WEIGHTED HLS INEQUALITIES FOR RADIAL FUNCTIONS AND STRICHARTZ ESTIMATES FOR WAVE AND SCHRÖDINGER EQUATIONS

KUNIO HIDANO AND YUKI KUROKAWA

ABSTRACT. This paper is concerned with derivation of the global or local in time Strichartz estimates for radially symmetric solutions of the free wave equation from some Morawetz-type estimates via weighted Hardy–Littlewood–Sobolev (HLS) inequalities. In the same way, we also derive the weighted end-point Strichartz estimates with gain of derivatives for radially symmetric solutions of the free Schrödinger equation.

The proof of the weighted HLS inequality for radially symmetric functions involves an application of the weighted inequality due to Stein and Weiss and the Hardy–Littlewood maximal inequality in the weighted Lebesgue space due to Muckenhoupt. Under radial symmetry, we get significant gains over the usual HLS inequality and Strichartz estimate.

1. Introduction

In this paper, we discuss the roles of the weighted Hardy–Littlewood–Sobolev (HLS, for short) inequalities for radially symmetric functions in the derivation of the Strichartz estimates for the free wave equation and the free Schrödinger equation.

In the first half of this paper, we prove the weighted HLS inequality for radially symmetric functions on $\mathbb{R}^n$ ($n \geq 2$) (see (2.2) below). The proof proceeds by writing out the Riesz potentials in polar coordinates, integrating out the angular coordinates, and reducing the argument to the one-dimensional setting. We then make use of the weighted inequality due to Stein and Weiss [33] and the Hardy–Littlewood maximal inequality in the weighted

Lebesgue space due to Muckenhoupt [26]. Naturally, the weighted HLS inequality thereby obtained has some similarity with the one-dimensional part of the weighted inequalities due to Stein and Weiss, except that the norm on the right-hand side of (2.2) involves such a homogeneous weight function as $|x|^{-(n-1)((1/p)-(1/q))}$. At the cost of the presence of such a singular weight function on the right-hand side, the weighted radial HLS inequality (2.2) holds even for the Riesz potential whose kernel has a rather singular form $|x|^{-n+\mu}$ with $\mu = \alpha + \beta + (1/p) - (1/q)$. (Compare it with the kernel $|x|^{-n+\tilde{\mu}}$, $\tilde{\mu} = \alpha + \beta + (n/p) - (n/q)$, of the Riesz potential in the usual weighted HLS inequality (2.15) below.)

In the second half of this paper, we discuss how the weighted radial HLS inequality is used to prove the global (in space and time) or local (in time) Strichartz estimate for radially symmetric solutions. It is well known that the range of admissible exponents in the global Strichartz estimate for the free wave equation can be significantly improved in the radial setting. (See Theorem 6.6.2 of Sogge [31], Proposition 4 of Klainerman and Machedon [19], Theorem 1.3 of Sterbenz [34], and Theorem 4 of Fang and Wang [5].) Adapting an argument of Vilela [39], we explain how to derive the global radial Strichartz estimate in space dimension $n \geq 3$ from the generalized Morawetz estimate (see (3.7) below) via the weighted radial HLS inequality. Our analysis therefore yields another proof of Theorem 1.3 of Sterbenz [34]. As for the local-in-time radial Strichartz estimate of the free wave equation, we extend the space–time L^q estimate due to Sogge in space dimension $n = 3$ into the space–time mixed-norm estimate in space dimension $n \geq 2$ (see (5.3) below). For that purpose, we exploit the local-in-time space–time L^2-estimate (5.7) below by combining it with the weighted radial HLS inequality. Such a method does not end with applications to the free wave equation. Combined with the global (in space and time) estimate of the local smoothing property (6.4) below, the weighted radial HLS inequality is useful in proving the weighted end-point Strichartz estimate for radially symmetric solutions to the free Schödinger equation (see (6.3) below). In the radial setting, we observe a significant gain of regularity over the end-point estimate due to Keel and Tao [18].

The authors have received a couple of very instructive suggestions from the referee. One is concerned with the flexibility in our approach. The approach to proving the Strichartz estimates used here does not rely upon explicit representations or parametrices for the solution. Therefore, it can provide Strichartz-type estimates for the large family of equations with defocusing radial potentials. Another is concerned with the weighted versions of the inhomogeneous Strichartz estimates. As was first observed by Kato [16] and has been explored by Oberlin [27], Harmse [8], Foschi [6], and Vilela [40], the (unweighted) inhomogeneous Strichartz estimates are known to hold for the larger range of exponent pairs. We now enjoy the approach based upon

the celebrated lemma of Christ and Kiselev [4], and the referee has kindly suggested that radial weighted analogs thereby obtained may turn out to be very useful for certain nonlinear problems. Indeed, by virtue of the Christ–Kiselev lemma one of the present authors has obtained some radial weighted analogs in order to study global existence of small solutions to nonlinear wave equations [12] and nonlinear Schrödinger equations [11] with radially symmetric data of scale-critical regularity.

We conclude this section by explaining the notation. By $L^p(\mathbb{R}^n, \omega(x)\,dx)$, we mean the Lebesgue space of all μ-measurable functions $(d\mu(x) = \omega(x)\,dx)$ on $\mathbb{R}^n$. We simply denote $L^p(\mathbb{R}^n, dx)$ by $L^p(\mathbb{R}^n)$. The mixed norm $\|u\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))}$ for functions u on $\mathbb{R} \times \mathbb{R}^n$ is defined as

$$\|u\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^p \, dx \right)^{q/p} dt \right)^{1/q}$$

with an obvious modification for $q = \infty$ or $p = \infty$. By p', we denote the exponent conjugate to p, that is, $(1/p) + (1/p') = 1$. The operator $|D_x|^s$ ($s \in \mathbb{R}$) is defined by using the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$, as usual. We denote by $\dot{H}_2^s(\mathbb{R}^n)$ the homogeneous Sobolev space $|D_x|^{-s} L^2(\mathbb{R}^n)$. The free evolution operators for the wave equation and the Schrödinger equation are defined as

$$(1.1) \qquad (W\varphi)(t,x) = W(t)\varphi(x) = \mathcal{F}^{-1} e^{it|\xi|} \mathcal{F}\varphi,$$

$$(1.2) \qquad (S\varphi)(t,x) = S(t)\varphi(x) = \mathcal{F}^{-1} e^{it|\xi|^2} \mathcal{F}\varphi,$$

respectively.

This paper is organized as follows. In the next section, we prove the weighted HLS inequality for radial functions. Section 3 is devoted to the proof of the global-in-time Strichartz estimate for radial solutions to the free wave equation. In Section 4, we draw our attention to the limiting case of the estimates obtained in Section 3. An adaptation of observations due to Agemi [1], Rammaha [29], and Takamura [37] shows the failure of such critical estimates. In Section 5, we are concerned with the local-in-time Strichartz estimate for radial solutions to the free wave equation. In the final section, we revisit the problem of deriving the end-point Strichartz estimate for radial solutions to the free Schrödinger equation from the global (in space) estimate of local smoothing property. Using the weighted radial HLS inequality, we show the weighted end-point Strichartz estimate with gain of derivatives for radial free solutions.

2. Weighted HLS inequality

We let

$$(2.1) \qquad (T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma} \, dy, \quad 0 < \gamma < n.$$

The purpose of this section is to prove the weighted Hardy–Littlewood–Sobolev (HLS) inequalities for radially symmetric functions. We show the following:

THEOREM 2.1. *Suppose $n \geq 2$. Let p, q, α and β satisfy $1 < p < q < \infty$, $\alpha < 1/p'$, $\beta < 1/q$ and $\alpha + \beta \geq 0$. Set $\mu = \alpha + \beta + (1/p) - (1/q)$. There exists a constant C depending only on n, p, q, α and β, and the inequality*

$$(2.2) \qquad \||x|^{-\beta} T_{n-\mu} v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^{\alpha - (n-1)(1/p - 1/q)} v\|_{L^p(\mathbb{R}^n)}$$

holds for radially symmetric $v \in L^p(\mathbb{R}^n, |x|^{p(\alpha - (n-1)(1/p - 1/q))} \, dx)$.

REMARK. Obviously, the number μ in Theorem 2.1 is strictly positive. Moreover, we should note that μ is strictly smaller than one. Indeed, by the assumption $\alpha < 1/p'$, $\beta < 1/q$, we see $\mu < 1/p' + 1/q + 1/p - 1/q = 1$.

