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FREE MARKOV PROCESSES AND STOCHASTIC DIFFERENTIAL EQUATIONS IN VON NEUMANN ALGEBRAS

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Abstract. Free Markov processes are investigated in Voiculescu's free probability theory. We show that Voiculescu's free Markov property implies a property called "weak Markov property", which is the classical Markov property in the commutative case; while, in the general case, the "weak Markov property" is the same as the Markov property defined by Bozejko, Kummer, and Speicher. We also show that a kind of stochastic differential equations driven by free Levy processes has solutions. The solutions are free Markov processes.

Introduction

The concept of *reduced free products* of von Neumann algebras was introduced by Ching [Ch] in 1973. Later, Voiculescu [V1] and Avitzour [Av] introduced the free product construction in the framework of C^*-algebras independently in 1980s. Since then, the study on free product operator algebras has been considerably developed and become an independent and important direction of research free probability theory. This theory has many interactions with other subjects in mathematics such that quantum probability, operator algebras, and operator spaces ([RX], [S], [V2], [VDN]).

The study on stochastic processes is a vast research area in free probability. The analogues of classical Brownian motion and Levy processes in free probability were introduced in 1990s. The concept of Markov processes of bounded self-adjoint operators in a (tracial) probability space (i.e., a finite von Neumann algebra with a normal tracial faithful state) was introduced by Bozejko, Kummer, and Speicher in [BKS]. They showed that a Markov process in a (noncommutative) probability space can be "realized" as a classical Markov

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process. Moreover, they defined transition functions for a Markov process (see 4.3 and 4.4 in [BKS]). Voiculescu [V3], [V4] defined Markovinian property for a parameterized family of $*$-homomorphisms from a unital C^*-algebra into a finite von Neumann algebra. Voiculescu's idea to define the Markovinian property is the free independence with amalgamation of the "future" algebra and the "last" algebra with respect to the "present" algebra. We adapt Voiculescu's definition for free Markov processes. So far, most of the research work on stochastic processes in free probability is on free Brownian motion and free Lévy processes (see [A], [BT1], [BT2], [BeP], [BeV], [Bi1], [Bi2], [Bi3], [BiS1], [BiS2], [Bo], [BS], [GM]).

In this article, we study the properties of free Markov processes and a kind of stochastic differential equations driven by free Lévy processes. We get some results in the following three aspects.

I. The properties of free Markov processes.

(1) We classify the relation between the free Markov property, the classical Markov property, and Markov processes defined in [BKS]. We show that a free Markov process has a weak Markov property (Theorem 2.5), that is, the "future" subalgebra and the "past" subalgebra are "conditionally perpendicular" with respect to the "present" algebra. In the case of Abelian algebras, the weak Markov property is the same as the classical Markov property (Theorem 2.4); while, in general case, the weak Markov property is the same as the Markov property defined in [BKS] (Theorem 2.6). Hence, we see that the concept of Markov processes defined in [BKS] is a classical version of the Markov property in a noncommutative probability space. Voiculescu's concept of free Markov property is a free version of the classical Markov property in free probability.

(2) Biane [Bi2] showed that every free increments process has Markov transition functions. [BKS] defined transition functions for a Markov process. But, there were no properties of the transition functions given in [BKS]. We now show that every weak Markov process has transition functions. The transition functions have very similar properties to those transition functions of a classical Markov process. In the commutative case, these transition functions determine the weak Markov property completely (Theorem 2.8).

II. Free stochastic differential equations. Certain free stochastic differential equations driven by free Brownian motion were studied by Biane and Speicher in 2001 (see [BiS2]). They showed that the free stochastic differential equations driven by free Brownian motion have solutions, and the solutions have the free Markov property (see [BiS2]).

We consider the similar free stochastic differential equations driven by *free Lévy processes.* We prove that the equations have solutions (Theorem 3.6).

Under certain conditions, the solutions are free Markov processes (Theorems 3.7 and 3.8). Our proofs rely on a free Burkholder–Gundy type inequality in L^2-norm (for the Lévy case) proved by M. Anshelevich [A]. A similar inequality in operator norm for stochastic integrals with respect to free Brownian motion was obtained in [BiS1]. Our results provide a method to find examples of free Markov processes of random variables with noncompactly supported distributions.

III. Free Ornstein–Uhlenbeck equations. Biane and Speicher [BiS2] studied the solution to the following stochastic differential equation (a special case of the free stochastic differential equations mentioned previously)

$$X_t = X_0 - \lambda \int_0^t X_s\, ds + S_t, \quad t \geq 0, \tag{0.1}$$

where $\lambda > 0$, $\{S_t : t \geq 0\}$ is free Brownian motion, and the initial variable X_0 and $\{S_t : t \geq 0\}$ are free. They proved that the unique solution to (0.1) has the following form

$$X_t = e^{-\lambda t} X_0 + e^{-\lambda t} \int_0^t e^{\lambda s}\, dS_s, \quad t \geq 0. \tag{0.2}$$

The process given in (0.2) is called a *free Ornstein–Uhlenbeck process* (briefly, a free OU process). They also showed that its limit distribution is a semicircular law. Barndorff-Nielsen and Thorbjornsen [BT2] mentioned free OU processes driven by free Lévy processes (but there were no details given).

In this paper, we study similar equations to (0.1), driven by free Lévy processes. It is shown that the solution of the equation has the same form as (0.2), a free OU process driven by a free Lévy process (Theorem 4.3). We show that a probability measure on $\mathbb{R}$ is freely self-decomposable if and only if it is the limit distribution of a free OU process driven by a free Lévy process (Theorem 4.4).

We study free OU processes in detail in [G].

The paper is organized as follows. In Section 1, we review some basic concepts and results in classical Markov processes, unbounded operators affiliated with a von Neumann algebra and operator Lipschitz functions. In Section 2, we study some basic properties of free Markov processes. Section 3 is devoted to the study of a kind of free stochastic differential equations driven by free Lévy processes. We prove the existence, uniqueness, and the free Markov property of the solution to this kind of systems of the equations. The purpose of Section 4 is to study free Ornstein–Uhlenbeck processes in some details.

1. Preliminaries

In this section, we briefly discuss relevant background materials on classical Markov processes, stochastic processes in noncommutative probability spaces, unbounded operators affiliated with a von Neumann algebra, the convergence in distribution for stochastic processes, and operator Lipschitz inequality. We refer to [BT2], [KR], and [VDN] for the general basics on free probability, operator algebras, and unbounded operators affiliated with a von Neumann algebra and the convergence of unbounded operators in distribution, respectively.

We first review some basic concepts in free stochastic processes (see, e.g., [BKS]).

Let $\mathcal{M}$ be a finite von Neumann algebra, τ be a faithful normal tracial state on $\mathcal{M}$. We call $(\mathcal{M}, \tau)$ a *(tracial) probability space,* (or a W^*-probability space). A *random variable* is a self-adjoint operator $X \in \mathcal{M}$. A *stochastic process* on $(\mathcal{M}, \tau)$ is a family $(X_t)_{t\geq 0}$ of random variables X_t in $\mathcal{M}$. The *distribution* of a random variable $X \in \mathcal{M}$ is a probability measure μ_X on the spectrum σ_X of X determined by the equation

$$\tau(X^n) = \int x^n \, d\mu_X(x) \quad \forall n = 0, 1, 2, \ldots.$$

Free Brownian motion ([BiS1], [BiS2]). Let $(\mathcal{M}, \tau)$ be a tracial probability space with filtration $\{\mathcal{M}_t : t \geq 0\}$ (that is, $\{\mathcal{M}_t : t \geq 0\}$ is a family of von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{M}_t \subseteq \mathcal{M}_s$, when $0 \leq t \leq s$). A stochastic process $\{S_t : t \geq 0\}$ is called $(\mathcal{M}_t)$-*free Brownian motion*, if $S_0 = 0$, and, for $0 \leq s < t$, $S_t - S_s$ and $\mathcal{M}_s$ are free, and $S_t - S_s$ has a semicircular distribution of mean zero and variance $t - s$.

To define free Lévy processes, we need some basics on unbounded operators (see [BT2]).

Unbounded operators and convergence in distribution. Let $(\mathcal{A},, \tau)$ be a tracial probability space with $\mathcal{A},$ acting on the Hilbert space $\mathcal{H}$ $(= L^2(\mathcal{A},, \tau))$ by left multiplications. A self-adjoint (unbounded) operator A defined on a dense subspace of $\mathcal{H}$ is said to *be affiliated with* $\mathcal{A},$, if all spectral projections of A lie in $\mathcal{A},$. Generally, a closed densely defined operator T on $\mathcal{H}$ is said to *be affiliated with* $\mathcal{A},$, if $T = UA$, for some U in $\mathcal{A},$, and self-adjoint operator A affiliated with $\mathcal{A},$, where $T = UA$ is the polar decomposition of T. Denoted by $\widetilde{\mathcal{A},}$ the algebra of all densely defined and closed (unbounded) operators affiliated with $\mathcal{A},$ (see [BT2], [KR], [MV], [N] for details). Elements in $\widetilde{\mathcal{A},}$ are called random variables, in general, with noncompactly supported distributions.

Let $\widetilde{\mathcal{A},}_{sa}$ be the set of all self-adjoint elements in $\widetilde{\mathcal{A},}$. Given X in $\widetilde{\mathcal{A},}_{sa}$, let $C^*(X)$ be the unital C^*-algebra generated by $\{f(X) : f \in BC(\mathbb{R})\}$, where $BC(\mathbb{R})$ is the space of all bounded continuous functions on $\mathbb{R}$. Let $W^*(X)$ be the von Neumann subalgebra of $\mathcal{A},$ generated by $C^*(X)$. Let $U|A|$ be the polar decomposition of the element A in $\widetilde{\mathcal{A},}$, $W^*(A)$ be the von Neumann subalgebra of $\mathcal{A},$ generated by U and $W^*(|A|)$. The family $\{X_i \in \widetilde{\mathcal{A},} : i \in \Lambda\}$

is said to be *free* if $\{W^*(X_i) : i \in \Lambda\}$ forms a free family. Similarly, we can define freeness with amalgamation for elements in $\widetilde{\mathcal{A}}$, (see [VDN]).

The distribution of element $X \in \widetilde{\mathcal{A}}_{sa}$, denoted by $\mu(X)$, is a linear functional on $BC(\mathbb{R})$, which maps function f in $BC(\mathbb{R})$ to $\tau(f(X))$. Let $A, B \in \widetilde{\mathcal{A}}_{,sa}$ be freely independent elements with distributions $\mu(A)$ and $\mu(B)$, respectively. We call the distribution μ of $A+B$ the *freely additive convolution* of $\mu(A)$ and $\mu(B)$, denoted by $\mu(A) \boxplus \mu(B)$. A probability measure on $\mathbb{R}$ is $\boxplus$ *(or free)-infinitely divisible*, if for every natural number n, there exists a probability measure μ_n on $\mathbb{R}$ such that

$$\mu = \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \quad \text{times}}.$$

In classical probability theory, there are very similar concepts. Let f and g be independent (classical) random variables on a (classical) probability space (Ω, Σ, μ) with distributions $\mu(f)$ and $\mu(g)$, respectively. The distribution $\mu(f+g)$ of $f+g$ is called the *convolution* of $\mu(f)$ and $\mu(g)$, denoted by $\mu(f) * \mu(g)$. A probability measure μ on $\mathbb{R}$ is *infinitely divisible* if for every natural number n, there is a probability measure μ_n such that

$$\mu = \underbrace{\mu_n * \cdots * \mu_n}_{n \quad \text{times}}.$$

We use $\mathcal{ID}(\boxplus)$ and $\mathcal{ID}(*)$ to denote the set of all $\boxplus$-infinitely divisible distributions on $\mathbb{R}$ and that of all infinitely divisible measures on $\mathbb{R}$, respectively.

