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SCATTERING LENGTH FOR STABLE PROCESSES

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ABSTRACT. Let $0 < \alpha < 2$ and X_t be the isotropic α-stable Lévy process. We define scattering length $\Gamma(v)$ of a positive potential v. We use the scattering length to find estimates for the first eigenvalue of the Schrödinger operator of the "Neumann" fractional Laplacian in a cube with a potential v.

1. Introduction

The scattering length has been studied for the Brownian motion and the classical Laplacian by many authors, see [7], [8], [12], [13]. The last two papers contain applications, for example, a bound for the first eigenvalue of the Schrödinger operator of the Neumann Laplacian in a cube. Scattering length is also important in mathematical physics where it arises in many situations, including the study of neutron scattering and general few-body systems (see, for example [11] and [1]).

This paper is the first attempt to define and study scattering length for processes different than the Brownian motion. As an application, we prove estimates for the first eigenvalue of the Schrödinger operator of the "Neumann" fractional Laplacian in a cube. This result is similar to the one obtained in [12] for the Laplacian.

Let X_t be the isotropic α-stable Lévy process in $\mathbb{R}^d$ with the characteristic function

$$\mathbf{E}^0(\exp(i\xi X_t)) = \exp(-t|\xi|^\alpha). \tag{1.1}$$

For simplicity, we assume that $d > \alpha$. It is well known that this process has the generator $\Delta^{\alpha/2} = -(-\Delta)^{\alpha/2}$, where Δ is the classical Laplacian on $\mathbb{R}^d$. For an overview of results, for the potential theory of this process, we refer

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the reader to [4]. The Dirichlet form for this process is given by

$$\mathcal{E}_X(u,u) = C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dx\, dy \tag{1.2}$$

and its domain is $W^{\alpha/2,2}(\mathbb{R}^d)$, the fractional Sobolev space. See [3] for details about Dirichlet forms and domains for the generators of stable processes.

The constant in front of the double integral depends on α and d, as do all other constants in this paper. Dependence on any other parameter will be indicated explicitly. We also adopt the convention that constants may change their values from line to line as long as they stay positive.

We can also define the "Neumann" fractional Laplacian $\Delta_N^{\alpha/2}$ on an open set Ω as the operator with the Dirichlet form

$$\mathcal{E}_Y(u,u) = C \int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\, dx\, dy \tag{1.3}$$

and the domain $W^{\alpha/2,2}(\Omega)$. Here, we also refer the reader to [3] for details about the definition of $\Delta_N^{\alpha/2}$. The stochastic process Y_t associated with this operator is the reflected stable process in Ω studied in [3].

Let v be a positive function and let U_v be the capacitory potential of v

$$U_v(x) = 1 - \mathbf{E}^x \exp\left(-\int_0^\infty v(X_s)\, ds\right). \tag{1.4}$$

Let μ_v be the capacitory measure of v

$$\mu_v = -\Delta^{\alpha/2} U_v. \tag{1.5}$$

We define the scattering length $\Gamma(v)$ by

$$\Gamma(v) = \int_{\mathbb{R}^d} d\mu_v(x). \tag{1.6}$$

The product of d intervals $(-1,1)^d$ will be called the cube in $\mathbb{R}^d$. The main result of the paper is the following.

THEOREM 1.1. *Let Ω be the cube in $\mathbb{R}^d$, $0 \le v \in L^1(\Omega)$, and let $\lambda_1(v)$ be the first eigenvalue of the operator $-\Delta_N^{\alpha/2} + v$ in Ω (the Schrödinger operator of the "Neumann" fractional Laplacian on Ω). Then there exists a constant $C_1(\Omega)$, such that*

$$C_1(\Omega)\Gamma(v) \le \lambda_1(v). \tag{1.7}$$

Furthermore, there exists a constant $\beta = \beta(\Omega) > 0$, such that whenever $\Gamma(v) \le \beta$, then

$$\lambda_1(v) \le C_2(\Omega)\Gamma(v). \tag{1.8}$$

REMARK 1.2. The second bound is valid for any bounded domain Ω.