Proof of Theorem 2.1. We start with the well-known formula:

$$(2.3) \qquad (T_{n-s} v)(x) = \frac{\omega_{n-1}}{r} \int_0^\infty \lambda^{n-2} w(\lambda) \, d\lambda$$
$$\times \int_{|r-\lambda|}^{r+\lambda} \rho^{-n+s+1} h(\rho, \lambda; r)^{(n-3)/2} \, d\rho$$

$(0 < s < n)$ for radially symmetric function $v(x) = w(r)$. Here, and in what follows, we use the notation $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$,

$$(2.4) \qquad h(\rho, \lambda; r) = 1 - \left(\frac{r^2 + \lambda^2 - \rho^2}{2\lambda r} \right)^2,$$

$\omega_1 = 2$, and ω_n $(n = 2, 3, \dots)$ is the area of $S^{n-1} = \{x \in \mathbb{R}^n \, | \, |x| = 1\}$. For the proof of (2.3) consult, e.g., John [15] on page 8. Since the function $h(\rho, \lambda; r)^{(n-3)/2}$ causes another singularity in the case of $n = 2$, let us first study the case $n \geq 3$. By virtue of the following proposition, our argument will be reduced to the special case of the weighted estimate of Stein and Weiss. (See Lemma 2.3 below.)

PROPOSITION 2.2. *Suppose $n \geq 3$, $1 < q < \infty$ and $0 < s < 1$. The inequality*

$$(2.5) \qquad r^{(n-1)/q} (T_{n-s} v)(x) \leq C \int_0^\infty \frac{\lambda^{(n-1)/q} w(\lambda)}{|r - \lambda|^{1-s}} \, d\lambda \quad (x \in \mathbb{R}^n)$$

holds for radially symmetric, nonnegative function $v(x) = w(r)$.

Proof. Set $I_i = I_i(r)$ $(i = 1, 2)$ as

$$(2.6) \quad \frac{1}{r} \int_0^{r/2} \lambda^{n-2} w(\lambda) \, d\lambda \int_{r-\lambda}^{r+\lambda} \rho^{-n+s+1} h(\rho, \lambda; r)^{(n-3)/2} \, d\rho$$

$$+ \frac{1}{r} \int_{r/2}^{\infty} \lambda^{n-2} w(\lambda) \, d\lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-n+s+1} h(\rho, \lambda; r)^{(n-3)/2} \, d\rho$$

$$=: I_1(r) + I_2(r).$$

For the estimate of I_1, we note that $-1 \le (r^2 + \lambda^2 - \rho^2)/(2\lambda r) \le 1$ for $|r - \lambda| \le \rho \le r + \lambda$, which implies $h(\rho, \lambda; r)^{(n-3)/2} \le 1$ by virtue of the assumption $n \ge 3$. We therefore obtain

$$(2.7) \quad I_1(r) \le \frac{1}{r} \int_0^{r/2} \lambda^{n-2} w(\lambda) \, d\lambda \int_{r-\lambda}^{r+\lambda} \rho^{-n+s+1} \, d\rho$$

$$\le \frac{C}{r^{n-s}} \int_0^{r/2} \lambda^{n-1} w(\lambda) \, d\lambda.$$

The last inequality is due to the fact that for $0 < s < 1$ and $0 \le \lambda \le r/2$

$$(2.8) \quad \int_{r-\lambda}^{r+\lambda} \rho^{-n+s+1} \, d\rho \le 2\lambda(r - \lambda)^{-n+s+1} \le C\lambda r^{-n+s+1}.$$

For the estimate of I_2, let us first observe $h(\rho, \lambda; r) = (\rho^2/\lambda^2) h(\lambda, \rho; r)$. Indeed, we see that

$$(2.9) \quad h(\rho, \lambda; r) = \frac{4\lambda^2 r^2 - (r^2 + \lambda^2 - \rho^2)^2}{4\lambda^2 r^2} = \frac{\{\rho^2 - (r - \lambda)^2\}\{(r + \lambda)^2 - \rho^2\}}{4\lambda^2 r^2}$$

$$= \frac{(\rho + r - \lambda)(\rho - r + \lambda)(r + \lambda + \rho)(r + \lambda - \rho)}{4\lambda^2 r^2}$$

$$= \frac{\{(\rho + r)^2 - \lambda^2\}\{\lambda^2 - (\rho - r)^2\}}{4\lambda^2 r^2}$$

$$= \frac{\rho^2}{\lambda^2} \left\{ 1 - \left(\frac{r^2 + \rho^2 - \lambda^2}{2\rho r} \right)^2 \right\}$$

$$= \frac{\rho^2}{\lambda^2} h(\lambda, \rho; r),$$

as desired. Since $-1 \le (r^2 + \rho^2 - \lambda^2)/(2\rho r) \le 1$ for $|r - \lambda| \le \rho \le r + \lambda$, we have $h(\rho, \lambda; r) \le \rho^2/\lambda^2$ and, therefore,

$$(2.10) \quad I_2(r) \le \frac{1}{r} \int_{r/2}^{\infty} \lambda^{n-2} w(\lambda) \, d\lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-n+s+1} \left(\frac{\rho^2}{\lambda^2} \right)^{(n-3)/2} \, d\rho$$

$$= \frac{1}{r} \int_{r/2}^{\infty} \lambda w(\lambda) \, d\lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-2+s} \, d\rho.$$

Keeping the assumption $0 < s < 1$ in mind, we proceed as

$$(2.11) \quad \int_{|r-\lambda|}^{r+\lambda} \rho^{-2+s} \, d\rho = \frac{1}{(1-s)|r-\lambda|^{1-s}} \left\{ 1 - \left(\frac{|r-\lambda|}{r+\lambda} \right)^{1-s} \right\}$$

$$\leq \frac{1}{(1-s)|r-\lambda|^{1-s}} \left(1 - \frac{|r-\lambda|}{r+\lambda} \right)$$

$$= \frac{2\min\{\lambda, r\}}{(1-s)|r-\lambda|^{1-s}(r+\lambda)}.$$

Combining (2.10) with (2.11), we get

$$(2.12) \quad I_2(r) \leq \frac{C}{r} \int_{r/2}^{\infty} \frac{r}{|r-\lambda|^{1-s}(r+\lambda)} \lambda w(\lambda) \, d\lambda \leq C \int_{r/2}^{\infty} \frac{w(\lambda)}{|r-\lambda|^{1-s}} \, d\lambda.$$

Therefore, we have obtained by (2.3), (2.6), (2.7), and (2.12)

$$(2.13) \quad (T_{n-s}v)(x) \leq \frac{C}{r^{n-s}} \int_0^{r/2} \lambda^{n-1} w(\lambda) \, d\lambda + C \int_{r/2}^{\infty} \frac{w(\lambda)}{|r-\lambda|^{1-s}} \, d\lambda.$$

We are in a position to complete the proof of (2.5). It follows from (2.13) that

$$(2.14) \quad r^{(n-1)/q}(T_{n-s}v)(x)$$

$$\leq C r^{((n-1)/q)-n+s} \int_0^{r/2} \lambda^{n-1} w(\lambda) \, d\lambda$$

$$+ C r^{(n-1)/q} \int_{r/2}^{\infty} \frac{w(\lambda)}{|r-\lambda|^{1-s}} \, d\lambda$$

$$\leq C \int_0^{r/2} \frac{1}{r^{1-s}} \left(\frac{\lambda}{r} \right)^{(n-1)(1-(1/q))} \lambda^{(n-1)/q} w(\lambda) \, d\lambda$$

$$+ C \int_{r/2}^{\infty} \frac{\lambda^{(n-1)/q} w(\lambda)}{|r-\lambda|^{1-s}} \, d\lambda \leq C \int_0^{\infty} \frac{\lambda^{(n-1)/q} w(\lambda)}{|r-\lambda|^{1-s}} \, d\lambda$$

as desired. The proof of Proposition 2.2 has been finished. $\qquad\square$

Once we have obtained the pointwise (in x) estimate (2.5), the three or higher dimensional part of Theorem 2.1 is an immediate consequence of the following lemma due to Stein and Weiss [33].

LEMMA 2.3. *Assume $n \geq 1$, $0 < \gamma < n$, $1 < p < \infty$, $\alpha < n/p'$, $\beta < n/q$, $\alpha + \beta \geq 0$, and $1/q = (1/p) + ((\gamma + \alpha + \beta)/n) - 1$. If $p \leq q < \infty$, then the inequality*

$$(2.15) \quad \||x|^{-\beta} T_\gamma v\|_{L^q(\mathbb{R}^n)} \leq C \||x|^\alpha v\|_{L^p(\mathbb{R}^n)}$$

holds for any $v \in L^p(\mathbb{R}^n, |x|^{p\alpha} \, dx)$.

It is easily seen that the three or higher dimensional part of Theorem 2.1 is a consequence of (2.5) with $s = \mu$ and (2.15) with $n = 1$. The proof of Theorem 2.1 has been finished for $n \geq 3$.