A probability measure μ on $\mathbb{R}$ is said to be *free (or $\boxplus$) self-decomposable* if, for any $c \in (0,1)$, there exists a probability measure μ_c on $\mathbb{R}$ such that $\mu = D_c\mu \boxplus \mu_c$, where measure $D_c\mu$ is defined by the formula $D_c\mu(B) = \mu(c^{-1}B)$, for Borel set $B \subseteq \mathbb{R}$. A sequence (σ_n) of finite measures on $\mathbb{R}$ is said to *converge weakly* to a finite measure σ on $\mathbb{R}$, denoted by $\sigma_n \xrightarrow{\text{w}} \sigma$, if for all f in $BC(\mathbb{R})$,

$$\int_{\mathbb{R}} f(t)\sigma_n(dt) \to \int_{\mathbb{R}} f(t)\sigma(dt),$$

as $n \to \infty$. For X_n, X in $\widetilde{\mathcal{A}}_{sa}$, $\{X_n\}_{n=1}^{\infty}$ is said to converge to X in distribution, denoted by $X_n \xrightarrow{\text{d}} X$, if $\mu(X_n) \xrightarrow{\text{w}} \mu(X)$. Given X_n, X in $\widetilde{\mathcal{A}}$, $\{X_n\}_{n=1}^{\infty}$ is said to converge to X in probability, denoted by $X_n \xrightarrow{\text{p}} X$, if $|X_n - X| \xrightarrow{\text{d}} 0$. By [BT2], for $X_n, X \in \widetilde{\mathcal{A}}_{sa}$, $X_n \xrightarrow{\text{p}} X$ if and only if $X_n - X \xrightarrow{\text{d}} 0$, and $X_n \xrightarrow{\text{p}} X$ implies that $X_n \xrightarrow{\text{d}} X$. For $X, Y \in \widetilde{\mathcal{A}}_{sa}$, $X \overset{\text{d}}{=} Y$ means $\mu(X) = \mu(Y)$.

Free Lévy processes. A family $\{S_t : t \geq 0\}$ of elements in $\widetilde{\mathcal{A}}_{,sa}$ is a *free Lévy process*, if $S_0 = 0$, it has free increments (that is, $S_{t_0}, S_{t_1} - S_{t_0}, \ldots, S_{t_n} - S_{t_{n-1}}$ are free, for $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$), it is stationary [that is, $\mu(S_{t+s} - S_s) = \mu(S_t)$, for $s, t \in (0, \infty)$] and $S_t \xrightarrow{\text{d}} 0$, as $t \to 0$ (see [A], [BT2], [BiS2]). We say that a free Lévy process $\{S_t : t \geq 0\}$ is adapted to the filtration $\{\mathcal{A}_{,t} : t \geq 0\}$

of $\mathcal{A}$ (we call it an $\mathcal{A}_t$*-free Lévy process*), if $W^*(S_t) \in \mathcal{A}_t$, for $t \geq 0$, and $S_t - S_s$ and $\mathcal{A}_s$ are free, for $0 \leq s < t$ (see [A]).

Classical Markov processes. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\{f_t : t \geq 0\}$ a family of measurable functions from $(\Omega, \mathcal{F}, \mu)$ into a locally compact Hausdorff space X with a Borel σ-algebra $\mathcal{B}$. Define $\mathcal{F}_{\leq t}$ to be the σ-subalgebra of $\mathcal{F}$ generated by $f_s^{-1}(B)$ for all Borel subsets B of X and $s \leq t$. Similarly, one may define $\mathcal{F}_{=t}$ and $\mathcal{F}_{\geq t}$. The family $\{f_t : t \geq 0\}$ is a *Markov process* if

$$\mathbf{P}(AB|\mathcal{F}_{=t}) = \mathbf{P}(A|\mathcal{F}_{=t})\mathbf{P}(B|\mathcal{F}_{=t}),$$

for all A in $F_{\leq t}$, B in $\mathcal{F}_{\geq t}$, where $\mathbf{P}(\cdot|\mathcal{F}_{=t})$ is the conditional probability with respect to $\mathcal{F}_{=t}$. Given $s \leq t$, $x \in X$, and Borel subset $\Gamma \subseteq X$, we can define a transition function $P(s,x,t,\Gamma) = P(f_t \in \Gamma | f_s = x)$. Then $\{f_t : t \geq 0\}$ is a Markov process if and only if $P(s,x,t,\Gamma)$ has the following properties (see 8.1.3 and 8.2.3 in [W]).

(1) When s,t,x are given, $P(s,x,t,\cdot)$ is a probability measure on $\mathcal{B}$.
(2) When s,t,Γ are given, $P(s,\cdot,t,\Gamma)$ is a measurable function on $(\mathbb{R},\mathcal{B})$.
(3) $P(s,x,s,\Gamma) = \chi_\Gamma(x)$.

Operator-valued Lipschitz functions. A map $Q : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ is called *Lipschitz* (or *operator-valued Lipschitz*) with respect to $\|\cdot\|_2$, if there exists a constant $C > 0$ such that

$$\|Q(X_1,\ldots,X_k) - Q(Y_1,\ldots,Y_k)\|_2 \leq C \sum_{i=1}^{k} \|X_i - Y_i\|_2, \tag{1.1}$$

for all operators $X_1, Y_1, \ldots, X_k, Y_k$ in $\mathcal{A}_{,sa}$. A map $Q : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ is *locally Lipschitz* (or *locally operator-valued Lipschitz*) with respect to $\|\cdot\|_2$, if for all $M > 0$, there exists a constant $C_M > 0$ such that (1.1) holds for all X_i, Y_i in $\mathcal{A}_{,sa}$ with $\|X_i\|_2$ and $\|Y_i\|_2$ less than M, $1 \leq i \leq k$. Similar definitions of (locally) Lipschitz maps with respect to operator norm can be found in Section 2.3 in [BiS2].

2. Free Markov processes

In this section, we study free Markov processes of (unbounded) random variables in a W^*-probability space. We classify the relation between the free Markov property, the classical Markov property, and the Markov property defined in [BKS]. Moreover, we show that every free Markov process has transition functions.

First, let us recall the definition of freeness with amalgamation for two subalgebras.

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra τ be a faithful normal tracial state on $\mathcal{M}$, $\mathcal{B} \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$ be von Neumann subalgebras of $\mathcal{M}$. Let

$\mathbb{E}_{\mathcal{B}} : \mathcal{M} \to \mathcal{B}$ be the trace preserving conditional expectation onto $\mathcal{B}$. We say that the subalgebras $\mathcal{M}_1$ and $\mathcal{M}_2$ are *free with amalgamation over* $\mathcal{B}$, or $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{B}$-free, if

$$\mathbb{E}_{\mathcal{B}}(a_1 a_2 \cdots a_n) = 0,$$

whenever $a_j \in \mathcal{M}_{i_j}$, $i_j \in \{1, 2\}$, $i_1 \neq i_2 \neq \cdots \neq i_n$, $\mathbb{E}_{\mathcal{B}}(a_j) = 0$, and $n \in \mathbb{N}$ (see Section 5.3 in [V4]).

A classical example of free subalgebras with amalgamation comes from the analogue concept in group theory. Let $H \subseteq G_1 \cap G_2$ be subgroups of a group G. We say that G_1 *and* G_2 *are free with amalgamation over* H *in* G if $g_1 g_2 \cdots g_n \neq e$, where e is the unit of group G, whenever $g_j \in G_{i_j} - H$, $i_1 \neq i_2 \neq \cdots \neq i_n$. Consider group von Neumann algebras $\mathcal{L}_H, \mathcal{L}_{G_1}, \mathcal{L}_{G_2}$ and $\mathcal{L}_G$. Then $\mathcal{L}_{G_1}$ and $\mathcal{L}_{G_2}$ are free with amalgamation over $\mathcal{L}_H$ in the finite von Neumann algebra $\mathcal{L}_G$ if and only if G_1 and G_2 are free with amalgamation over H in G (see Section 5.3 in [V4]).

By [V3] and [V4], we have the following.

Definition 2.1. Let $\{X_t : t \geq 0\}$ be a family of (unbounded) operators in $\widetilde{\mathcal{A}},$. Let $\mathcal{A},_{\leq t}$ be the von Neumann subalgebra of $\mathcal{A}$ generated by $\{W^*(X_s),\ 0 \leq s \leq t\}$, $\mathcal{A},_{\geq t}$ be the von Neumann subalgebra generated by $\{W^*(X_s), s \geq t\}$ and $\mathcal{A},_{=t} = W^*(X_t)$, for $t \geq 0$. We say that the random process $\{X_t : t \geq 0\}$ is a free Markov process, if, for $t \geq 0$, $\mathcal{A},_{\leq t}$ and $\mathcal{A},_{\geq t}$ are $\mathcal{A},_{=t}$-free.

We generalized it to a more general case.

Definition 2.2. Let $\{X_t = (X_{1,t}, \ldots, X_{k,t}) \in \widetilde{\mathcal{A}}^k : t \geq 0\}$ be a family of k-tuples of random variables. Let $\mathcal{A}_{\leq t}$, $\mathcal{A}_{=t}$, respectively, $\mathcal{A}_{\geq t}$ be the von Neumann subalgebras of $\mathcal{A}$ generated by $\{W^*(X_{i,s}) : 0 \leq s \leq t, i = 1, 2, \ldots, k\}$, $\{W^*(X_{i,t}) : i = 1, 2, \ldots, k\}$, respectively, $\{W^*(X_{i,s}), s \geq t, i = 1, 2, \ldots, k\}$. We say random process $\{X_t : t \geq 0\}$ is a free Markov process, if $\mathcal{A}_{\leq t}$ and $\mathcal{A}_{\geq t}$ are $\mathcal{A}_{=t}$-free.

Voiculescu pointed out in [V4] that every process with free increments is a free Markov process.

An analogue of the classical Markov property in noncommutative probability spaces is the following "weak Markov property".

Definition 2.3. Let $(\mathcal{A},, \tau)$ be a W^*-probability space. Let $(X_t)_{t \geq 0}$ be a family of self-adjoint operators in $\mathcal{A},$. Let $\mathcal{A},_{\leq t} = W^*\{X_s : s \leq t\}$, $\mathcal{A}_{=t} = W^*(X_t)$ and $\mathcal{A},_{\geq t} = W^*\{X_s : s \geq t\}$. We say $\{X_t : t \geq 0\}$ is a weak Markov process (or it has weak Markov property) in $(\mathcal{A},, \tau)$, if

$$\mathbb{E}_{=t}(AB) = \mathbb{E}_{=t}(A)\mathbb{E}_{=t}(B), \quad \forall A \in \mathcal{A},_{\leq t}, B \in \mathcal{A},_{\geq t},$$

where $\mathbb{E}_{=t} : \mathcal{A}, \to \mathcal{A},_{=t}$ is the trace preserving conditional expectation onto $\mathcal{A},_{=t}$.

The following two results show the relation between the weak Markov property, the classical Markov property, and the free Markov property.

THEOREM 2.4. *A family* $\{f_t \in \mathcal{A},_{as} : t \geq 0\}$ *in* $\mathcal{A}, = L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ *is a weak Markov process if and only if the random process* $\{f_t : t \geq 0\}$ *is a Markov process in classical sense.*

Proof. Let $f \in \mathcal{A}, = L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ be a real valued random variable. Then

$$W^*(f) \cong L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df),$$

where $W^*(f)$ is the von Neumann subalgebra generated by f, $\mathcal{B}_{\sigma(f)}$ is the Borel algebra on $\sigma(f)$, the image (i.e., the spectrum) of f, df (or μ_f) is the distribution of random variable f, and $\mathcal{M} \cong \mathcal{N}$ means that $\mathcal{M}$ is $*$-isomorphic to $\mathcal{N}$, as von Neumann algebras. Let $\mathcal{F}_{=t} = \{f_t^{-1}(B) : B \in \mathcal{B}\}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Then $\mathcal{F}_{=t}$ is a σ-subalgebra of $\mathcal{F}$. Define

$$\pi : L^\infty\big(\sigma(f), \mathcal{B}_{\sigma(f)}, df\big) \to L^\infty(\Omega, \mathcal{F}_{=t}, \mathbf{P})$$

by the formula $\pi(g) = g \circ f$, for $g \in L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$. It is obvious that $\pi(g) = g \circ f \in L^\infty(\Omega, \mathcal{F}_{=t}, \mathbf{P})$. Given $g_1, g_2 \in L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$, $x \in \Omega$, and $\lambda_1, \lambda_2 \in \mathbb{C}$, we have

$$(\lambda_1 g_1 + \lambda_2 g_2) \circ f(x) = \lambda_1 g_1(f(x)) + \lambda_2 g_2(f(x)),$$

and

$$g_1(f(x)) \cdot g_2(f(x)) = (g_1 g_2)(f(x)), \qquad \overline{g_1(f(x))} = \overline{g}(f(x)).$$

