Illinois Journal of Mathematics
Volume 52, Number 1, Spring 2008, Pages 1–15
S 0019-2082

ON B-INJECTORS OF THE COVERING GROUPS OF A_N

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ABSTRACT. A B-injector in an arbitrary finite group G is defined as a maximal nilpotent subgroup of G, containing a subgroup A of G of maximal order satisfying $\mathrm{class}(A) \leq 2$. The aim of this paper is to determine the B-injector of the covering groups of A_n.

1. Introduction

Let G be a finite group. A subgroup $U \leq G$ is an N-injector of G, if for every subnormal subgroup S of G, $U \cap S$ is a maximal nilpotent subgroup of S. N-injectors for nonsolvable groups have been introduced first by Mann [8]. He extended Fischer's results to N-constrained groups, that is, to groups G, such that $C_G(F(G)) \subseteq F(G)$, where $F(G)$ denotes the Fitting subgroup of G. It is well known that a solvable group is always N-constrained. In [5], Fischer, Gaschutz, and Hartley proved that if G is solvable, then N-injectors exist and any two of them are conjugate. It was (Förster [6], Iranso and Perez-Monasor [7]) who proved that N-injectors exist in all finite groups. Arad and Chillag [2] proved that if G is an N-constrained group, then A is an N-injector of G if and only if A is a maximal nilpotent subgroup of G containing an element of $a_2(G)$ where $a_2(G)$ is the set of all nilpotent subgroups of G of class at most 2 and having order $d_2(G)$ where $d_2(G)$ denotes the maximum of the orders of all nilpotent subgroups of class at most 2. A subgroup A of G is called a B-injector of G if A is a maximal nilpotent subgroup of G containing an element of $a_2(G)$. This definition has been used here and in [1]. In N-constrained groups the definition of N-injectors and the definition of B-injectors yield the same class of subgroups. If U is a B-injector of G, then U contains every nilpotent subgroup of G which is normalized by U [2]. In [9], Neumann proved that in any finite group G, B-injectors are N-injectors. The motivation behind this work is that B-injectors will lead to theorems similar

Received February 16, 2006; received in final form February 27, 2008.
2000 *Mathematics Subject Classification.* 20E28.

to Glauberman's ZJ-theorem and it is hoped that they provide tools and arguments for a modified and shortened the proof of the classification theorem of finite simple groups, in particular where the Thompson factorization theorem might fail [11]. The B-injectors of S_n and A_n have been determined in [3] and [4]. In [10], it is proved that the B-injectors of S_n and A_n are conjugate apart from some trivial cases which can be enumerated.

2. Preliminaries and notations

Our notation is fairly standard: throughout all groups are finite. If G is a group, $Z(G)$ denotes the center of G. If H and X are subsets of G, $C_H(X)$ and $N_H(X)$ denote respectively the centralizer and normalizer of X in H.

The generalized Fitting group $F^*(G)$ is defined by $F^*(G) = F(G)E(G)$ where $E(G) = \langle L \mid L \lhd \lhd\ G \text{ and } L \text{ is quasisimple}\rangle$ is a subgroup of G, A group L is called quasisimple if and only if $L' = L$ where L' is the derived group of L, and $L'/Z(L)$ is non-Abelian simple. $O_p(G)$ denotes the unique maximal normal p-subgroup of G, it is the Sylow p-subgroup of $F(G)$ and $O_{p'}(G) = \prod O_q(G)$, $q \neq p$ and q is prime. If $\Omega = \{1, 2, \ldots, n\}$, S_Ω will denote the symmetric group of degree n. Sometimes we write S_n for S_Ω. As is customary, we shall denote the alternating group on n points by A_n. Let $\Phi(G)$ denotes the Frattini subgroup of G, the intersection of all maximal subgroups of G. The integer part of the real number x is denoted by $[x]$. We denoted by $a_{2,p}(G)$ the set of p-subgroups, of class at most two and of largest possible order, of G.