Remarks 3.6, 3.7, and 3.9 give applications of this result. The idea of the proof is the following. We choose an appropriate representative of stable-like processes (see [2] and the definition in Section 4) for which the proof is similar to the Brownian case. The main result follows from the fact that all stable-like processes have the same bounds for eigenvalues.

The rest of the paper is organized as follows. In Section 2, we give the precise definition of scattering length. In Section 3, we prove some properties of capacitory potential and scattering length. The proofs in this section are easy and they carry over from the Brownian case to the stable case with minimal changes. We present them here for the sake of completeness. Section 4 contains the proof of Theorem 1.1.

2. Definitions

We start with the definition of the potential operator U.

DEFINITION 2.1. For any nonnegative function f define U by

$$U[f](x) = \mathbf{E}^x\left(\int_0^\infty f(X_s)\,ds\right). \tag{2.1}$$

Note that the definition of this operator can be naturally extended to positive measures on $\mathbb{R}^d$.

It is well known that for the symmetric stable processes this potential operator is given by the Riesz kernel (see e.g., [2])

$$U[f](x) = C\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-\alpha}}\,dy. \tag{2.2}$$

We have

LEMMA 2.2. *If $f \in L^1(\mathbb{R}^d)$, then $U[f](x)$ is finite for almost all $x \in \mathbb{R}^d$. If f is in L^∞, then $U[f](x)$ is finite everywhere.*

Proof. Set $g(x) = C/|x|^{d-\alpha}$.

$$U[f](x) = (f*g)(x) = \big(f*\big(g1_{B(0,1)}\big)\big)(x) + \big(f*\big(g1_{B^c(0,1)}\big)\big)(x). \tag{2.3}$$

The first term is the convolution of two L^1 functions, hence, also in L^1. The second term is bounded above by $\|f\|_1$ since $g \le C$ outside of the ball $B(0,1)$. The result follows. □

Let $v \in L^1(\mathbb{R}^d)$ be positive, and

$$1 - U_v^t(x) = e^{t(\Delta^{\alpha/2}-v)}1(x) = \mathbf{E}^x\big(e^{-\int_0^t v(X_s)\,ds}\big). \tag{2.4}$$

Using U_v^t, we can define the capacitory potential of v by

$$U_v(x) = \lim_{t\longrightarrow\infty} U_v^t(x) = \mathbf{E}^x\big(1 - e^{-\int_0^\infty v(X_s)\,ds}\big). \tag{2.5}$$

Let the capacitory measure of v equal

$$\mu_v(x)\,dx = v(x)\bigl(1 - U_v(x)\bigr)\,dx. \tag{2.6}$$

We want to show that $U_v(x) = U[\mu_v](x)$. We have

$$\begin{aligned} U[\mu_v](x) &= \mathbf{E}^x\left(\int_0^\infty v(X_s)\mathbf{E}^{X_s}\bigl(e^{-\int_0^\infty v(X_r)\,dr}\bigr)\,ds\right) \\ &= \mathbf{E}^x\left(\int_0^\infty v(X_s)\mathbf{E}\bigl(e^{-\int_0^\infty v(X_r)\,dr}\circ\theta_s|\mathcal{F}_s\bigr)\right) \\ &= \int_0^\infty \mathbf{E}^x\mathbf{E}\bigl(v(X_s)e^{-\int_0^\infty v(X_r)\,dr}\circ\theta_s|\mathcal{F}_s\bigr)\,ds. \end{aligned} \tag{2.7}$$

The last equality follows from Fubini theorem and the fact that $v(X_s)$ is $\mathcal{F}_s$ measurable. Hence,

$$U[\mu_v](x) = \mathbf{E}^x\left(\int_0^\infty v(X_s)e^{-\int_s^\infty v(X_r)\,dr}\,ds\right). \tag{2.8}$$