Thus, π is a $*$-homomorphism. Moreover, the image of f (i.e., the spectrum of f) is the domain of elements in $L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$. Hence, π is injective. For any simple function $s = \sum_{i=1}^k \lambda_i \chi_{B_i} \in L^\infty(\Omega, \mathcal{F}_{=t}, P)$, let $g = \sum_{i=1}^k \lambda_i \chi_{f(B_i)}$. Then $g \in L^\infty(\sigma(f), \mathcal{B}_{\sigma(f)}, df)$ and $s = g \circ f$. It implies that the image of π is dense in $L^\infty(\Omega, \mathcal{F}_{=t}, \mathbf{P})$. Hence, π is a *-isomorphism. To prove

$$\mathcal{A},_{\leq t} \cong L^\infty(\Omega, \mathcal{F}_{\leq t}, P), \qquad \mathcal{A},_{\geq t} = L^\infty(\Omega, \mathcal{F}_{\geq t}, \mathbf{P}),$$

we first note that $\mathcal{A},_{\leq t}$ is generated, as a von Neumann algebra, by $\{X_s : s \leq t\}$, and we have proved that $W^*(X_s)$ is *-isomorphic to $L^\infty(\Omega, \mathcal{F}_{=t}, \mathbf{P})$. Thus, up to *-isomorphisms, we can assume that $\mathcal{A},_{\leq t}$ is the von Neumann algebra generated by elements in $L^\infty(\Omega, \mathcal{F}_{=s}, \mathbf{P})$, $s \leq t$, and it is enough to show that $L^\infty(\Omega, \mathcal{F}_{\leq t}, \mathbf{P})$ is generated by $L^\infty(\Omega, \mathcal{F}_{=s}, \mathbf{P})$, $s \leq t$. In fact, given a sequence $t \geq s_1 \geq s_2 \geq \cdots$, and $B_1, B_2, \ldots \in \mathcal{B}$, we have

$$\chi_{\bigcap_{i=1}^\infty f_{s_i}^{-1}(B_i)} = \lim_{n\to\infty} \chi_{f_{s_1}^{-1}(B_1)} \cdots \chi_{f_{s_n}^{-1}(B_n)} \in \mathcal{A},_{\leq t}.$$

Moreover, let $S_1 = f_{s_1}^{-1}(B_1)$, and

$$S_j = f_{s_j}^{-1}(B_j) - \left(\bigcup_{i=1}^{j-1} f_{s_i}^{-1}(B_i)\right), \quad j = 2, 3, \ldots,$$

we have

$$\chi_{\bigcup_{i=1}^{\infty} f_{s_i}^{-1}(B_i)} = \chi_{\bigcup_{i=1}^{\infty} S_j} = \sum_{i=1}^{\infty} \chi_{S_i} \in \mathcal{A}_{,\leq t}.$$

Hence, for $S \in \mathcal{F}_{\leq t}$, $\chi_S \in \mathcal{A}_{,\leq t}$. Hence, $L^\infty(\Omega, \mathcal{F}_{\leq t}, P) \subseteq \mathcal{A}_{,\leq t}$. Conversely, it is obvious that $\mathcal{A}_{,\leq t} \subseteq L^\infty(\Omega, \mathcal{F}_{\leq t}, \mathbf{P})$ (up to *-isomorphism). Hence, $\mathcal{A}_{,\leq t} \cong L^\infty(\Omega, \mathcal{F}_{\leq t}, \mathbf{P})$. Similarly, $\mathcal{A}_{,\geq t} \cong L^\infty(\Omega, \mathcal{F}_{\geq t}, P)$.

Let $\{f_t : t \geq 0\}$ be a weak Markov process in sense of Definition 2.3. For all $t \geq 0$, $A \in \mathcal{F}_{\leq t}$, and $B \in \mathcal{F}_{\geq t}$, we have $\chi_A \in \mathcal{A}_{,\leq t}, \chi_B \in \mathcal{A}_{,\geq t}$. Hence,

$$\mathbf{P}(AB|f_t) = \mathbb{E}_{=t}(\chi_A \chi_B) = \mathbb{E}_{=t}(\chi_A)\mathbb{E}_{=t}(\chi_B) = \mathbf{P}(A|f_t)\mathbf{P}(B|f_t).$$

It follows that $\{f_t : t \geq 0\}$ is a classical Markov process.

Conversely, suppose $\{f_t : t \geq 0\}$ is a classical Markov process. By the previous discussion, $\mathbb{E}_{=t}(PQ) = \mathbb{E}_{=t}(P)\mathbb{E}_{=t}(Q), \forall t \geq 0$, for projections P and Q in $\mathcal{A}_{,\leq t}$ and $\mathcal{A}_{,\geq t}$, respectively. Thus, for $\lambda_i, \lambda_i' \in \mathbf{C}$, $P_i \in \mathcal{A}_{,\leq t}, Q_i \in \mathcal{A}_{,\geq t}$, and $xX = \sum_{i=1}^n \lambda_i P_i$, $Y = \sum_{i=1}^n \lambda_i' q Q_i$, we have

$$\mathbb{E}_{=t}(XY) = \sum_{i,j=1}^n \lambda_i \lambda_j' \mathbb{E}_{=t}(P_i Q_j) = \mathbb{E}_{=t}(X)\mathbb{E}_{=t}(Y).$$

Note that conditional expectation $\mathbb{E}_{=t}$ is norm continuous and the linear span of all projections is norm dense in a von Neumann algebra, so we have

$$\mathbb{E}_{=t}(AB) = \mathbb{E}_{=t}(A)\mathbb{E}_{=t}(B), \quad \forall A \in \mathcal{A}_{,\leq t}, B \in \mathcal{A}_{,\geq t}.$$

It follows that $\{f_t : t \geq 0\}$ is a weak Markov process. $\square$

THEOREM 2.5. *Let $\{X_t : t \geq 0\}$ be a free Markov process of elements in $\mathcal{A}_{,sa}$. Then $\{X_t : t \geq 0\}$ is a weak Markov process in W^*-probability space $(\mathcal{A}_{,}, \tau)$.*

Proof. For any $t_0 \geq 0$, let $\mathcal{A}_{,\leq t} = W^*\{X_t : t \leq t_0\}$, $\mathcal{A}_{,=t_0} = W^*(X_{t_0})$, and $\mathcal{A}_{,\geq t_0} = W^*\{X_t : t \geq t_0\}$. Let $\mathbb{E}_{t_0}$ be the trace-preserving conditional expectation on $\mathcal{A}_{,=t_0}$. For $A \in \mathcal{A}_{,\leq t_0}$ and $B \in \mathcal{A}_{,\geq t_0}$, we have

$$\begin{aligned}
\mathbb{E}_{t_0}(AB) &= \mathbb{E}_{t_0}\big(\big(A - \mathbb{E}_{t_0}(A) + \mathbb{E}_{t_0}(A)\big)\big(B - \mathbb{E}_{t_0}(B) + \mathbb{E}_{t_0}(B)\big)\big) \\
&= \mathbb{E}_{t_0}\big(\big(A - \mathbb{E}_{t_0}(A)\big)\big(B - \mathbb{E}_{t_0}(B)\big)\big) + \big(\mathbb{E}_{t_0}(A)\mathbb{E}_{t_0}\big(\big(B - \mathbb{E}_{t_0}(B)\big)\big) \\
&\quad + \mathbb{E}_{t_0}\big(\big(A - \mathbb{E}_{t_0}(A)\big)\mathbb{E}_{t_0}(B)\big) + \mathbb{E}_{t_0}(A)\mathbb{E}_{t_0}(B) \\
&= \big(\mathbb{E}_{t_0}(A)\mathbb{E}_{t_0}\big(\big(B - \mathbb{E}_{t_0}(B)\big)\big) + \mathbb{E}_{t_0}\big(\big(A - \mathbb{E}_{t_0}(A)\big)\mathbb{E}_{t_0}(B)\big) \\
&\quad + \mathbb{E}_{t_0}(A)\mathbb{E}_{t_0}(B) \\
&= \mathbb{E}_{t_0}(A)\mathbb{E}_{t_0}(B),
\end{aligned}$$

where the third equality holds true because of the free Markov property of $\{X_t : t \geq 0\}$. $\square$

Let $(\mathcal{A}_{,}, \tau)$ be a (tracial) probability space, and $\{X_t : t \geq 0\}$ of self-adjoint operators in $\mathcal{A}$, be a stochastic process. Denoted by

$$\mathcal{A}_{,\leq t} = W^*(\{X_u : u \leq t\}), \qquad \mathcal{A}_{,\geq t} = W^*(\{X_u : u \geq t\}), \qquad \mathcal{A}_{,=t} = W^*(X_t).$$

In [BKS], $\{X_t : t \geq 0\}$ is called a *Markov process* if

$$\mathbb{E}_{\leq s}(X) \in \mathcal{A}_{,=s}, \quad \forall X \in \mathcal{A}_{,=s}, s \leq t.$$

The authors of [BKS] pointed out that there is another canonical version for the Markov property as follows. A stochastic process $\{X_t : t \geq 0\}$ has the Markov property if

$$\mathbb{E}_{\leq s}(X) \in \mathcal{A}_{,=s}, \quad \forall X \in \mathcal{A}_{,\geq s}, s \leq t. \tag{2.1}$$

The following result gives some sufficient and necessary conditions for a process to be a weak Markov process.

Theorem 2.6. *Let $(\mathcal{A}_{,}, \tau)$ be a W^*-probability space. Let $(X_t)_{t \geq 0}$ be a family of self-adjoint operators in $\mathcal{A}_{,}$. Then the following are equivalent.*

(1) *The process $\{X_t : t \geq 0\}$ is a weak Markov process.*
(2) *For all $t \geq 0$, $\mathbb{E}_{\leq t}(A) = \mathbb{E}_{=t}(A), \forall A \in \mathcal{A}_{,\geq t}$, where $\mathbb{E}_{\leq t} : \mathcal{A}_{,} \to \mathcal{A}_{,\leq t}$ is the trace preserving conditional expectation onto $\mathcal{A}_{,\leq t}$.*
(3) *For all $t \geq 0$, $\mathbb{E}_{\geq t}(A) = \mathbb{E}_{=t}(A), \forall A \in \mathcal{A}_{,\leq t}$, where $\mathbb{E}_{\geq t} : \mathcal{A}_{,} \to \mathcal{A}_{,\geq t}$ is the trace preserving conditional expectation onto $\mathcal{A}_{,\geq t}$.*
(4) *For all $0 \leq s \leq t$, let $\mathcal{A}_{,s,t} = W^*\{X_r : s \leq r \leq t\}$ and $\mathbb{E}_{s,t} : \mathcal{A}_{,\leq t} \to \mathcal{A}_{,\leq s}$ be the trace preserving conditional expectation. Then $\mathbb{E}_{s,t}(\mathcal{A}_{,s,t}) \subseteq \mathcal{A}_{,=s}$.*

Proof. $(1) \Rightarrow (2)$ Without loss of generality, we can assume that von Neumann algebra $\mathcal{A}$, acts on the Hilbert space $L^2(\mathcal{A}_{,}, \tau)$. Then τ is the vector state associated to identity element I of $\mathcal{A}_{,}$. Thus, τ is continuous with respect to WOT (weak operator topology). Note that the linear span $\mathcal{L}$ of the set $\{X_{t_1} \cdots X_{t_n} : t_j \geq t, j = 1, 2, \ldots, n, n = 1, 2, \ldots\}$ is dense in $\mathcal{A}_{,\geq t}$ with respect to WOT. If we can prove

$$\mathbb{E}_{\leq t}(X_{t_1} \cdots X_{t_n}) = \mathbb{E}_{=t}(X_{t_1} \cdots X_{t_n}), \quad \forall t_j \geq t, j = 1, 2, \ldots, n, \ n \in \mathbb{N}, \tag{2.2}$$

then we have $\mathbb{E}_{\leq t}(X) = \mathbb{E}_{=t}(X), \forall X \in \mathcal{L}$. Moreover, for $A \in \mathcal{A}_{,\geq t}$, there is a net $\{X_\lambda : \lambda \in \Lambda\}$ in $\mathcal{L}$ such that $\lim_\lambda X_\lambda = A$, where the limit is with respect to WOT. Hence, for $B \in \mathcal{A}_{,\leq t}$, we have

$$\begin{aligned} \tau(\mathbb{E}_{=t}(A)B) &= \lim_\lambda \tau(\mathbb{E}_{=t}(X_\lambda)B) = \lim_\lambda \tau(\mathbb{E}_{\leq t}(X_\lambda)B) \\ &= \lim_\lambda \tau(X_\lambda B) = \tau(\mathbb{E}_{\leq t}(A)B). \end{aligned}$$