To show Theorem 2.1 in the case of $n = 2$, we set $J_i = J_i(r)$ ($i = 1, 2, 3$) for radially symmetric function $v(x) = w(r)$:

$$(2.16) \qquad \frac{1}{r} \int_0^{r/2} w(\lambda)\, d\lambda \int_{r-\lambda}^{r+\lambda} \rho^{-1+s} h(\rho, \lambda; r)^{-1/2}\, d\rho$$

$$+ \frac{1}{r} \int_{r/2}^{2r} w(\lambda)\, d\lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-1+s} h(\rho, \lambda; r)^{-1/2}\, d\rho$$

$$+ \frac{1}{r} \int_{2r}^{\infty} w(\lambda)\, d\lambda \int_{\lambda-r}^{\lambda+r} \rho^{-1+s} h(\rho, \lambda; r)^{-1/2}\, d\rho$$

$$=: J_1(r) + J_2(r) + J_3(r).$$

It is possible to show the counterpart of Proposition 2.2 for J_1 and J_3. It is J_2 that we must handle quite differently from before. Let us begin with the proof of the following proposition.

PROPOSITION 2.4. *Suppose $1 < q < \infty$ and $0 < s < 1$. The inequality*

$$(2.17) \qquad r^{1/q} J_i(r) \leq C \int_0^{\infty} \frac{\lambda^{1/q} w(\lambda)}{|r - \lambda|^{1-s}}\, d\lambda \quad (i = 1, 3)$$

holds for nonnegative w.

Proof. We use the property of the beta function $B(\cdot, \cdot)$:

$$(2.18) \qquad \int_a^b \frac{2\rho}{\sqrt{\rho^2 - a^2}\sqrt{b^2 - \rho^2}}\, d\rho = B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$$

Observing

$$(2.19) \qquad \frac{1}{r} \int_{|r-\lambda|}^{r+\lambda} \rho^{-1+s} h(\rho, \lambda; r)^{-1/2}\, d\rho$$

$$= \lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-2+s} \frac{2\rho}{\sqrt{\rho^2 - (r-\lambda)^2}\sqrt{(r+\lambda)^2 - \rho^2}}\, d\rho$$

$$\leq \frac{\lambda}{|r - \lambda|^{2-s}} B\left(\frac{1}{2}, \frac{1}{2}\right),$$

we are led to

$$(2.20) \qquad r^{1/q} J_1(r) \leq C \int_0^{r/2} \left(\frac{r^{1/q} \lambda^{1-(1/q)}}{r - \lambda}\right) \frac{\lambda^{1/q} w(\lambda)}{(r - \lambda)^{1-s}}\, d\lambda$$

and

$$(2.21) \qquad r^{1/q} J_3(r) \leq C \int_{2r}^{\infty} \left(\frac{r^{1/q} \lambda^{1-(1/q)}}{\lambda - r}\right) \frac{\lambda^{1/q} w(\lambda)}{(\lambda - r)^{1-s}}\, d\lambda.$$

Since $r^{1/q}\lambda^{1-(1/q)}/(r-\lambda) \le C$ for $0 \le \lambda \le r/2$ and $r^{1/q}\lambda^{1-(1/q)}/(\lambda-r) \le C$ for $2r \le \lambda$, the inequality (2.17) is a consequence of (2.20) and (2.21). We have finished the proof of Proposition 2.4. □

It remains to show the bound for J_2.

PROPOSITION 2.5. *Let p, q, α, β, and μ be the same as in Theorem* 2.1. *The inequality*

$$(2.22) \quad \left\| r^{(1/q)-1} \int_{r/2}^{2r} w(\lambda)\,d\lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-1+\mu} h(\rho,\lambda;r)^{-1/2}\,d\rho \right\|_{L^q((0,\infty),r^{-q\beta}dr)}$$
$$\le C\|r^{1/q}w\|_{L^p((0,\infty),r^{p\alpha}dr)}$$

holds.

Proof. Without loss of generality, we may assume that w is nonnegative. Identifying the dual space of $L^q((0,\infty),r^{-q\beta}\,dr)$ with $L^{q'}((0,\infty),r^{q'\beta}\,dr)$ and reversing the order integration twice, we have

$$(2.23) \quad \left\| r^{(1/q)-1} \int_{r/2}^{2r} w(\lambda)\,d\lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-1+\mu} h(\rho,\lambda;r)^{-1/2}\,d\rho \right\|_{L^q((0,\infty),r^{-q\beta}dr)}$$
$$= \sup \int_0^\infty r^{(1/q)-1} g(r)\,dr \int_{r/2}^{2r} w(\lambda)\,d\lambda \int_{|r-\lambda|}^{r+\lambda} \rho^{-1+\mu} h(\rho,\lambda;r)^{-1/2}\,d\rho$$
$$= \sup \int_0^\infty w(\lambda)\,d\lambda \int_{\lambda/2}^{2\lambda} r^{(1/q)-1} g(r)\,dr \int_{|r-\lambda|}^{r+\lambda} \rho^{-1+\mu} h(\rho,\lambda;r)^{-1/2}\,d\rho$$
$$= \sup \Bigg(\int_0^\infty w(\lambda)\,d\lambda \int_0^{\lambda/2} \rho^{-1+\mu}\,d\rho$$
$$\times \int_{\lambda-\rho}^{\lambda+\rho} r^{(1/q)-1} h(\rho,\lambda;r)^{-1/2} g(r)\,dr$$
$$+ \int_0^\infty w(\lambda)\,d\lambda \int_{\lambda/2}^{\lambda} \rho^{-1+\mu}\,d\rho \int_{\lambda/2}^{\lambda+\rho} r^{(1/q)-1} h(\rho,\lambda;r)^{-1/2} g(r)\,dr$$
$$+ \int_0^\infty w(\lambda)\,d\lambda \int_{\lambda}^{3\lambda/2} \rho^{-1+\mu}\,d\rho \int_{\lambda/2}^{2\lambda} r^{(1/q)-1} h(\rho,\lambda;r)^{-1/2} g(r)\,dr$$
$$+ \int_0^\infty w(\lambda)\,d\lambda \int_{3\lambda/2}^{3\lambda} \rho^{-1+\mu}\,d\rho \int_{\rho-\lambda}^{2\lambda} r^{(1/q)-1} h(\rho,\lambda;r)^{-1/2} g(r)\,dr \Bigg)$$
$$=: \sup(L_1 + L_2 + L_3 + L_4).$$

Here, the supremum is taken over all nonnegative $g \in L^{q'}((0,\infty),r^{q'\beta}dr)$ with $\|g\|_{L^{q'}((0,\infty),r^{q'\beta}dr)} = 1$.

In what follows, we shall often use the identity

$$(2.24) \quad r^{-1} h(\rho,\lambda;r)^{-1/2} = \frac{2\lambda}{\sqrt{(\rho-r+\lambda)(\rho+r-\lambda)(r+\lambda-\rho)(r+\lambda+\rho)}}$$

as well as the inequality $2\lambda \leq r + \lambda + \rho \leq 6\lambda$ for $\lambda/2 \leq r \leq 2\lambda$, $|r - \lambda| \leq \rho \leq r + \lambda$. To begin with, we first estimate L_1. Observing $r + \lambda - \rho \leq (\lambda + \rho) + \lambda - \rho = 2\lambda$, $r + \lambda - \rho \geq (\lambda - \rho) + \lambda - \rho \geq \lambda$ for $\lambda - \rho \leq r \leq \lambda + \rho$ and $0 \leq \rho \leq \lambda/2$, we obtain for $0 \leq \rho \leq \lambda/2$

$$(2.25) \quad \int_{\lambda-\rho}^{\lambda+\rho} r^{(1/q)-1} h(\rho, \lambda; r)^{-1/2} g(r) \, dr$$

$$\leq C\lambda^{1/q} \int_{\lambda-\rho}^{\lambda+\rho} \frac{1}{\sqrt{(\rho - r + \lambda)(\rho + r - \lambda)}} g(r) \, dr$$

$$\leq C\lambda^{1/q} \left(\int_{\lambda-\rho}^{\lambda} \frac{1}{\sqrt{\rho(r - \lambda + \rho)}} g(r) \, dr \right.$$

$$\left. + \int_{\lambda}^{\lambda+\rho} \frac{1}{\sqrt{\rho(\lambda + \rho - r)}} g(r) \, dr \right)$$

$$\leq C\lambda^{1/q} \left(\frac{1}{2\rho} \int_{(\lambda-\rho)-\rho}^{(\lambda-\rho)+\rho} \left| \frac{\rho}{\eta - (\lambda - \rho)} \right|^{1/2} g^*(\eta) \, d\eta \right.$$

$$\left. + \frac{1}{2\rho} \int_{(\lambda+\rho)-\rho}^{(\lambda+\rho)+\rho} \left| \frac{\rho}{\lambda + \rho - \eta} \right|^{1/2} g^*(\eta) \, d\eta \right)$$

$$\leq C\lambda^{1/q} \left(\sup_{\sigma>0} \frac{1}{2\sigma} \int_{(\lambda-\rho)-\sigma}^{(\lambda-\rho)+\sigma} \left| \frac{\sigma}{\eta - (\lambda - \rho)} \right|^{1/2} g^*(\eta) \, d\eta \right.$$

$$\left. + \sup_{\sigma>0} \frac{1}{2\sigma} \int_{(\lambda+\rho)-\sigma}^{(\lambda+\rho)+\sigma} \left| \frac{\sigma}{\lambda + \rho - \eta} \right|^{1/2} g^*(\eta) \, d\eta \right)$$

$$= C\lambda^{1/q} (\mathcal{M}_{1/2} g^*)(\lambda - \rho) + C\lambda^{1/q} (\mathcal{M}_{1/2} g^*)(\lambda + \rho).$$

