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STRUCTURE OF THE BRAUER RING OF A FIELD EXTENSION

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Abstract. In 1986, Jacobson has defined the Brauer ring $B(E,D)$ for a finite Galois field extension E/D, whose unit group canonically contains the Brauer group of D. In 1993, Cheng Xiang Chen determined the structure of the Brauer ring in the case where the extension is trivial. He revealed that if the Galois group G is trivial, the Brauer ring of the trivial extension E/E becomes naturally isomorphic to the group ring of the Brauer group of E. In this paper, we generalize this result to any finite group G via the theory of the restriction functor, by means of the well-understood functor $-_{+}$. More generally, we determine the structure of the F-Burnside ring for any additive functor F. We construct a certain natural isomorphism of Green functors, which induces the above result with an appropriate F related to the Brauer group. This isomorphism will enable us to calculate Brauer rings for some extensions. We illustrate how this isomorphism provides Green-functor-theoretic meanings for the properties of the Brauer ring shown by Jacobson, and compute the Brauer ring of the extension $\mathbb{C}/\mathbb{R}$.

1. Introduction

For the general theory of Mackey and Green functors, see [2]. Throughout this paper, we fix a finite group G, and use the following notation:

- $H \leq G$ means that H is a subgroup of G.

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- For any $K \le H \le G$ and any $g \in G$, ${}^gH := gHg^{-1}$, $H^g := g^{-1}Hg$, and $\ell_{g,H} : G/{}^gH \to G/H$ is the G-map defined by $\ell_{g,H}(g' \cdot {}^gH) = g'g \cdot H$, $p^H_K : G/K \to G/H$ is the canonical projection.
- $Mack(G)$ and $Green(G)$ denote the category of Mackey functors and Green functors, respectively.
- For any group M, $\mathbb{Z}[M]$ denotes its group ring over $\mathbb{Z}$, and similarly for $\mathbb{Q}$.

Most of the following arguments will work well even if the codomain of Green (and several other) functors is the category of R-modules $R-Mod$ instead of $Ab = \mathbb{Z}-Mod$, for any commutative ring R with 1. But we restrict ourselves to the case of $R = \mathbb{Z}$, for the sake of simplicity. Monoids, rings, and Green functors are equipped with 1, but not assumed to be commutative, unless otherwise specified.

For a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[d] \\ Y \arrow[r] & W \end{tikzcd},$$

we use a small square $\square$ to indicate that it is a pull-back diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ Y \arrow[r] & W \end{tikzcd}$$

The Brauer ring $B(E,D)$ of a finite Galois field extension E/D was defined by Jacobson in [4]. $B(E,D)$ can be regarded as an example of the F-Burnside ring, where F is an additive functor $F : \mathcal{G} \to Ab$. By using Chen's result (Corollary 3.4) in [3], for any trivial field extension E/E, we can see that the Brauer ring $B(E,E)$ is naturally isomorphic to the group ring of the Brauer group $Br(E)$:

$$B(E,E) \cong \mathbb{Z}[Br(E)]$$

(see also Proposition 2.8 and Remark 2.9 in this paper).

In the following, we will define several types of additive functors, and by the adjoint properties concerning these functors, we will see the structure of the F-Burnside ring as follows.

Theorem 3.13. *For any $F \in Ob(Add(G))$, there is a natural isomorphism of Green functors*

$$(\mathbb{Z}[\mathcal{R}_F])_+ \xrightarrow{\cong} \mathcal{A}_F.$$

As a corollary, the structure of the Brauer ring $B(E,D)$ can be seen as follows.

COROLLARY 4.1. *For any finite Galois extension E/D of fields with Galois group G, we have a ring isomorphism*

$$B(E,D) \cong \Big(\bigoplus_{H \le G} \mathbb{Z}[Br(E^H)] \Big) \Big/ \Big(I(\mathbb{Z}[G]) \cdot \bigoplus_{H \le G} \mathbb{Z}[Br(E^H)] \Big).$$

This is a generalization of the above isomorphism $B(E,E) \cong \mathbb{Z}[Br(E)]$.

2. Definition of the additive functor and the Brauer ring

In this section, we recall the construction of the F-Burnside ring, defined by Jacobson [4], and introduce the Brauer ring. In fact, we make a generalization of the definition in [4], which is mostly due to the referee.

We fix a finite group G, and let $\mathcal{G}$ be the category of finite G-sets and G-maps. Set denotes the category of small sets. A contravariant functor $E : \mathcal{G} \to Set$ is said to be additive if the canonical map $(E(i_X), E(i_Y)) : E(X \coprod Y) \to E(X) \times E(Y)$ induced by the inclusions i_X, i_Y is bijective for any $X, Y \in Ob(\mathcal{G})$. Let $j_{X,Y}$ denote the inverse bijection. $E(\emptyset)$ consists of one element. $\mathcal{S}add(G)$ denotes the category of additive functors from $\mathcal{G}$ to Set, whose morphisms are natural transformations.

DEFINITION 2.1. Let E be in $Ob(\mathcal{S}add(G))$. For any $S \in Ob(\mathcal{G})$, category (G, S, E) is defined as follows:
$Ob(G,S,E) = \{(Y, \phi, u) \mid Y \in Ob(\mathcal{G}), \phi \in \mathcal{G}(Y,S), u \in E(Y)\}$,
$Morph_{(G,S,E)}((Y,\phi,u),(Z,\psi,v)) = \{\alpha \in \mathcal{G}(Y,Z) \mid \phi = \psi \circ \alpha, E(\alpha)(v) = u\}$.

$$\begin{tikzcd} Y \arrow[rr, "\alpha"] \arrow[dr, "\phi"'] & \arrow[d, phantom, "\circlearrowleft"] & Z \arrow[dl, "\psi"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} E(\alpha) : E(Z) \arrow[r] & E(Y) \\ v \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & u \arrow[u, phantom, "\in" sloped] \end{tikzcd}$$

For any $(Y, \phi, u), (Z, \psi, v) \in Ob(G, S, E)$, we define their sum as follows:
Sum: $(Y,\phi,u) + (Z,\psi,v) := (Y \coprod Z, \phi \cup \psi : Y \coprod Z \to S, u \coprod v)$, where $u \coprod v := j_{Y,Z}((u,v))$. With this sum, we define a group $\mathcal{M}_E(S)$ as the Grothendieck group of the category (G, S, E). For any object (Y, ϕ, u), we write its image in $\mathcal{M}_E(S)$ as $[Y, \phi, u]$.

REMARK 2.2. $\mathcal{M}_E$ becomes a Mackey functor by the following definition:
Covariant part: For any $f \in \mathcal{G}(S,T)$, $\mathcal{M}_{E*}(f) : \mathcal{M}_E(S) \to \mathcal{M}_E(T)$, $[Y, \phi, u] \mapsto [Y, f \circ \phi, u]$.