By Definition 2.1 and Lemma 2.2, we have

$$\int_0^\infty v(X_s)\,ds < \infty \quad \text{a.s.} \tag{2.9}$$

for almost every starting points $x \in \mathbb{R}^d$. Therefore, the function

$$f(s) = \int_s^\infty v(X_r)\,dr \tag{2.10}$$

is absolutely continuous for almost all paths of the process X_s and so is $e^{-f(s)}$. By the fundamental theorem of calculus,

$$\begin{aligned} U[\mu_v](x) &= \mathbf{E}^x\left(\int_0^\infty \frac{d}{ds}e^{-\int_s^\infty v(X_r)\,dr}\,ds\right) \\ &= \mathbf{E}^x\bigl(1 - e^{-\int_0^\infty v(X_r)\,dr}\bigr) = U_v(x), \end{aligned} \tag{2.11}$$

for almost every $x \in \mathbb{R}^d$.

Note that if v is also bounded than by the second part of the Lemma 2.2 the last equality holds for all x. Since $e^{-f(s)}$ is nondecreasing, its derivative exists almost everywhere and $U_v(x) = U[\mu_v](x)$ for all $x \in \mathbb{R}^d$.

Finally, we define the scattering length of v as

$$\Gamma(v) = \int_{\mathbb{R}^d}\mu_v(x)\,dx = \int_{\mathbb{R}^d} v(x)\bigl(1 - U_v(x)\bigr)\,dx. \tag{2.12}$$

If we assume that the potential $v(x)$ is in $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, then $\mu_v(x)$ is also in $L^2(\mathbb{R}^d)$. In such a case, we get

$$-\Delta^{\alpha/2}U_v = \mu_v. \tag{2.13}$$

The notion of the scattering length generalizes the capacity of sets. Let K be a Kac regular set (see [10] for details). Informally, the set K is Kac regular if after entering the set K the process will stay there for a positive amount of time. Put

$$v_K = \infty \text{ on } K, \text{ and } 0 \text{ outside.} \tag{2.14}$$

Under the assumption of Kac regularity, the notion of the capacitory potential U_{v_K} coincides with the capacitory potential $U[\mu_K]$ of the set K, where μ_K is the equilibrium measure on K (see [10]). Analogously to (2.11)

$$U_K(x) := U_{v_K}(x) = U[\mu_K](x). \tag{2.15}$$

In such a case, this potential is also equal to the probability that the process X_t starting from x ever hits K.

Similarly to (2.13), we also have

$$-\Delta^{\alpha/2} U_K = \mu_K. \tag{2.16}$$

The total mass of the equilibrium measure is called the capacity of the set K

$$\mathrm{Cap}(K) = \int d\mu_K. \tag{2.17}$$

We see that $\mathrm{Cap}(K) = \Gamma(v_K)$. Proposition 3.8 shows that the capacity is also a limit of scattering lengths.

3. Properties of scattering length

In this section, we prove several useful properties of the scattering length and the capacitory potential. We start with upper bounds for U_v and $\Gamma(v)$.

Proposition 3.1. *Let $v \in L^1(\mathbb{R}^d)$. Then*

(1) $\Gamma(v) \le \|v\|_1$,

(2) *if $B \subset \mathbb{R}^d$ is bounded, then*

$$\int_B U_v(x)\,dx \le C(B)\Gamma(v).$$

Proof. The first inequality follows from (2.6) and $U_v \le 1$. For the second, we have

$$\begin{aligned}\int_B U_v(x)\,dx &= C \int_B \int_{\mathbb{R}^d} \frac{d\mu_v(y)}{|x-y|^{d-\alpha}}\,dx \\ &\le C\left(\sup_y \int_B \frac{dx}{|x-y|^{d-\alpha}}\right)\Gamma(v) \\ &= C(B)\Gamma(v).\end{aligned}$$

□

Next, we prove some monotonicity and convergence properties of the scattering length.

Proposition 3.2. *Let $v, v_n, w \in L^1(\mathbb{R}^d)$ be positive. Then*

(1) *if $v \le w$ a.e. then $U_v \le U_w$ a.e. and $\Gamma(v) \le \Gamma(w)$,*

(2) *if* $v_n(x)$ *is a.e. nondecreasing and converges a.e. to* v *then* U_{v_n} *is a.e. nondecreasing and converges a.e. to* U_v*, and* $\Gamma(v_n)$ *is nondecreasing and converges to* $\Gamma(v)$*.*

Proof. The monotonicity and convergence of capacitory potentials follow directly from (2.5).