Hence, it is sufficient to show (2.2). For $t_j \geq t, j = 1, 2, \ldots, n$ and $B \in \mathcal{A}_{,\leq t}$, we have

$$\begin{aligned} \tau(X_{t_1} \cdots X_{t_n} B) &= \tau(\mathbb{E}_{=t}(X_{t_1} \cdots X_{t_n} B)) \\ &= \tau(\mathbb{E}_{=t}(X_{t_1} \cdots X_{t_n})\mathbb{E}_{=t}(B)) \\ &= \tau(\mathbb{E}_{=t}(X_{t_1} \cdots X_{t_n})B), \end{aligned}$$

where the second equality holds because of Definition 2.3. Hence,

$$\mathbb{E}_{\leq t}(X_{t_1}\cdots X_{t_n}) = \mathbb{E}_{=t}(X_{t_1}\cdots X_{t_n}).$$

$(2)\Rightarrow(1)$ For $A\in\mathcal{A}_{,\leq t}, B\in\mathcal{A}_{,\geq t}$, and $C\in\mathcal{A}_{,=t}$, we have

$$\begin{aligned}\tau(ABC) &= \tau(CA\mathbb{E}_{\leq t}(B)) = \tau(CA\mathbb{E}_{=t}(B))\\ &= \tau(A\mathbb{E}_{=t}(B)C) = \tau(E_{=t}(A)\mathbb{E}_{=t}(B)C).\end{aligned}$$

Hence,

$$\mathbb{E}_{=t}(AB) = \mathbb{E}_{=t}(A)\mathbb{E}_{=t}(B).$$

The proof of the equivalence of (1) and (3) is the same as that of $(1)\Leftrightarrow(2)$.
$(4)\Rightarrow(2)$ It is enough to show that

$$\mathbb{E}_{\leq t}(X_{t_1}\cdots X_{t_n})\in\mathcal{A}_{,=t},\quad \forall t_j\geq t, j=1,2,\ldots,n,\ n\in\mathbb{N}.$$

Let $u=\max\{t_j : j=1,2,\ldots,n\}$. Then $X_{t_1}\cdots X_{t_n}\in\mathcal{A}_{,t,u}$. Hence, by (4),

$$\mathbb{E}_{\leq t}(X_{t_1}\cdots X_{t_n}) = \mathbb{E}_{s,t}(X_{t_1}\cdots X_{t_n})\in\mathcal{A}_{,=s}.$$

$(2)\Rightarrow(4)$ It is enough to show that $\mathbb{E}_{s,t}(X_{r_1}\cdots X_{r_n})\in\mathcal{A}_{,=s}$, for all $s\leq r_j\leq t$. Since $X_{r_1}\cdots X_{r_n}\in\mathcal{A}_{,\geq s}\cap\mathcal{A}_{,\leq t}$,

$$\mathbb{E}_{s,t}(X_{r_1}\cdots X_{r_n}) = \mathbb{E}_{\leq s}(X_{r_1}\cdots X_{r_n})\in\mathcal{A}_{,=s},$$

by (2). □

Remark 2.7. The above result shows that the concept of Markov processes (2.1) defined in [BKS] is the same as that of weak Markov processes, a weak version of the free Markov property. Combining Theorem 2.4, we see that the concept of Markov processes in [BKS] is a "classical" Markov property in noncommutative probability spaces. The concept of free Markov processes defined in [V3, V4] and our Definitions 2.1 and 2.2 is a free version of the classical Markov property in free probability.

The following result shows that a free Markov process has transition functions, which have very similar properties to those of a classical Markov process.

Theorem 2.8. *Let $\{X_t : t\geq 0\}$ be a stochastic process in W^*-probability space $(\mathcal{A},,\tau)$. Then the following statements hold.*

(1) *If $\{X_t : t\geq 0\}$ has the weak Markov property, then there is an operator*

$$\mathcal{K}_{s,t} : L^\infty(\mathbb{R})\to L^\infty(\mathbb{R}),$$

for $0\leq s\leq t$, such that

  (a) *the map $k_{s,t}(x,\cdot):\Gamma\to k_{s,t}(x,\Gamma)=\mathcal{K}_{s,t}(\chi_\Gamma)(x)$ is a probability measure on the Borel σ-algebra $\mathcal{B}_{\sigma(X_s)}$ over the spectrum $\sigma(X_s)$ of operator X_s, for almost all $x\in\sigma(X_s)$ with respect to the spectral distribution $dX_s:\Gamma\to\tau(\Delta(\Gamma))$, where $\Delta(\Gamma)$ is the spectral projection of X_s corresponding to the Borel set Γ of $\sigma(X_s)$.*
  (b) *when $t=s$, we have $K_{s,s}(\chi_\Gamma)=\chi_\Gamma$, for every Borel set $\Gamma\in\sigma(X_s)$.*
  (c) *$\mathbb{E}_{\leq s}(\varphi(X_t))=\mathcal{K}_{s,t}(\varphi)(X_s),\forall\varphi\in L^\infty(\mathbb{R})$.*

(2) *If* $\{X_t : t \geq 0\}$ *is a commutative process of operators in* $\mathcal{A}_{,sa}$ *(i.e.,* $X_tX_s = X_sX_t$*, for all* $t, s \geq 0$*), and there is an operator*

$$\mathcal{K}_{s,t} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}),$$

for $0 \leq s \leq t$*, satisfies conditions* (a), (b), *and* (c) *in* (1), *then* $\{X_t : t \geq 0\}$ *is a weak Markov process.*

Proof. By (4) in Theorem 2.6, $\mathbb{E}_{s,t}(\mathcal{A}_{,=t}) \subseteq \mathcal{A}_{,=s}$, for $0 \leq s < t$. Note that there is a *-isomorphism

$$\pi_t : \mathcal{A}_{,=t} \to L^\infty(\sigma(X_t), \mathcal{B}_{\sigma(X_t)}, dX_t),$$

where dX_t is the (spectral) distribution of X_t with respect to τ. For $0 \leq s < t$, define

$$\mathcal{K}_{s,t}(f)(x) = \pi_s \mathbb{E}_{s,t}(f(X_t))(x), \quad \forall f \in L^\infty(\mathbb{R}), x \in \mathbb{R}.$$

Then $\mathcal{K}_{s,t} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, and

$$\mathbb{E}_{\leq s}(f(X_t)) = \pi_s^{-1}(\mathcal{K}_{s,t}(f)) = \mathcal{K}_{s,t}(f)(X_s), \quad \forall f \in L^\infty(\mathbb{R}).$$

This shows that $\mathcal{K}_{s,t}$ satisfies condition (c). Now, we show that $\mathcal{K}_{s,t}$ satisfies the properties (a) and (b). It is obvious that function $\mathcal{K}_{s,t}(f)(x)$ is measurable, since

$$\mathcal{K}_{s,t}(f) \in L^\infty(\sigma(X_s), \mathcal{B}_{\sigma(X_s)}, dX_s).$$

For $0 \leq s \leq t, x \in \mathbb{R}$, a Borel set $F = \bigcup_{i \geq 1} F_i \in \mathcal{B}, F_i \cap F_j = \emptyset$, $\forall i \neq j, i, j = 1, 2, \ldots$, $\forall G \in \mathcal{B}_{\sigma(X_s)}$, and $k_{s,t}(x, F) = \mathcal{K}_{s,t}(\chi_F)(x)$, we have

$$\begin{aligned}
\int_G \mathcal{K}_{s,t}(\chi_F)\, dX_s &= \int_{\sigma(X_s)} (\mathcal{K}_{s,t}(\chi_F)\chi_G)\, dX_s = \tau(\mathbb{E}_{s,t}(\chi_F(X_t))\chi_G(X_s)) \\
&= \tau(\chi_F(X_t)\chi_G(X_s)) = \tau\left(\sum_{i=1}^\infty \chi_{F_i}(X_t)\chi_G(X_s)\right) \\
&= \sum_{i=1}^\infty \tau(\chi_{F_i}(X_t)\chi_G(X_s)) = \sum_{i=1}^\infty \int_G \mathcal{K}_{s,t}(\chi_{F_i})\, dX_s \\
&= \int_G \left(\sum_{i=1}^\infty \mathcal{K}_{s,t}(\chi_{F_i})\right) dX_s.
\end{aligned}$$

It follows that

$$k_{s,t}(F, x) = \sum_{i=1}^\infty k_{s,t}(F_i, x),$$

for almost all $x \in \sigma(X_s)$ with respect to dX_s. Moreover,

$$k_{s,t}(x, \sigma(X_s)) = \mathcal{K}_{s,t}\chi_{\sigma(X_s)}(x) = \pi_s E_{s,t}\big(\chi_{\sigma(X_t)}(X_t)\big)(x) = 1.$$

Hence, $k_{s,t}(x, \cdot)$ is a probability measure on $\sigma(X_s)$, for almost all $x \in \sigma(X_s)$. This completes the proof of (a). Property (b) is obvious.

Conversely, by property (c) of operator $\mathcal{K}_{s,t}$, we have $\mathbb{E}_{s,t}(\mathcal{A}_{,=t}) \subseteq \mathcal{A}_{,=s}$, for $0 \le s < t$. Now we show that $E_{s,t}(\mathcal{A}_{,s,t}) \subseteq \mathcal{A}_{,=s}$, for $0 \le s < t$. Since $\mathcal{A}_{,s,t}$ is Abelian, the linear span $\mathcal{L}$ of elements in $\{X_{r_1} \cdots X_{r_n} : s \le r_1 \le \cdots \le r_n \le t, n \in \mathbb{N}\}$ is dense in $\mathcal{A}_{,s,t}$ with respect to WOT. Hence, it is sufficient to show that

$$\tau(X_{r_1} \cdots X_{r_n} B) = \tau(\mathbb{E}_{=s}(X_{r_1} \cdots X_{r_n}) B),$$

for all $B \in \mathcal{A}_{,\le s}$. We shall prove it by induction in n. For $n = 1$, we have

$$\mathbb{E}_{s,t}(f(X_{t_1})) \in \mathcal{A}_{,=s}, \quad \forall f \in L^\infty(\mathbb{R}),$$

since $\mathbb{E}_{s,t}(\mathcal{A}_{,=t}) \subseteq \mathcal{A}_{,=s}$. Suppose

$$\mathbb{E}_{s,t}(f_1(X_{t_1}) \cdots f_n(X_{t_n})) \in \mathcal{A}_{,=s}, \quad \forall f_1, \ldots, f_n \in L^\infty(\mathbb{R}), s \le t_1 \le \cdots \le t_n \le t.$$

Now for $f_1, \ldots, f_{n+1} \in L^\infty(\mathbb{R}), s \le t_1 \le \cdots \le t_{n+1} \le t$, and $B \in \mathcal{A}_{,\le s}$, we have

$$\begin{aligned}
&\tau(f_1(X_{t_1}) \cdots f_{n+1}(X_{t_{n+1}}) B) \\
&\quad = \tau(f_1(X_{t_1}) \cdots f_n(X_{t_n}) \mathbb{E}_{\le t_n}(f_{n+1}(X_{t_{n+1}})) B) \\
&\quad = \tau(f_1(X_{t_1}) \cdots (f_n(X_{t_n}) \mathbb{E}_{\le t_n}(f_{n+1}(X_{t_{n+1}}))) B) \\
&\quad = \tau(\mathbb{E}_{=s}(f_1(X_{t_1}) \cdots (f_n(X_{t_n}) \mathbb{E}_{\le t_n}(f_{n+1}(X_{t_{n+1}})))) B) \\
&\quad = \tau(\mathbb{E}_{=s}(f_1(X_{t_1}) \cdots f_n(X_{t_n}) f_{n+1}(X_{t_{n+1}})) B).
\end{aligned}$$

It implies that $\mathbb{E}_{s,t}(f_1(X_{t_1}) \cdots f_{n+1}(X_{t_{n+1}})) \in \mathcal{A}_{,=s}$. We have proved (4) in Theorem 2.6. Hence, $\{X_t : t \ge 0\}$ is a weak Markov process in W^*-probability space $(\mathcal{A}, , \tau)$. $\square$

3. Free stochastic differential equations

In this section, we study a kind of system of stochastic differential equations (3.3), and the free Markov property of its solution. Our results generalize Biane and Speicher's work on free differential equations driven by free Brownian processes (see [BiS2]) to a more general case of the free stochastic differential equations driven by free Lévy processes. On the other hand, our results provide a way to get free Markov processes of random variables with uncompact supported distributions.