Contravariant part: For any $f \in \mathcal{G}(S,T)$, $\mathcal{M}_E^*(f) : \mathcal{M}_E(T) \to \mathcal{M}_E(S)$, $[Z,\psi,v] \mapsto [S \underset{T}{\times} Z, \pi_S, E(\pi_Z)(v)]$.

$$\begin{tikzcd} S \times_T Z \arrow[r, "\pi_Z"] \arrow[d, "\pi_S"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\psi"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

We abbreviate $\mathcal{M}_E(G/H)$ to $\mathcal{M}_E(H)$ for any $H \leq G$. The correspondence $E \mapsto \mathcal{M}_E$ is a functor from $\mathcal{S}add(G)$ to $Mack(G)$. Indeed, for any morphism $\eta : E_1 \to E_2$ in $\mathcal{S}add(G)$, we obtain a sum-preserving functor $(G,S,E_1) \to (G,S,E_2)$ for any $S \in Ob(\mathcal{G})$, and thus obtain a set of homomorphisms $\mathcal{M}_\eta(H) : \mathcal{M}_{E_1}(H) \to \mathcal{M}_{E_2}(H)$ $(H \leq G)$, which form a morphism of Mackey functors $\mathcal{M}_\eta : \mathcal{M}_{E_1} \to \mathcal{M}_{E_2}$.

Let $\mathcal{E}$ denote the forgetful functor from $Mack(G)$ to $\mathcal{S}add(G)$; so if M is a Mackey functor for G and if X is a finite G-set, then $\mathcal{E}(M)(X)$ is the set $M(X)$, and if $f : X \to Y$ is a map in $\mathcal{G}$, then $\mathcal{E}(M)(f) : M(Y) \to M(X)$ is the map $M^*(f)$.

Proposition 2.3. *The functor $E \mapsto \mathcal{M}_E$ is left adjoint to $\mathcal{E}$.*

Proof. Let $E \in Ob(\mathcal{S}add(G))$ and $M \in Ob(Mack(G))$. A morphism of Mackey functors $\Phi : \mathcal{M}_E \to M$ is a collection of group homomorphisms $\Phi_S : \mathcal{M}_E(S) \to M(S)$ for all finite G-sets S, which are compatible with the Mackey structure. This implies

$$\Phi_S([Y,\phi,u]) = M_*(\phi) \circ \Phi_Y([Y,id,u])$$

for any $Y \in Ob(\mathcal{G})$, $u \in E(Y)$ and $\phi \in \mathcal{G}(Y,S)$.

$$\begin{tikzcd} \mathcal{M}_E(Y) \arrow[r, "\Phi_Y"] \arrow[d, "\mathcal{M}_{E*}(\phi)"'] \arrow[dr, phantom, "\circlearrowright"] & M(Y) \arrow[d, "M_*(\phi)"] \\ \mathcal{M}_E(S) \arrow[r, "\Phi_S"'] & M(S) \end{tikzcd}$$

It follows that if we define $\theta_Y : E(Y) \to M(Y)$ by $\theta_Y(u) = \Phi_Y([Y, id_Y, u])$, then Φ is determined by θ as

$$\Phi_S([Y,\phi,u]) = M_*(\phi)(\theta_Y(u)). \tag{2.1}$$

Conversely, for a given set of maps $\theta = (\theta_Y)_{Y \in Ob(\mathcal{G})}$, define $\Phi = (\Phi_S)_{S \in Ob(\mathcal{G})}$ by (2.1). Then Φ is a morphism of Mackey functors if and only if θ is a morphism in $\mathcal{S}add(G)$. To see this, since Φ defined by (2.1) is always natural with respect to the covariant part of the Mackey functors, it suffices to show that the following (A) and (B) are equivalent.

$$\text{(A)} \quad \Phi_S\mathcal{M}_E^*(f)([Z,\psi,v]) = M^*(f)\Phi_T([Z,\psi,v])$$
$$\big(\forall f\in\mathcal{G}(S,T), \forall[Z,\psi,v]\in\mathcal{M}_E(T)\big),$$
$$\text{(B)} \quad \theta_S(E(f)(v)) = M^*(f)(\theta_T(v)) \quad \big(\forall f\in\mathcal{G}(S,T), \forall v\in E(T)\big)$$

Since

$$\begin{aligned}\Phi_S\mathcal{M}_E^*(f)([Z,\psi,v]) &= \Phi_S\big([S\times_T Z, \pi_S, E(\pi_Z)(v)]\big)\\ &= M_*(\pi_S)(\theta_{S\times_T Z}(E(\pi_Z)(v)))\end{aligned}$$

and

$$\begin{aligned}M^*(f)\Phi_T([Z,\psi,v]) &= M^*(f)M_*(\psi)\theta_Z(v)\\ &= M_*(\pi_S)M^*(\pi_Z)(\theta_Z(v)),\end{aligned}$$

$$\begin{tikzcd} S\times_T Z \arrow[r, "\pi_Z"] \arrow[d, "\pi_S"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\psi"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

we have

$$\text{(A)} \quad \Leftrightarrow \quad M_*(\pi_S)(\theta_{S\times_T Z}(E(\pi_Z)(v))) = M_*(\pi_S)(M^*(\pi_Z)\theta_Z(v)).$$

Obviously, this follows from (B), and conversely (B) follows from this equality if we put $Z=T$ and $\psi = id_T$. $\square$

Let $\mathcal{M}add(G)$ denote the category of additive contravariant functors from $\mathcal{G}$ to the category Mon of monoids.

Remark 2.4. Let $F\in Ob(\mathcal{S}add(G))$. The following are equivalent:
(1) $F\in Ob(\mathcal{M}add(G))$.
(2) F is equipped with cross product maps

$$F(X)\times F(Y)\ni(u,v)\mapsto u\times v\in F(X\times Y)$$

which are functorial in an obvious way in both X and Y, and associative. Moreover, there exists a unit element $\varepsilon_F\in F(\bullet)$ ($\bullet$ denotes the one-element set).

Proof. (1)⇒(2)
For any X, $Y\in Ob(\mathcal{G})$, by using the product in the monoid $F(X\times Y)$, we define

$$u\times v := F(p_X)(u)\cdot F(p_Y)(v),$$

where $p_X : X\times Y\to X$ and $p_Y : X\times Y\to Y$ are the projections.
(2)⇒(1)
For any $X\in Ob(\mathcal{G})$, we define the monoid structure on $F(X)$ by

$$u\cdot v := F(\Delta_X)(u\times v),$$

where $\Delta_X : X \to X \times X$ is the diagonal map. Zero element is given by $F(X \to \bullet)(\varepsilon_F)$. $\square$

Now, if F is in $Ob(\mathcal{M}add(G))$, then $\mathcal{M}_F$ has an additional Green functor structure: In the category (G, S, F), we can define the product of two objects (Y, ϕ, u) and (Z, ψ, v) by $(Y, \phi, u) \cdot (Z, \psi, v) := (Y \times_S Z, \phi \circ \pi_Y = \psi \circ \pi_Z, F(\pi_Y)(u) \cdot F(\pi_Z)(v))$, where π_Y and π_Z are the projections of the fiber product $Y \times_S Z$ of Y, Z over S, and $F(\pi_Y)(u) \cdot F(\pi_Z)(v)$ is the product of $F(\pi_Y)(u)$ and $F(\pi_Z)(v)$ in the monoid $F(Y \times_S Z)$.

$$\begin{tikzcd} Y \times_S Z \arrow[r, "\pi_Z"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\psi"] \\ Y \arrow[r, "\phi"'] & S \end{tikzcd}$$

Thus, $\mathcal{M}_F(S)$ has a natural ring structure, defined by

$$[Y, \phi, u] \cdot [Z, \psi, v] := [(Y, \phi, u) \cdot (Z, \psi, v)].$$

Equivalently, in the view of Remark 2.4, we can describe the Green functor structure on $\mathcal{M}_F$ by the maps

$$\begin{array}{ccc} \mathcal{M}_F(S) \times \mathcal{M}_F(T) & \longrightarrow & \mathcal{M}_F(S \times T) \\ \Cup & & \Cup \\ ([Y, \phi, u], [Z, \psi, v]) & \longmapsto & [Y \times Z, \phi \times \psi, u \times v] \end{array}$$

$(\forall S, T \in Ob(\mathcal{G}))$ (cf. Section 2.2 in [2]), where $u \times v$ is the cross product of $u \in F(Y)$, $v \in F(Z)$. From now on, if F is an object of $\mathcal{M}add(G)$, the Green functor $\mathcal{M}_F$ will be denoted by $\mathcal{A}_F$. $\mathcal{A}_F$ is called the F-Burnside ring functor [4]. If F is commutative, i.e., $F(X)$ is a commutative monoid for each $X \in Ob(\mathcal{G})$, then $\mathcal{A}_F$ becomes a commutative Green functor.