Here, we have set $g^*(\eta) = g(\eta)$ for $\eta \geq 0$, $g^*(\eta) = g(-\eta)$ for $\eta < 0$, and for $t \in \mathbb{R}$

$$(2.26) \quad (\mathcal{M}_{1/2} f)(t) = \sup_{\sigma>0} \frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} \left| \frac{\sigma}{\eta - t} \right|^{1/2} |f(\eta)| \, d\eta.$$

It follows from the observation of Lindblad and Sogge that the maximal function $\mathcal{M}_{1/2} f$, which is a singular variant of the Hardy–Littlewood maximal function

$$(2.27) \quad (\mathcal{M} f)(t) = \sup_{\sigma>0} \frac{1}{2\sigma} \int_{t-\sigma}^{t+\sigma} |f(\eta)| \, d\eta,$$

has the pointwise estimate

$$(2.28) \quad (\mathcal{M}_{1/2} f)(t) \leq C(\mathcal{M} f)(t)$$

(see page 1,062 of [23]). Combining (2.25) with (2.28), we see that L_1 has the bound such as

$$(2.29) \quad L_1 \leq C \int_0^\infty \lambda^{1/q} w(\lambda) \, d\lambda \int_0^{\lambda/2} \rho^{-1+\mu} (\mathcal{M} g^*)(\lambda - \rho) \, d\rho$$

$$+ C \int_0^\infty \lambda^{1/q} w(\lambda) \, d\lambda \int_0^{\lambda/2} \rho^{-1+\mu} (\mathcal{M} g^*)(\lambda + \rho) \, d\rho$$

$$\leq C \| \lambda^{1/q} w \|_{L^p((0,\infty), \lambda^{p\alpha} \, d\lambda)} \| T_{1-\mu} (\mathcal{M} g^*) \|_{L^{p'}((0,\infty), \lambda^{-p'\alpha} \, d\lambda)}$$

$$\leq C \| \lambda^{1/q} w \|_{L^p((0,\infty), \lambda^{p\alpha} \, d\lambda)} \| \mathcal{M} g^* \|_{L^{q'}(\mathbb{R}, |t|^{q'\beta} \, dt)}.$$

At the last inequality, we have used the one-dimensional part of Lemma 2.3. To finish the estimate of L_1 we need the following lemma.

LEMMA 2.6. *Suppose that $1 < p < \infty$ and $-1 < a < p - 1$. The operator $\mathcal{M}$ enjoys the boundedness*

$$(2.30) \qquad \qquad \| \mathcal{M} f \|_{L^p(\mathbb{R}, |t|^a \, dt)} \leq C \| f \|_{L^p(\mathbb{R}, |t|^a \, dt)}.$$

The original proof of (2.30) is due to Muckenhoupt [26]. See also Chapter 5 of Stein [32] for further references. Before we use Lemma 2.6 to bound $\| \mathcal{M} g^* \|_{L^{q'}(\mathbb{R}, |t|^{q'\beta} dt)}$, let us see that the condition $-1 < q'\beta < q' - 1$ is satisfied. The condition $q'\beta < q' - 1$ is equivalent to $\beta < 1/q$ which is supposed in Theorem 2.1. Moreover, to see that the condition $-1 < q'\beta$ is also satisfied, we note that the assumption $(1/q) - (1/p) - \beta + \mu = \alpha < 1/p'$ implies $\mu < (1/q') + \beta$. Since μ is positive, we finally find that $-1/q' < \beta$, as desired. We may therefore use Lemma 2.6 to proceed as

$$(2.31) \qquad L_1 \leq C \| \lambda^{1/q} w \|_{L^p((0,\infty), \lambda^{p\alpha} \, d\lambda)} \| g^* \|_{L^{q'}(\mathbb{R}, |t|^{q'\beta} dt)}$$

$$\leq C \| \lambda^{1/q} w \|_{L^p((0,\infty), \lambda^{p\alpha} \, d\lambda)} \| g \|_{L^{q'}((0,\infty), r^{q'\beta} \, dr)}.$$

The estimate of L_1 has been completed.

We next consider the estimate of L_2. Observing $r + \lambda - \rho \leq (\rho + \lambda) + \lambda - \rho = 2\lambda$, $r + \lambda - \rho \geq (\lambda/2) + \lambda - \lambda = \lambda/2$ for $\lambda/2 \leq r \leq \lambda + \rho$ and $\lambda/2 \leq \rho \leq \lambda$, we obtain for $\lambda/2 \leq \rho \leq \lambda$

$$(2.32) \qquad \int_{\lambda/2}^{\lambda+\rho} r^{(1/q)-1} h(\rho, \lambda; r)^{-1/2} g(r) \, dr$$

$$\leq C \lambda^{1/q} \int_{\lambda/2}^{\lambda+\rho} \frac{1}{\sqrt{(\rho - r + \lambda)(\rho + r - \lambda)}} g(r) \, dr$$

$$\leq C \lambda^{(1/q)-1} \left(\int_{\lambda/2}^{\lambda} \sqrt{\frac{\lambda}{r - (\lambda - \rho)}} g(r) \, dr \right.$$

$$\left. + \int_{\lambda}^{\lambda+\rho} \sqrt{\frac{\lambda}{(\lambda + \rho) - r}} g(r) \, dr \right)$$

$$\leq C\lambda^{1/q}\left(\frac{1}{2\rho}\int_{(\lambda-\rho)-\rho}^{(\lambda-\rho)+\rho}\left|\frac{\rho}{\eta-(\lambda-\rho)}\right|^{1/2}g^*(\eta)\,d\eta\right.$$

$$\left.+\frac{1}{2\rho}\int_{(\lambda+\rho)-\rho}^{(\lambda+\rho)+\rho}\left|\frac{\rho}{(\lambda+\rho)-\eta}\right|^{1/2}g^*(\eta)\,d\eta\right)$$

$$\leq C\lambda^{1/q}(\mathcal{M}_{1/2}g^*)(\lambda-\rho)+C\lambda^{1/q}(\mathcal{M}_{1/2}g^*)(\lambda+\rho)$$

as in (2.25). Note that, at the second inequality, we have used

$$\frac{\lambda}{2}\leq\rho-\lambda+\lambda\leq\rho-r+\lambda\leq\lambda-\frac{\lambda}{2}+\lambda=\frac{3}{2}\lambda$$

for $\lambda/2\leq r\leq\lambda$ and $\lambda/2\leq\rho\leq\lambda$, and

$$\frac{\lambda}{2}\leq\rho+\lambda-\lambda\leq\rho+r-\lambda\leq\lambda+(\lambda+\rho)-\lambda\leq2\lambda$$

for $\lambda\leq r\leq\lambda+\rho$ and $\lambda/2\leq\rho\leq\lambda$. By virtue of the estimate (2.32), we can obtain

$$(2.33)\qquad L_2\leq C\|\lambda^{1/q}w\|_{L^p((0,\infty),\lambda^{p\alpha}\,d\lambda)}\|g\|_{L^{q'}((0,\infty),r^{q'\beta}\,dr)}$$

as in (2.29) and (2.31). The estimate of L_2 has been completed.