To prove the results about the scattering length, we need another formula for $\Gamma(v)$ if $\operatorname{supp} v$ is bounded. Consider a compact set K such that $v \subset\subset K$. Let U_K be its capacitory potential and μ_K its equilibrium measure (see (2.15)). Note, that $U_K = 1$ on $\operatorname{supp} v$, and

$$\Gamma(v) = \int U_K(x)\mu_v(dx) = \int U[\mu_K](x)\mu_v(dx) = \int U[\mu_v](x)\, d\mu_K(x). \tag{3.1}$$

If we take K large enough to have $(\operatorname{supp} v) \cup (\operatorname{supp} w) \subset\subset K$, the monotonicity of the scattering length follows from the monotonicity of the capacitory potentials.

Now, suppose that v is any positive $L^1(\mathbb{R}^d)$ function. Let v_n be a.e. nondecreasing sequence of functions with bounded supports such that v_n converges a.e. to $v \in L^1(\mathbb{R}^d)$. Suppose that $w \le v$ a.e., and let $w_n = \min\{v_n, w\}$. We have

$$\begin{aligned}\int_{\mathbb{R}^d} v_n(x)\big(1 - U_{v_n}(x)\big)\, dx = \Gamma(v_n) \ge \Gamma(w_n)\\ = \int_{\mathbb{R}^d} w_n(x)\big(1 - U_{w_n}(x)\big)\, dx.\end{aligned} \tag{3.2}$$

Both integrands are bounded above by v, hence, by the dominated convergence theorem

$$\Gamma(v) = \int_{\mathbb{R}^d} v(x)\big(1 - U_v(x)\big)\, dx \ge \int_{\mathbb{R}^d} w(x)\big(1 - U_w(x)\big)\, dx = \Gamma(w). \tag{3.3}$$

The second part of the proposition follows from the first one, and the dominated convergence theorem. ☐

PROPOSITION 3.3. *For* $r > 0$ *and nonnegative* $v, w \in L^1(\mathbb{R}^d)$ *we have*

(1) $U_{v+w} \le U_v + U_w$ *and* $\Gamma(v + w) \le \Gamma(v) + \Gamma(w)$ *and*
(2) *if* $w(x) = r^\alpha v(rx)$*, then* $U_w(x) = U_v(rx)$ *and* $\Gamma(w) = r^{\alpha-d}\Gamma(v)$*.*

Proof. The first inequality follows from the inequality $1 - e^{-a-b} \le (1 - e^{-a}) + (1 - e^{-b})$ which is valid for any nonnegative numbers a and b. The second inequality follows from the first one and the monotonicity of the potentials.

The second part of the proposition can be easily verified by a direct calculations. ☐

It is interesting to see what is the behavior of the scattering length for very small and very large potentials.

PROPOSITION 3.4. *Let* $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$. *Assume that* $v \in L^p(\mathbb{R}^d)$ *and* $\operatorname{supp} v \subset B$ *bounded. For any* $\varepsilon > 0$,

$$\Gamma(\varepsilon v) = \varepsilon \|v\|_1 - O(\varepsilon^{1+1/q} \|v\|_p \|v\|_1^{1/q}).$$

Proof. By Proposition 3.1 and by definition, we have

$$\|U_v\|_{L^1(B)} \leq C(B)\Gamma(v), \tag{3.4}$$

$$\|U_v\|_\infty \leq 1. \tag{3.5}$$

Hence,

$$\|U_v\|_{L^q(B)} = \left(\int_B |U_v|^q \right)^{1/q} \leq \left(\int_B |U_v| \right)^{1/q} \leq C(B)\Gamma(v)^{1/q}.$$

But,

$$\begin{aligned} \int v U_v \, dx &\leq \|v\|_p \|U_v\|_{L^q(B)} \\ &\leq C(B) \|v\|_p \Gamma(v)^{1/q} \\ &\leq C(B) \|v\|_p \|v\|_1^{1/q}. \end{aligned}$$