Let $\mathcal{F} : Green(G) \to \mathcal{M}add(G)$ be the forgetful functor, i.e., for any $A \in Ob(Green(G))$, $\mathcal{F}(A)(X) = A(X)$ $(\forall X \in Ob(\mathcal{G}))$, $\mathcal{F}(A)(f) = A^*(f)$ $(\forall f : X \to Y$ in $\mathcal{G})$, and the cross product on $\mathcal{F}(A)$ is the cross product on A.

PROPOSITION 2.5 (cf. Theorem 5.11 in [4]). *The functor $F \mapsto \mathcal{A}_F$ from $\mathcal{M}add(G)$ to $Green(G)$ is left adjoint to $\mathcal{F}$.*

Proof. Let $F \in Ob(\mathcal{M}add(G))$ and $A \in Ob(Green(G))$. By Proposition 2.3, there is a one-to-one correspondence between $\Phi \in Mack(G)(\mathcal{A}_F, A)$ and $\theta \in \mathcal{S}add(G)(F, \mathcal{F}A)$. So, it suffices to show that under this correspondence, Φ is a morphism of Green functors if and only if θ is a morphism in $\mathcal{M}add(G)$.

Since

$$\Phi_S([Y, \phi, u]) \times \Phi_T([Z, \psi, v]) = A_*(\phi)(\theta_Y(u)) \times A_*(\psi)(\theta_Z(v)),$$
$$\Phi_{S \times T}([Y \times Z, \phi \times \psi, u \times v]) = A_*(\phi \times \psi)\big(\theta_{Y \times Z}(u \times v)\big),$$

for any S, $T \in Ob(\mathcal{G})$, $[Y,\phi,u] \in \mathcal{A}_F(S)$, $[Z,\psi,v] \in \mathcal{A}_F(T)$, Φ is a morphism of Green functors if and only if

$$A_*(\phi)(\theta_Y(u)) \times A_*(\psi)(\theta_Z(v)) = A_*(\phi \times \psi)\big(\theta_{Y\times Z}(u \times v)\big)$$

for any $[Y,\phi,u] \in \mathcal{A}_F(S)$, $[Z,\psi,v] \in \mathcal{A}_F(T)$. This is equivalent to

$$\theta_{Y\times Z}(u \times v) = \theta_Y(u) \times \theta_Z(v)$$
$$(\forall Y, Z \in Ob(\mathcal{G}), \forall u \in F(Y), \forall v \in F(Z)),$$

which is equal to the fact that θ is a morphism in $\mathcal{M}add(G)$. □

Let $\mathcal{G}add(G)$ be the category of additive contravariant functors from $\mathcal{G}$ to the category Grp of groups. If F is an object of $\mathcal{M}add(G)$, then F belongs to $Ob(\mathcal{G}add(G))$ if and only if

$$F(X) \in Ob(Grp) \quad \big(\forall X \in Ob(\mathcal{G})\big).$$

For any $F \in Ob(\mathcal{M}add(G))$, if we define

$$\mathcal{U}F(X) := \{u \in F(X) \mid u \text{ is invertible}\}$$

for any X, then $\mathcal{U}F = (\mathcal{U}F(X))_{X\in Ob(\mathcal{G})}$ naturally forms an element $\mathcal{U}F \in Ob(\mathcal{G}add(G))$. Moreover, for any $F_1 \in Ob(\mathcal{G}add(G))$ and $F_2 \in Ob(\mathcal{M}add(G))$, we have a natural isomorphism

$$\mathcal{G}add(G)(F_1, \mathcal{U}F_2) \cong \mathcal{M}add(G)(F_1, F_2).$$

Thus, if we abbreviate $R^\times := \mathcal{U} \circ \mathcal{F}(R)$ for any $R \in Ob(Green(G))$, we obtain the next corollary of Proposition 2.5.

Corollary 2.6. *For any $F \in Ob(\mathcal{G}add(G))$ and any $R \in Ob(Green(G))$, there is a natural isomorphism*

$$\mathcal{G}add(G)(F, R^\times) \cong Green(G)(\mathcal{A}_F, R)$$

Let $Add(G)$ denote the category of additive contravariant functor from $\mathcal{G}$ to the category Ab of Abelian groups. Morphisms are natural transformations.

For $F = \widetilde{Br}_{E/D}$ constructed below, its F-Burnside ring is called the Brauer ring.

Example 2.7. Let E/D be a finite Galois extension of fields with Galois group G. For any $S \in Ob(\mathcal{G})$, put $\widetilde{Br}_{E/D}(S) := Br(\mathcal{G}(S,E))$ where $\mathcal{G}(S,E)$ is regarded as a commutative ring by the pointwise operations, and $Br(\mathcal{G}(S,E))$ is its Brauer group. Recall that by taking the Brauer group of commutative rings, we obtain a covariant functor $Br : (CommRng) \to Ab$ from the category of commutative rings $(CommRng)$ to Ab. For any $f \in \mathcal{G}(S,T)$, we have a ring homomorphism $f^* : \mathcal{G}(T,E) \to \mathcal{G}(S,E)$ defined by the pullback, and if we put $\widetilde{Br}_{E/D}(f : S \to T) := (Br(f^*) : \widetilde{Br}_{E/D}(T) \to \widetilde{Br}_{E/D}(S))$, we obtain an additive functor $\widetilde{Br}_{E/D} \in \mathcal{M}add(G)$ (in fact, $\widetilde{Br}_{E/D} \in Add(G)$). As in [4], we abbreviate the $\widetilde{Br}_{E/D}$-Burnside ring functor $\mathcal{A}_{\widetilde{Br}_{E/D}}$ to $\mathcal{A}_{Br}$, and we call this functor the Brauer ring functor. In particular, we write its value at G as $B(E,D) := \mathcal{A}_{Br}(G)$.

When the extension is trivial (i.e., G is trivial, $E = D$), we have the following structure theorem by Chen [3].

Proposition 2.8 (Corollary 3.4 in [3]). *There is a natural isomorphism $\mathbb{Z}[Br(E)] \cong B(E,E)$ ($B(E,E)$ is denoted by $B(E)$ in [3]), compatible with the inclusions of $Br(E)$ into the multiplicative unit groups.*

$$\begin{tikzcd} & Br(E) \arrow[dl, hook'] \arrow[dr, hook] \arrow[d, phantom, "\circlearrowleft"] & \\ \mathbb{Z}[Br(E)] \arrow[rr, "\cong"] & {} & B(E,E) \end{tikzcd}$$

Remark 2.9. Indeed, Chen defined the Brauer ring $B(R)$ for any commutative ring R, and showed $\mathbb{Z}[Br(R)] \cong B(R)$ for any connected ring R (the word connected means that $Spec(R)$ is connected).

