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RAPIDLY GROWING ENTIRE FUNCTIONS WITH THREE SINGULAR VALUES

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ABSTRACT. We settle the problem of finding an entire function with three singular values whose Nevanlinna characteristic dominates an arbitrarily prescribed function.

1. Introduction

Let f be a transcendental meromorphic function in the plane $\mathbb{C}$. A *critical point* of f is a point at which the spherical derivative of f vanishes. The value of f at a critical point is called a *critical value*. A point a in the sphere $\overline{\mathbb{C}}$ is called an *asymptotic value* of f if there exists a curve $\gamma : [0,1) \to \mathbb{C}$ such that

$$\gamma(t) \to \infty \quad \text{and} \quad f(\gamma(t)) \to a \quad \text{as } t \to 1.$$

A point a in $\overline{\mathbb{C}}$ is a *singular value* of f if it is either a critical or an asymptotic value. In this paper we study the growth behavior of entire and meromorphic functions which have finitely many singular values. The class of such functions is usually denoted by $\mathcal{S}$, after Speiser [19], [20].

If f is an arbitrary meromorphic function in the plane, the Nevanlinna characteristic of f is defined as (see [9], [17])

$$T(r,f) = N(r,f) + m(r,f),$$

where

$$N(r,f) = \int_0^r \frac{n(t,f)}{t}\,dt, \qquad m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\varphi})|\,d\varphi,$$

and $n(t,f)$ is the number of poles of f in $\{|z| < t\}$. Here we assumed that 0 is not a pole of f. If f is a rational function of degree d, then its Nevanlinna

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characteristic $T(r,f)$ grows like $d\log r$, as $r\to\infty$. If f is a transcendental meromorphic function, then $T(r,f)$ grows faster than any multiple of $\log r$, but it is easy to see that for any $a>1$ one can find a transcendental f for which $T(r,f)$ grows slower than $\log^a r$, as $r\to\infty$.

The question of slowest possible growth of the Nevanlinna characteristic for meromorphic functions with finitely many singular values has been studied in recent years, notably by Eremenko [6], and Langley [14], [15]. In particular, it was proved that if f is a transcendental meromorphic function with three singular values, then

$$\liminf_{r\to\infty}\frac{T(r,f)}{\log^2 r}\geq\frac{\sqrt{3}}{\pi},$$

and the constant on the right-hand side is sharp. Langley established the existence of an absolute constant for the right-hand side, and Eremenko found the exact value for this constant. If f is a transcendental entire function with three singular values, then $\liminf_{r\to\infty}T(r,f)/\log^2 r$ is infinite. In fact, the Nevanlinna characteristic $T(r,f)$ of such a function dominates a positive multiple of $\sqrt{r}$.

In general, if f is a transcendental meromorphic function which has finitely many singular values, then Langley showed that

$$\liminf_{r\to\infty}\frac{T(r,f)}{\log^2 r}>0,$$

but the left-hand side can be as small as one wishes if the number of singular values is greater than three.

Here we investigate the question of arbitrarily rapid growth.

Theorem 1. *For every $\mathbb{R}$-valued function $M(r)$, $r\geq 0$, there exists an entire function f with three singular values* 0, 1, *and* ∞ *such that*

$$T(r,f)\geq M(r),\quad \textit{for } r\geq r_0,$$

and some $r_0>0$.

Our proof of this theorem is based on a combinatorial construction of a Riemann surface spread over the sphere which branches over three points. The desired map is obtained as a composition of a uniformizing map of this Riemann surface and the projection map to the sphere. One of the key steps in proving Theorem 1 is to establish a quantitative control on the volume growth of a graph in terms of the combinatorial modulus. This is done in Lemma 1.

A meromorphic function whose Nevanlinna characteristic dominates an arbitrarily prescribed function is easier to produce. Indeed, there is more flexibility in constructing surfaces spread over the sphere that correspond to meromorphic functions, since one does not need to worry about ∞ being an omitted value. The construction is outlined in Section 6.

2. Graphs

A graph G is a pair (V_G, E_G), where V_G is a set of *vertices* and E_G is a subset of unordered pairs of elements in V_G, called *edges*. If $v_1, v_2 \in V_G$, and $\{v_1, v_2\} \in E_G$, we say that v_1 and v_2 are *connected by an edge*, and write $v_1 \sim v_2$. We assume that no vertex is connected to itself by an edge. A graph is called *bipartite* if the vertices can be subdivided into two disjoint sets, say A and B, and every edge connects a vertex from A to one from B. A *subgraph* G' of a graph G is a graph whose vertex set forms a subset of V_G, and if two vertices of G' are connected by an edge in G', then they are connected by an edge in G. If A is a subset in V_G, we denote by $|A|$ the cardinality of A, where $|A| = \infty$ if the set A is infinite.

The *valence* of $v \in V_G$ is $|\{u \in V_G : u \sim v\}|$. The *valence* of G is the supremum of the valences over all vertices of G. A graph G is called *locally finite*, if the valence of each vertex is finite. A graph is said to have a *finite valence*, if there is a uniform bound on the valence at each vertex. A graph is called *homogeneous* of valence q if every vertex has the same valence q.

A *chain* in G is a sequence $(\dots, x_{-1}, x_0, x_1, \dots)$ of vertices, finite or infinite in one or both directions, such that $\cdots \sim x_{-1} \sim x_0 \sim x_1 \sim \cdots$. We also refer to a chain as a sequence of vertices along with the edges connecting them. A chain $(\dots, y_{-1}, y_0, y_1, \dots)$ is a *subchain* of a chain $(\dots, x_{-1}, x_0, x_1, \dots)$, if $y_j = x_{k(j)}$ for some monotone increasing sequence $(k(j))$. We say that a chain $(x_1, \dots, x_n)$ *connects* two subsets A and B of V_G, if $x_1 \in A$ and $x_n \in B$. A chain $(x_1, x_2, \dots)$ *connects* a finite set A to ∞, if $x_1 \in A$ and it eventually leaves every finite set, i.e., for every finite subset K of V_G there exists $k \in \mathbb{N}$ such that $x_j \notin K$ for $j \geq k$. A set B in V_G is said to *separate* a set $A \subset V_G$ from ∞ if every chain connecting A to ∞ has a vertex in B.

A *loop* in a graph is a finite chain $(x_1, \dots, x_n)$, such that $x_1 = x_n$ and all other vertices of the chain are distinct. A *tree* is a graph that does not contain any loops $(x_1, \dots, x_n)$ with $n > 3$. A *subtree* is a subgraph of a tree.

If G is a graph and V is a subset of the vertex set V_G, we consider the subgraph G' of G *determined by the vertex set* V to be the graph whose vertex set is V, and two vertices v_1 and v_2 are connected by an edge in G' if and only if they are connected by an edge in G.