Let $(\mathcal{A}, \tau)$ be a W^*-probability space with filtration $\{\mathcal{A}_t : t \ge 0\}$. For each $1 \le i \le k$, $\{S_{i,t} : t \ge 0\}$ is $\mathcal{A}_t$-free Brownian motion, and $\{S_{1,t} : t \ge 0\}, \ldots, \{S_{k,t} : t \ge 0\}$ are free in $(\mathcal{A}, \tau)$. In [BiS2], Biane and Speicher proved the following result.

Theorem 3.1 (Theorem 3.1, Proposition 3.3 in [BiS2]). *Let $Q_1, \ldots, Q_k : \mathcal{A}_{sa}^k \to \mathcal{A}$ be k locally operator-valued Lipschitz functions (with respect to operator norm) such that each Q_i maps $(\mathcal{A}_s)_{sa}^k$ into $(\mathcal{A}_s)_{sa}$ for all $s \ge 0$. If there exist constants $a \in \mathbb{R}, b > 0$ such that*

$$\sum_{i=1}^{k} \big(Q_i(X_1, \ldots, X_k) X_i + X_i Q_i(X_1, \ldots, X_k) + 1\big) \le a \sum_{i=1}^{k} X_i^2 + b, \tag{3.1}$$

for all $X_1,\dots,X_k\in\mathcal{A}_{sa}$. Then, for $X_{i,0}\in\mathcal{A}_0, i=1,2,\dots,k$, the system

$$dX_{i,t}=Q_i(X_{1,t},\dots,X_{k,t})\,dt+dS_{i,t},\quad i=1,\dots,k,t\geq 0 \tag{3.2}$$

has a unique solution $X(t)=(X_{1,t},\dots,X_{k,t})$ for all $t\geq 0$. Furthermore, we have $X_{i,t}\in\mathcal{A}_t$ for all $i=1,\dots,k,t\geq 0$, the maps $t\to X_{i,t}$ are norm continuous. Moreover, let $\mathcal{B}_{\leq t}=W^*\{X_{i,0},S_{i,s}:s\leq t,1\leq i\leq k\}$, $\mathcal{B}_{\geq t}=W^*\{X_{i,t}, S_{i,s}-S_{i,t}:s\geq t,1\leq i\leq k\}$ and $\mathcal{B}_{=t}=W^*\{X_{i,t}:1\leq i\leq k\}$, then $(\mathcal{B}_{\leq t},\mathcal{B}_{=t},\mathcal{B}_{\geq t})$ is a free Markovinian triple (i.e., $\mathcal{B}_{\leq t}$ and $\mathcal{B}_{\geq t}$ are $\mathcal{B}_{=t}$-free).

It is obvious that (3.1) is equivalent to

$$\sum_{i=1}^k\big(Q_i(X_1,\dots,X_k)X_i+X_iQ_i(X_1,\dots,X_k)\big)\leq a\sum_{i=1}^k X_i^2+b, \tag{3.1$'$}$$

for all $X_1,\dots,X_k\in\mathcal{A}_{sa}$ and some $a,b>0$. In this section, we consider a system similar to (3.2) as follows.

$$dX_i(t)=Q_i(X_{1,t},\dots,X_{k,t})\,dt+dS_{i,t},\quad i=1,\dots,k,t\geq 0, \tag{3.3}$$

where $\{S_{i,t}:t\geq 0\}$ $(i=1,\dots,k)$ are $\mathcal{A}_t$-free Lévy processes of elements in $\mathcal{A}_{sa}$ (by [1, Lemma 1], the function $t\to S_{i,t}$ is continuous in $L^n(\mathcal{A},,\tau)$, for all $n\in\mathbb{N}$), and $\{S_{1,t}:t\geq 0\},\dots,\{S_{k,t}:t\geq 0\}$ are free in $(\mathcal{A},\tau)$. We shall prove that under conditions similar to Theorem 3.1, the system (3.3) has a unique solution $X_t=(X_{1,t},\dots,X_{k,t})\in L^2(\mathcal{A},\tau)$, and $\{X_t:t\geq 0\}$ is a free Markov process.

LEMMA 3.2. *For $1\leq i\leq k$, let $Q_i:\mathcal{A}_{sa}^k\to\mathcal{A}_{sa}$ be a Lipschitz maps with respect to $\|\cdot\|_2$, such that $Q_i:(\mathcal{A}_s)_{sa}^k\to(\mathcal{A}_s)_{sa}$, for $i=1,\dots,k,s\geq 0$. Then given arbitrary initial conditions $X_{i,0}\in\mathcal{A}_0$, $i=1,2,\dots,k$, (3.3) has a unique solution $\{X_t=(X_{1,t},\dots,X_{k,t}):t\geq 0\}$. Furthermore, we have $X_{i,t}\in L^2(\mathcal{A}_{t,sa},\tau)$ for all $i=1,\dots,k,t\geq 0$, and $t\to X_{i,t}$ is continuous with respect to $\|\cdot\|_2$.*

Proof. The solution to (3.3) is a process $\{X_{i,t}\in L^2(\mathcal{A}_{sa},\tau):t\geq 0\}$ such that

$$X_{i,t}=X_{i,0}+\int_0^t Q_i(X_{1,s},\dots,X_{k,s})\,ds+S_{i,t},\quad \forall t\geq 0,1\leq i\leq k. \tag{3.4}$$

We use Picard iteration method to get the solution. Since Q_i is Lipschitz, there exists $C>0$ such that

$$\|Q_j(X_1,\dots,X_k)-Q_j(Y_1,\dots,Y_k)\|_2\leq C\sum_{i=1}^k\|X_i-Y_i\|_2,$$

for all $X_i,Y_i\in\mathcal{A}_{sa},1\leq i,j\leq k$. Take $T>0$ such that $kCT<1$. For $0\leq t\leq T$, let $X_{i,t}^{(0)}=X_{i,0},1\leq i\leq k$, and

$$X_{i,t}^{(n+1)}=X_{i,0}+\int_0^t Q_i\big(X_{1,s}^{(n)},\dots,X_{k,s}^{(n)}\big)\,ds+S_{i,t},\quad n=1,2,\dots. \tag{3.5}$$

Then $X_{i,t}^{(0)} \in \mathcal{A}_{t,sa}$ and the function $X_{i,t}^{(0)} \in \mathcal{A}_{t,sa}$ is continuous with respect to $\|\cdot\|_2$, for $1 \le i \le k$. Assume $X_{i,t}^{(n)} \in L^2(\mathcal{A}_{t,sa}, \tau)$ and $t \to X_{i,t}^{(n)} \in L^2(\mathcal{A}_{sa}, \tau)$ is continuous with respect to $\|\cdot\|_2$. Then $Q_i(X_{1,s}^{(n)}, \dots, X_{k,s}^{(n)}) \in L^2(\mathcal{A}_{s,sa}, \tau)$ and $s \to Q_i(X_{1,s}^{(n)}, \dots, X_{k,s}^{(n)})$ is continuous with respect to $\|\cdot\|_2$, since $Q_i : L^2(\mathcal{A}, \tau)^k \to L^2(\mathcal{A}, \tau)$ is continuous. It implies that $X_{i,t}^{(n+1)} = X_{i,0} + \int_0^t Q_i(X_{1,s}^{(n)}, \dots, X_{k,s}^{(n)})\, ds + S_{i,t} \in L^2(\mathcal{A}_{t,sa}, \tau)$ and $t \to X_{i,t}^{(n+1)} \in L^2(\mathcal{A}_{sa}, \tau)$ is continuous. By induction, $X_{i,t}^{(n)} \in \mathcal{A}_{t,sa}$ and $t \to X_{i,t}^{(n)} \in L^2(\mathcal{A}_{sa}, \tau)$ is continuous with respect to $\|\cdot\|_2$.

$$\begin{aligned}
\|X_{i,t}^{(n+1)} - X_{i,t}^{(n)}\|_2 &= \left\| \int_0^t \big(Q_i\big(X_{1,s}^{(n)}, \dots, X_{k,s}^{(n)}\big) - Q_i\big(X_{1,s}^{(n-1)}, \dots, X_{k,s}^{(n-1)}\big)\big)\, ds \right\|_2 \\
&\le \int_0^t \big\|Q_i\big(X_{1,s}^{(n)}, \dots, X_{k,s}^{(n)}\big) - Q_i\big(X_{1,s}^{(n-1)}, \dots, X_{k,s}^{(n-1)}\big)\big\|_2\, ds \\
&\le C \int_0^t \sum_{i=1}^k \|X_{i,s}^{(n)} - X_{i,s}^{(n-1)}\|_2\, ds.
\end{aligned}$$

Let $\mathbf{D}_n = \sup_{0 \le t \le T} \sum_{i=1}^k \|X_{i,t}^{(n)} - X_{i,t}^{(n-1)}\|_2$, we have

$$\mathbf{D}_n \le kTC\mathbf{D}_{n-1} \le \dots \le (KTC)^{n-1}\mathbf{D}_1.$$

It follows that $\{X_{i,t}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence with respect to $\|\cdot\|_2$, since $0 < kTC < 1$. Therefore, there exist $X_{i,t} \in L^2(\mathcal{A}_{t,sa}, \tau)$, for $0 \le t \le T, i = 1, 2, \dots, k$, such that $X_{i,t} = \lim_{n\to\infty} X_{i,t}^{(n)}$ where the limit is taken in the topology of norm $\|\cdot\|_2$. Note that $Q_i : L^2(\mathcal{A}_{sa}, \tau)^k \to L^2(\mathcal{A}_{sa}, \tau)$ is continuous with respect to $\|\cdot\|_2$. Let n approach ∞ in (3.5), we get (3.4). Hence, $X_t = (X_{1,t}, \dots, X_{k,t})$ is a solution to (3.3), and $X_{i,t} \in L^2(\mathcal{A}_{t,sa}, \tau)$, for $0 \le t \le T$. Now we show that $t \to X_{i,t} \in L^2(\mathcal{A}_{sa}, \tau)$ is continuous. For $0 \le s, t$, we have

$$\begin{aligned}
\|X_{i,s} - X_{i,t}\|_2 &\le \|X_{i,s} - X_{i,s}^{(n)}\|_2 + \|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2 + \|X_{i,t} - X_{i,t}^{(n)}\|_2 \\
&= \lim_{m\to\infty} \|X_{i,s}^{(m)} - X_{i,s}^{(m)}\|_2 + \|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2 \\
&\quad + \lim_{m\to\infty} \|X_{i,t}^{(m)} - X_{i,t}^{(n)}\|_2 \\
&\le 2 \sum_{m=n}^\infty (kCT)^m (KTC)^{n-1}\mathbf{D}_1 + \|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2.
\end{aligned}$$

Since $\lim_{m\to\infty} \sum_{m=n}^\infty (kCT)^m (KTC)^{n-1}\mathbf{D}_1 = 0$, for $\varepsilon > 0$, there exists n such that

$$\sum_{m=n}^\infty (kCT)^m (KTC)^{n-1}\mathbf{D}_1 < \varepsilon/4.$$

Note also that $t \mapsto X_{i,t}^{(n)}$ is continuous, for the above $\varepsilon > 0$, and $t \in [0,T]$, there exists $\delta > 0$ such that $\|X_{i,s}^{(n)} - X_{i,t}^{(n)}\|_2 < \varepsilon/2$, whenever, $|t-s| < \delta$. Hence, we have

$$\|X_{i,s} - X_{i,t}\|_2 \leq \varepsilon,$$

whenever $|t-s| < \delta$. It follows that $t \to X_{i,t} \in L^2(\mathcal{A}_{sa}, \tau)$ is continuous. For $T < t \leq 2T$, (3.4) can be rewritten as

$$X_{i,t} = X_{i,T} + \int_T^t Q_i(X_{1,s}, \dots, X_{k,s})\, ds + S_{i,t} - S_{i,T}.$$

Let $X_{i,t}^{(0)} = X_{i,T}$ and

$$X_{i,t}^{(n+1)} = X_{i,T} + \int_T^t Q_i\big(X_{1,s}^{(n)}, \dots, X_{k,s}^{(n)}\big)\, ds + S_{i,t} - S_{i,T}, \quad n = 1, 2, \dots.$$

As the above proof, we can prove that (3.3) has solution $X_t = (X_{1,t}, \dots, X_{k,t})$, for $T < t \leq 2T$. Generally, for $t > 0$, there exists $n \in \mathbb{N}$ such that $nT < t \leq (n+1)T$. Thus, after doing the above process n times, we get a solution of (3.3). Hence, by the construction of $X_t = (X_{1,t}, \dots, X_{k,t})$, $(X_{1,t}, \dots, X_{k,t}) \in L^2(\mathcal{A}_{t,sa}, \tau)$ and $t \to X_{i,t}$ is continuous with respect to $\|\cdot\|_2$.