Hence,

$$\Gamma(\varepsilon v) = \int \varepsilon v (1 - U_{\varepsilon v}) \, dx = \varepsilon \|v\|_1 - O(\varepsilon^{1+1/q} \|v\|_p \|v\|_1^{1/q}).$$ □

PROPOSITION 3.5. *If* $v \in L^1(\mathbb{R}^d)$, *then*

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \Gamma(\varepsilon v) = \|v\|_1.$$

Proof. By the definition of scattering length, we have

$$\|v\|_1 - \frac{1}{\varepsilon}\Gamma(\varepsilon v) = \int v U_{\varepsilon v} \, dx. \tag{3.6}$$

By Proposition 3.1, $\|U_{\varepsilon v}\|_{L^1(B)} \leq C(B)\|v\|_1 \varepsilon$. Therefore, $U_{\varepsilon v} \to 0$ in measure, and the same is true for $vU_{\varepsilon v}$. Since $0 \leq U_{\varepsilon v} \leq 1$, $\int v U_{\varepsilon v} \to 0$ and this completes the proof. □

The last two propositions together with the main result give an asymptotic formula for the first eigenvalue of the Neumann Laplacian with a small perturbation.

REMARK 3.6. Let $v \in L^1(\Omega)$ be positive. We have

$$C_1(\Omega) \leq \lim_{\varepsilon \to 0} \frac{\lambda_1(\varepsilon v)}{\varepsilon \|v\|_1} \leq C_2(\Omega). \tag{3.7}$$

REMARK 3.7. Proposition 3.4 can be used to improve the above if $v \in L^p(\Omega)$.

If the potential v is large on its support, the scattering length is close to the capacity of the support of v.

PROPOSITION 3.8. *Consider $v \geq 0$ with compact support K. Let us also assume that K is Kac regular. Then*

$$\Gamma(v) \leq \operatorname{Cap} K.$$

If $v_i(x) \nearrow v_K(x)$ (see (2.14)), then

$$\Gamma(v_i) \nearrow \operatorname{Cap} K.$$

Proof. Suppose that $v \geq 0$ is supported in K, where K is Kac regular. Let B be a ball such that K is contained in the interior of B. By (3.1)

$$\Gamma(v) = \int U_v(x)\, d\mu_B(x),$$

and

$$\operatorname{Cap} K = \int U_K(x)\, d\mu_B(x).$$

But $U_v \leq U_K$, so $\Gamma(v) \leq \operatorname{Cap} K$. The second part of the proposition follows from monotone convergence theorem and from the first part. □

The main result combined with the properties of the scattering length gives the dependence of the first eigenvalue on the scaling of the potential.

REMARK 3.9. *Let v be a positive $L^1(\Omega)$ function. Suppose that $w_\beta(x) = r^\beta v(rx)$. We have*

$$C_1(\Omega) \leq \lim_{r\to\infty} \frac{\lambda_1(w_\alpha)}{r^{\alpha-d}\Gamma(v)} \leq C_2(\Omega). \tag{3.8}$$

Moreover, if $\beta > \alpha$,

$$C_1(\Omega) \leq \lim_{r\to\infty} \frac{\lambda_1(w_\beta)}{r^{\alpha-d}\operatorname{Cap}(\operatorname{supp} v)} \leq C_2(\Omega), \tag{3.9}$$

and if $\beta < \alpha$

$$C_1(\Omega) \leq \lim_{r\to\infty} \frac{\lambda_1(w_\beta)}{r^{\beta-d}\|v\|_1} \leq C_2(\Omega). \tag{3.10}$$

Proof. Follows from Propositions 3.3, 3.5, and 3.8. □

4. Proof of Theorem 1.1

First we prove the upper bound (1.8) by using a variational characterization of eigenvalues. The first eigenvalue $\lambda_1(v)$ can be calculated using Rayleigh quotient

$$\lambda_1(v) = \inf_{\varphi\in W^{\alpha/2,2}(\Omega)} \frac{\mathcal{E}_Y(\varphi,\varphi) + \int_\Omega v\varphi^2}{\int_\Omega \varphi^2}. \tag{4.1}$$