A *domain* D in a graph G is a subset of the vertex set V_G which is *connected* in the sense that every two vertices in D can be connected by a chain all of whose vertices are in D. The *boundary* of D in G, denoted by $\partial_G D$, or ∂D if the graph is understood, is the set of all vertices in V_G that are not in D, and each of which is connected by an edge in E_G to a vertex in D. An *annulus* in a graph G is a subset of V_G whose complement in V_G consists of two disjoint domains. Not every graph contains an annulus. A sequence of annuli (A_k) is called *nested* if the annuli are pairwise disjoint, and A_{k+1} separates A_k from ∞.

In this paper we only consider *planar* graphs, i.e., graphs embedded in the plane $\mathbb{R}^2$. If we fix an embedding of a graph into $\mathbb{R}^2$, then we can speak of *faces* of the graph. These are complementary components of the image of the graph in the plane. A *side* of a face is a part of its boundary that is the image of an edge under the embedding. If G is a planar graph, one can also define its *dual* G^*. The vertices of G^* are in one to one correspondence with the faces of G. Two vertices of G^* are connected by an edge if and only if the boundaries of the corresponding faces of G share a side.

A connected graph can be viewed as a metric space if one declares that every edge is isometric to a unit interval on the real line. This metric restricts to the space whose elements are vertices of the graph, in which case it is said that the graph is endowed with the *word metric*. Thus, we can speak of *geodesics* in a graph, i.e., chains connecting two vertices or two sets and having the smallest lengths among all such chains. If A and B are two subsets of V_G, we denote by $\delta(A,B)$ the word distance between A and B, i.e., the number of edges in a geodesic connecting A and B. If A is a one vertex set $\{v\}$, we write $\delta(v,B)$ instead of $\delta(\{v\},B)$. Similarly, $\delta(v,w)$ stands for $\delta(\{v\},\{w\})$.

3. Surfaces of Speiser class

A *surface spread over the sphere* is an equivalence class of pairs $[(X,\pi)]$, where X is an open, i.e., non-compact, simply connected topological surface and $\pi : X \to \overline{\mathbb{C}}$ is a continuous, open, and discrete map. Two pairs (X_1,π_1) and (X_2,π_2) are *equivalent* if there exists a homeomorphism $h : X_1 \to X_2$ such that $\pi_1 = \pi_2 \circ h$. The map π is called the *projection*.

In a neighborhood of each point x in X, the map π is given in some local coordinates (for neighborhoods of x and $\pi(x)$) by $z \mapsto z^k$, where $k = k(x) \in \mathbb{N}$ is called the *local degree* of f at x. If $k \geq 2$, then x is called a *critical point* of f, and in this case the value $f(x)$ is called a *critical value*. As in the case $X = \mathbb{C}$ and π a meromorphic function, $a \in \overline{\mathbb{C}}$ is called an *asymptotic value* if there exists a curve $\gamma : [0,1) \to X$, such that

$$\gamma(t) \to \infty \quad \text{and} \quad \pi(\gamma(t)) \to a \quad \text{as } t \to 1.$$

Here, $\gamma(t) \to \infty$ means that $\gamma(t)$ leaves every compact set of X as $t \to 1$. A point a in $\overline{\mathbb{C}}$ is a *singular value* of π if it is either a critical or an asymptotic value.

According to Stoïlow [21], X supports a complex structure, the *pullback structure*, in which the map π is holomorphic. A surface spread over the sphere is said to have *parabolic type*, or is called *parabolic*, if X endowed with the pullback structure is conformally equivalent to the plane. Otherwise it is said to have *hyperbolic type*. The homeomorphism h in the definition of equivalence is a conformal map in these pullback structures, and therefore the conformal type of a surface spread over the sphere is well defined. For simplicity, below we refer to a pair (X,π), rather than an equivalence class, as a surface spread over the sphere. If g is a uniformizing map for X defined in the complex plane or the unit disc, then $f = \pi \circ g$ is a meromorphic function. If π omits the value ∞, then f is holomorphic. The surface spread over the sphere (X, π) is classically referred to as the "surface of f^{-1}."

A surface spread over the sphere belongs to *Speiser class* $\mathcal{S}$ if π has only finitely many singular values. If $\{a_1, \dots, a_q\}$ is the set of singular values of π, then π restricted to $\pi^{-1}(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_q\})$ is a covering map. Surfaces spread over the sphere of class $\mathcal{S}$ have combinatorial representations in terms of Speiser graphs.

Assuming that $(X, \pi) \in \mathcal{S}$ and π has q singular values $a_1, \dots, a_q$, we fix an oriented Jordan curve L in $\overline{\mathbb{C}}$, visiting the points $a_1, \dots, a_q$ in cyclic order of increasing indices. This curve decomposes the sphere into two simply connected regions. Let L_i, $i \in \{1, 2, \dots, q\}$, be the arc of L from a_i to a_{i+1} (with indices taken modulo q). Let us fix points p_1 and p_2 in the two complementary components of L, and choose q Jordan arcs $\gamma_1, \dots, \gamma_q$ in $\overline{\mathbb{C}}$, such that each arc γ_i has p_1 and p_2 as its endpoints, and has a unique point of intersection with L, which is in L_i. These arcs are chosen to be interiorwise disjoint, that is, $\gamma_i \cap \gamma_j = \{p_1, p_2\}$ when $i \neq j$. Let Γ' denote the graph embedded in $\overline{\mathbb{C}}$, whose vertices are p_1 and p_2, and whose edges are γ_i, $i = 1, \dots, q$, and let $\Gamma = \pi^{-1}(\Gamma')$. We identify Γ with its image in $\mathbb{R}^2$ under an orientation-preserving homeomorphism of X onto $\mathbb{R}^2$. The graph Γ is infinite, connected, homogeneous of valence q, and bipartite. The vertices that project to p_1 are labelled $\times$ and the ones that project to p_2 are labelled $\circ$. A graph, properly embedded in the plane and having these properties is called a *Speiser graph*. Two Speiser graphs Γ_1, Γ_2 are said to be *equivalent*, if there is an orientation-preserving homeomorphism of the plane which takes Γ_1 to Γ_2. Each face of the Speiser graph Γ is labelled by the corresponding element of the set $\{a_1, \dots, a_q\}$.