Next, let us consider the estimate of L_3. Note that $\rho+r-\lambda\leq5\lambda/2$, $\rho+r-\lambda\geq\lambda+r-\lambda\geq\lambda/2$ for $\lambda/2\leq r\leq2\lambda$ and $\lambda\leq\rho\leq3\lambda/2$. Using (2.24), we hence have for $\lambda\leq\rho\leq3\lambda/2$

$$(2.34)\qquad\int_{\lambda/2}^{2\lambda}r^{(1/q)-1}h(\rho,\lambda;r)^{-1/2}g(r)\,dr$$

$$\leq C\lambda^{1/q}\int_{\lambda/2}^{2\lambda}\frac{1}{\sqrt{(\rho-r+\lambda)(r+\lambda-\rho)}}g(r)\,dr$$

$$\leq C\lambda^{(1/q)-1}\left(\int_{\lambda/2}^{\lambda}\sqrt{\frac{\lambda}{r-(\rho-\lambda)}}g(r)\,dr\right.$$

$$\left.+\int_{\lambda}^{2\lambda}\sqrt{\frac{\lambda}{(\rho+\lambda)-r}}g(r)\,dr\right)$$

$$\leq C\lambda^{1/q}\left(\frac{1}{2\lambda}\int_{(\rho-\lambda)-\lambda}^{(\rho-\lambda)+\lambda}\left|\frac{\lambda}{\eta-(\rho-\lambda)}\right|^{1/2}g^*(\eta)\,d\eta\right.$$

$$\left.+\frac{1}{3\lambda}\int_{(\rho+\lambda)-(3\lambda/2)}^{(\rho+\lambda)+(3\lambda/2)}\left|\frac{3\lambda/2}{(\rho+\lambda)-\eta}\right|^{1/2}g^*(\eta)\,d\eta\right)$$

$$\leq C\lambda^{1/q}(\mathcal{M}_{1/2}g^*)(\rho-\lambda)+C\lambda^{1/q}(\mathcal{M}_{1/2}g^*)(\rho+\lambda).$$

This leads us to the estimate

$$(2.35)\qquad L_3\leq C\|\lambda^{1/q}w\|_{L^p((0,\infty),\lambda^{p\alpha}\,d\lambda)}\|g\|_{L^{q'}((0,\infty),r^{q'\beta}\,dr)}$$

as before. The estimate of L_3 has been completed.

It remains to bound L_4. Note that for $\rho - \lambda \leq r \leq 2\lambda$ and $3\lambda/2 \leq \rho \leq 3\lambda$, we have $\rho - r + \lambda \leq \rho - (\rho - \lambda) + \lambda = 2\lambda$, $\rho - r + \lambda \geq (3\lambda/2) - 2\lambda + \lambda = \lambda/2$ and $\rho + r - \lambda \leq 3\lambda + 2\lambda - \lambda = 4\lambda$, $\rho + r - \lambda \geq \rho + (\rho - \lambda) - \lambda \geq \lambda$. We therefore obtain for $3\lambda/2 \leq \rho \leq 3\lambda$

$$(2.36) \qquad \int_{\rho-\lambda}^{2\lambda} r^{(1/q)-1} h(\rho, \lambda; r)^{-1/2} g(r)\, dr$$

$$\leq C\lambda^{(1/q)-1} \int_{\rho-\lambda}^{2\lambda} \sqrt{\frac{\lambda}{r - (\rho - \lambda)}} g(r)\, dr$$

$$\leq C\lambda^{1/q} \frac{1}{4\lambda} \int_{(\rho-\lambda)-2\lambda}^{(\rho-\lambda)+2\lambda} \left| \frac{2\lambda}{\eta - (\rho - \lambda)} \right|^{1/2} g^*(\eta)\, d\eta$$

$$\leq C\lambda^{1/q} (\mathcal{M}_{1/2} g^*)(\rho - r),$$

which yields

$$(2.37) \qquad L_4 \leq C \|\lambda^{1/q} w\|_{L^p((0,\infty),\, \lambda^{p\alpha}\, d\lambda)} \|g\|_{L^{q'}((0,\infty),\, r^{q'\beta}\, dr)}$$

as before. Combining (2.31), (2.33), (2.35), (2.37) with (2.23), we have shown (2.22). The proof of Proposition 2.5 has been completed. $\qquad\square$

We are in a position to complete the proof of Theorem 2.1 for $n = 2$. This is a direct consequence of (2.3), (2.17), (2.15) with $n = 1$, and (2.22). The proof of Theorem 2.1 has been completed for all $n \geq 2$. $\qquad\square$

REMARK. The inequality (2.2) with $\alpha = \beta = 0$ is just the one Vilela has used in [39]. Vilela has shown the inequality by employing some ideas in Stein and Weiss [33]. (See [39] on page 369.) Now that we have completed the proof of Theorem 2.1, it is obvious that we can show (2.2) for $\alpha = \beta = 0$ by employing the classical Hardy–Littlewood inequality and the Hardy–Littlewood maximal inequality in the standard $L^p(\mathbb{R}^n)$ space. Hence, it is also possible to show (2.2) without results in [33], as far as the case $\alpha = \beta = 0$ is concerned. It is in the case $\alpha \neq 0$ or $\beta \neq 0$ that our proof of (2.2) essentially relies upon the result of Stein and Weiss [33].

3. Strichartz estimates for radial solutions

Adapting an argument of Vilela [39], we explain how the weighted Hardy–Littlewood–Sobolev inequality (2.2) is used to prove the Strichartz estimate for the free wave equation with radially symmetric data. Let us start our consideration with global-in-time estimates. Recalling the definition of the operator W (see (1.1)), we shall show the following theorem.

THEOREM 3.1. *Suppose $n \geq 3$ and $1/2 < (n-1)((1/2)-(1/p)) < (n-1)/2$. There exists a constant C depending on n and p, and the estimate*

$$(3.1) \qquad \|W\varphi\|_{L^2(\mathbb{R};L^p(\mathbb{R}^n))} \leq C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)},$$

$$\frac{1}{2} + \frac{n}{p} = \frac{n}{2} - s$$

holds for radially symmetric $\varphi \in \dot{H}_2^s(\mathbb{R}^n)$.

It should be mentioned that Sterbenz has proved (3.1) in a completely different way (see Proposition 1.2 of [34]). As has been done in [34], we can actually obtain the following result by the interpolation between (3.1) and the energy identity. For any integer $n \geq 3$, we define

$$(3.2) \qquad D_n := \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, 0 < x \leq \frac{1}{2}, 0 < y \leq \frac{1}{2}, \right.$$

$$\left. \frac{n-1}{2}\left(\frac{1}{2}-y\right) < x < (n-1)\left(\frac{1}{2}-y\right)\right\}$$

and

$$(3.3) \qquad A_n := D_n \cup \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x=0 \text{ and } y = \frac{1}{2}\right\}.$$

COROLLARY 3.2. *Suppose $n \geq 3$ and $(1/q, 1/p) \in A_n$. There exists a constant C depending on n, p, q, and the estimate*

$$(3.4) \qquad \|W\varphi\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))} \leq C\||D_x|^s\varphi\|_{L^2(\mathbb{R}^n)},$$

$$\frac{1}{q} + \frac{n}{p} = \frac{n}{2} - s$$

holds for radially symmetric $\varphi \in \dot{H}_2^s(\mathbb{R}^n)$.

REMARK. Without the assumption of radial symmetry, the Strichartz estimate (3.4) holds, provided that

$$(3.5) \qquad n \geq 2, \qquad 0 \leq \frac{1}{q} \leq \frac{1}{2}, \qquad 0 \leq \frac{1}{p} \leq \frac{1}{2},$$

$$\left(\frac{1}{q}, \frac{1}{p}\right) \neq (0,0), \qquad \frac{2}{q} \leq (n-1)\left(\frac{1}{2} - \frac{1}{p}\right),$$

$$\left(\frac{1}{q}, \frac{1}{p}\right) \neq \left(\frac{1}{4}, 0\right) \quad \text{if } n=2, \qquad \left(\frac{1}{q}, \frac{1}{p}\right) \neq \left(\frac{1}{2}, 0\right) \quad \text{if } n \geq 3.$$

See [7], [20], [22], [28], [35], and [18] for the proof. We note that the condition $2/q \leq (n-1)(1/2 - 1/p)$ of (3.5) is necessary. Otherwise, it is well known that using the method of Knapp, one can indeed choose a sequence $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^n)$ of *nonradial* data for which the existence of such a uniform constant $C = C(n, p, q)$ as in (3.4) is forbidden. Keeping in mind that some nonradial

solutions yield this counterexample, we mention that radial symmetry vastly improves on the range of the admissible pairs $(1/q, 1/p)$. Indeed, it has turned out by the works of Klainerman and Machedon [19], Sterbenz [34], and Fang and Wang [5] (see also Sogge [31] on page 125) that one actually has the Strichartz estimate (3.4) under the assumption of radial symmetry in the case of

$$(3.6) \qquad n \geq 2, \qquad 0 \leq \frac{1}{q} \leq \frac{1}{2}, \qquad 0 \leq \frac{1}{p} \leq \frac{1}{2},$$

$$\left(\frac{1}{q}, \frac{1}{p}\right) \neq (0,0), \qquad \frac{1}{q} < (n-1)\left(\frac{1}{2} - \frac{1}{p}\right),$$

in addition to the obvious case $(1/q, 1/p) = (0, 1/2)$.

In Section 4, we shall show the condition $1/q < (n-1)((1/2) - (1/p))$ of (3.6) is necessary for the global-in-time estimate (3.4) to hold for radially symmetric data. The prime purpose of this section is to explain how we can prove Proposition 1.2 of Sterbenz [34] using the weighted Hardy–Littlewood–Sobolev inequality (2.2).

Proof of Theorem 3.1. We use the following result which is a generalization of the classical estimate of Morawetz [25].