Remark 2.10. For any $H \leq G$, $\mathcal{G}(G/H,E)$ is naturally isomorphic to the fixed field E^H. With this identification, we can easily show that $(\ell_{g,H})^* : E^H \to E^{({}^gH)} = g\cdot(E^H)$ is equal to the multiplication by g (we write this as $(\ell_{g,H})^* = g : E^H \to g\cdot(E^H)$) for any $g \in G$. So, we have $\widetilde{Br}_{E/D}(\ell_{g,H}) = Br(g) : Br(E^H) \to Br(g\cdot(E^H))$.

3. Structure of the F-Burnside ring

We recall the definition of a restriction functor from [1].

Definition 3.1. A restriction functor is a triple $(\mathcal{R}, c, res)$ where $\mathcal{R}, c, res$ are

$\mathcal{R}$: a family of Abelian groups $(\mathcal{R}(H))_{H\leq G}$,
c : a family of conjugation homomorphisms $c_{g,H} : \mathcal{R}(H) \to \mathcal{R}({}^gH)$ ($g \in G$, $H \leq G$),
res : a family of restriction homomorphisms $res^H_K : \mathcal{R}(H) \to \mathcal{R}(K)$ ($K \leq H \leq G$), which satisfy the following conditions:

$(R1)$ $c_{h,H} = res^H_H = id_{\mathcal{R}(H)}$ $(\forall H \leq G, \forall h \in H)$,
$(R2)$ $c_{g'g,H} = c_{g',{}^gH} \circ c_{g,H}$ $(\forall g, g' \in G, \forall H \leq G)$,
$(R3)$ $c_{g,K} \circ res^H_K = res^{{}^gH}_{{}^gK} \circ c_{g,H}$ $(\forall g \in G, \forall K \leq H \leq G)$.

We sometimes abbreviate $(\mathcal{R}, c, res)$ to $\mathcal{R}$. A morphism $\Phi : \mathcal{R} \to \mathcal{S}$ of restriction functors is a family $(\Phi_H : \mathcal{R}(H) \to \mathcal{S}(H))_{H\leq G}$ of Abelian group homomorphisms, compatible with conjugations and restrictions. We write the category of restriction functors $Res(G)$.

Definition 3.2. Let $\mathcal{R}$ be a restriction functor. A stable basis of $\mathcal{R}$ is a family of subsets $\mathcal{B} = (\mathcal{B}(H))_{H \le G}$ such that $\mathcal{B}(H) \subset \mathcal{R}(H)$ is a basis for each $H \le G$, and $c_{g,H}(\mathcal{B}(H)) = \mathcal{B}({}^gH)$ for any $g \in G$ and any $H \le G$.

There is a correspondence between additive functors and restriction functors.

Proposition 3.3. *Let F be an object in $Add(G)$. If we put $\mathcal{R}_F(H) := F(G/H)$, $c_{g,H} = F(\ell_{g,H})$, $res^H_K := F(p^H_K)$ for each $g \in G$ and $K \le H \le G$, then $(\mathcal{R}_F, c, res)$ is a restriction functor.*

Proof. (R1) is trivial. (R2) and (R3) follows from the compatibility of corresponding $\ell_{g,H}$'s and p^H_K's. □

For any F_1, $F_2 \in Ob(Add(G))$ and any $\varphi \in Add(G)(F_1, F_2)$, define $\mathcal{R}_\varphi \in Res(G)(\mathcal{R}_{F_1}, \mathcal{R}_{F_2})$ by $(\mathcal{R}_\varphi)_H = \varphi_{G/H}$. Thus, we obtain a functor $Add(G) \to Res(G)$. We claim this functor gives an equivalence of the categories. A similar argument seems to be well known in the case of Mackey functors, but we include this proof for the reader's convenience. We remark that the author's proof was fairly improved by the referee's suggestion.

Proposition 3.4. *The above functor $F \mapsto \mathcal{R}_F$, $\varphi \mapsto \mathcal{R}_\varphi$ gives an equivalence of categories $Add(G) \xrightarrow{\cong} Res(G)$.*

Proof. We construct a quasi-inverse functor from $Res(G)$ to $Add(G)$ as follows. Suppose that $\mathcal{R}$ is a restriction functor. If X is a finite G-set, then G acts on the Abelian group $V = V_{\mathcal{R}}(X) := \bigoplus_{x \in X} \mathcal{R}(G_x)$, where G_x denotes the stabilizer group of x in X: If $x \in X$ and $u \in \mathcal{R}(G_x)$, denote by u_x the image of u in V, and set $g \cdot u_x = (c_{g,G_x}(u))_{gx}$. This makes sense since $G_{gx} = {}^gG_x$. Then define

$$F_{\mathcal{R}}(X) := (V_{\mathcal{R}}(X))_G,$$

as the group of coinvariants, i.e., the quotient of V by the subgroup generated by the elements $(c_{g,G_x}(u))_{gx} - u_x$, for $g \in G$ and $x \in X$. Denote by $[u_x]$ the image of u_x in this quotient.

If $f : X \to Y$ is a morphism in $\mathcal{G}$, then define $F_{\mathcal{R}}(f) : F_{\mathcal{R}}(Y) \to F_{\mathcal{R}}(X)$ by

$$F_{\mathcal{R}}(f)([u_y]) = \sum_{x \in [G_y \backslash f^{-1}(y)]} [(res^{G_y}_{G_x}(u))_x],$$

where $[G_y \backslash f^{-1}(y)]$ is a set of representatives of G_y-orbits of $f^{-1}(y)$. This makes sense since $G_x \le G_y$ if $f(x) = y$. The right-hand side does not depend on the choice of a set of representatives $[G_y \backslash f^{-1}(y)]$, since for any $x \in f^{-1}(y)$ and any $g \in G_y$, we have

$$\begin{aligned}[(res^{G_y}_{G_{gx}}(u))_{gx}] &= [(res^{{}^gG_y}_{{}^gG_x} \circ c_{g,G_y}(u))_{gx}] \\ &= [(c_{g,G_x} \circ res^{G_y}_{G_x}(u))_{gx}] = [(res^{G_y}_{G_x}(u))_x].\end{aligned}$$

$F_{\mathcal{R}}(f)([u_y])$ is well defined, i.e., $F_{\mathcal{R}}(f)([u_y])$ does not depend on the choice of a representative of $[u_y]$. Indeed, for any $g\in G$ we have

$$\begin{aligned}F_{\mathcal{R}}(f)([(c_{g,G_y}(u))_{gy}]) &= \sum_{x\in[G_y\backslash f^{-1}(y)]}[(res^{G_{gy}}_{G_{gx}}\circ c_{g,G_y}(u))_{gx}]\\ &= \sum_{x\in[G_y\backslash f^{-1}(y)]}[(c_{g,G_x}\circ res^{G_y}_{G_x}(u))_{gx}]\\ &= \sum_{x\in[G_y\backslash f^{-1}(y)]}[(res^{G_y}_{G_x}(u))_x] = F_{\mathcal{R}}(f)([u_y]).\end{aligned}$$

Here, we used the fact that $\{\, gx \mid x\in[G_y\setminus f^{-1}(y)]\}$ is a set of representatives of $G_{gy}\setminus f^{-1}(gy)$ for any fixed $g\in G$.