The above construction of a Speiser graph from a surface spread over the sphere of class $\mathcal{S}$ is reversible. Suppose we are given a Speiser graph Γ whose faces are labelled by $a_1, \dots, a_q$. A necessary condition for existence of a surface spread over the sphere of class $\mathcal{S}$ with singular values $a_1, \dots, a_q$ and whose Speiser graph is Γ is that the labels should satisfy a certain compatibility condition. Namely, when going counterclockwise around a vertex $\times$, the indices are encountered in their cyclic order, and around $\circ$ in the reversed cyclic order. We fix a simple closed curve $L \subset \overline{\mathbb{C}}$ passing through $a_1, \dots, a_q$. Let H_1, H_2 be the complementary regions whose common boundary is L, and let $L_1, \dots, L_q$ be as above. Let Γ^* be the planar dual of Γ. The vertices of Γ^* are naturally labelled by $a_1, \dots, a_q$. If e is an edge of Γ^* connecting a_j and a_{j+1}, let π map e homeomorphically onto the corresponding arc L_j of L. This defines a map π on the edges and vertices of Γ^*. We then extend π to map the faces of Γ^* homeomorphically to H_1 or H_2, depending on the orientation of the boundaries. This defines a surface spread over the sphere $(\mathbb{R}^2, \pi) \in \mathcal{S}$. The corresponding labelled Speiser graph is the graph Γ with the prescribed labels. Thus, up to a choice of the curve L, we have a one to one correspondence between surfaces spread over the sphere of class $\mathcal{S}$ and equivalence classes of labelled Speiser graphs. See [7] and [17] for further details.

4. Type problem

A long studied problem is the one of recognizing the conformal type of a surface spread over the sphere of class $\mathcal{S}$ from its Speiser graph. An infinite locally finite connected graph is called *parabolic* if the simple random walk on it is recurrent. Otherwise it is called *hyperbolic*. Doyle [3] gave a criterion of type for a surface spread over the sphere of class $\mathcal{S}$ in terms of a so-called extended Speiser graph.

Let $\mathbb{Z}_+$ denote the set of non-negative integers. A *half-plane lattice* Λ is the graph embedded in $\mathbb{R}^2$ whose vertices form the set $\mathbb{Z} \times \mathbb{Z}_+$, and $(x', y') \sim (x'', y'')$ if and only if $(x'' - x', y'' - y') = (\pm 1, 0)$ or $(0, \pm 1)$. The *boundary* of the half-plane lattice Λ is the infinite connected subgraph of Λ determined by the vertex set $\mathbb{Z} \times \{0\}$. There is an action of $\mathbb{Z}$ on Λ by horizontal shifts. A *half-cylinder lattice* Λ_n is $\Lambda/n\mathbb{Z}$. The *boundary* of Λ_n is the induced boundary from Λ.

Suppose that Γ is a Speiser graph and let $n \in \mathbb{N}$ be given. If we replace each face of Γ with $2k$ edges on the boundary, $k \geq n$, by the half-cylinder lattice Λ_{2k}, and each face with infinitely many edges on the boundary by the half-plane lattice Λ, identifying the boundaries of the faces with the boundaries of the corresponding lattices along the edges and vertices, we obtain an *extended Speiser graph* Γ_n. The graph Γ_n is an infinite connected graph embedded in the plane, containing Γ as a subgraph. It has a finite valence, and all faces of Γ_n have no more than $\max\{2(n-1), 4\}$ sides.

THEOREM A ([3]). *A surface spread over the sphere $(X, \pi) \in \mathcal{S}$ is parabolic if and only if Γ_1 is parabolic.*

In [16] we proved a slight modification of Theorem A.

THEOREM B ([16]). *Let $n \in \mathbb{N}$ be fixed. A surface spread over the sphere $(X, \pi) \in \mathcal{S}$ is parabolic if and only if Γ_n is parabolic.*

Doyle's arguments are probabilistic and electrical, whereas [16] employs geometric methods, using results of Kanai [12], [13]. Below we derive Theorem B from Theorem A using results from [4].

Kanai shows the invariance of type under quasi-isometries for spaces with bounded geometry. A map $\Phi : (X_1, d_1) \to (X_2, d_2)$ between two metric spaces is called a *quasi-isometry*, if the following conditions are satisfied:

1. for some $\varepsilon > 0$, the ε-neighborhood of the image of Φ in X_2 covers X_2;

2. there are constants $k \geq 1$, $C \geq 0$, such that for all x_1, $x_2 \in X_1$,

$$k^{-1} d_1(x_1, x_2) - C \leq d_2(\Phi(x_1), \Phi(x_2)) \leq k d_1(x_1, x_2) + C.$$

The metric space (X_1, d_1) is *quasi-isometric* to the metric space (X_2, d_2) if there exists a quasi-isometry from X_1 to X_2. This is an equivalence relation. The notion of quasi-isometry, or rough isometry, was introduced by Gromov [8].

A Riemannian surface has *bounded geometry* if it is complete, the Gaussian curvature is bounded from below, and the radius of injectivity is positive. The latter means that there exists $\delta > 0$ such that every open ball whose radius is at most δ is homeomorphic to a Euclidean ball. Kanai proves that if a Riemannian surface has bounded geometry and is quasi-isometric to a finite valence graph with the word metric, then the surface and the graph have the same type. Likewise, two quasi-isometric graphs with finite valence have the same type.

Proof of Theorem B. By Theorem A one needs to show that Γ_n is parabolic if and only if Γ_1 is. Assume first that Γ_1 is parabolic. The graph Γ_n is obtained from Γ_1 by cutting the edges that connect the vertices of Γ, viewed as a subgraph of Γ_1 using the obvious embedding, on the boundary of faces of Γ with $2k$ edges, $k < n$, to the vertices of Λ_{2k}. Therefore, this direction follows from the Cutting Law [4], page 100. For the other direction, assume that Γ_1 is hyperbolic. We consider a new graph $\tilde{\Gamma}_1$, obtained from Γ_1 by shorting all nonboundary vertices of every half-cylinder lattice Λ_{2k}, $k < n$, that have replaced a face of Γ. Here, *shorting* a set of vertices means identifying them. By the Shorting Law [4], page 100, $\tilde{\Gamma}_1$ is also hyperbolic. But $\tilde{\Gamma}_1$ has finite valence and is quasi-isometric to Γ_n. The quasi-isometry is given by an embedding of Γ_n into $\tilde{\Gamma}_1$ induced from the obvious embedding of Γ_n into Γ_1. Therefore, Γ_n is hyperbolic. □

Due to the nature of a construction, as in our case below, it is often easier to establish the type for the dual graph Γ_n^* to the extended Speiser graph Γ_n.

THEOREM C. *Let $n \in \mathbb{N}$ be fixed. A surface spread over the sphere $(X, \pi) \in \mathcal{S}$ is parabolic if and only if Γ_n^* is parabolic.*

Proof. The graph Γ_n in question and its dual have finite valences. A map Φ that sends every vertex v of Γ_n^* to any vertex on the boundary of the face of Γ_n corresponding to v is a quasi-isometry. Indeed, the first condition for quasi-isometry follows since every vertex of Γ_n is on the boundary of a face and there is a uniform bound on the number of sides of each face since Γ_n^* has finite valence. Therefore, every vertex of Γ_n is within a uniformly bounded distance from an image of a vertex in Γ_n^* under Φ.