LEMMA 3.3. *Suppose* $n \geq 2$ *and* $1/2 < \alpha < n/2$. *There exists a constant* C *depending on* n *and* α, *and the estimate*

$$(3.7) \qquad \||x|^{-\alpha} W \varphi\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq C \||D_x|^{\alpha - (1/2)} \varphi\|_{L^2(\mathbb{R}^n)}$$

holds for $\varphi \in \dot{H}_2^{\alpha - (1/2)}(\mathbb{R}^n)$.

The proof of (3.7) uses the trace inequality in the Fourier space

$$(3.8) \quad \sup_{\lambda > 0} \lambda^{(n/2) - s} \int_{S^{n-1}} |\hat{w}(\lambda \omega)|^2 \, d\sigma \leq C \||D_\xi|^s \hat{w}\|_{L^2(\mathbb{R}^n)} = C' \||x|^s w\|_{L^2(\mathbb{R}^n)}$$

which holds for $1/2 < s < n/2$. See Ben-Artzi [2], Ben-Artzi and Klainerman [3], Hoshiro [13] for the proof of (3.7) via the trace inequality such as (3.8) and the duality argument. For the proof of (3.8), see, e.g., (2.45) of Li and Zhou [21] and the Appendix of Hidano [9].

We are in a position to complete the proof of Theorem 3.1. We follow the argument of Vilela [39]. Fix any p satisfying $1/2 < (n-1)((1/2) - (1/p)) < (n-1)/2$. It follows from Theorem 2.1 with $\alpha = \beta = 0$ that the Sobolev-type inequality

$$(3.9) \qquad \|v\|_{L^p(\mathbb{R}^n)} \leq C \||x|^{-(n-1)((1/2) - (1/p))} |D_x|^{(1/2) - (1/p)} v\|_{L^2(\mathbb{R}^n)}$$

holds for radially symmetric v. The estimate (3.1) is an immediate consequence of (3.7) and (3.9). Indeed, we see that

$$(3.10) \quad \|W\varphi\|_{L^2(\mathbb{R};L^p(\mathbb{R}^n))}$$

$$\leq C\big\|\,|x|^{-(n-1)((1/2)-(1/p))}|D_x|^{(1/2)-(1/p)}W\varphi\big\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$

$$= C\big\|\,|x|^{-(n-1)((1/2)-(1/p))}W\big(|D_x|^{(1/2)-(1/p)}\varphi\big)\big\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$

$$\leq C\big\|\,|D_x|^{((n-1)/2)-(n/p)}\varphi\big\|_{L^2(\mathbb{R}^n)}$$

as desired. The proof of Theorem 3.1 has been finished. □

4. Failure of the critical estimate

The problem to be discussed in this section is whether the Strichartz estimate (3.4) holds under the assumption of radial symmetry of data even for the limiting pair $(1/q, 1/p) \in (0, 1/2] \times [0, 1/2)$ with $1/q = (n-1)((1/2) - (1/p))$. If it were true, we would enjoy

$$(4.1) \qquad \|W\varphi\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))} \leq C\big\|\,|D_x|^{(1/2)-(1/p)}\varphi\big\|_{L^2(\mathbb{R}^n)}$$

for radially symmetric data φ, and the estimate (4.1) would imply the estimate

$$(4.2) \quad \|u\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))}$$

$$\leq C\big(\big\|\,|D_x|^{(1/2)-(1/p)}f\big\|_{L^2(\mathbb{R}^n)} + \big\|\,|D_x|^{-(1/2)-(1/p)}g\big\|_{L^2(\mathbb{R}^n)}\big)$$

for the solution u to the wave equation $\Box u = 0$ in $\mathbb{R} \times \mathbb{R}^n$ with radially symmetric data (f, g). We shall show that the estimate (4.2) is false in the limiting case $(1/q, 1/p) \in (0, 1/2] \times [0, 1/2)$ with $1/q = (n-1)((1/2) - (1/p))$, though $\mathcal{S}(\mathbb{R}^n) \subset \dot{H}_2^{-(1/2)-(1/p)}(\mathbb{R}^n)$ $(n \geq 2)$. The key to such a result is the following lemma.

LEMMA 4.1. *Let $n \geq 2$ and $r = |x|$. Suppose $g(x)$ is a smooth, nonnegative function with $\operatorname{supp} g \subset \{x \in \mathbb{R}^n \,|\, |x| \leq R\}$ for some $R > 0$. Suppose also that g is a radially symmetric function written as $g(x) = \psi(r)$ for an even function $\psi \in C_0^\infty(\mathbb{R})$. Let u be the solution to $\Box u = 0$ in $\mathbb{R} \times \mathbb{R}^n$ with data $(0, g)$ at $t = 0$. There exists a positive constant δ depending only on n such that the estimate*

$$(4.3) \qquad u(t, x) \geq \frac{1}{4r^{(n-1)/2}} \int_{r-t}^{\min\{R, r+t\}} \lambda^{(n-1)/2} \psi(\lambda) \, d\lambda$$

holds for any (t, x) with $R/(1 + \delta) \leq r - t \leq R$, $t > 0$.

Let us postpone the proof of Lemma 4.1 for the moment and see how it can be used to prove the following theorem.

THEOREM 4.2. *Let $n \geq 2$ and fix the constant $\delta > 0$ given by Lemma 4.1. Suppose that $g(x) \geq 0$ is a smooth, radially symmetric function with $\operatorname{supp} g \subset$*

$\{x \in \mathbb{R}^n \mid |x| \leq 1\}$ *which is written as* $g(x) = \psi(r)$ *for an even function* $\psi \in C_0^\infty(\mathbb{R})$ *satisfying the condition that the function* Ψ *defined as*

$$(4.4) \qquad \Psi(\rho) := \int_\rho^1 \lambda^{(n-1)/2} \psi(\lambda) \, d\lambda$$

does not vanish identically for $\rho \in (1/(1+\delta), 1)$.

Let $(1/q, 1/p) \in (0, 1/2] \times [0, 1/2)$ *satisfy* $1/q = (n-1)(1/2 - 1/p)$. *Then for the solution to* $\Box u = 0$ *with data* $(0, g)$ *at* $t = 0$,

$$(4.5) \qquad \lim_{T \to +\infty} \|u\|_{L^q((0,T);L^p(\mathbb{R}^n))} = +\infty.$$

Proof. We separate two cases: $(1/q, 1/p) \in (0, 1/2] \times (0, 1/2)$ satisfying $1/q = (n-1)(1/2 - 1/p)$ for $n \geq 2$ and $(1/q, 1/p) = (1/2, 0)$ for $n = 2$. We start with the former. Employing (4.3) and writing $u(t,x) = v(t,r)$, we have for $t \in (\delta/(2(1+\delta)), T)$ by the change of variables $\rho = r - t$

$$(4.6) \qquad \int_{t+(1/(1+\delta))}^{t+1} v^p(t,r) r^{n-1} \, dr$$

$$\geq \frac{1}{4^p} \int_{t+(1/(1+\delta))}^{t+1} \left(\frac{1}{r^{(n-1)/2}} \int_{r-t}^1 \lambda^{(n-1)/2} \psi(\lambda) \, d\lambda \right)^p r^{n-1} \, dr$$

$$= \frac{1}{4^p} \int_{1/(1+\delta)}^1 \frac{1}{(t+\rho)^{((n-1)/2)p - (n-1)}} \Psi^p(\rho) \, d\rho$$

$$\geq \frac{1}{4^p} \frac{1}{(t+1)^{((n-1)/2)p - (n-1)}} \int_{1/(1+\delta)}^1 \Psi^p(\rho) \, d\rho.$$

Setting a strictly positive constant A as

$$A := \left(\int_{1/(1+\delta)}^1 \Psi^p(\rho) \, d\rho \right)^{1/p},$$

we then find

$$(4.7) \qquad \|u\|_{L^q((0,T);L^p(\mathbb{R}^n))}^q$$

$$\geq \int_{\delta/(2(1+\delta))}^T \left(\int_{t+(1/(1+\delta))}^{t+1} v^p(t,r) r^{n-1} \, dr \right)^{q/p} \, dt$$

$$\geq \frac{A^q}{4^q} \int_{\delta/(2(1+\delta))}^T \left(\frac{1}{(t+1)^{(n-1)/2 - (n-1)/p}} \right)^q \, dt$$

$$= \frac{A^q}{4^q} \int_{\delta/(2(1+\delta))}^T \frac{1}{t+1} \, dt = \frac{A^q}{4^q} \log \frac{T+1}{\frac{\delta}{2(1+\delta)} + 1}$$

for all $T \geq \delta/(2(1+\delta))$.