$F_{\mathcal{R}}$ is a contravariant functor, since for any $X\xrightarrow{f}Y\xrightarrow{f'}Z$ in $\mathcal{G}$, we have

$$\begin{aligned}\big(F_{\mathcal{R}}(f)\circ F_{\mathcal{R}}(f')\big)([u_z]) &= \sum_{y\in[G_z\backslash f'^{-1}(z)]}\ \sum_{x\in[G_y\backslash f^{-1}(y)]}[(res^{G_z}_{G_x}(u))_x]\\ &= \sum_{x\in[G_z\backslash (f'\circ f)^{-1}(z)]}[(res^{G_z}_{G_x}(u))_x]\\ &= \big(F_{\mathcal{R}}(f'\circ f)\big)([u_z]) \quad \big(\forall z\in Z,\forall u\in\mathcal{R}(G_z)\big).\end{aligned}$$

$F_{\mathcal{R}}$ is additive, since for any sum diagram $X\overset{i_X}{\hookrightarrow}X\amalg Y\overset{i_Y}{\hookleftarrow}Y$ in $\mathcal{G}$, we have

$$F_{\mathcal{R}}(X\amalg Y)=\big(V_{\mathcal{R}}(X\amalg Y)\big)_G=\big(V_{\mathcal{R}}(X)\oplus V_{\mathcal{R}}(Y)\big)_G=V_{\mathcal{R}}(X)_G\oplus V_{\mathcal{R}}(Y)_G$$

and

$$\begin{aligned}F_{\mathcal{R}}(i_X): F_{\mathcal{R}}(X\amalg Y) &\longrightarrow F_{\mathcal{R}}(X),\\ [u_x]&\longmapsto [u_x] \quad \big(\forall x\in X,\forall u\in\mathcal{R}(G_x)\big),\\ [v_y]&\longmapsto 0 \quad \big(\forall y\in Y,\forall v\in\mathcal{R}(G_y)\big).\end{aligned}$$

This assignment $\mathcal{R}\mapsto F_{\mathcal{R}}$ gives in fact a functor $Res(G)\to Add(G)$. Indeed, for any morphism $\Phi=(\Phi_H:\mathcal{R}(H)\to\mathcal{S}(H))_{H\le G}\in Res(G)(\mathcal{R},\mathcal{S})$, we have a natural set of morphisms

$$V_{\Phi,X}:V_{\mathcal{R}}(X)\to V_{\mathcal{S}}(X)$$

defined simply by the direct sum, and since $V_{\Phi,X}$ is compatible with G-action on $V_{\mathcal{R}}(X)$ and $V_{\mathcal{S}}(X)$, we obtain a natural transformation

$$F_{\Phi}=\big(F_{\Phi,X}:F_{\mathcal{R}}(X)\to F_{\mathcal{S}}(X)\big)_{X\in Ob(\mathcal{G})}$$

induced by $V_{\Phi,X}$.

This functor $\mathcal{R}\mapsto F_{\mathcal{R}}$ is a quasi-inverse of the functor $F\mapsto\mathcal{R}_F$. Indeed, since

$$\mathcal{R}_{F_{\mathcal{R}}}(H)=F_{\mathcal{R}}(G/H)=\bigg(\bigoplus_{x\in G/H}\mathcal{R}(G_x)\bigg)_G$$

for any $H \leq G$, the natural morphism

$$\mathcal{R}(H) \xrightarrow{\cong} \mathcal{R}(G_{1_G \cdot H}) \hookrightarrow \bigoplus_{x \in G/H} \mathcal{R}(G_x) \overset{\text{quotient}}{\twoheadrightarrow} \Big(\bigoplus_{x \in G/H} \mathcal{R}(G_x) \Big)_G$$

gives a natural isomorphism $\mathcal{R} \overset{\cong}{\to} \mathcal{R}_{F_{\mathcal{R}}}$. Here, the first isomorphism is the identification of $\mathcal{R}(H)$ with the component of $\bigoplus_{x \in G/H} \mathcal{R}(G_x)$ at $x = 1_G \cdot H \in G/H$. And conversely, since

$$F_{\mathcal{R}_F}(X) = \Big(\bigoplus_{x \in X} \mathcal{R}_F(G_x) \Big)_G = \Big(\bigoplus_{x \in X} F(G/G_x) \Big)_G,$$

any set of representatives $\{x_1, \ldots, x_\ell\}$ of G-orbits of X defines a morphism

$$F(X) \overset{\cong}{\to} \bigoplus_{1 \leq i \leq \ell} F(G/G_{x_i}) \hookrightarrow \bigoplus_{x \in X} F(G/G_x) \twoheadrightarrow \Big(\bigoplus_{x \in X} F(G/G_x) \Big)_G,$$

which gives a natural isomorphism $F \overset{\cong}{\to} F_{\mathcal{R}_F}$. Note that this morphism does not depend on the choice of $\{x_1, \ldots, x_\ell\}$, since

$$[u_{gx_i}] = [(c_{g, G_{x_i}}(u))_{gx_i}] = [u_{x_i}]$$

for any $g \in G_{x_i}$ and $u \in F(G/G_{x_i}) = F(G/G_{gx_i})$. $\square$

Finally, let $\mathcal{R}add(G)$ be the category of additive contravariant functors from $\mathcal{G}$ to the category of rings: The word additive for such a functor R means that for any object X and Y of $\mathcal{G}$, the map

$$(R(i_X), R(i_Y)) : R(X \amalg Y) \to R(X) \times R(Y)$$

is a ring isomorphism. Equivalently, R is an object in $Add(G)$, together with cross product maps

$$R(X) \times R(Y) \to R(X \times Y)$$

for any $X, Y \in Ob(\mathcal{G})$, which are natural in X and Y, bilinear, and associative. There is a unit element $\varepsilon \in R(\bullet)$.

Definition 3.5. A restriction functor $(\mathcal{R}, c, res)$ is an algebra restriction functor if $\mathcal{R}(H)$ is a ring for each $H \leq G$, and conjugation and restriction homomorphisms are ring homomorphisms.

In the definition of a morphism $\Phi : \mathcal{R} \to \mathcal{S}$ of restriction functors, if moreover $\mathcal{R}, \mathcal{S}$ are algebra restriction functors and Φ_H are ring homomorphisms for all $H \leq G$, Φ is said to be a morphism of algebra restriction functors. Thus, we have the category of algebra restriction functors $Res_{alg}(G)$. From each restriction functor $(\mathcal{R}, c, res)$, we can construct an algebra restriction functor $(\mathbb{Z}[\mathcal{R}], c, res)$ by putting $(\mathbb{Z}[\mathcal{R}])(H) := \mathbb{Z}[\mathcal{R}(H)]$ for each $H \leq G$. Conjugation and restriction homomorphisms of $\mathbb{Z}[\mathcal{R}]$ are canonically induced by those of $\mathcal{R}$. In the same way as $Res(G) \overset{\cong}{\to} Add(G)$, $\mathcal{R}add(G)$ is shown to be equivalent to $Res_{alg}(G)$.