The second condition follows since both graphs have finite valence. Let γ^* be a geodesic chain in Γ_n^* connecting two vertices v_1 and v_2. By tracing the boundaries of faces corresponding to the vertices of γ^*, one can find a chain in Γ_n connecting $\Phi(v_1)$ and $\Phi(v_2)$, and whose length is at most C_1 times the length of γ, where C_1 depends only on the valences of Γ_n and Γ_n^*. Conversely, for every geodesic chain γ in Γ_n connecting two vertices $\Phi(v_1)$ and $\Phi(v_2)$, one can find a chain in Γ_n^* that connects v_1 and v_2 by following the faces that contain γ on their boundaries, such that the length of this new chain is at most C_2 times the length of γ. The constant C_2 depends only on the valences of Γ_n and Γ_n^*.

Since the graphs Γ_n and Γ_n^* are quasi-isometric and have finite valences, they have the same type. Now the result follows from Theorem B. □

5. Combinatorial modulus

In 1962, Duffin [5] introduced a combinatorial modulus for chain families in graphs. In his setting, the masses are assigned to the edges of the graph. Parabolicity of a locally finite graph is equivalent to the condition that the modulus of the family of chains connecting a fixed vertex to infinity is zero. For our purposes, it is more convenient to use a different notion of modulus, introduced more recently by Cannon [2], where masses are assigned to vertices rather than edges. This approach leads to certain combinatorial uniformization results, see e.g., [18]. If a graph has finite valence, as in our case below, it does not matter which definition of combinatorial modulus one uses when establishing parabolicity. This can be seen by distributing masses from vertices to edges and *vice versa*.

A *mass distribution* for a graph G is a non-negative function on V_G. Let $\mathcal{C}$ be a family of chains in G. We say that a mass distribution m is *admissible* for $\mathcal{C}$, if for each chain $(\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{C}$, its *weighted length* $\sum m(x_j) \geq 1$. We denote by $\operatorname{mod}_G \mathcal{C}$ the *combinatorial modulus* of the chain family $\mathcal{C}$, namely

$$\operatorname{mod}_G \mathcal{C} = \inf\Big\{\sum m(v)^2\Big\},$$

where the infimum is taken with respect to all admissible mass distributions, and the sum is over all vertices in V_G. We write $\operatorname{mod} \mathcal{C}$ if the graph is understood. To distinguish, the conformal modulus of a curve family on a surface will be denoted by Mod. If $\mathcal{C}$ is the family of all chains connecting sets A and B, or a set A to ∞, we denote $\operatorname{mod} \mathcal{C}$ by $\operatorname{mod}(A, B)$ or $\operatorname{mod}(A, \infty)$, respectively. If A is an annulus in a graph G, then $\operatorname{mod} A$ denotes the modulus of the family of all chains that connect the complementary components of A in V_G.

As for the classical conformal modulus, if $\mathcal{C}$ and $\mathcal{C}'$ are two families of chains, such that every chain in $\mathcal{C}$ contains a subchain which is in $\mathcal{C}'$, then $\operatorname{mod} \mathcal{C} \leq \operatorname{mod} \mathcal{C}'$. Also, if (A_k) is a sequence of (disjoint) nested annuli, then

$$\operatorname{mod}(\{v_0\}, \infty) \leq \frac{1}{\sum 1/\operatorname{mod} A_k},$$

for any vertex v_0 that is separated from ∞ by every A_k. The first property follows immediately from the definition. A proof of the inequality mimics that for the classical conformal modulus. Now, as in the classical case, to show parabolicity of a finite valence graph, it is enough to exhibit a sequence (A_k) of (disjoint) nested annuli, such that

$$\sum \frac{1}{\operatorname{mod} A_k} = \infty.$$

This will be used in the proof of Lemma 1.

6. Meromorphic example

Since later we prove that the Nevanlinna characteristic of an entire function dominates an arbitrarily prescribed function, here we only give an outline that such a meromorphic function exists.

Consider the infinite half-strip in the plane

$$S = \{z = x + iy : 0 \le x \le 2, 0 \le y < \infty\},$$

subdivided into squares

$$\{z : j \le x \le j+1, n \le y \le n+1\}, \quad j = 0, 1, \; n = 0, 1, 2, \ldots.$$

For each even $n = 2k$, we attach $N(k)$ Euclidean squares with side length 1 to the edge

$$e_k = \{z : x = 1, 2k \le y \le 2k+1\},$$

so that all of these squares share the side e_k, and are otherwise disjoint. More specifically, we cut the strip S along e_k, take a two-sided unit square cut along one of its edges, and glue the square to the strip along a cut. We repeat this operation if necessary, attaching more squares to e_k. What results can be thought of as a book spread open along its spine.

The result of the gluing of all the squares is a simply connected Riemann surface Y with boundary, which corresponds to the boundary of S. Now we consider the double X of Y across the boundary. This means that X is obtained from two copies of Y by identifying every boundary point of one copy with the point of the other copy that corresponds to the same point of Y. This is a simply connected Riemann surface without boundary. For each $n = 0, 1, 2, \ldots$, let A_n denote the annulus in X that consists of all points corresponding to the points of the horizontal rectangle $\{n \le y \le n+1\}$ of S and all points of squares attached to $e_{n/2}$ if n is even. Each surface X is parabolic since it contains a sequence of annuli (A_n), where n is odd, of fixed modulus. Using a modulus estimate, one can show that if F is a uniformizing map of $\mathbb{C}$ onto X, then the image I_r under F of the disc D_r centered at 0 of radius r contains a ball (in the intrinsic metric of X) of radius

$$L(r) \ge C \log r,$$

where C is a constant not depending on the sequence $(N(k))$. Indeed, let s denote the set in X that corresponds to the segment in S connecting $(0,0)$ to $(2,0)$, and let s_F be the preimage of s under F. The set s_F is homeomorphic to a line segment. Suppose that $n(r)$ is the smallest natural number so that the annulus $A_{n(r)}$ is not contained in I_r. The conformal modulus of the curve family consisting of curves in D_r that separate s_F from the boundary of D_r grows like $\log r/(2\pi)$ as $r \to \infty$. On the other hand, the conformal modulus of the image family in X is bounded above by $C'n(r)$, where C' is a constant independent of $(N(k))$. This can be seen by choosing a weight function equal $1/2$ at all points of the annuli $A_0, A_1, \dots, A_{n(r)+1}$ that correspond to points of S, and equal 0 at all other points of these annuli. From the invariance of modulus under conformal maps we obtain that

$$n(r) \geq \log r/(2\pi C'),$$

which immediately gives the desired estimate for $L(r)$.

Now, by choosing $N(k)$ to grow sufficiently rapidly, one can arrange arbitrarily rapid growth of the areas, with respect to the radii, of the intrinsic balls of X centered at some point. Arbitrarily rapid growth of the areas implies arbitrarily rapid growth of the Nevanlinna characteristic (see Ahlfors–Shimizu characteristic in [9]). A similar fact is based on the First Main Theorem of Nevanlinna and it will be discussed in Section 8.

By subdividing each square of the surface X into four triangles using diagonals, and considering the Speiser graph which is dual to such a triangulation, we obtain a meromorphic function with three singular values that has the desired properties.