Our strategy is to choose a function φ which will give a desired bound. We claim that $\varphi = 1 - U_v$ will do the job. The space $W^{\alpha/2,2}(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of all functions f for which the Dirichlet form $\mathcal{E}_Y(f,f)$ is finite. Therefore, the infimum may be taken over the whole $L^2(\Omega)$. Since φ is bounded and Ω is also bounded, this function belongs to $L^2(\Omega)$. We get

$$\begin{aligned}\lambda_1(v)\int_\Omega \varphi^2 &\le \mathcal{E}_Y(\varphi,\varphi) + \int_\Omega v\varphi^2 = C\int_{\Omega\times\Omega} \frac{(U_v(x)-U_v(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy + \int_\Omega v\varphi^2\\ &\le C\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(U_v(x)-U_v(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy + \int_\Omega v\varphi^2.\end{aligned}$$

Let

$$v_n(x) = 1_{B(0,n)}(x)\min(v(x),n).$$

The sequence v_n is nondecreasing and converges to v. By Proposition 3.2, we get $U_{v_n} \nearrow U_v$ and $\Gamma(v_n) \nearrow \Gamma(v)$. The sequence v_n belongs to $L^2(\mathbb{R}^d)$, hence, $-\Delta_N^{\alpha/2} U_{v_n} = \mu_{v_n}$. Therefore,

$$\begin{aligned}\lambda_1(v)\int_\Omega \varphi^2 &\le C\int_{\mathbb{R}^d\times\mathbb{R}^d} \liminf_{n\to\infty} \frac{(U_{v_n}(x)-U_{v_n}(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy\\ &\qquad + \int_\Omega \lim_{n\to\infty} v_n(1-U_{v_n})^2\\ &\le C\liminf_{n\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(U_{v_n}(x)-U_{v_n}(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy\\ &\qquad + \lim_{n\to\infty}\int_\Omega v_n(1-U_{v_n})^2\\ &= \liminf_{n\to\infty}\left(\mathcal{E}_X(U_{v_n},U_{v_n}) + \int_\Omega v_n(1-U_{v_n})^2\right)\\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^d}\big(-U_{v_n}\Delta^{\alpha/2}U_{v_n} + v_n(1-U_{v_n})^2\big)\\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^d}\big(v_nU_{v_n}(1-U_{v_n}) + v_nU_{v_n}(U_{v_n}-1) - v_n(U_{v_n}-1)\big)\\ &= \liminf_{n\to\infty}\Gamma(v_n) = \Gamma(v).\end{aligned}$$

On the other hand,

$$\int_\Omega \varphi^2 \ge |\Omega| - 2C(\Omega)\Gamma(v). \tag{4.2}$$

Hence, if $\Gamma(v) \le |\Omega|/(4C(\Omega)) = \beta(\Omega)$ the last expression is comparable to the volume of Ω. This completes the proof of (1.8).

Let B_t be a Brownian motion running at twice the usual speed, and U_t be a reflected Brownian motion in the cube Ω. That is, U_t is the process generated by the Laplacian with Neumann boundary conditions in the cube. We will use a subordination technique (see [9]) to obtain stable processes from these processes. Let A_t be a positive $\alpha/2$-stable subordinator independent of B_t and U_t. If we subordinate a Brownian motion with A_t we get an isotropic α-stable process. In other words, $X_t = B_{A_t}$. Let V_t be the process U_t subordinated with the same subordinator A_t. The resulting process is a stable-like process (see [5]). However, it is not the same as a the reflected stable process Y_t (see [3]).

The following lemma gives a comparison between expected values of the multiplicative potentials of X_t and V_t.