Here, we recall the definition of the functor $-_+ : \mathit{Res}(G) \to \mathit{Mack}(G)$. For a restriction functor $(\mathcal{R}, c, \mathit{res})$, put $S_{\mathcal{R}}(H) := \bigoplus_{K \le H} \mathcal{R}(K)$. Then H acts on $S_{\mathcal{R}}(H)$ by ${}^h x := c_{h,K}(x)$ $(\forall x \in \mathcal{R}(K) \subset S_{\mathcal{R}}(H), \forall h \in H)$ and we put $\mathcal{R}_+(H) := S_{\mathcal{R}}(H)_H := S_{\mathcal{R}}(H)/(I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H))$ for any $H \le G$, where $I(\mathbb{Z}[H]) \subset \mathbb{Z}[H]$ is the augmentation ideal defined by $I(\mathbb{Z}[H]) = \{\sum_{h \in H} m_h h \mid \sum_{h \in H} m_h = 0, m_h \in \mathbb{Z}\}$. We write $[K, x]_H := x + I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H)$ for any $x \in \mathcal{R}(K) \subset S_{\mathcal{R}}(H)$.

Remark 3.6. The submodule $I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H) \subset S_{\mathcal{R}}(H)$ is generated by $\{x - {}^h x \mid x \in \mathcal{R}(K), h \in H\}$.

Definition 3.7. For any restriction functor $(\mathcal{R}, c, \mathit{res})$, $\mathcal{R}_+ \in \mathit{Mack}(G)$ is defined as follows:

$\mathcal{R}_+(H) = S_{\mathcal{R}}(H)/(I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H))$ as above.
$c_{+g,H} : \mathcal{R}_+(H) \to \mathcal{R}_+({}^g H), [K, x]_H \mapsto [{}^g K, {}^g x]_{{}^g H}$.
$\mathit{res}_{+}{}^H_K : \mathcal{R}_+(H) \to \mathcal{R}_+(K), [L, x]_H \mapsto \sum_{h \in K \backslash H / L} [K \cap {}^h L, \mathit{res}^{{}^h L}_{K \cap {}^h L}({}^h x)]_K$.
$\mathit{ind}_{+}{}^H_K : \mathcal{R}_+(K) \to \mathcal{R}_+(H), [L, x]_K \mapsto [L, x]_H$.

With an appropriate definition for morphisms (see [1]), we obtain a functor $-_+ : \mathit{Res}(G) \to \mathit{Mack}(G)$, which restricts to a functor $-_+ : \mathit{Res}_{alg}(G) \to \mathit{Green}(G)$, and makes the following diagram commutative:

$$\begin{tikzcd}
\mathit{Res}_{alg}(G) \arrow[r, "-_+"] \arrow[d, "\text{forgetful}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathit{Green}(G) \arrow[d, "\text{forgetful}"] \\
\mathit{Res}(G) \arrow[r, "-_+"'] & \mathit{Mack}(G)
\end{tikzcd}$$

Here, for $\mathcal{R} \in \mathit{Ob}(\mathit{Res}_{alg}(G))$, the ring structure on $\mathcal{R}_+(H)$ is defined by

$$[K, x]_H \cdot [L, y]_H := \sum_{h \in K \backslash H / L} [K \cap {}^h L, \mathit{res}^K_{K \cap {}^h L}(x) \cdot \mathit{res}^{{}^h L}_{K \cap {}^h L}({}^h y)]_H \tag{3.1}$$

for each $H \le G$.

Remark 3.8. When a restriction functor $\mathcal{S}$ has a stable basis $\mathcal{B} = (\mathcal{B}(H))_{H \le G}$, if we choose a set of representatives R_H for the H-orbits of the H-sets $\{(K, x) \mid K \le H, x \in \mathcal{B}(K)\}$, then for each $H \le G$, $\mathcal{S}_+(H)$ is a free $\mathbb{Z}$-module with a basis $\{[K, x]_H \mid (K, x) \in R_H\}$.

Now, when $\mathcal{S} = \mathbb{Z}[\mathcal{R}]$, if we take $\mathcal{B}(H) := \mathcal{R}(H)$, then $\mathcal{B}$ is a stable basis for $\mathcal{S}$. As a corollary, we obtain a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathcal{R}]_+(H)$ as follows.

Corollary 3.9. *For each $H \le G$, $\mathbb{Z}[\mathcal{R}]_+(H)$ is a free $\mathbb{Z}$-module over the basis $\{[K, x]_H \mid (K, x) \in R_H\}$, where R_H is a set of representatives for the H-orbits of $\{(K, x) \mid K \le H, x \in \mathcal{R}(K)\}$.*

For the functor $-_+$, the following adjoint property is known.

Remark 3.10 (Proposition I.4.1 in [1]). The functor

$$-_+ : Res(G) \longrightarrow Mack(G)$$
$$\big(\text{resp.} \ : Res_{alg}(G) \longrightarrow Green(G)\big)$$

is left adjoint to the forgetful functor

$$\mathcal{O} : Mack(G) \longrightarrow Res(G)$$
$$\big(\text{resp.} \ : Green(G) \longrightarrow Res_{alg}(G)\big).$$

There is a forgetful functor $gr : Green(G) \to \mathcal{R}add(G)$, obtained by forgetting the covariant part of the structure of Green functors. In the same way, we obtain a commutative diagram of categories and forgetful functors

$$\begin{tikzcd} Green(G) \arrow[r, "gr"] \arrow[d, "gm"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{R}add(G) \arrow[r, "rm"] \arrow[d, "ra"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}add(G) \arrow[d, "ms"] \\ Mack(G) \arrow[r, "ma"] & Add(G) \arrow[r, "as"] & \mathcal{S}add(G). \end{tikzcd} \tag{3.2}$$

Remark 3.11. Let $\mathcal{R}$ be a restriction functor for G, and set $F := F_{\mathcal{R}}$. Then for any $X \in Ob(\mathcal{G})$, the module $\mathcal{R}_+(X)$ is isomorphic to the quotient of $\mathcal{A}_F(X)$ by the elements of the form

$$(Z, \phi, u+v) - (Z, \phi, u) - (Z, \phi, v),$$

where $\phi : Z \to X$ is a morphism in $\mathcal{G}$, and where $u, v \in \mathcal{R}(Z)$. Moreover, the family of projection maps

$$\pi_X : \mathcal{A}_F(X) \to \mathcal{R}_+(X)$$

is a morphism of Mackey functors $\mathcal{A}_F \to \mathcal{R}_+$.