It remains to deal with $(1/q, 1/p) = (1/2, 0)$ for $n = 2$. We naturally modify the argument in (4.6) and (4.7) as follows. Fix a constant c_0 satisfying $1/(1+\delta) < c_0 < 1$ so that $\Psi(c_0) > 0$. We see, noting $\|v(t, \cdot)\|_{L^\infty(t+(1/(1+\delta))<r<t+c_0)} \geq v(t, t+c_0)$,

$$(4.8) \qquad \int_{\delta/(2(1+\delta))}^{T} \|v(t, \cdot)\|_{L^\infty(t+(1/(1+\delta))<r<t+c_0)}^2 \, dt$$

$$\geq \int_{\delta/(2(1+\delta))}^{T} \frac{1}{4^2(t+c_0)} \left(\int_{c_0}^1 \lambda^{1/2} \psi(\lambda) \, d\lambda \right)^2 dt$$

$$= \frac{1}{4^2} \Psi^2(c_0) \log \frac{T + c_0}{\frac{\delta}{2(1+\delta)} + c_0}$$

for all $T \geq \delta/(2(1+\delta))$. We have completed the proof. $\qquad \square$

Using the solution u described in Theorem (4.2), we easily obtain the following result by scaling argument.

COROLLARY 4.3. *Let $n \geq 2$ and $(1/q, 1/p) \in (0, 1/2] \times [0, 1/2)$ satisfy $1/q = (n-1)(1/2 - 1/p)$. Then for the solution u_h to $\Box u = 0$ with radially symmetric data $(0, h)$ at $t = 0$*

$$\sup \left\{ \frac{\|u_h\|_{L^q((0,1);L^p(\mathbb{R}^n))}}{\||D_x|^{-(1/2)-(1/p)}h\|_{L^2(\mathbb{R}^n)}} \right|$$

$$h \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \text{ and } h \text{ is radially symmetric} \right\} = +\infty.$$

This shows that the estimate (4.2) is false even if the global-in-time norm is replaced by the local-in-time norm on the left-hand side. The proof of Corollary (4.3) is straightforward and, therefore, we leave it to the reader.

Proof of Lemma 4.1. We must establish Lemma 4.1. The proof is essentially based on Rammaha's way for Lemma 2 of [29] together with the treatment of fundamental solutions in even space dimensions in Agemi [1], which is summarized in Takamura [38]. Following [1], [29], and [37]–[38], we show Lemma 4.1.

By the representations (6a) and (6b) of radial solutions in [29], u is expressed as

$$(4.9) \qquad u(t, x) = \frac{1}{2r^m} \int_{|r-t|}^{r+t} \lambda^m \psi(\lambda) P_{m-1} \left(\frac{\lambda^2 + r^2 - t^2}{2r\lambda} \right) d\lambda,$$

if $n = 2m + 1$, and

$$(4.10) \quad u(t, x)$$

$$= \frac{2}{\pi r^{m-1}} \int_0^t \frac{\rho \, d\rho}{\sqrt{t^2 - \rho^2}} \int_{|r-\rho|}^{r+\rho} \frac{\lambda^m \psi(\lambda)}{\sqrt{G(\lambda, r, \rho)}} T_{m-1} \left(\frac{\lambda^2 + r^2 - \rho^2}{2r\lambda} \right) d\lambda$$

$$= \frac{2}{\pi r^{m-1}} \int_{r-t}^{r+t} \lambda^m \psi(\lambda) \, d\lambda$$

$$\times \int_{|r-\lambda|}^{t} \frac{\rho}{\sqrt{G(\rho, r, \lambda)}\sqrt{t^2 - \rho^2}} T_{m-1}\left(\frac{\lambda^2 + r^2 - \rho^2}{2r\lambda}\right) d\rho,$$

if $n = 2m$ and $r > t$, where

$$G(\lambda, r, \rho) = \left(\lambda^2 - (r-\rho)^2\right)\left((r+\rho)^2 - \lambda^2\right) = G(\rho, r, \lambda),$$

and P_k, T_k are the Legendre and Tschebyscheff polynomials, respectively, defined by

$$P_k(z) = \frac{1}{2^k k!} \frac{d^k}{dz^k}(z^2 - 1)^k,$$

$$T_k(z) = \frac{(-1)^k}{(2k-1)!!}(1 - z^2)^{1/2} \frac{d^k}{dz^k}(1 - z^2)^{k-1/2}.$$

See also [37] for details.

As is well known, P_k and T_k have the properties: $|P_k(z)|$, $|T_k(z)| \leq 1$ ($|z| \leq 1$) and $P_k(1) = T_k(1) = 1$ for all $k = 1, 2, \ldots$ (see Magnus, Oberhettinger, and Soni [24], pages 227, 237, 256–267). By these properties together with the continuity of the two functions, one can choose a small constant δ depending on n so that

$$(4.11) \qquad P_{m-1}(z), \qquad T_{m-1}(z) \geq \frac{1}{2} \quad \text{for} \quad \frac{1}{1+\delta} \leq z \leq 1, m \in \mathbb{N},$$

in the same manner as Takamura did in Lemma 2.5 of [37].

In what follows, we assume that $R/(1+\delta) \leq r - t \leq R$ with $t > 0$. Note that the upper limit of the λ-integrals in (4.9) and (4.10) can be replaced with $\min\{R, r+t\}$ by virtue of the support property of data. We then have

$$(4.12) \qquad 1 \geq \frac{\lambda^2 + r^2 - \rho^2}{2r\lambda} \geq \frac{\lambda^2 + r^2 - t^2}{2r\lambda}$$

$$\geq \frac{(r-t)^2 + r^2 - t^2}{2rR} = \frac{r-t}{R} \geq \frac{1}{1+\delta}$$

for $r - t \leq \lambda \leq \min\{R, r+t\}$ and $0 \leq \rho \leq t$. It therefore follows from (4.9)–(4.11) that

$$(4.13) \qquad u(t, x) \geq \begin{cases} \dfrac{1}{4r^m} \displaystyle\int_{r-t}^{\min\{R, r+t\}} \lambda^m \psi(\lambda) \, d\lambda & \text{if } n = 2m+1, \\[4mm] \dfrac{1}{\pi r^{m-1}} \displaystyle\int_{r-t}^{\min\{R, r+t\}} \lambda^m \psi(\lambda) \, d\lambda \\[4mm] \qquad \times \displaystyle\int_{|r-\lambda|}^{t} \dfrac{\rho \, d\rho}{\sqrt{G(\rho, r, \lambda)}\sqrt{t^2 - \rho^2}} & \text{if } n = 2m, \end{cases}$$

provided $\psi \geq 0$ on the support. Therefore, the odd dimensional case has been proved. For the even dimensional case, the ρ-integral in (4.13) is estimated

as follows:

$$(\rho\text{-integral}) \geq \frac{1}{2\sqrt{r\lambda}} \int_{|r-\lambda|}^{t} \frac{\rho\,d\rho}{\sqrt{\rho^2 - (r-\lambda)^2}\sqrt{t^2 - \rho^2}}$$

$$= \frac{B(2^{-1}, 2^{-1})}{4\sqrt{r\lambda}} = \frac{\pi}{4\sqrt{r\lambda}}.$$

Here, by $B(\cdot,\cdot)$, we have meant the beta function as in Section 2. This completes the proof for the even dimensional case. $\square$

REMARK. During the preparation of this article, the authors found that arguing in a way similar to Takamura [37], Jiao and Zhou had already obtained an estimate which is a bit less precise than (4.3) (see Lemma 2 of [14]).

5. Local-in-time Strichartz estimates

For any integer $n \geq 2$, we define

$$(5.1) \qquad \Omega_n := \left\{ (x,y) \in \mathbb{R}^2 \,\Big|\, 0 < x \leq \frac{1}{2}, 0 < y \leq \frac{1}{2}, x > (n-1)\left(\frac{1}{2} - y\right) \right\}$$

and

$$(5.2) \qquad \Lambda_n := \Omega_n \cup \left\{ (x,y) \in \mathbb{R}^2 \,\Big|\, x = 0 \text{ and } y = \frac{1}{2} \right\}.$$

The main result of this section is the following.

THEOREM 5.1. *Suppose $n \geq 2$ and $(1/q, 1/p) \in \Lambda_n$. Let T be an arbitrary positive number. There exists a constant C depending only on n, p, and q, and the estimate*

$$(5.3) \qquad \|W\varphi\|_{L^q((0,T), L^p(\mathbb{R}^n))} \leq CT^\theta \big\| |D_x|^{(1/2)-(1/p)} \varphi \big\|_{L^2(\mathbb{R}^n)},$$

$$\frac{1}{q} + \frac{n}{p} = \theta + \frac{n}{2} - \left(\frac{1}{2} - \frac{1}{p}\right)$$

holds for radially symmetric data $\varphi \in \dot{H}_2^{(1/2)-(1/p)}(\mathbb{R}^n)$.