LEMMA 4.1. *Let* $\operatorname{supp}(v) \subset \Omega$, *where* Ω *is a cube. Then*

$$\mathbf{E}^x\left\{\exp\left(-\int_0^t v(V_s)\,ds\right)\right\} \le \mathbf{E}^x\left\{\exp\left(-\int_0^t v(X_s)\,ds\right)\right\}. \tag{4.3}$$

Proof. Define g as follows:

$$g(x) = \begin{cases} x - 2n, & \text{if } x \in [2n, 2n+1),\ n \in Z, \\ 2n - x, & \text{if } x \in [2n-1, 2n),\ n \in Z. \end{cases} \tag{4.4}$$

One can think about g as a function that continuously folds a real line into a unit interval. Using a tensor product, we can define

$$f(x_1, x_2, \ldots, x_d) = g(x_1) \otimes g(x_2) \otimes \cdots \otimes g(x_d).$$

We have $U_t = f(B_t)$ for a reflected Brownian motion U_t on $[0,1]^d$. Since 1-dimensional components of B_t (and U_t on a cube) are independent of each other and are transition invariant, we have

$$U_t = f(B_t). \tag{4.5}$$

This gives

$$V_t = U_{A_t} = f(B_{A_t}) = f(X_t).$$

Now, we can define $\tilde{v}(x) = v(f(x))$ so that $\tilde{v} = v$ on $\operatorname{supp}(v)$. We have:

$$\begin{aligned}
\mathbf{E}^x\left\{\exp\left(-\int_0^t v(V_s)\,ds\right)\right\} &= \mathbf{E}^x\left\{\exp\left(-\int_0^t v(f(X_s))\,ds\right)\right\} \\
&= \mathbf{E}^x\left\{\exp\left(-\int_0^t \tilde{v}(X_s)\,ds\right)\right\} \\
&\le \mathbf{E}^x\left\{\exp\left(-\int_0^t v(X_s)\,ds\right)\right\}. \qquad \square
\end{aligned}$$

The processes X_t and Y_t are examples of a larger class of processes defined in [5], called stable-like processes Z_t. These processes have generators with quadratic forms

$$\mathcal{E}_Z(u,u) = \int_\Omega \int_\Omega c(x,y) \frac{(u(x) - u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy. \tag{4.6}$$

Here, $c(x,y)$ is a symmetric function satisfying $0 < c \le c(x,y) \le C < \infty$ for all x, y where c and C are constants independent of x and y. The domain of this form is the same as the domain of the "Neumann" fractional Laplacian, namely $W^{\alpha/2,2}(\Omega)$. For more detail about this class, we refer the reader to [5]. The first eigenvalues of the Schrödinger operators of the generators of two arbitrary stable-like processes are comparable. In particular, we have

LEMMA 4.2. *Let λ_1^V be the first eigenvalue of the operator $-A+v$, where A is the generator of V_t. Let also $\lambda_1(v)$ be as in Theorem 1.1, i.e., the first eigenvalue of the Schrödinger operator for the "Neumann" fractional Laplacian. Then*

$$c\lambda_1^V \le \lambda_1(v) \le C\lambda_1^V, \tag{4.7}$$

where c and C are positive constants.

Proof. Process V_t is a stable-like process, hence,

$$\mathcal{E}_V(u,u) = \int_\Omega\int_\Omega c_V(x,y)\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy \tag{4.8}$$

and $c \le c_V(x,y) \le C$ for some constants c and C. We can assume that $c \le 1$ and $C \ge 1$. Given any positive potential v,

$$\frac{1}{C}\left(\mathcal{E}_V(u,u) + \int vu^2\right) \le \mathcal{E}_Y(u,u) + \int vu^2 \le \frac{1}{c}\left(\mathcal{E}_V(u,u) + \int vu^2\right). \tag{4.9}$$

By (4.1), we get the inequality between the eigenvalues. □

Now, we are ready to prove the lower bound (1.7). By Lemma 4.2, it is enough to prove the lower bound for the process V_t. Let A be its generator. It is enough to prove that there exists t, such that

$$\left\|e^{t(A-v)}\right\|_2 \le e^{-C\Gamma(v)}. \tag{4.10}$$

There exists a kernel function $u_A(t,x,y)$, such that

$$e^{t(A-v)}f(x) = \int_\Omega u_A(t,x,y)f(y)\,dy, \tag{4.11}$$

for every bounded f (see [6] for existence and properties of such kernels). Using the Feynman–Kac formula we get

$$\int_\Omega u_A(t,x,y)\,dy = \mathbf{E}^x\left(e^{-\int_0^\infty v(V_s)\,ds}\right). \tag{4.12}$$