Uniqueness, if (3.3) has two solutions $X_{i,t}$ and $Y_{i,t}$ in $L^2(\mathcal{A}_{sa}, \tau)$, for $1 \leq i \leq k$, then we have

$$\sup_{0 \leq s \leq t} \sum_{i=1}^k \|X_{i,s} - Y_{i,s}\|_2 \leq kCt \sup_{0 \leq s \leq t} \sum_{i=1}^k \|X_{i,s} - Y_{i,s}\|_2. \tag{3.5'}$$

Let $f(t) := \sum_{i=1}^k \|X_{i,s} - Y_{i,s}\|_2$. Then $f(t)$ is a continuous nonnegative function on $[0, \infty)$ such that

$$\sup_{0 \leq s \leq t} f(s) \leq kCt \sup_{0 \leq s \leq t} f(s), \quad \forall t \geq 0.$$

Hence, let t_0 be $\frac{1}{kC}$, we have $f(t) = 0$, for $t \in [0, t_0]$. That is, $X_{i,s} = Y_{i,s}$, for $1 \leq i \leq k$, and $0 \leq s \leq t_0$. When $t > t_0$, we have

$$X_{i,t} = X_{i,t_0} + \int_{t_0}^t Q(X_{1,s}, \dots, X_{k,s})\, ds + S_{i,t} - S_{i,t_0},$$

$$Y_{i,t} = Y_{i,t_0} + \int_{t_0}^t Q(Y_{1,s}, \dots, Y_{k,s})\, ds + S_{i,t} - S_{i,t_0}.$$

Thus, we get an equation similar to (3.5′)

$$\sup_{t_0 \leq s \leq t} \sum_{i=1}^k \|X_{i,s} - Y_{i,s}\|_2 \leq kC(t - t_0) \sup_{t_0 \leq s \leq t} \sum_{i=1}^k \|X_{i,s} - Y_{i,s}\|_2.$$

It implies that $X_{i,t} = Y_{i,t}$, for $t \in [t_0, 2t_0]$. Finally, we get that $X_{i,t} = Y_{i,t}$, for all $t \geq 0$, and $i = 1, 2, \dots, k$. $\square$

LEMMA 3.3. *Let $Q : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$ be a locally operator-valued Lipschitz function, and $h : [0,\infty) \to [0,1]$ be a continuous function with the following property: there is a $R > 0$ such that $h|_{[0,R]} = 1$, $h|_{[2R,\infty)} = 0$ and there is a $C > 0$ such that $|h(t) - h(s)| \leq C|t-s|, \forall t, s \geq 0$. Let*

$$f(X_1,\dots,X_k) = Q(X_1,\dots,X_k)h\left(\sum_{i=1}^k \|X_i\|_2\right), \quad \forall X_1,\dots,X_k \in L^2(\mathcal{A}_{sa},\tau).$$

Then f is Lipschitz.

Proof. The proof is the same as that of Lemma 3.2 in [BiS2]. □

We need the following well-known result.

LEMMA 3.4. *Let $A \in L^2(\mathcal{A},_{sa},\tau) \subseteq \widetilde{\mathcal{A}}_{sa}$. Then $A^2 \in L^1(\mathcal{A},_{sa},\tau)$ and $\|A\|_2 = \tau(A^2)^{1/2}$.*

To prove the existence of the solution to (3.3), we need the following lemma. First, we introduce some notions (see [A] for details).

Let $\mathcal{A},^{op}$ be the opposite algebra of $\mathcal{A}$, (i.e., the von Neumann algebra obtained by defining $A \cdot B = BA$, for $A, B \in \mathcal{A}$, and preserving all other operations in $\mathcal{A}$,). Given $0 \leq t_1 \leq \cdots \leq t_{n+1} < \infty$ and $A_1, B_1, \dots, A_n,\ B_n \in \mathcal{A},$, the function $U(t) = \sum_{i=1}^n A_i \otimes B_i \chi_{[t_i,t_{i+1})}$ is called a simple biprocess. A simple biprocess $U(t)$ is adapted with a filtration $\{\mathcal{A},_t : t \geq 0\}$, if $U(t) \in \mathcal{A},_t \otimes \mathcal{A},_t^{op}$, for all $t \geq 0$. The space of all $\mathcal{A},_t$-adapted simple biprocesses is denoted by $\mathcal{B}$. For $U(t) = \sum_{i=1}^n A_i \otimes B_i \chi_{[t_i,t_{i+1})} \in \mathcal{B}$, we can define

$$\int_0^\infty U(s)\sharp\, dS_s := \sum_{i=1}^n A_i(S_{t_{i+1}} - S_{t_i})B_i.$$

Denoted by m the multiplication map $\mathcal{A}, \otimes \mathcal{A},^{op} \to \mathcal{A},$. Then $m(U(t)) = A_i B_i$, if $U(t) = \sum_{j=1}^n A_j \otimes B_j \chi_{[t_j,t_{j+1})}(t)$ and $t_i \leq t < t_{i+1}$. Given $a > 0$, we can define the norm

$$\|U\|_{2,a}' = \left(\int_0^\infty \|U(s)\|_2^2\, ds\right)^{1/2} + a\left\|\int_0^\infty m(U(s))\, ds\right\|_2,$$

for $U \in \mathcal{B}$. The completion of $\mathcal{B}$ with respect to $\|\cdot\|_{2,a}'$ is denoted by $\mathcal{B}_2^{2,a}$.

LEMMA 3.5. *Let $t \to X_t$ be a continuous function in $L^2(\mathcal{A},,\tau)$, $\{S_t : t \geq 0\}$ be a $\mathcal{A},_t$-free Lévy process of elements in $\mathcal{A},_{sa}$, and $r_1 = |\tau(S_1)|$. Then*

$$\max\left\{\left\|\int_0^t X_s\, dS_s\right\|_2, \left\|\int_0^t dS_s X_s\right\|_2\right\} \leq \left\|X_\cdot \chi_{[0,t]}(\cdot)\right\|_{2,r_1}'.$$

Proof. By Proposition 6 in [A], for $X_t \in \mathcal{B}_2^{2,r_1}$, $\|\int_0^\infty X_s \sharp\, dS_s\|_2 \leq \|X\|_{2,r_1}'$. Thus, it is enough to show that $X_s\chi_{[0,t]}(s) \in \mathcal{B}_2^{2,r_1}$, for all $t > 0$. In fact, for

$n \in \mathbb{N}$, let $U_{n,s} = \sum_{i=1}^{n} X_{\frac{i}{n}t} \chi_{[\frac{(i-1)t}{n}, \frac{it}{n})}(s)$. Then $U_n \in \mathcal{B}$. Moreover,

$$\begin{aligned}
&\left\| X_{\cdot} \chi_{[0,t]} - U_n \right\|'_{2,r_1} \\
&\quad = \left(\sum_{i=1}^{n} \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} \|X_s - X_{\frac{it}{n}}\|_2^2 \, ds \right)^{1/2} + \left\| \sum_{i=1}^{n} \int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} (X_s - X_{\frac{it}{n}}) \, ds \right\|_2 \\
&\quad \le \sum_{i=1}^{n} \left(\sup_{\frac{(i-1)t}{n} \le s \le \frac{it}{n}} \|X_s - X_{\frac{it}{n}}\|_2^2 \frac{t}{n} \right)^{1/2} \\
&\qquad + \sum_{i=1}^{n} \sup_{\frac{(i-1)t}{n} \le s \le \frac{it}{n}} \|X_s - X_{\frac{it}{n}}\|_2 \frac{t}{n} \\
&\quad \le \sum_{i=1}^{n} \sup_{0 \le s, s' \le t, |s-s'| \le \frac{t}{n}} \|X_s - X_{s'}\|_2 (t^{1/2} + t) \\
&\quad \to 0,
\end{aligned}$$

as $n \to \infty$, where we have used the fact that $s \to X_s$ is uniformly continuous as a function from $[0,t]$ into $L^2(\mathcal{A},,\tau)$. Hence, $X_{\cdot} \chi_{[0,t]}(\cdot) \in \mathcal{B}_2^{2,r_1}$. □

THEOREM 3.6. *Let $Q_i : \mathcal{A}_{sa}^k \to \mathcal{A}_{sa}$, $(i = 1, \ldots, k)$ be k local Lipschitz maps with respect to $\|\cdot\|_2$ such that $Q_i : \mathcal{A}_{s,sa}^k \to \mathcal{A}_{s,sa}$, for $i = 1, \ldots, k$, $s \ge 0$, and there exist constants $a, b > 0$ such that (3.1′) holds. Then for initial conditions $X_{i,0} \in \mathcal{A}_0$ $(i = 1, 2, \ldots, k)$, the system (3.3) has a unique solution $X_t = (X_{1,t}, \ldots, X_{k,t})$ for $t \ge 0$. Furthermore, we have $X_{i,t} \in L^2(\mathcal{A}_{t,sa}, \tau)$ for $i = 1, \ldots, k, t \ge 0$, and $t \to X_{i,t}$ is continuous with respect to $\|\cdot\|_2$.*

Proof. For $R > 0$, take a function h_R as that in Lemma 3.3, and let

$$f_i(X_1, \ldots, X_k) = Q_i(X_1, \ldots, X_k) h_R\left(\sum_{i=1}^{k} \|X_i\|_2 \right),$$

for all $X_1, \ldots, X_k \in L^2(\mathcal{A}_{sa}, \tau)$ and $1 \le i \le k$. By Lemmas 3.2 and 3.3, the following system

$$X_{i,t} = X_{i,0} + \int_0^t f_i(X_{1,s}, \ldots, X_{k,s}) \, ds + S_{i,t}, \quad 1 \le i \le k$$

has a unique solution $X_t^R = (X_{1,t}^R, \ldots, X_{k,t}^R)$. Note that if $\sum_{i=1}^{k} \|X_{i,t}\|_2 \le R$, we have $f_i = Q_i, 1 \le i \le k$. So, X_t^R is a solution to (3.3). Let $T_R = \inf\{t : \sum_{i=1}^{k} \|X_{i,t}^R\|_2 > R\}$, then X_t^R is a solution to (3.3), if $t < T_R$. Hence, we shall be done if we can prove that

$$\lim_{R \to \infty} T_R = \infty.$$

By [A, Corollary 12]

$$(S_{i,t})^2 = \int_0^t dS_{i,s}S_{i,s} + \int_0^t S_{i,s}\,dS_{i,s} + \Delta_{i,2}(t),$$

where $\Delta_{i,2}(t) = \lim_{N\to\infty}\sum_{j=1}^N (S_{i,\frac{j}{N}t} - S_{i,\frac{j-1}{N}t})^2$, where the limit is taken in the topology of the operator norm (see Definition 3 in [A]). By Lemma 2 in [A], $\{\Delta_{i,k}(t) : t \geq 0\}$ is an $\mathcal{A}_{,t}$-free Lévy process. Hence,

$$d(S_{i,t}^2) = dS_{i,t}S_{i,t} + S_{i,t}\,dS_{i,t} + d\Delta_{i,2}(t).$$

Let $X_t^R = (X_{1,t}^R, \ldots, X_{k,t}^R)$, we have

$$\begin{aligned}
d((X_{i,t}^R)^2) &= d\left(X_{i,0}^2 + X_{i,0}\int_0^t Q_i(X_s^R)\,ds\right) + X_{i,0}S_{i,t} \\
&\qquad + \int_0^t Q(X_s)\,dsX_{i,0} + \left(\int_0^t Q(X_s)\,ds\right)^2 + \int_0^t Q(X_s)\,dsS_{i,t} \\
&\qquad + S_{i,t}X_{i,0} + S_{i,t}\int_0^t Q(X_s)\,ds + (S_{i,t})^2 \\
&= X_{i,0}\,dX_{i,t}^R + Q(X_t^R)\,dtX_{i,t}^R + \int_0^t Q(X-s)\,ds\,dX_{i,t}^R \\
&\qquad + dS_{i,t}X_{i,0} + dS_{i,t}\int_0^t Q(X_s)\,ds + S_{i,t}Q(X_t^R)\,dt + d(S_{i,t}^2) \\
&= X_{i,t}^R\,dX_{i,t}^R + dX_{i,t}^R X_{i,t}^R + d\Delta_{i,2}(t).
\end{aligned}$$