Proof. By letting $\mathcal{R}_\sharp(X)$ be the quotient of $\mathcal{A}_F(X)$ as above, we obtain a quotient Mackey functor $\mathcal{R}_\sharp$ of $\mathcal{A}_F$. Remark that there is a commutative diagram

$$\begin{tikzcd} Mack(G) \arrow[rr, "ma"] \arrow[rrrr, bend left, "\varepsilon"] \arrow[dr, "\mathcal{O}"'] \arrow[rr, phantom, bend left=20, "\circlearrowleft"] & \arrow[d, phantom, "\circlearrowleft"] & Add(G) \arrow[rr, "as"] & & \mathcal{S}add(G). \\ & Res(G) \arrow[ur, "\mathcal{R}\mapsto F_{\mathcal{R}}"'] & & & \end{tikzcd}$$

By Proposition 2.3, there is a functorial isomorphism

$$\begin{array}{ccc} Mack(G)(\mathcal{A}_{F_{\mathcal{R}}}, M) & \xrightarrow{\cong} & \mathcal{S}add(G)(F_{\mathcal{R}}, \mathcal{E}M) \\ \Cup & & \Cup \\ \Phi & \longmapsto & \theta \end{array}$$

in the notation in the proof of Proposition 2.3. Since

$$\Phi([Z,\phi,u]) = M_*(\phi)\theta_Z(u),$$

we have

$$\begin{aligned}
&\Phi \text{ factors } \mathcal{R}_\sharp\\
&\quad\Leftrightarrow\quad \Phi_S([Z,\phi,u+v]-[Z,\phi,u]-[Z,\phi,v])=0\\
&\qquad\qquad \big(\forall\phi\in\mathcal{G}(Z,S),\forall u,v\in F_{\mathcal{R}}(Z)\big)\\
&\quad\Leftrightarrow\quad M_*(\phi)\theta_Z(u+v)-M_*(\phi)\theta_Z(u)-M_*(\phi)\theta_Z(v)=0\\
&\qquad\qquad \big(\forall\phi\in\mathcal{G}(Z,S),\forall u,v\in F_{\mathcal{R}}(Z)\big)\\
&\quad\Leftrightarrow\quad \theta_Z(u+v)-\theta_Z(u)-\theta_Z(v)=0 \quad \big(\forall Z\in Ob(\mathcal{G}),\forall u,v\in F_{\mathcal{R}}(Z)\big)\\
&\quad\Leftrightarrow\quad \theta\in Add(G)(F_{\mathcal{R}},ma(M)).
\end{aligned}$$

Thus, we obtain a functorial isomorphism

$$\begin{aligned}
Mack(G)(\mathcal{R}_\sharp,M) &\xrightarrow{\cong} Add(G)(F_{\mathcal{R}},ma(M))\\
&= Add(G)\big(F_{\mathcal{R}},F_{\mathcal{O}(M)}\big)\\
&\cong Res(G)(\mathcal{R},\mathcal{O}(M)).
\end{aligned}$$

So, the functor $-_\sharp : Res(G)\longrightarrow Mack(G)$ is left adjoint to $\mathcal{O}$, and must agree with $-_+$. □

In diagram (3.2), the composition $as\circ ma$ is the forgetful functor $\mathcal{E}$. So, the left adjoint of $\mathcal{E}$ is the composition of the left adjoint of as, followed by the left adjoint of ma. The left adjoint of as is the "free Abelian group functor," sending an object E of $\mathcal{S}add(G)$ to the additive functor $\mathbb{Z}[E]$, defined in the obvious way by $(\mathbb{Z}[E])(X)=\mathbb{Z}[E(X)]$, for any G-set X. The left adjoint of ma is the composition

$$Add(G)\xrightarrow{\simeq} Res(G)\xrightarrow{-_+} Mack(G).$$

By the uniqueness of the left adjoint of $\mathcal{E}$, it follows that for any additive contravariant functor $E\in Ob(\mathcal{S}add(G))$, there is a natural isomorphism of Mackey functors

$$\big(\mathcal{R}_{\mathbb{Z}[E]}\big)_+\xrightarrow{\cong}\mathcal{M}_E.$$

Similarly, the composition $rm\circ gr$ is equal to the forgetful functor $\mathcal{F}$. A similar argument shows that for any $F\in Ob(\mathcal{M}add(G))$, there is a natural isomorphism of Green functors

$$\big(\mathcal{R}_{\mathbb{Z}[F]}\big)_+\xrightarrow{\cong}\mathcal{A}_F.$$

Thus, we obtained the following adjoint isomorphisms.

PROPOSITION 3.12. (1) *For any $E\in Ob(\mathcal{S}add(G))$, there is a natural isomorphism of Mackey functors*

$$\big(\mathcal{R}_{\mathbb{Z}[E]}\big)_+\xrightarrow{\cong}\mathcal{M}_E.$$

(2) *For any $F \in Ob(\mathcal{M}add(G))$, there is a natural isomorphism of Green functors*

$$\left(\mathcal{R}_{\mathbb{Z}[F]}\right)_+ \xrightarrow{\cong} \mathcal{A}_F.$$

Since obviously $\mathcal{R}_{\mathbb{Z}[F]} \cong \mathbb{Z}[\mathcal{R}_F]$ for any $F \in Ob(Add(G))$, we have the following structure theorem for F-Burnside rings.

THEOREM 3.13. *For any $F \in Ob(Add(G))$, there is a natural isomorphism of Green functors*

$$(\mathbb{Z}[\mathcal{R}_F])_+ \xrightarrow{\cong} \mathcal{A}_F.$$

4. Applications

We state some results obtained from Theorem 3.13.

First, we see the structure of the Brauer ring. By Theorem 3.13, especially we have $\mathbb{Z}[\mathcal{R}_F]_+(G) \overset{\cong}{\to} \mathcal{A}_F(G)$. By putting $F = \widetilde{Br}_{E/D}$, we obtain the next corollary.

COROLLARY 4.1. *For any finite Galois extension E/D of fields with Galois group G, we have a ring isomorphism*

$$B(E,D) \cong \left(\bigoplus_{H \leq G} \mathbb{Z}[Br(E^H)]\right) \Big/ \left(I(\mathbb{Z}[G]) \cdot \bigoplus_{H \leq G} \mathbb{Z}[Br(E^H)]\right),$$

where the ring structure of the right-hand side is defined by (3.1) in Definition 3.7. When G is trivial, this is nothing other than Proposition 2.8.

As mentioned in [4], if F is the trivial functor, then $\mathcal{A}_F$ is canonically isomorphic to the (ordinary) Burnside ring functor Ω. We can also induce this isomorphism from Theorem 3.13, since there is a canonical isomorphism $\Omega \cong \underline{\mathbb{Z}}_+$, where $\underline{\mathbb{Z}}$ is the constant algebra restriction functor with value $\mathbb{Z}$ (see Example I.2.3 in [1]).

Theorem 3.13 gives us the structure of the F-Burnside ring functors, and allows us to deduce some properties of them. We also remark here that conversely this isomorphism gives an explicit categorical meaning (Definition 2.1) to the functor $\mathcal{S}_+$, in the case where $\mathcal{S} = \mathbb{Z}[\mathcal{R}]$ for a certain $\mathcal{R}$.

For any algebra restriction functor $\mathcal{A}$ (in fact, being an algebra conjugation functor is enough (cf. [1])), we have a Green functor defined by $\mathcal{A}^+(H) = (\prod_{K \leq H} \mathcal{A}(K))^H$. Here H acts on $\prod_{K \leq H} \mathcal{A}(K)$ by conjugation, similarly *as* in the definition of $\mathcal{A}_+$. And $\mathcal{A}^+(H)$ has a canonical ring structure, induced by the componentwise multiplication of $\prod_{K \leq H} \mathcal{A}(K)$. This construction gives us a functor $-^+ : Res_{alg}(G) \to Green(G)$. There is a natural Green functor morphism $\rho^{\mathcal{A}} : \mathcal{A}_+ \to \mathcal{A}^+$, called the mark morphism. As in Proposition I.3.2 in [1], for any $H \leq G$, there exists a map $\sigma^{\mathcal{A}}_H : \mathcal{A}^+(H) \to \mathcal{A}_+(H)$ such that $\sigma^{\mathcal{A}}_H \circ \rho^{\mathcal{A}}_H = |H| \cdot id$, $\rho^{\mathcal{A}}_H \circ \sigma^{\mathcal{A}}_H = |H| \cdot id$. Since $\mathbb{Z}[\mathcal{R}_F]_+(H)$ is free for any $H \leq G$ (and so, it has no $|H|$-torsion), we obtain the following proposition.