We introduce the following definition.

Definition 1. Let G be a finite group, a nilpotent subgroup $U$of G is called a BG-injector of G if it contains every nilpotent subgroup it normalizes.

It is clear that BG-injector is maximal nilpotent and containing $F(G)$. Also, if U is a BG-injector of G and if $U \leq H \leq G$, then U is a BG-injector of H. Also, B-injectors are BG-injectors [9]. Schur [12] showed that if G is a non-Abelian simple group, then there exists a unique quasisimple group $\hat{G}$ such that $\hat{G}/Z(\hat{G}) \cong G$, and given any quasisimple group H with $H/Z(H) \cong G$, then H is isomorphic to $\hat{G}/Z$ for some subgroup $Z \subseteq Z(\hat{G})$, $Z(\hat{G})$ is called the Schur multiplier of G and denotes by $M(G)$ and $H \cong \hat{G}/Z$ is called a universal covering group of G. The Schur multipliers $M(A_n)$ for alternating groups A_n, have been determined in [12] and they are

$$M(A_n) = \begin{cases} Z_6, & n = 6, 7, \\ Z_2, & n \geq 5, n \neq 6, 7. \end{cases}$$

Hence, the universal covering groups of A_n, are $6A_6, 6A_7$, and $2A_n$ where $n \neq 6, 7$. Schur showed that there are two types of groups of shape $2S_n$ which denoted by $2\overset{+}{S}_n, 2\overset{-}{S}_n$, and $2A_n$ is then the commutator group of any of these.

So, $2A_n = (2\overset{+}{S}_n)' = (2\overset{-}{S}_n)'$ where G' denotes the commutator group of G. $2\overset{+}{S}_n$ can be easily described by defining relations.

So, let $H = 2\overset{+}{S}_n$ and denote $Z(H) = \langle -1 \rangle$, then we have the following. If $t \in S_n$ is a transposition and T is its preimage in $H = 2\overset{+}{S}_n$, then $T^2 = -1$ and if s, t are two transpositions in S_n and disjoint support with preimages S, T in H, then $[s, T] = -1$. So, $H = 2\overset{+}{S}_n$ is uniquely determined by these two relations. Also, if s, t are two pairwise commuting transpositions with preimages $T_1, T_2, \ldots, T_m$, then

$$(T_1, T_2, \ldots, T_m)^2 = (-1)^{\binom{m+1}{2}}.$$

Let Ω be a finite set, and let $\pi = (A_1, A_2, \ldots, A_m)$ be a partition of Ω into pairwise disjoint nonempty subsets of Ω, we denote its stabilizer by Y_π, Y_π is also called the Young subgroup of π, that is,

$$Y_\pi = \{g \in S_\Omega \mid A_i^g = A_i \text{ for all } i\}.$$

It is obvious that

$$Y_\pi = Y_{A_1} \times Y_{A_2} \times \cdots \times Y_{A_m} \leq S_\Omega,$$

where $Y_{A_i} = \{g \in S_\Omega \mid g \text{ fixes all points not in } A_i\}$ and $Y_{A_i} \equiv S_{A_i}$.

Furthermore, we define $Y^*_{A_i} \equiv Y_{A_i} \cap A_\Omega$, where A_Ω is the alternating group of Ω and we have

$$Y^*_\pi = \langle Y^*_{A_1}, Y^*_{A_2}, \ldots, Y^*_{A_m} \rangle = Y^*_{A_1} \times Y^*_{A_2} \times \cdots \times Y^*_{A_m} \leq A_\Omega.$$

NOTE 1. If $\sigma : K \longrightarrow A_\Omega$ be a surjective homomorphism, where $K = (2\overset{+}{S}_n)'$, then $\ker \sigma = \langle -1 \rangle$ and for any subgroup $X \leq A_\Omega$ we have the preimage $\hat{X} = \{x \in K \mid x^\sigma \in X\}$.

We prove the following lemma.