Let $Z_t = (\sum_{i=1}^k (X_{i,t}^R)^2)^{1/2}$. Then

$$\begin{aligned}
d(e^{-at}Z_t^2) &= -ae^{-at}\left(\sum_{i=1}^k (X_{i,t}^R)^2\right) \\
&\qquad + e^{-at}\sum_{i=1}^k \big(dX_{i,t}^R \cdot X_{i,t}^R + X_{i,t}^R \cdot dX_{i,t}^R + (d\Delta_{i,2}(t))\big) \\
&= -ae^{-at}\left(\sum_{i=1}^k (X_{i,t}^R)^2\right) + e^{-at}\sum_{i=1}^k \big(f_i(X_{1,t}^R, \ldots, X_{k,t}^R)X_{i,t}^R \\
&\qquad + X_{i,t}^R f_i(X_{1,t}^R, \ldots, X_{k,t}^R)\big) + e^{-at}\sum_{i=1}^k (dS_{i,t}X_{i,t}^R + X_{i,t}^R dS_{i,t}) \\
&\qquad + e^{-at}\sum_{i=1}^k (d\Delta_{i,2}(t)).
\end{aligned}$$

By Lemma 3.2, $t \to X_{i,t}^R$ is continuous with respect to $\|\cdot\|_2$. Therefore, $T_R > 0$, if R is big enough. Moreover, $\{t : \sum_{i=1}^k \|X_{i,t}^R\|_2 > R\}$ is open. So, for $t \leq T_R$, we have X_t^R is a solution to (3.3). Hence, we have

$$\begin{aligned}
Z_t^2 = e^{at}\Bigg(& Z_0^2 - a\int_0^t e^{-au}\sum_{i=1}^k (X_{i,u}^R)^2\,du \\
& + \int_0^t e^{-au}\left(\sum_{i=1}^k f_i(X_{1,u}^R,\ldots,X_{k,u}^R)X_{i,u}^R + X_{i,u}^R f_i(X_{1,u}^R,\ldots,X_{k,u}^R)\right)du \\
& + \int_0^t e^{-au}\sum_{i=1}^k (dS_{i,u}X_{i,u}^R + X_{i,u}^R dS_{i,u})\,du\Bigg) + e^{at}\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u) \\
\le{}& e^{at}Z_0^2 + e^{at}\int_0^t be^{-au}\,du + e^{at}\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u) \\
& + e^{at}\int_0^t e^{-au}\sum_{i=0}^k (dS_{i,u}X_{i,u}^R + X_{i,u}^R\,dS_{i,u}),
\end{aligned}$$

where the inequality holds because of (3.1′). Let

$$r = \max\{|\tau(S_{i,1}| : 1 \le i \le k\},$$

we have

$$\begin{aligned}
\tau(Z_t^2) \le{}& e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau\left(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)\right) \\
& + e^{at}\sum_{i=1}^k \left|\tau\left(\int_0^t e^{-au}(dS_{i,u}X_{i,u}^R + X_{i,u}^R\,dS_{i,u})\right)\right| \\
\le{}& e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau\left(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)\right) \\
& + e^{at}\sum_{i=1}^k \left\|\int_0^t e^{-au}\,dS_{i,u}X_{i,u}^R\right\|_2 + e^{at}\sum_{i=1}^k \left\|\int_0^t e^{-au}X_{i,u}^R\,dS_{i,u}\right\|_2 \\
\le{}& e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau\left(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)\right) \\
& + 2e^{at}\sum_{i=1}^k\left(\int_0^t \|X_{i,u}^R\|_2^2 e^{-2au}\,du\right)^{\frac{1}{2}} + 2re^{at}\sum_{i=1}^k\left\|\int_0^t e^{-au}X_{i,u}^R\,du\right\|_2 \\
\le{}& e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau\left(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_i(u)\right) \\
& + 2e^{at}\sup_{0\le u\le t}\sum_{i=1}^k \|X_{i,u}^R\|_2\left(\left(\int_0^t e^{-2au}\,du\right)^{\frac{1}{2}} + r\int_0^t e^{-au}\,du\right)
\end{aligned}$$

$$\leq e^{at}\|Z_0\|_2^2 + \frac{b}{a}(e^{at}-1) + e^{at}\tau\left(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)\right)$$
$$+ 2ke^{at}\sup_{0\leq u\leq t}\left(\sum_{i=1}^k \|X_{i,u}^R\|_2^2\right)^{\frac{1}{2}}\left(\left(\int_0^t e^{-2au}\,du\right)^{\frac{1}{2}} + r\int_0^t e^{-au}\,du\right),$$

where the third inequality holds by Lemma 3.5. Let

$$\varphi(t) = \sup\{\tau(Z_u^2) : 0 \leq u \leq t\} = \sup_{0\leq u\leq t}\sum_{i=1}^k \|X_{i,u}^R\|_2^2,$$

we have

$$\varphi(t) \leq e^{at}\varphi(0) + \frac{b}{a}(e^{at}-1) + e^{at}\tau\left(\int_0^t e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)\right)$$
$$+ 2k\left[\left(\frac{e^{2at}-1}{2a}\right)^{\frac{1}{2}} + \frac{r(e^{at}-1)}{a}\right]\varphi(t)^{\frac{1}{2}}.$$

Note that $\sum_{i=1}^k \|X_{i,T_R}^R\|_2 = R$, so $\max_{1\leq i\leq k}\|X_{i,T_R}^R\|_2 \geq R/k$. It follows that

$$\varphi(T_R)^{1/2} = \left(\sup_{0\leq u\leq T_R}\sum_{i=1}^k \|X_{i,u}^R\|_2^2\right)^{1/2} \geq R/k.$$

It implies that

$$R^2/k^2 \leq \varphi(T_R)$$
$$\leq e^{aT_R}\varphi(0) + \frac{b}{a}(e^{aT_R}-1) + e^{aT_R}\tau\left(\int_0^{T_R} e^{-au}\sum_{i=1}^k d\Delta_{i,2}(u)\right)$$
$$+ 2k\left[\left(\frac{e^{2aT_R}-1}{2a}\right)^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a}\right]\varphi(T_R)^{\frac{1}{2}}.$$

Moreover,

$$\varphi(T_R) = \sup_{0\leq u\leq T_R}\sum_{1\leq i\leq k}\|X_{i,u}^R\|_2^2 \leq \sup_{0\leq u\leq T_R}\left(\sum_{1\leq i\leq k}\|X_{i,u}^R\|_2\right)^2 \leq R^2.$$

Hence, let $r_1' = \max\tau(\Delta_{i,2}(1)) : 1 \leq i \leq k$, we have

$$R/k^2 \leq \frac{\varphi(0)}{R}e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR}$$
$$+ e^{aT_R}\frac{1}{R}\left|\tau\int_0^{T_R}\sum_{i=1}^k e^{-au}(d\Delta_{i,2}(u))\right|$$
$$+ 2k\left(\left(\frac{e^{2aT_R}-1}{2a}\right)^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a}\right)$$

$$
\begin{aligned}
&\leq \frac{\varphi(0)}{R}e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR} + e^{aT_R}\frac{1}{R}\sum_{i=1}^{k}\left\| \int_0^{T_R} e^{-au}\, d\Delta_{i,2}(u)\right\|_2 \\
&\quad + 2k\left(\left(\frac{e^{2aT_R}-1}{2a}\right)^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a}\right) \\
&\leq \frac{\varphi(0)}{R}e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR} + e^{aT_R}\frac{1}{R}\sum_{i=1}^{k}\left(\left(\int_0^{T_R} e^{-2au}\, du\right)^{1/2}\right. \\
&\quad \left. + r_1'\int_0^{T_R} e^{-au}\, du\right) + 2k\left(\left(\frac{e^{2aT_R}-1}{2a}\right)^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a}\right) \\
&\leq \frac{\varphi(0)}{R}e^{aT_R} + \frac{b(e^{aT_R}-1)}{aR} \\
&\quad + e^{aT_R}\frac{k}{R}\left(\left(\frac{1}{2a}\right)^{1/2} + r_1' a^{-1}\right) + 2k\left(\left(\frac{e^{2aT_R}-1}{2a}\right)^{\frac{1}{2}} + \frac{r(e^{aT_R}-1)}{a}\right).
\end{aligned}
$$

It is obvious that map $R \to T_R$ is increasing. Thus, if $\lim_{R\to\infty} T_R \neq \infty$, the right-hand side of the inequality above is upper bounded. On the other hand, the left-hand side is upper unbounded as $R \to \infty$. This gives rise of a contradiction. Hence, $\lim_{R\to\infty} T_R = \infty$. We finish the proof of the existence of solution to (3.3). Moreover, for $t \geq 0$, we can take $R > 0$ such that $t \leq T_R$, so, $X_{i,t} = X_{i,t}^R$. Hence, $X_{i,t} \in L^2(\mathcal{A}_{t,sa}, \tau)$ and $t \to X_{i,t}$ is continuous with respect to $\|\cdot\|_2$, by Lemma 3.2.

Uniqueness. This result follows from the uniqueness of solutions to (3.4) (Lemma 3.2). ☐

We shall show that the solution $\{X_t : t \geq 0\}$ to (3.3) is a free Markov process in $L^2(\mathcal{A},, \tau)$.

THEOREM 3.7. *Under the hypotheses of Theorem 3.6, and the condition that $Q : \mathcal{A}_{sa}^k \to \mathcal{A}$ is polynomial of k noncommutative unknown variables. Then the solution $\{X_t : t \geq 0\}$ is a free Markov process.*

Proof. Let

$$
\begin{aligned}
&\mathcal{B}_{\leq t} = W^*\{X_{i,0}, S_{i,s} : s \leq t, 1 \leq i \leq k\}, \\
&\mathcal{B}_{\geq t} = W^*\{X_{i,t}, S_{i,s} - S_{i,t} : s \geq t, 1 \leq i \leq k\}, \\
&\mathcal{C}_{\leq t} = W^*\{X_{i,s} : s \leq t, 1 \leq i \leq k\}, \mathcal{C}_{\geq t} = W^*\{X_{i,s} : s \geq t, 1 \leq i \leq k\},
\end{aligned}
$$

and

$$\mathcal{C}_{=t} = W^*\{X_{i,t} : 1 \leq i \leq k\}.$$

We want to show that

$$\mathcal{C}_{\leq t} \subseteq \mathcal{B}_{\leq t}, \mathcal{C}_{\geq t} \subseteq \mathcal{B}_{\geq t}. \tag{3.6}$$