7. Entire functions with three singular values

If f is a transcendental entire function with three singular values $0, 1$, and ∞, then $f^{-1}([0,1])$ forms a locally finite, infinite tree T embedded in $\mathbb{R}^2$. The vertices are the preimages of 0 and 1, and the edges are the preimages of $[0,1]$. Indeed, the graph is connected since f restricted to $f^{-1}(\mathbb{C} \setminus \{0,1\})$ is a covering map. The valence of each vertex is the local degree of f at the corresponding point. The graph is infinite since f is transcendental. Finally, it is a tree because otherwise there would exist a complementary component of $f^{-1}([0,1])$ that is compactly contained in $\mathbb{C}$. This is impossible since such a component would have to contain a preimage of ∞, but f is assumed to be entire.

Conversely, suppose we are given an arbitrary locally finite, infinite, embedded tree T, whose vertices are labelled 0 and 1, and each edge connects 0 and 1. We construct a surface spread over the sphere (X, π) with three singular values as follows. For every vertex v in V_T of valence k, we consider k non-homotopic, non-intersecting Jordan arcs in $\mathbb{R}^2 \setminus T$ that originate at v and escape to infinity. We can choose the arcs corresponding to different vertices to be disjoint. This gives a triangulation T' of $\mathbb{R}^2$, with each triangle having an ideal vertex at infinity. Every triangle of T' has an edge of T and two arcs escaping to infinity as its sides. Each vertex of T' has an even valence, and it receives a label 0 or 1 from the corresponding label of T. The ideal vertices at infinity are labelled by ∞.

Consider the dual graph to T', denoted Γ. The graph Γ is an infinite connected graph, properly embedded in the plane. It has valence three at each vertex, and every face of Γ has an even (or infinite) number of vertices on its boundary, so Γ is bipartite. Therefore Γ is a Speiser graph. Let (X,π) denote a surface spread over the sphere that corresponds to Γ with the induced labels from T', which are 0, 1, and ∞. These are the singular values of π, and π omits the value ∞. Thus, the composition of a uniformizing map of X with π is a holomorphic function. We proceed by explicitly describing (X,π) up to conformal equivalence.

Let

$$\alpha = \{(x,y) : 0 \le x, 0 \le y \le 1\}$$

be a half-strip in the plane. To each triangle t of T' we associate a copy of α, which we denote by $\alpha(t)$, so that under an orientation-preserving homeomorphism of the plane the side of t contained in T corresponds to the segment joining $(0,0)$ and $(0,1)$, and the sides of t that are in $T' \setminus T$ correspond to two horizontal rays. If t_1 and t_2 are adjacent triangles, we glue $\alpha(t_1)$ and $\alpha(t_2)$ along the corresponding sides using the identity map. The result of the gluing is a simply connected open Riemann surface, which we denote by $S(T)$. A tree isomorphic to T embeds in $S(T)$, and we identify this tree with T. Now we consider the conformal map, continuously extended to the boundary, from the half-strip

$$\alpha^o = \{(x,y) : 0 < x, 0 < y < 1\}$$

to the lower half-plane that takes $(0,0), (0,1)$, and ∞ to 0, 1, and ∞, respectively. This map extends by reflection to a conformal map from the Riemann surface $S(T)$ to the surface spread over the sphere (X,π) with the pullback complex structure. The tree T is isomorphic to $\pi^{-1}([0,1])$ with the natural graph structure.

Since we need to consider an extended Speiser graph in deciding the type of a surface spread over the sphere, the following subdivision of $S(T)$ is useful. We subdivide α into squares

$$\alpha_k = \{(x+k,y) : 0 \le x \le 1, 0 \le y \le 1\}, \quad k = 0,1,2,\dots.$$

The subdivision of α by $\alpha_k,\ k=0,1,2,\dots$, induces a subdivision of $S(T)$ into squares, a *square subdivision.* The 1-skeleton of this subdivision considered as a graph will be denoted by $\sigma = \sigma(T)$. The tree T is a subgraph of σ. In the case when the tree T has valence n, as in our example below with $n=3$, the graph $\sigma(T)$ is the dual graph Γ_n^* to the extended Speiser graph Γ_n. According to Theorem C, the surface spread over the sphere (X,π) is parabolic if and only if σ is.

8. Volume growth

The First Main Theorem of Nevanlinna (see [9], [17]) asserts that for every $a\in\mathbb{C}$,

$$T(r,f)=N\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-a}\right)+O(1),\quad r\to\infty.$$

Therefore, by choosing a to be either 0 or 1, we conclude that in order to find f with $T(r,f)$ growing arbitrarily rapidly, it is sufficient to find an embedded tree T with the following properties. The corresponding surface $S(T)$ is parabolic, and if $M(r),\ r\geq 0$, is a prescribed function, and g a uniformizing map from $\mathbb{C}$ to $S(T)$, then the number of vertices of $g^{-1}(T)$ in the disc of radius r about 0 is greater than $M(r)$, for all $r\geq r_0>0$. In this case, the first term $N(r,1/(f-a))$ alone dominates $M(r)$.

Assuming that $S(T)$ is parabolic and g is a uniformizing map from $\mathbb{C}$ to $S(T)$, we denote by $n(r,T,g)$ the number of vertices of $g^{-1}(T)$ contained in the disc of radius r centered at 0. This is an analog of the counting function $n(r,f)$ in the definition of Nevanlinna characteristic $T(r,f)$. Theorem 1 follows from the following theorem, proved in Section 10.

THEOREM 2. *Given any $\mathbb{R}$-valued function $M(r),\ r\geq 0$, there exists a locally finite, infinite tree T, embedded in the plane, such that $S(T)$ is parabolic, and $n(r,T,g)\geq M(r),\ r\geq r_0$, for any uniformizing map g and $r_0=r_0(g)>0$. Moreover, we can choose T to be a subtree of the regular tree of valence three, denoted T_3.*

The tree T_3 is homogeneous of valence 3, and we think of T_3 as being embedded in the plane. Let v_0 be a fixed vertex in V_{T_3}, and ε_0 denote the combinatorial modulus $\operatorname{mod}_{T_3}(\{v_0\},\infty)$, which is a positive number because T_3 is hyperbolic, as is well known. The complement of T_3 in the plane has infinitely many components, three of which have v_0 on their boundaries. We consider one of these three components, denoted D, and let $c=(\dots,v_{-1},v_0,v_1,\dots)$ be the chain in T_3 such that $v_j\neq v_k$ for $j\neq k$, and c together with the edges that connect its vertices bounds D.

