E}[\exp(\eta (W_1^*)^{-2})] < +\infty$ (this can be deduced e.g., from [5, Chapter II, Exercice 3.10]), so that the random variable $(W_1^*)^{-1}$ has finite moments of all orders. Note also that the conclusions of both part (i) and part (ii) are trivial in the case where $p = m$. In the rest of the proof, we will therefore assume that $p \neq m$.

To prove part (i), we shall apply Theorem 2 and Proposition 3 in the following framework: $X_t = |W_t|^m$ and $\phi(x) = x^{\frac{p-m}{m}} = x^{\frac{p}{m}-1}$. In this case, the martingale $M_t = |W_t|^m - A_{m,t}$ is such that $\langle M \rangle_t = m^2 \int_0^t W_s^{2m-2}\, ds$, $t \geq 0$, and $\Phi(x,z) = -\int_z^x y\phi'(y)\, dy = -(\frac{p}{m} - 1)\int_z^x y^{\frac{p}{m}-1}\, dy = \frac{m-p}{p}(x^{\frac{p}{m}} - z^{\frac{p}{m}})$. Also, for every $T > \varepsilon > 0$,

$$\begin{aligned}
(15)\qquad & \mathbb{E}\left[\int_\varepsilon^T \phi(X_s^*)^2\, d\langle M \rangle_s\right] \\
&= m^2 \mathbb{E}\left[\int_\varepsilon^T (W_s^*)^{2p-2m} W_s^{2m-2}\, ds\right] \\
&\leq m^2 \mathbb{E}\left[\int_\varepsilon^T (W_s^*)^{2p-2}\, ds\right] = m^2 \mathbb{E}[(W_1^*)^{2p-2}] \int_\varepsilon^T s^{\frac{p}{2}-1}\, ds,
\end{aligned}$$

so that $\phi$ verifies (2) and (9). Relations (8) and (11) are trivially satisfied. To see that (10) holds, use the relations

$$\begin{aligned}
&\mathbb{E}\{|\phi(X_\varepsilon^*)(X_\varepsilon - A_\varepsilon)|\} \\
&\quad = \mathbb{E}\{|(W_\varepsilon^*)^{p-m}[|W_\varepsilon|^m - A_{m,\varepsilon}]|\} \\
&\quad = \mathbb{E}\{|(W_\varepsilon^*)^{p-m} M_\varepsilon|\} \leq \mathbb{E}\{(W_\varepsilon^*)^{2p-2m}\}^{1/2} \mathbb{E}\{\langle M \rangle_\varepsilon\}^{1/2} \\
&\quad = m \mathbb{E}\{W_1^{2m-2}\}^{1/2} \mathbb{E}\{(W_1^*)^{2p-2m}\}^{1/2} \varepsilon^{\frac{p}{2}-\frac{m}{2}} \left(\int_0^\varepsilon s^{m-1}\, ds\right)^{1/2} \\
&\quad \to 0, \quad \text{as } \varepsilon \downarrow 0.
\end{aligned}$$

From part (i) of Proposition 3, we therefore deduce that the process $Z(t)$ defined as $Z(0) = 0$ and, for $t > 0$,

$$\begin{aligned}
(16)\qquad Z(t) &= \phi((W_t^*)^m)[|W_t|^m - A_{m,t}] + \alpha \Phi((W_t^*)^m, 0) \\
(17)\qquad &= (W_t^*)^{p-m}[|W_t|^m - A_{m,t}] + \alpha \frac{m-p}{p}(W_t^*)^p,
\end{aligned}$$

is a $\mathcal{F}_t^W$-submartingale for every $\alpha \geq 1$. By writing $c = \alpha \frac{m-p}{p}$ in the previous expression, and by using the fact that $\frac{m-p}{p} \geq 0$ by assumption, we deduce immediately that $J_t(m,c;p)$ is a submartingale for every $c \geq \frac{m-p}{p}$. Now, suppose $c < \frac{m-p}{p}$. One can use formulae (6), (16), and (17) to prove that

$$\begin{aligned}
J_t(m,c;p) &= \int_0^t \phi(X_s^*)\,dM_s + \int_0^t [-A_{m,s}\phi'((W_s^*)^m)]\,d(W_s^*)^m \\
&\quad + (\alpha-1)\Phi((W_t^*)^m,0) \\
&= \int_0^t (W_s^*)^{p-m}\,dM_s \\
&\quad + \left(\frac{p}{m}-1\right)\int_0^t [(1-\alpha)(W_s^*)^m - A_{m,s}](W_s^*)^{p-2m}\,d(W_s^*)^m,
\end{aligned}$$

where $1-\alpha = 1-pc/(m-p) > 0$. Note that $\int_0^t (W_s^*)^{p-m}\,dM_s$ is a square-integrable martingale, due to (15). To conclude that in this case, $J_t(m,c;p)$ cannot be a submartingale (nor a supermartingale), it is sufficient to observe that (for every $m \geq 1$ and every $\alpha < 1$) the paths of the finite variation process

$$t \mapsto \int_0^t [(1-\alpha)(W_s^*)^m - A_{m,s}](W_s^*)^{p-2m}\,d(W_s^*)^m$$

are neither nondecreasing nor nonincreasing, with $\mathbb{P}$-probability one.