Let $u(t,x,y)$ be a heat kernel associated with the Schrödinger operator of the fractional Laplacian. Then

$$\int_{\mathbb{R}^d} u(t,x,y)\,dy = \mathbf{E}^x\left(e^{-\int_0^\infty v(X_s)\,ds}\right). \tag{4.13}$$

By Lemma 4.1, it is now enough to prove that

$$1 - U_v^t(x) = \int_{\mathbb{R}^d} u(t,x,y)\,dy \le e^{-C\Gamma(v)}.$$

First, we need an upper bound for the capacitory potential

$$U_v(x) \ge C \int \frac{d\mu_v(y)}{|x-y|^{d-\alpha}} \ge C\Gamma(v)(\mathrm{diam}(\Omega))^{\alpha-d}. \tag{4.14}$$

Using the semigroup property of $u(t,x,y)$,

$$\begin{aligned}\int u(t,x,y)U_v^s(y)\,dy &= \int u(t,x,y)\left(1 - \int u(y,z,s)\,dz\right)dy\\ &= \int u(t,x,y)\,dy - \int u(t+s,x,z)\,dz\\ &= U_v^{t+s}(x) - U_v^t(x).\end{aligned}$$

If we let s tend to ∞, we get

$$U_v(x) - U_v^t(x) = \int u(t,x,y)U_v(y)\,dy. \tag{4.15}$$

Let $p(t,x,y)$ be a heat kernel associated with the process X_t. Since our potentials v are nonnegative, we have

$$u(t,x,y) \le p(t,x,y). \tag{4.16}$$

Using this inequality, we obtain

$$\begin{aligned}U_v(x) - U_v^t(x) &= \int u(t,x,y)U_v(y)\,dy \le \int p(t,x,y)U_v(y)\,dy\\ &= C\int\int \frac{d\mu_v(z)}{|z-y|^{d-\alpha}} p(t,x,y)\,dy\\ &= C\int\int \frac{p(t,x,y)}{|z-y|^{d-\alpha}}\,dy\,d\mu_v(z)\\ &\le C\Gamma(v) \sup_{z\in\Omega, x\in\mathbb{R}^d} \int \frac{p(t,x,y)}{|z-y|^{d-\alpha}}\,dy.\end{aligned}$$

We need to show that the supremum tends to 0 as t tends to ∞. Then we can take t_0 large enough so that

$$U_v - U_v^{t_0} \le U_v/2. \tag{4.17}$$

And, using (4.14)

$$1 - U_v^{t_0}(x) \le 1 - U_v(x)/2 \le 1 - C/2\Gamma(v)(\mathrm{diam}\,\Omega)^{\alpha-d} = e^{-C\Gamma(v)}.$$

The only thing left to prove is the following.

LEMMA 4.3. *Let $p(t,x,y)$ be the heat kernel associated with the process X_t. Then*

$$\lim_{t\longrightarrow\infty}\sup_{z,x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{p(t,x,y)}{|y-z|^{d-\alpha}}\,dy=0.$$

Proof. The Riesz kernel (see (2.2)) satisfies

$$\frac{C}{|z-y|^{d-\alpha}}=\int_0^\infty p(t,y,z)\,dt.$$

By the semigroup property,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{p(t,x,y)}{|y-z|^{d-\alpha}}\,dy&=C\int_{\mathbb{R}^d}\int_0^\infty p(t,x,y)p(s,y,z)\,ds\,dy\\&=C\int_0^\infty p(t+s,x,z)\,ds=C\int_t^\infty p(s,x,z)\,ds\\&\le C\int_t^\infty s^{-d/\alpha}\,ds=Ct^{-d/\alpha+1}\to 0.\end{aligned}$$

□

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