If $k\in\mathbb{N}$, then $T_3\setminus\{v_k,v_{-k}\}$ is a union of five disjoint domains, one of which contains v_0, and each of the four others is bounded by either v_k or v_{-k}. For each $k\in\mathbb{N}\cup\{0\}$, let $\mathcal{C}_k$ be the family of all chains $(x_1,x_2,\dots)$ in T_3 that connect $\{v_0\}$ to ∞, and such that all but finitely many of x_j's are contained in one of the domains into which $T_3\setminus\{v_k,v_{-k}\}$ splits, that does not contain v_0. The family $\mathcal{C}_0$ consists of all chains connecting v_0 to ∞. If $k>0$, each chain of $\mathcal{C}_k$ should have all but finitely many of its vertices to lie in one of the four domains of $T_3\setminus\{v_k,v_{-k}\}$ that does not contain v_0. In other words, every chain in $\mathcal{C}_k$ should escape to infinity through either v_k or v_{-k}. It is easy to see that the sequence (ε_k) defined by $\varepsilon_k = \operatorname{mod}_{T_3} \mathcal{C}_k$ decreases, $0 < \varepsilon_k \le \varepsilon_0$ for every $k \in \mathbb{N}$, and $\lim \varepsilon_k = 0$.

For two quantities a and b, we use the notation $a \lesssim b$ if there exists a constant $C > 0$ which depends only on the data of an underlying space, such that $a \le Cb$. The key step in the proof of Theorem 2 is the following lemma.

LEMMA 1. *Let $c, \mathcal{C}_k$, and ε_k be as above, $k \in \mathbb{N} \cup \{0\}$. Let $L(\varepsilon)$, $0 < \varepsilon \le \varepsilon_0$, be a positive decreasing function, $L(\varepsilon_0) \ge 1$. Let B'_k be the subset of vertices of T_3 defined by*

$$B'_k = \{v \in V_{T_3} : \delta(v, v_0) = \delta(v, c) + k\},$$

and let B_k be the subset of B'_k given by

$$B_k = \{v \in B'_k : \delta(v, c) \le L(\varepsilon_{k+1})\}.$$

Then the subtree T of T_3, determined by the vertex set

$$V_T = \bigcup_{k=0}^{\infty} B_k,$$

satisfies the property that for every $\varepsilon \in (0, \varepsilon_0]$, and every domain D in T with $v_0 \in D$, we have

$$\operatorname{mod}_T(\{v_0\}, \partial D) < \varepsilon \quad \Rightarrow \quad |D| > L(\varepsilon). \tag{1}$$

Moreover, if

$$2^{[L(\varepsilon_1)]} + \cdots + 2^{[L(\varepsilon_k)]} \le C 2^{[L(\varepsilon_{k+1})]}, \quad k = 1, 2, \ldots, \tag{2}$$

where C is a positive constant, then $S(T)$ is parabolic.

Proof. It follows from the definition that B'_k, $k = 0, 1, 2, \ldots$, are disjoint, $\bigcup_{k=0}^{\infty} B'_k = V_{T_3}$, and every chain in $\mathcal{C}_k$ has all but finitely many of its vertices in $\bigcup_{l \ge k} B'_l$.

Suppose that $\varepsilon \in (0, \varepsilon_0]$, and let D be a domain in T with $v_0 \in D$, and such that $\operatorname{mod}_T(\{v_0\}, \partial D) < \varepsilon$. There exists $k \in \mathbb{N} \cup \{0\}$ such that $\varepsilon_{k+1} < \varepsilon \le \varepsilon_k$. Assume for contradiction that $|D| \le L(\varepsilon)$. Since L is decreasing, $|D| \le L(\varepsilon_{k+1})$, and therefore every chain in $\mathcal{C}_k$ contains a subchain in T that connects $\{v_0\}$ to $\partial_T D$, the boundary of D in T. Indeed, D can also be considered as a domain in T_3, and it cannot contain vertices of B'_l, $l \ge k$, that are more than distance $[L(\varepsilon_{k+1})] - 1$ away from c because $|D| \le L(\varepsilon_{k+1})$. Thus every boundary vertex of $D \subset T_3$ contained in $\bigcup_{l \ge k} B'_l$ is a boundary vertex of $D \subset T$. Since $v_0 \in D$, every chain c' in $\mathcal{C}_k$ has a subchain connecting v_0 to some boundary vertex v' of D in T_3. Furthermore, c' contains a subchain connecting v_0 to $v' \in \partial_T D$. If not, let v'' be the last vertex of c' that belongs to the boundary of D in T_3. Since D is a domain, and hence is connected, and T_3 is a tree, v'' either belongs to c or is contained in $\bigcup_{l \ge k} B'_l$. But c is contained in T, and in the latter case v'' belongs to T as a boundary vertex of

$D \subset T_3$ contained in $\bigcup_{l \geq k} B'_l$. The desired subchain is obtained by removing edges of c' that connect vertices outside of V_T.

Now, we have

$$\varepsilon_k = \operatorname{mod}_{T_3} \mathcal{C}_k \leq \operatorname{mod}_T(\{v_0\}, \partial D) < \varepsilon.$$

This last estimate contradicts our understanding that $\varepsilon \leq \varepsilon_k$, and proves (1).

It remains to prove that under condition (2), $S(T)$ is parabolic. The tree T has an axis of symmetry passing through v_0 so that under the symmetry transformation the vertex v_k is mapped to v_{-k} and *vice versa*, and each B_k as well as the chain c are invariant. One should think of this axis of symmetry as being orthogonal to c. Let $\sigma = \sigma(T)$ be the 1-skeleton of the square subdivision of $S(T)$ that was created using the α_k's. The graph σ has also an axis of symmetry, denoted a, induced by the axis of symmetry of T. We claim that σ is parabolic. For that purpose, we exhibit a sequence of nested annuli (A_k) and verify that $\sum 1/\operatorname{mod} A_k = \infty$.

For each $k = 1, 2, \ldots$, we consider an annulus A_k in σ obtained as follows. The vertices of T separate those of σ into two groups, which we call V_+ and V_-. The sets V_+ and V_- form the sets of vertices of the upper and lower square grids $\{(m,n) : m \in \mathbb{Z}, n \in \mathbb{N}\}$ and $\{(m,n) : m \in \mathbb{Z}, -n \in \mathbb{N}\}$, respectively, so that for each of these sets the vertices with coordinates $(0,n)$ are located on the symmetry axis a. Each A_k consists of the vertices of the set $B_k \subset V_T$ defined above, vertices (m,n) in V_+ such that $\max\{|m|,|n|\} = k$, and vertices (m,n) in V_- such that $a_k \leq \max\{|m|,|n|\} \leq b_k$, where a_k and b_k are chosen as follows. The number a_k is the least one such that the vertex $(a_k, -1)$ of V_- is connected by an edge to v_k, and b_k is the largest number such that $(b_k, -1)$ is connected by an edge to v_k. A direct calculation gives

$$a_k = 2^{[L(\varepsilon_1)]} + 2\big(2^{[L(\varepsilon_2)]} + \cdots + 2^{[L(\varepsilon_k)]}\big) - k + 1,$$
$$b_k = 2^{[L(\varepsilon_1)]} + 2\big(2^{[L(\varepsilon_2)]} + \cdots + 2^{[L(\varepsilon_{k+1})]}\big) - k - 1.$$

Indeed, for each $l > 0$, the number of vertices of B_l lying to one side of the axis of symmetry a is $2^{[L(\varepsilon_{l+1})]}$, and the total number of vertices v of V_- to one side of a, such that v is connected to a vertex in B_l, is $2^{[L(\varepsilon_{l+1})]+1} - 1$. Adding the latter terms for $l = 1, 2, \ldots, k-1$ and for $l = 1, 2, \ldots, k$ together, each along with $2^{[L(\varepsilon_1)]}$, contributed by B_0, we obtain the quantities a_k and $b_k + 1$, respectively.