LEMMA 1. *$\hat{Y}^*_\pi = \hat{Y}^*_{A_1} \circ \hat{Y}^*_{A_2} \circ \cdots \circ \hat{Y}^*_{A_m}$, is the central product of $\hat{Y}^*_{A_1}, \hat{Y}^*_{A_2}, \ldots, \hat{Y}^*_{A_m}$, where $\hat{Y}^*_\pi$ is the preimage of Y^*_π and $\hat{Y}^*_{A_i}$ is the preimage of $Y^*_{A_i}$, $i = 1, 2, \ldots, m$, A_i, Ω and $Y^*_{A_i}$ are defined above.*

Proof. Let $\sigma : K \longrightarrow A_\Omega$ be a surjective homomorphism and let $x \in \hat{Y}^*_\pi$, then $x^\sigma \in Y^*_\pi$, so $x^\sigma = y_1 y_2 \cdots y_m$ for $y_i \in Y^*_{A_i}$. Choose $x_i \in \hat{Y}^*_{A_i}$ such that $x_i^\sigma = y_i$. Thus, $(x_1, x_2, \ldots, x_m) \in K$ and

$$(x_1, x_2, \ldots, x_m)^\sigma = x_1^\sigma x_2^\sigma \cdots x_m^\sigma = y_1 y_2 \cdots y_m = x^\sigma,$$

so $x^\sigma = (x_1 x_2 \cdots x_m)^\sigma$, it follows that $[(x_1 x_2 \cdots x_m) x^{-1}]^\sigma = 1$. This implies that $(x_1 x_2 \cdots x_m) x^{-1} \in \ker \sigma = \langle -1 \rangle$, thus $x_1 x_2 \cdots x_m = x$ or $-x$. It remains to prove that $[\hat{Y}^*_{A_i}, \hat{Y}^*_{A_j}] = 1$, for $i \neq j$.

Let $g \in Y_{A_i}$, $h \in Y_{A_j}$, then $g = t_1 t_2 \cdots t_k$ where t_i's are transpositions in Y_{A_i} and $h = s_1 s_2 \cdots s_m$ where s_i's are transpositions in Y_{A_j}. If T_i, S_i

are the corresponding preimages of t_i, s_i respectively, then $[T_i, S_i] = -1$ and $\hat{g} = T_1T_2\cdots T_k$, $\hat{h} = S_1S_2\cdots S_m$ are the preimages of g, h, respectively. So, $[\hat{g}, \hat{h}] = \hat{g}^{-1}(\hat{g})^{\hat{h}} = (T_1T_2\cdots T_k)^{S_1S_2\cdots S_m} = (-1)^{mk}(T_1T_2\cdots T_k)^{-1}T_1T_2\cdots T_k = (-1)^{mk}$ as

$$T_i^{S_1S_2\cdots S_m} = (-1)^m T_i.$$

So,

$$[\hat{g}, \hat{h}] = \begin{cases} -1, & \text{if } g, h \in S_\Omega \setminus A_\Omega, \\ 1, & \text{otherwise} \end{cases}$$

and it follows that $[\hat{Y}^*_{A_i}, \hat{Y}^*_{A_j}] = 1$ for $i \neq j$. This completes the proof of the lemma. □

NOTE 2. If Ω is a set of size n, and $\pi = (A_1, A_2, \ldots, A_m)$ is a partition of Ω, then the preimage $\hat{Y}^*_{A_i}$ of the Young subgroup $Y^*_{A_i}$ is isomorphic to:

(i) $2A_{n_i}$, if $|A_i| = n_i \geq 5$.
(ii) Z_2, if $n_i = 1, 2$.
(iii) Z_6, if $n_i = 3$ or $\mathrm{SL}(2,3)$ if $|A_i| = 4$.