By the proofs of Lemma 3.2 and Theorem 3.6,

$$\lim_{n\to\infty} \big\|X_{i,t} - X_{i,t}^{(n)}\big\|_2 = 0, \quad 1 \le i \le k,$$

where $X_{i,t}^{(0)} = X_{i,0} \in \mathcal{A}_0$, and $X_{i,t}^{(n)}$ $(n \ge 1)$ are defined by (3.5). Let $\mathcal{H}_{\le t} = L^2(\mathcal{B}_{\le t}, \tau)$. Then $X_{0,t} \in \mathcal{H}_{\le t}$. Let $f_i = Q_i h$ (see Lemma 3.3 for the definition of function h). Assume $X_{i,s}^{(n)} \in \mathcal{B}_{\le t}, 1 \le i \le k, s \le t$. Let $X_{i,s}^{(n)} = \lim_{m\to\infty} X_{i,s}^{(m,n)}$ in norm $\|\cdot\|_2$, where $X_{i,s}^{(m,n)} \in (\mathcal{B}_{\le t})_{sa}, 1 \le i \le k$. Then $f_i(X_{1,s}^{(m,n)}(s), \ldots, X_{k,s}^{(m,n)}) \in \mathcal{B}_{\le t}$, since Q_i is a polynomial. Note that $Q_i : \mathcal{A},_{sa}^k \to \mathcal{A},_{sa}$ is continuous with respect to $\|\cdot\|_2$. It implies that the $\|\cdot\|_2$ limit $f_i(X_{1,s}^{(n)}, \ldots, X_{k,s}^{(n)})$ of $f_i(X_{1,s}^{(m,n)}, \ldots, X_{k,s}^{(m,n)})$ is in $\mathcal{H}_{\le t}$, for $s \le t, 1 \le i \le k$. Hence,

$$X_{i,t}^{(n+1)} = X_{i,0} + \int_0^t f_i\big(X_{1,s}^{(n)}, \ldots, X_{k,s}^{(n)}\big)\, ds + S_{i,t} \in \mathcal{H}_{\le t}.$$

By induction, $X_{i,t}^{(n)} \in \mathcal{H}_{\le t}$. Hence, $X_{i,t} = \lim_{n\to\infty}(X_{i,t}^{(n)}) \in \mathcal{H}_{\le t}$. It follows that $\mathcal{C}_{\le t} \subseteq \mathcal{B}_{\le t}$. For $s \ge t$,

$$X_{i,s} = X_{i,t} + \int_t^s f_i(X_{1,u}, \ldots, X_{k,u})\, du + S_s - S_t.$$

By the above proof and the uniqueness of the solutions to (3.3), $\mathcal{C}_{\ge t} \subseteq \mathcal{B}_{\ge t}$. Now we show that $\mathcal{B}_{\le t}$ and $\mathcal{B}_{\ge t}$ are $\mathcal{C}_{=t}$-free. Note that $W^*\{X_0, S_s : s \le t\}$ and $W^*\{S_u - S_t : u \ge t\}$ are free in $(\mathcal{A}, \tau)$, and $\mathcal{C}_{=t} \subseteq W^*\{X_0, S_s : s \le t\}$. By Lemma 2.1 in [BiS2], $\mathcal{B}_{\le t}$ and $\mathcal{B}_{\ge t}$ are $\mathcal{C}_{=t}$-free. Therefore, $\mathcal{C}_{\le t}$ and $\mathcal{C}_{\ge t}$ are $\mathcal{C}_{=t}$-free. By Definition 2.1, $\{X_t : t \ge 0\}$ is a free Markov process in $L^2(\mathcal{A},, \tau) \subseteq \widetilde{\mathcal{A}},$. □

For $k = 1$, we can get more general condition on Q so that the solution is a free Markov process.

Theorem 3.8. *Under the hypotheses of Theorem 3.6, and the conditions that $k = 1$ and $Q : \mathbb{R} \to \mathbb{R}$ is Borel measurable, the solution $\{X_t : t \ge 0\}$ is a free Markov process.*

Proof. We use the same notation (with $k = 1$) as that in Theorem 3.7. Assume $X_{n,s} \in \mathcal{H}_{\le t}$, then $f(X_{n,s}) \in \mathcal{B}_{\le t}$, since $f = Qh$ is bounded measurable function. Hence,

$$X_{n+1,t} = X_0 + \int_0^t f(X_{n,s})\, ds + S_t \in \mathcal{H}_{\le t}.$$

The rest of the proof is the same as that of Theorem 3.7. □

4. Free Ornstein–Uhlenbeck process

In this section, we consider a special case of (3.3). Let $k=1$, $Q(X)=-\lambda X$, $\lambda>0$, and $\{S_t : t\geq 0\}$ be $\mathcal{A}_t$-free Lévy process of operators in $\mathcal{A}_{sa}$. We consider the following equation

$$X_t = X_0 - \lambda\int_0^t X_s\,ds + S_t, \quad t\geq 0, \tag{4.1}$$

where self-adjoint operator $X_0\in\mathcal{A}_0$. We call

$$X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-u)}\,dS_u, \quad t\geq 0$$

a *free OU process*, where $\int_0^t e^{-\lambda(t-u)}\,dS_u$ is defined by Theorem 6.1 in [BT2] (Generally, we can define a free OU process $\{X_t : t\geq 0\}$ by the formula above in the case that $\{S_t : t\geq 0\}$ is a free Lévy process of self-adjoint operators in $\widetilde{\mathcal{A}},$, and X_0 is affiliated with $\mathcal{A}_{,0}$). We show that the free *OU* process is the unique solution to (4.1) and the limit distribution of X_t, as $t\to\infty$, is free self-decomposable.

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function. For $n\in\mathbb{N}$, and $a=t_{n,0}<t_{n,1}<\cdots<t_{n,k_n}=b$ a partition of $[a,b]$, let $f_n(t)=\sum_{i=1}^{k_n} a_{n,i}\chi_{[t_{n,i-1},t_{n,i})}(t)$, $f_n(b)=f(b)$ be a step function such that $f_n(t)\rightrightarrows f(t)$ for $t\in[a,b]$. Then*

$$\lim_{n\to\infty}\left\|\int_a^b \big(f(t)-f_n(t)\big)\,dS_t\right\|_2 = 0.$$

Proof. By Lemma 3.5, $f-f_n\in\mathcal{B}_2^{2,a}$. Hence,

$$\begin{aligned}\|f_n-f\|_2 &\leq \|f_n-f\|_{L^2([a,b])} + |\tau(S_1)|\cdot\|f_n-f\|_{L^1([a,b])}\\ &\leq \|f_n-f\|_{L^\infty([a,b])}(b-a)\big(1+|\tau(S_1)|\big)\\ &\to 0,\end{aligned}$$

as $n\to\infty$, since $f_n\rightrightarrows f$ on $[a,b]$. □

The following lemma gives some kind of Fibini theorem. We omit the proof of the lemma, because it is very similar to that of Proposition 35 in [RS].

Lemma 4.2. *Let f and g be continuous functions on $[a,b]$,*

$$X=\int_a^b g(s)\int_a^s f(u)\,dS_u\,ds, \qquad Y=\int_a^b f(u)\int_u^b g(s)\,ds\,dS_u.$$

Then $X=Y$.

Theorem 4.3. *Let $X_t=e^{-\lambda t}X_0+\int_0^t e^{-\lambda(t-u)}\,dS_u$. Then $t\to X_t$ is continuous with respect to $\|\cdot\|_2$, and $\{X_t : t\geq 0\}$ is the unique continuous solution to* (4.1).

Proof. It is obvious that $t \to X_t$ is continuous. Moreover,

$$\begin{aligned}-\lambda \int_0^t X_u\,du &= e^{-\lambda t}X_0 - X_0 - \lambda \int_0^t e^{-\lambda s}\int_0^s e^{\lambda u}\,dS_u\,ds\\ &= e^{-\lambda t}X_0 - X_0 - \lambda \int_0^t e^{\lambda u}\int_u^t e^{-\lambda s}\,ds\,dS_u\\ &= e^{-\lambda t}X_0 - X_0 + \int_0^t e^{-\lambda(t-u)}\,dS_u - S_t\\ &= X_t - X_0 - S_t,\end{aligned}$$

where the second equality holds because of Lemma 4.2.

Uniqueness. Since (4.1) is a special case of (3.3), by Lemma 3.2, equation (4.1) has a unique solution. □

Now we study the limit distribution of the process $\{X_t : t \geq 0\}$. Let $\{S_t : t \geq 0\}$ be a free Lévy process of (unbounded) operators. Then $\mu(S_1)$ is $\boxplus$-infinitely divisible. By 2.7 and 2.8 in [BT2], there are a real number γ and finite measure σ on $\mathbb{R}$ such that the Voiculescu transform $\phi_{\mu_{S(1)}}$ (see [BT2] for Voiculescu transform) can be given by

$$\phi_{\mu_{S_1}}(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\sigma(dt).$$

(γ,σ) is called the free generating pair of $\mu(S_1)$.

THEOREM 4.4. *If the measure σ in the free generating pair (γ,σ) of $\mu(S_1)$ in the Lévy process $\{S_t \in \widetilde{\mathcal{A}}_{,sa} : t \geq 0\}$ satisfies*

$$\int_{|t|\geq 1} \log(1+|t|)\sigma(dt) < \infty, \tag{4.2}$$

then the limit distribution of X_t, as $t \to \infty$, is $\boxplus$-self-decomposable.

Conversely, if μ_0 is a $\boxplus$-self-decomposable distribution on $\mathbb{R}$, there is a free OU process $\{X_t | t \geq 0\}$ such that the limit distribution of X_t is μ_0.

Proof. Since $X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-u)}\,dS_u$, it is enough to show that the limit distribution of $\int_0^t e^{-\lambda(t-u)}\,dS_u$ is $\boxplus$ self-decomposable. Let

$$t_{n,j} = jt/n, \quad j = 0,1,\ldots,n, \quad T_n = \sum_{j=1}^n e^{-\lambda(t-t_{n,j})}(S_{t_{n,j}} - S_{t_{n,j-1}}).$$

Then $T_n \xrightarrow{\mathrm{p}} \int_0^t e^{-\lambda(t-u)}\,dS_u$, by Theorem 6.1 in [BT2]. On the other hand,

$$\begin{aligned}T_n &= \sum_{j=1}^n e^{-\lambda(t-t_j)}\big(S_{t-(t-t_{n,j})} - S_{t-(t-t_{n,j-1})}\big)\\ &= \sum_{j=1}^n e^{-\lambda r_{n,n-j}}\big(S_{t-r_{n,n-j}} - S_{t-r_{n,n-j+1}}\big)\end{aligned}$$

$$\stackrel{\mathrm{d}}{=} \sum_{j=1}^{n} e^{-\lambda r_{n,n-j}} (S_{r_{n,n-j+1}} - S_{r_{n,n-j}}) = \sum_{i=1}^{n} e^{-\lambda r_{n,i-1}} (S_{r_{n,i}} - S_{r_{n,i-1}})$$
$$\xrightarrow{\mathrm{p}} \int_0^t e^{-\lambda u}\, dS_u.$$

Hence, we have

$$\int_0^t e^{-\lambda(t-u)}\, dS_u \stackrel{\mathrm{d}}{=} \int_0^t e^{-\lambda u}\, dS_u = \int_0^{t\lambda} e^{-u}\, dS_{u/\lambda}.$$

Let $\widetilde{S}_t = S_{t/\lambda}, \forall t \geq 0$. It is obvious that $\widetilde{S}_t$ is a $\mathcal{A}_t$-free Lévy process. Let $\phi_{\mu_1}(z)$ be the Voiculescu transform of $\mu(S_1)$. By [BT2], $\phi_{\mu(S_t)}(t) = t\phi_{\mu_1}(z)$. Let (γ, σ) be the free generating pair of μ_1. Then $(t\gamma, t\sigma)$ is the free generating pair of $\mu(S(t))$. Hence, $\mu(\widetilde{S}_1) = \mu(S(\frac{1}{\lambda}))$ has free generating pair $(\frac{1}{\lambda}\gamma, \frac{1}{\lambda}\sigma)$. It follows that the finite measure $\frac{1}{\lambda}\sigma$ in $(\frac{1}{\lambda}\gamma, \frac{1}{\lambda}\sigma)$ satisfies (4.3). By Theorem 6.5 in [BT2], there is a self adjoint operator $X \in \widetilde{\mathcal{A}}$ such that

$$\int_0^t e^{-\lambda(t-u)}\, dS_u \stackrel{\mathrm{d}}{=} \int_0^{t\lambda} e^{-u}\, dS_{u/\lambda} \xrightarrow{\mathrm{d}} X,$$

as $t \to \infty$, and X has a $\boxplus$ self-decomposable distribution.

Suppose μ_0 is a free self-decomposable distribution on $\mathbb{R}$. By Theorem 6.5 in [BT2], there is free Lévy process S_t satisfying (4.2) and $\mu(\int_0^\infty e^{-t}\, dS_t) = \mu_0$. Let

$$X_t = e^{-t} \int_0^t e^{s}\, dS_s, \quad t \geq 0.$$

By the proof above, the limit distribution of X_t, as $t \to \infty$, is μ_0. □

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