Now, we assign mass 1 to all vertices in $A_k \cap V_+$, mass $1/2^{l-1}$ to vertices v in B_k such that $\delta(v,c) = l,\ l = 1, 2, \ldots, [L(\varepsilon_{k+1})]$, and mass $1/2^{[L(\varepsilon_{k+1})]-1}$ to the vertices in $A_k \cap V_-$. This is an admissible mass distribution for the family of chains that connect the two components of $V_\sigma \setminus A_k$. Indeed, if a chain contains a vertex in $A_k \cap V_+$, we are done. If a chain only contains vertices of

$A_k \cap V_-$, then its weighted length is at least

$$\frac{b_k - a_k}{2^{[L(\varepsilon_{k+1})]-1}} = 4\left(1 - \frac{1}{2^{[L(\varepsilon_{k+1})]}}\right) \geq 1,$$

since we assumed that $L(\varepsilon_0) \geq 1$, and L is decreasing. A chain that contains only vertices of B_k has weighted length at least 1, because the subgraph of σ determined by the vertex set B_k is a tree, and hence such a chain has to contain the vertex v_k. The remaining case is when a chain γ contains vertices of $A_k \cap V_-$ as well as vertices in B_k. It is easy to see that then there is a chain that contains only vertices of $A_k \cap V_-$, and whose weighted length is comparable to that of γ, with absolute constants. Such a chain is obtained by replacing each vertex v of γ that belongs to B_k by a chain of vertices in $A_k \cap V_-$ of the form $(m, -1)$, so that the first and the last vertices of this chain are connected by edges in σ to v. Multiplying the mass distribution by an appropriate constant produces an admissible mass distribution.

The mass bound is

$$\begin{aligned}
&\lesssim k + \sum_{l=1}^{[L(\varepsilon_{k+1})]} \frac{2^l}{2^{2(l-1)}} + \frac{(2b_k)^2 - (2a_k)^2}{2^{2([L(\varepsilon_{k+1})]-1)}} \\
&\lesssim k + 1 + \left(1 + 2\frac{2^{[L(\varepsilon_1)]} + \cdots + 2^{[L(\varepsilon_k)]}}{2^{[L(\varepsilon_{k+1})]}}\right) \lesssim k, \quad k = 1, 2, \ldots.
\end{aligned}$$

Since $\sum^\infty 1/k = \infty$, we conclude that σ is parabolic. □

9. Comparison of moduli

The results of this section are essentially contained in [1], Section 8.

A pathwise connected metric measure space (X, d, μ) is an *n-Loewner space* if

$$\inf\{\operatorname{Mod}_n(E, F) : \Delta(E, F) \leq t\}$$

is a positive function for all $t > 0$, where $\operatorname{Mod}_n(E, F)$ denotes the n-modulus of a curve family connecting two disjoint continua E and F in X, and

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}$$

is called the *relative distance* between E and F. Loewner spaces were introduced in [11], see also [10].

Recall that $\sigma = \sigma(T)$ is the 1-skeleton of the square subdivision of $S(T)$. Let $\mathcal{U} = \{U_v : v \in V_\sigma\}$ be an open cover of $S(T)$, where U_v is the interior of the union of all squares in σ that have a vertex at $v \in V_\sigma$. If $J > 0$, we define the *J-star* of $v \in V_\sigma$ as

$$St_J(v) = \bigcup \{U_u : u \in V_\sigma, \delta(u, v) < J\}.$$

Note that $St_1(v) = U_v$. Since T is a tree, it is easy to see that $St_J(v)$ is an open, connected, and simply connected subset of $S(T)$. For a set A in $S(T)$ we denote by V_A the set of vertices v such that $U_v \cap A \neq \emptyset$.

LEMMA 2. *Assume that the valence k of T is finite. Let v be a vertex of σ, and ρ be an arbitrary Borel measurable non-negative function on $St_2(v)$. If Y_1, $Y_2 \subset S(T)$ are continua with $Y_i \cap U_v \neq \emptyset$, and $\operatorname{diam}(Y_i) \geq c_0 > 0$, $i = 1, 2$, then there is a rectifiable curve η in $St_2(v)$ connecting Y_1 and Y_2, such that*

$$\int_\eta \rho\, ds \leq C_0 \left(\int_{St_2(v)} \rho^2\, d\mu \right)^{1/2},$$

where $C_0 > 0$ depends only on c_0 and k.

Proof. The result follows from the observation that there are only finitely many, depending on k, different possibilities for $St_2(v)$ that can occur, and from Theorem 6.13 in [11], which implies that $St_2(v)$ is a 2-Loewner space. Indeed, the Loewner property gives that the conformal modulus $\operatorname{Mod}(Y_1, Y_2) \geq c > 0$, where c depends on c_0 and k only. This means that for every Borel measurable non-negative function ρ on $St_2(v)$ we have

$$\int_{St_2(v)} \rho^2\, d\mu \geq c \inf_\gamma \left(\int_\gamma \rho\, ds \right)^2,$$

where the infimum is taken over all curves γ in $St_2(v)$ that connect Y_1 and Y_2. Thus, for every $\varepsilon > 0$ there exists a rectifiable curve $\eta \subset St_2(v)$ connecting Y_1 and Y_2 such that

$$\left(\int_\eta \rho\, ds \right)^2 \leq \frac{1}{c} \int_{St_2(v)} \rho^2\, d\mu + \varepsilon.$$

Choosing $\varepsilon = \frac{1}{c} \int_{St_2(v)} \rho^2\, d\mu$ completes the proof in the case when ρ is not zero almost everywhere on $St_2(v)$. The latter case is trivial. □

LEMMA 3. *If T is an infinite embedded tree of valence k, then there exists a constant $C_1 \geq 1$, depending only on k, such that if $A, B \subset S(T)$ are two continua not contained in any set $St_2(v)$ for v a vertex of σ, then*

$$\operatorname{mod}_\sigma(V_A, V_B) \leq C_1 \operatorname{Mod}(A, B). \tag{3}$$

Proof. Let $\rho : S(T) \to [0, \infty]$ be an admissible Borel function for the pair (A, B), i.e.,

$$\int_\gamma \rho\, ds \geq 1,$$

for every rectifiable curve γ that connects A and B. We consider the mass distribution on σ defined by

$$m(v) = \left(\int_{St_2(v)} \rho^2\, d\mu \right)^{1/2}.$$

To prove (3) we need to establish a mass bound and verify admissibility. The mass bound is

$$\sum_{v\in V} m(v)^2 \le \sum_{v\in V}\Bigg(\sum_{u\,:\;\delta(u,v)<2}\int_{U_u}\rho^2\,d\mu\Bigg)$$
$$\lesssim \sum_{v\in V}\int_{U_v}\rho^2\,d\mu \lesssim \int_{S(T)}\rho^2\,d\mu,$$

where the constants understood depend only on k.