To prove part (ii), one can argue in exactly the same way, and use part (ii) of Proposition 3 to obtain that the process $Z(t)$ defined as $Z(0)=0$, and for $t>0$,

$$Z(t) = (W_t^*)^{p-m}[|W_t|^m - A_{m,t}] + \alpha\frac{m-p}{p}(W_t^*)^p$$

is a $\mathcal{F}_t^W$-supermartingale for every $\alpha \geq 1$. By writing once again $c = \alpha\frac{m-p}{p}$ in the previous expression, and since $\frac{m-p}{p} \leq 0$, we immediately deduce that $J_t(m,c;p)$ is a supermartingale for every $c \leq \frac{m-p}{p}$. One can show that $J_t(m, c;p)$ cannot be a supermartingale, whenever $c > \frac{m-p}{p}$ by using arguments analogous to those displayed in the last part of the proof of part (i). □

The following result is obtained by specializing Theorem 4 to the case $m=1$ (via Tanaka's formula).

COROLLARY 5. *Denote by* $\{\ell_t : t \geq 0\}$ *the local time at zero of the Brownian motion* $W$. *Then the process*

$$\begin{aligned}
J_t(p) &= (W_t^*)^{p-1}[|W_t| - \ell_t] + c(W_t^*)^p, \quad t>0, \\
J_0(p) &= 0,
\end{aligned}$$

*is such that:* (i) *for* $p \in (0,1]$, $J_t(p)$ *is a* $\mathcal{F}_t^W$*-submartingale, if and only if,* $c \geq 1/p - 1$, *and* (ii) *for* $p \in [1,+\infty)$, $J_t(p)$ *is a* $\mathcal{F}_t^W$*-supermartingale, if and only if,* $c \leq 1/p-1$.

## 4. Burkholder–Davis–Gundy (BDG) inequalities

We reproduce an argument taken from [3, p. 314], showing that the first part of Theorem 4 can be used to obtain a strong version of the BDG inequalities (see e.g., [5, Chapter IV, Section 4]).

Fix $p \in (0,2)$ and define $c = (2-p)/p = 2/p - 1$. Since, according to the first part of Theorem 4, $Y_t = Y_t(c,p)$ is a $\mathcal{F}_t^W$-submartingale starting from zero, we deduce that, for every bounded and strictly positive $\mathcal{F}_t^W$-stopping time $\tau$, one has $\mathbb{E}(Y_\tau) \geq 0$. In particular, this yields

$$\mathbb{E}\left(\frac{\tau}{(W_\tau^*)^{2-p}}\right) \leq \frac{2}{p}\mathbb{E}((W_\tau^*)^p). \tag{18}$$

Formula (18), combined with an appropriate use of Hölder's inequality, entails finally that for $0 < p < 2$,

$$\mathbb{E}(\tau^{\frac{p}{2}}) \leq \left[\frac{2}{p}\mathbb{E}((W_\tau^*)^p)\right]^{\frac{p}{2}} [\mathbb{E}((W_\tau^*)^p)]^{\frac{2-p}{2}} = \left[\frac{2}{p}\right]^{\frac{p}{2}} \mathbb{E}((W_\tau^*)^p). \tag{19}$$

Of course, relation (19) extends to general stopping times $\tau$ (not necessarily bounded) by monotone convergence (*via* the increasing sequence $\{\tau \wedge n : n \geq 1\}$).

REMARK 5. Let $\{\mathfrak{A}_n : n \geq 0\}$ be a discrete filtration of the reference $\sigma$-field $\mathfrak{A}$, and consider a $\mathfrak{A}_n$-adapted sequence of measurable random elements $\{f_n : n \geq 0\}$ with values in a Banach space $\mathbf{B}$. We assume that $f_n$ is a martingale, i.e., that for every $n$, $\mathbb{E}[f_n - f_{n-1}|\mathfrak{A}_{n-1}] = \mathbb{E}[d_n|\mathfrak{A}_{n-1}] = 0$, where $d_n := f_n - f_{n-1}$. We note

$$S_n(f) = \sqrt{\sum_{k=0}^{n} |d_k|^2} \quad \text{and} \quad f_n^* = \sup_{0 \leq m \leq n} |f_m|,$$