PROPOSITION 4.2. *For each $H \le G$, the component of the mark morphism at H*

$$\rho_H : \mathbb{Z}[\mathcal{R}_F]_+(H) \to \mathbb{Z}[\mathcal{R}_F]^+(H)$$

is injective.

As the (componentwise) scalar extension of $\mathbb{Z}[\mathcal{R}_F]$ by $\mathbb{Q}$, the functor $\mathbb{Q}[\mathcal{R}_F]$ has a simpler structure *as* in [4]. We can realize this with the mark morphism. Since $|H|$ is invertible in $\mathbb{Q}$ for any $H \le G$, we obtain the following isomorphism of Green functors.

PROPOSITION 4.3. *The mark morphism $\rho : \mathbb{Q}[\mathcal{R}_F]_+ \to \mathbb{Q}[\mathcal{R}_F]^+$ is an isomorphism.*

In view of Theorem 3.13, this is nothing other than the following corollary.

COROLLARY 4.4 (Theorem 3.13 in [4]). *For any additive functor F, there is an isomorphism*

$$\mathcal{A}_F(G) \otimes \mathbb{Q} \cong \prod_{a \in P(G)} \mathbb{Q}[F(G/G_a)]^{N_H(G_a)},$$

where $N_G(K)$ denotes the normalizer of K in G for each $K \le G$.

In Proposition 4.3, the domain $\mathbb{Q}[\mathcal{R}_F]_+$ is naturally isomorphic to $\mathbb{Z}[\mathcal{R}_F]_+ \otimes \mathbb{Q}$ (see Lemma I.5.1 in [1]). As for the codomain $\mathbb{Q}[\mathcal{R}_F]^+$, we have an isomorphism

$$\begin{aligned}
&\mathbb{Q}[\mathcal{R}_F]^+(H) \\
&\quad = \Big(\prod_{K \le H} \mathbb{Q}[\mathcal{R}_F](K)\Big)^H \\
&\quad = \Big\{ (x_K)_{K \le H} \in \prod_{K \le H} \mathbb{Q}[\mathcal{R}_F](K) \mid {}^h(x_K) = x_{({}^hK)}\ (\forall K \le H, \forall h \in H) \Big\} \\
&\quad \cong \Big\{ (x_a)_{a \in P(H)} \in \prod_{a \in P(H)} \mathbb{Q}[\mathcal{R}_F](H_a) \mid {}^h(x_a) = x_a\ \big(\forall h \in N_H(H_a)\big) \Big\} \\
&\quad = \prod_{a \in P(H)} (\mathbb{Q}[\mathcal{R}_F](H_a))^{N_H(H_a)}
\end{aligned}$$

for each $H \le G$. Thus, we obtain

$$\begin{aligned}
\mathcal{A}_F(G) \otimes \mathbb{Q} &\cong (\mathbb{Z}[\mathcal{R}_F]_+(G)) \otimes \mathbb{Q} \\
&\cong \prod_{a \in P(G)} (\mathbb{Q}[\mathcal{R}_F](G_a))^{N_H(G_a)} = \prod_{a \in P(G)} \mathbb{Q}[F(G/G_a)]^{N_H(G_a)}.
\end{aligned}$$

Theorem 3.13 also enables us to calculate the Brauer ring for some (nontrivial) finite Galois extensions. Here, we consider the case of $\mathbb{C}/\mathbb{R}$.

Corollary 4.5. *We have*

$$B(\mathbb{C},\mathbb{R}) \cong \mathbb{Z}[X,Y]/(X^2-1, Y^2-2Y, XY-Y).$$

Proof. Since $Gal(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ and $Br(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, we can write them as $Gal(\mathbb{C}/\mathbb{R}) = \{1,\sigma\}$ and $Br(\mathbb{R}) = \{1,h\}$. By Corollary 4.1, we have

$$B(\mathbb{C},\mathbb{R}) \cong \big(\mathbb{Z}[Br(\mathbb{R})] \oplus \mathbb{Z}[Br(\mathbb{C})]\big)/I(\mathbb{Z}[G]) \cdot \big(\mathbb{Z}[Br(\mathbb{R})] \oplus \mathbb{Z}[Br(\mathbb{C})]\big)$$

(we abbreviate $\widetilde{Br}_{\mathbb{C}/\mathbb{R}}$ to $\widetilde{Br}$). Here, we have

$$I(\mathbb{Z}[G]) = \{k\cdot 1 + \ell\cdot\sigma \mid k,\ell \in \mathbb{Z}, k+\ell = 0\} = \{k\cdot(1-\sigma) \mid k \in \mathbb{Z}\}.$$

By the definition of the conjugation of $\mathcal{R}_{\widetilde{Br}}$, for any $H \le G$ we have $c_{\sigma,H} = \widetilde{Br}(\ell_{\sigma,H}) = Br(\sigma) : Br(\mathbb{C}^H) \xrightarrow{\cong} Br(\sigma\cdot(\mathbb{C}^H))$. So both the maps

$$\widetilde{Br}(\ell_{\sigma,G}) : Br(\mathbb{R}) \to Br(\mathbb{R}) \quad (\cong \mathbb{Z}/2\mathbb{Z}),$$
$$\widetilde{Br}(\ell_{\sigma,\{1\}}) : Br(\mathbb{C}) \to Br(\mathbb{C}) \quad (= 0)$$

are identities, and we obtain $I(\mathbb{Z}[G])\cdot(\mathbb{Z}[Br(\mathbb{R})] \oplus \mathbb{Z}[Br(\mathbb{C})]) = 0$. Thus, $B(\mathbb{C},\mathbb{R})$ is equal to $\mathbb{Z}[Br(\mathbb{R})] \oplus \mathbb{Z}[Br(\mathbb{C})]$ as a module.

Finally, we compute its ring structure. To distinguish, let e and f denote the unit element of $Br(\mathbb{R})$ and $Br(\mathbb{C})$, respectively. Then in the notation after Definition 3.2, we have $B(\mathbb{C},\mathbb{R}) = \mathbb{Z}\cdot[G,e]_G \oplus \mathbb{Z}\cdot[G,h]_G \oplus \mathbb{Z}\cdot[\{1\},f]_G$. And for this basis $\{[G,e], [G,h], [\{1\},f]\}$ of $B(\mathbb{C},\mathbb{R})$ (we omit the subscript G), their multiplications are calculated by the formula (3.1) in Definition 3.7 *as* follows:

$$[G,e]^2 = [G,e], \qquad [G,h]^2 = [G,e],$$
$$[G,e]\cdot[G,h] = [G,h], \qquad [G,h]\cdot[\{1\},f] = [\{1\},f],$$
$$[G,e]\cdot[\{1\},f] = [\{1\},f], \qquad [\{1\},f]^2 = 2[\{1\},f].$$

So, if we put $X = [G,h]$ and $Y = [\{1\},f]$, $B(\mathbb{C},\mathbb{R})$ becomes isomorphic to $\mathbb{Z}[X,Y]/(X^2-1, Y^2-2Y, XY-Y)$. ∎

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