To show admissibility, we let $v_1, v_2, \ldots, v_k$ be vertices of a chain in σ that connect V_A and V_B. Then $U_{v_1}\cap A \ne \emptyset$, $U_{v_k}\cap B\ne\emptyset$, and $U_{v_{i-1}}\cap U_{v_i}\ne\emptyset$. We set $\lambda_1 = A$, $\lambda_{k+1}=B$, and for $i=2,\ldots,k$, let λ_i be a square in the square subdivision σ with two of the vertices being v_{i-1} and v_i. Then for $i=2,\ldots,k$, we have $\lambda_i\in U_{v_{i-1}}\cap U_{v_i}$, and $\operatorname{diam}\lambda_i=\sqrt{2}$. Also, since A and B are not contained in any $St_2(v)$, there exists an absolute constant $c_0>0$ such that $\operatorname{diam} A\ge c_0$ and $\operatorname{diam} B\ge c_0$. Using Lemma 2 we can inductively find rectifiable curves $\eta_1,\ldots,\eta_k$, satisfying the condition

$$\int_{\eta_i}\rho\,ds \le C_0 m(v_i),$$

and such that η_i connects $\lambda_1\cup\eta_1\cup\cdots\cup\eta_{i-1}$ and λ_{i+1}. The constant C_0 depends only on c_0 and k. The union $\eta_1\cup\cdots\cup\eta_k$ contains a rectifiable curve η connecting A and B, and having the property

$$1\le\int_\eta \rho\,ds\le C_0\sum_{i=1}^k m(v_i).$$

Thus $C_0 m$ is an admissible mass distribution for the pair (V_A, V_B), and the proof is complete. □

10. Proof of Theorem 2

Let $M(r)$, $r\ge 0$, be an arbitrary $\mathbb{R}$-valued function, and $L(\varepsilon)$ be a function that satisfies the conditions of Lemma 1, and such that $L(4\pi C_1/\log r)\ge M(r)$, where C_1 is the constant from Lemma 3 when $k=3$. Let T be the subtree of T_3 given by Lemma 1. Then $S(T)$ is parabolic, and let g be a uniformizing map from $\mathbb{C}$ to $S(T)$. Let $A_{r'}$ and B_r be the images under g of circles $\mathcal{C}_{r'}$ and $\mathcal{C}_r$ centered at 0 of radii r' and r, respectively, $1<r'<r$. We choose r' such that $A_{r'}$ is not contained in any set $St_2(v)$, $v\in V_\sigma$. Using Lemma 3 and the conformal invariance of Mod, we obtain that

$$\operatorname{mod}_\sigma(V_{A_{r'}}, V_{B_r}) \le C_1 \operatorname{Mod}(\mathcal{C}_{r'},\mathcal{C}_r) < \frac{4\pi C_1}{\log r},\quad r\ge r_0=(r')^3.$$

Since T is a subgraph of σ, from monotonicity we have

$$\mathrm{mod}_T(V_{A_{r'}}, V_{B_r}) < \frac{4\pi C_1}{\log r}, \quad r \geq r_0.$$

If D is the domain in T which is the connected component of $V_T \setminus V_{B_r}$ containing v_0, then $\mathrm{mod}_T(\{v_0\}, \partial D) < 4\pi C_1/\log r$. Therefore, by Lemma 1, $|D| > L(4\pi C_1/\log r) \geq M(r)$. The proof is complete.

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References

[1] M. Bonk and B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres*, Invent. Math. **150** (2002), 127–183. MR 1930885

[2] J. W. Cannon, *The combinatorial Riemann mapping theorem*, Acta Math. **173** (1994), 155–234. MR 1301392

[3] P. G. Doyle, *Random walk on the Speiser graph of a Riemann surface*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), 371–377. MR 0752802

[4] P. G. Doyle and J. L. Snell, *Random walks and electric networks*, Math. Assoc. Amer., Washington, DC, 1984. MR 0920811

[5] R. J. Duffin, *The extremal length of a network,* J. Math. Anal. Appl. **5** (1962), 200–215. MR 0143468

[6] A. Eremenko, *Transcendental meromorphic functions with three singular values*, Illinois J. Math. **48** (2004), 701–709. MR 2085435

[7] A. A. Goldberg and I. V. Ostrovskii, *Distribution of values of meromorphic functions*, Nauka, Moscow, 1970. (In Russian.) MR 0280720

[8] M. Gromov, *Hyperbolic manifolds, groups and actions*, Ann. Math. Stud., vol. 97, Princeton Univ. Press, 1981, pp. 183–213. MR 0624814

[9] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964. MR 0164038

[10] J. Heinonen, *Lectures on analysis on metric spaces*, Springer-Verlag, New York, 2001. MR 1800917

[11] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61. MR 1654771

[12] M. Kanai, *Rough isometries, and combinatorial approximations of geometries of noncompact Riemannian manifolds*, J. Math. Soc. Japan **37** (1985), 391–413. MR 0792983

[13] M. Kanai, *Rough isometries and the parabolicity of Riemannian manifolds*, J. Math. Soc. Japan **38** (1986), 227–238. MR 0833199

[14] J. K. Langley, *On differential polynomials, fixpoints and critical values of meromorphic functions*, Result. Math. **35** (1999), 284–309. MR 1694909

[15] J. K. Langley, *Critical values of slowly growing meromorphic functions*, Comput. Methods Funct. Theory **2** (2002), 537–547. MR 2038137

[16] S. Merenkov, *Determining biholomorphic type of a manifold using combinatorial and algebraic structures,* Ph.D. thesis, 2003.

[17] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer-Verlag, Berlin, 1936 (and also 1974). Translated as *Analytic functions*, Springer-Verlag, 1970. MR 0344426

[18] O. Schramm, *Square tilings with prescribed combinatorics*, Israel J. Math. **84** (1993), 97–118. MR 1244661

[19] A. Speiser, *Über Riemannsche Flächen,* Comment. Math. Helv. **2** (1930), 254–293.

[20] A. Speiser, *Probleme aus dem Gebiet der ganzen transzendenten Funktionen*, Comment. Math. Helv. **1** (1929), 289–312. MR 1509398

[21] S. Stoïlow, *Leçons sur les Principes Topologiques de la Théorie des Fonctions Analytiques,* Paris, 1956.

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