Theorem 5.1 is an extension of the intriguing result of Sogge (Proposition 6.3 on the page 125 of [31]) who proved the estimate (5.3) for $n = 3$ and

$$(5.4) \qquad \left(\frac{1}{q}, \frac{1}{p}\right) \in \left\{ (x,y) \in \mathbb{R}^2 \,\Big|\, 0 < x \leq y \leq \frac{1}{2}, x > 2\left(\frac{1}{2} - y\right) \right\}$$

$$\cup \left\{ (x,y) \in \mathbb{R}^2 \,\Big|\, x = 0 \text{ and } y = \frac{1}{2} \right\}.$$

Actually, Sogge himself proved the estimate (5.3) for $n = 3$, $1/3 < 1/q = 1/p \leq 1/2$. By the interpolation between his estimate and the energy estimate, we easily get (5.3) for $n = 3$ and $(1/q, 1/p)$ satisfying (5.4).

We should explain the significance of the local-in-time estimate (5.3). If the radially symmetric estimate (4.1) were true even for the limiting pair $(1/q, 1/p) \in (0, 1/2] \times [0, 1/2)$ with $1/q = (n-1)((1/2) - (1/p))$, our estimate (5.3) would be a trivial consequence of (4.1) and the Hölder inequality in time. The fact is that the estimate (4.1), even if localized in time, is false for any limiting pair $(1/q, 1/p) \in (0, 1/2] \times [0, 1/2)$ with $1/q = (n-1)((1/2) - (1/p))$ as we have seen in Section 4, and one can get nothing but a coarse estimate

$$(5.5) \qquad \|W\varphi\|_{L^q((0,T), L^p(\mathbb{R}^n))} \le CT^\theta \||D_x|^{(1/2)-(1/p)+\varepsilon}\varphi\|_{L^2(\mathbb{R}^n)},$$

$$\frac{1}{q} + \frac{n}{p} = \theta + \frac{n}{2} - \left(\frac{1}{2} - \frac{1}{p} + \varepsilon\right),$$

$$\varepsilon > 0 \text{ (sufficiently small)}$$

for any $(1/q, 1/p) \in \Omega_n$ with $(1/q, 1/p) \in (0, 1/2] \times (0, 1/2)$ by using both the Strichartz estimate (3.4) for $(1/q, 1/p)$ permitted in (3.6) and the Hölder inequality in time. As we have just mentioned, Sogge proved the sharper estimate (5.3) in the case of $n = 3$, $1/3 < 1/q = 1/p \le 1/2$, and the key to his proof was a clever use of the identity

$$(5.6) \qquad \widehat{d\sigma}(|\xi|) = \int_{S^2} e^{-i\omega\cdot\xi}\, d\sigma = 4\pi \frac{\sin|\xi|}{|\xi|}$$

$$\left(\omega \in S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}, d\sigma = d\sigma(\omega)\right).$$

Though the formula of $\widehat{d\sigma}(|\xi|)$ in terms of the Bessel function is well known for $n = 2$ or $n \ge 4$, the authors do not know whether such a formula is useful in proving our estimate (5.3). In the rest of this section, we see how the weighted inequality (2.2) is used to prove the local-in-time estimate (5.3).

Proof of Theorem 5.1. We use the following result.

LEMMA 5.2. *Suppose* $n \ge 1$ *and* $0 \le \alpha < 1/2$. *Let* T *be an arbitrary positive number. There exists a constant* C *depending on* n *and* α, *and the estimate*

$$(5.7) \qquad \||x|^{-\alpha}W\varphi\|_{L^2((0,T)\times\mathbb{R}^n)} \le CT^{(1/2)-\alpha}\|\varphi\|_{L^2(\mathbb{R}^n)}$$

holds for all $\varphi \in L^2(\mathbb{R}^n)$.

By scaling the proof of (5.7) can be reduced to the case $T = 1$. For $T = 1$, the estimate (5.7) has been shown in [10] as a direct consequence of integrability (in time) of the local energy [30]

$$(5.8) \qquad \|W\varphi\|_{L^2(\mathbb{R}\times\{x\in\mathbb{R}^n\mid|x|<1\})} \le C\|\varphi\|_{L^2(\mathbb{R}^n)},$$

scaling, and the energy estimate.

We are in a position to complete the proof of Theorem 5.1. Fix any p $(0 < 1/p \le 1/2)$ satisfying $1/2 > (n-1)((1/2) - (1/p))$. It follows from the

Sobolev-type estimate (3.9) and (5.7) that

$$(5.9) \quad \|W\varphi\|_{L^2((0,T);L^p(\mathbb{R}^n))}$$
$$\leq C \big\| |x|^{-(n-1)((1/2)-(1/p))} |D_x|^{(1/2)-(1/p)} W\varphi \big\|_{L^2((0,T)\times\mathbb{R}^n)}$$
$$\leq C T^{(1/2)-(n-1)((1/2)-(1/p))} \big\| |D_x|^{(1/2)-(1/p)} \varphi \big\|_{L^2(\mathbb{R}^n)}.$$

Our estimate (5.3) is a consequence of the interpolation between (5.9) and the energy estimate. We have finished the proof of Theorem 5.1. $\qquad\square$

6. End-point estimates for Schrödinger equations

The final section is devoted to the study of the Strichartz estimate for the Schrödinger equation

$$(6.1) \qquad\qquad i\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^n$$

subject to the initial data $u(0,x) = \varphi(x)$. The estimate

$$(6.2) \qquad \|S\varphi\|_{L^2(\mathbb{R};L^{2n/(n-2)}(\mathbb{R}^n))} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)} \quad (n\geq 3),$$

which was proved by Keel and Tao [18], is called an end-point estimate. (See (1.2) for the definition of the operator S.) As Vilela has explained in Section 3 of [39], it is possible to prove (6.2) for radially symmetric data via the weighted inequality (2.2) with $\alpha = \beta = 0$. We revisit the problem of showing (6.2) for radially symmetric data. Using our weighted inequality (2.2) with $-\beta = \alpha$, we prove

THEOREM 6.1. *Suppose $n \geq 3$ and $-(1/2) + (1/n) < \alpha < (1/2) - (1/n)$. There exists a constant C depending on n, α, and the estimate*

$$(6.3) \qquad \big\| |x|^\alpha |D_x|^\alpha S\varphi \big\|_{L^2(\mathbb{R};L^{2n/(n-2)}(\mathbb{R}^n))} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)}$$

holds for radially symmetric data $\varphi \in L^2(\mathbb{R}^n)$.

Proof. We need the following lemma.

LEMMA 6.2. *Suppose $n \geq 2$ and $1/2 < \gamma < n/2$. There exists a constant C depending on n, γ, and the estimate*

$$(6.4) \qquad \big\| |x|^{-\gamma} |D_x|^{1-\gamma} S\varphi \big\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)}$$

holds.

Large part of Lemma 6.2 was proved by Kato and Yajima [17], Ben-Artzi and Klainerman [3], independently. Later, their results were not only complemented but also generalized by Sugimoto [36] and Vilela [39]. For the proof of (6.4), see Section 4 of Sugimoto [36] or Section 1 of Vilela [39].

In what follows, we denote $2n/(n-2)$ by p_0. We note that

$$\frac{1}{2} < (n-1)\left(\frac{1}{2} - \frac{1}{p_0}\right) - \alpha < \frac{n}{2} \quad \Longleftrightarrow \quad 1 - \frac{1}{n} - \frac{n}{2} < \alpha < \frac{1}{2} - \frac{1}{n}$$

and that the inequality

$$1 - \frac{1}{n} - \frac{n}{2} \le -\frac{1}{2} + \frac{1}{n}$$

is true for all $n \ge 3$ (actually, for all $n \ge 1$). Taking account of the obvious fact

$$-\alpha < \frac{1}{p_0} \quad \Longleftrightarrow \quad -\frac{1}{2} + \frac{1}{n} < \alpha,$$

we can employing (2.2) with $-\beta = \alpha$ first and (6.4) secondly to have for radially symmetric φ

$$(6.5) \qquad \big\| |x|^\alpha |D_x|^\alpha S\varphi \big\|_{L^2(\mathbb{R}; L^{p_0}(\mathbb{R}^n))}$$
$$\le C \big\| |x|^{-(n-1)((1/2)-(1/p_0))+\alpha} |D_x|^{\alpha+(1/2)-(1/p_0)} \varphi \big\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$
$$\le C \|\varphi\|_{L^2(\mathbb{R}^n)}.$$

It is only at the last inequality above that the choice of $p_0 = 2n/(n-2)$ is essential. The proof of Theorem 6.1 has been completed. $\qquad\square$

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KUNIO HIDANO, DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, MIE UNIVERSITY, 1577 KURIMA-MACHIYA-CHO, TSU, MIE 514-8507, JAPAN

   E-mail address: hidano@edu.mie-u.ac.jp

YUKI KUROKAWA, GENERAL EDUCATION, YONAGO NATIONAL COLLEGE OF TECHNOLOGY, 4448 HIKONA-CHO, YONAGO, TOTTORI 683-8502, JAPAN

   E-mail address: kurokawa@yonago-k.ac.jp