and write $S(f)$ and $f^*$, respectively, to indicate the pointwise limits of $S_n(f)$ and $f_n^*$, as $n \to +\infty$. In [2], Burkholder proved that

$$\mathbb{E}(S(f)) \leq \sqrt{3}\mathbb{E}(f^*), \tag{20}$$

where $\sqrt{3}$ is the best possible constant, in the sense that for every $\eta \in (0, \sqrt{3})$ there exists a Banach space-valued martingale $f_{(\eta)}$, such that $\mathbb{E}(S(f_{(\eta)})) > \eta\mathbb{E}(f_{(\eta)}^*)$. As observed in [3], Burkholder's inequality (20) should be compared with (19) for $p = 1$, which yields the relation $\mathbb{E}(\tau^{1/2}) \leq \sqrt{2}\mathbb{E}(W_\tau^*)$ for every stopping time $\tau$. This shows that in such a framework, involving uniquely continuous martingales, the constant $\sqrt{3}$ is no longer optimal.

## 5. Balayage

Keeping the assumptions and notation of Section 2 and Theorem 2, fix $\varepsilon > 0$ and consider a finite variation function $\psi : (0, +\infty) \mapsto \mathbb{R}$. In this section, we focus on the formula

$$\begin{aligned}
&\psi(X_t^*)(X_t - A_t) - \psi(X_\varepsilon^*)(X_\varepsilon - A_\varepsilon) \\
&\quad = \int_\varepsilon^t \psi(X_s^*)\, d(X_s - A_s) + \int_\varepsilon^t (X_s^* - A_s)\, d\psi(X_s^*),
\end{aligned} \tag{21}$$

where $\varepsilon > 0$. Note that by choosing $\psi = \phi$ in (21), where $\phi \in C^1$ is monotone, one recovers formula (6), which was crucial in the proof of Theorem 2. We shall now show that (21) can be obtained by means of the *balayage formulae* proved in [6].

To see this, let $U = \{U_t : t \geq 0\}$ be a continuous $\mathcal{F}_t$-semimartingale issued from zero. For every $t > 0$, we define the random time

$$\sigma(t) = \sup\{s < t : U_s = 0\}. \tag{22}$$

The following result is a particular case of [6, Theorem 1].

PROPOSITION 6 (Balayage formula). *Consider a stochastic process $\{K_t : t > 0\}$ such that the restriction $\{K_t : t \geq \varepsilon\}$ is locally bounded and $\mathcal{F}_t$-predictable on $[\varepsilon, +\infty)$ for every $\varepsilon > 0$. Then for every fixed $\varepsilon > 0$, the process $K_{\sigma(t)}$, $t \geq \varepsilon$, is locally bounded and $\mathcal{F}_t$-predictable, and moreover*

$$U_t K_{\sigma(t)} = U_\varepsilon K_{\sigma(\varepsilon)} + \int_\varepsilon^t K_{\sigma(s)}\, dU_s. \tag{23}$$

To see how (21) can be recovered from (23), set $U_t = X_t - X_t^*$ and $K_t = \psi(X_t^*)$. Then $K_t = K_{\sigma(t)} = \psi(X_{\sigma(t)}^*)$ by construction, where $\sigma(t)$ is defined as in (22). As a consequence, (23) gives

$$\psi(X_t^*)(X_t - X_t^*) = \psi(X_\varepsilon^*)(X_\varepsilon - X_\varepsilon^*) + \int_\varepsilon^t \psi(X_s^*)\, d(X_s - X_s^*).$$

Finally, a standard integration by parts applied to $\psi(X_t^*)(X_t^* - A_t)$ yields

$$\begin{aligned}
\psi(X_t^*)(X_t - A_t) &= \psi(X_t^*)(X_t - X_t^*) + \psi(X_t^*)(X_t^* - A_t) \\
&= \psi(X_\varepsilon^*)(X_\varepsilon - X_\varepsilon^*) + \int_\varepsilon^t \psi(X_s^*)\, d(X_s - X_s^*) \\
&\quad + \psi(X_\varepsilon^*)(X_\varepsilon^* - A_\varepsilon) + \int_\varepsilon^t \psi(X_s^*)\, d(X_s^* - A_s) \\
&\quad + \int_\varepsilon^t (X_s^* - A_s)\, d\psi(X_s^*),
\end{aligned}$$

which is equivalent to (21).

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Giovanni Peccati, Centre de Recherche Modal'X, Université Paris Ouest Naterre La Défense, 92000 Nanterre, France

*E-mail address*: giovanni.peccati@gmail.com

Marc Yor, Laboratoire de Probabilités et Modèles Aléatoires, Universités Paris VI and Paris VII, France and Institut Universitaire de France