LEMMA 2. *Let G be a finite group and U be a BG-injector of it.*

(i) *If $Z \leq Z(G)$, then $Z \leq U$ and U/Z is a BG-injector of G/Z.*
(ii) *If $F^*(G) = O_p(G)$, for some prime p, then U is a Sylow p-subgroup of G.*
(iii) *If G is a central product of two subgroups G_1, G_2 of G, that is, $G = G_1G_2, [G_1, G_2] = 1$, then $U = (U \cap G_1)(U \cap G_2)$ and $U \cap G_i$ is a BG-injector of G_i, for $i = 1, 2$.*

Proof. The proof is easy and is omitted. □

REMARK 1 ([6]). Let H be a finite group such that $H \cong Z_p \wr S_k$; the Wreath product of the cyclic group Z_p, p a prime, with S_k, then $F^*(H) = O_p(H)$.

REMARK 2. If Ω is a finite set, we denote by S_Ω, A_Ω the corresponding symmetric and alternating group of Ω. For a partition $\Sigma = (A_1, A_2, \ldots, A_m)$ of Ω into pairwise disjoint nonempty subsets of Ω,

$$Y_\Sigma = \{g \in S_\Omega \mid {A_i}^g = A_i, 1 \leq i \leq m\}$$

denotes the Young subgroup of Ω. It is obvious that

$$Y_\Sigma = Y_{A_1} \times Y_{A_2} \times \cdots \times Y_{A_m} \leq S_\Omega,$$

where $Y_{A_i} = \{g \in S_\Omega \mid g$ fixes all points not in $A_i\}$ and $Y_{A_i} \cong S_{A_i}$. We define $Y^*_{A_i} \cap A_\Omega$ and $Y^*_\Sigma = \langle Y^*_{A_1}, Y^*_{A_2}, \ldots, Y^*_{A_m}\rangle = Y^*_{A_1} \times Y^*_{A_2} \times \cdots \times Y^*_{A_m} \leq A_\Omega$. Consider an element $g \in S_\Omega$ of prime order $p \neq 2$. Let $A = \{\alpha \in \Omega | \alpha^g \neq \alpha\}$, $\Gamma = \{\alpha \in \Omega | \alpha^g = \alpha\}$. So $\Sigma = (A, \Gamma)$ is a partition of Ω. If $|A| = pk$, then g is a product of k pairwise commuting p-cycles $t_1, t_2, \ldots, t_k$ and $t_i \in Y_A$ corresponding to the orbits of g in A. Since $C_{S_\Omega}(g)$ permutes these t_i's, and in particular normalizes $V = \langle t_1, t_2, \ldots, t_k\rangle \cong {Z_p}^k$.

We infer that $V \subseteq O_p(C_{S_\Omega}(g))$, and $C_{S_\Omega}(g) \leq Y_z = Y_A \times \Gamma$, hence:

$$C_{S_\Omega}(g) = C_{Y_A}(g) \times Y_\Gamma.$$

As $C_{Y_A}(g) \cong Z_p \wr S_k$, by Remark 1, it follows that

$$F^*(C_{Y_A}(g)) = O_p(C_{Y_A}(g))$$

and

$$C(V) = V \times Y_\Gamma.$$

Lemma 3. *Let U be a BG-injector in A_Ω and let $g \in Z(U)$ with $o(g) = p \neq 2$, p prime, where $o(g)$ denotes the order of g. Then*

$$U = \big(U \cap C_{Y_A^*}(g)\big) \times (U \cap Y_\Gamma^*).$$

Proof. Since $g \in Z(U)$, $U \leq C_{A_\Omega}(g) \leq C_{S_\Omega}(g) = C_{Y_A}(g) \times Y_\Gamma \leq Y_A \times Y_\Gamma$. If V is as defined above, it follows that

$$V \subseteq O_p(C_{S_\Omega}(g)) = O_p(C_{A_\Omega}(g)) = F^*(C_{A_\Omega}(g)),$$

as p is odd.

As U is a BG-injector of $C_{A_\Omega}(g)$, this implies that $V \subseteq O_p(C_{A_\Omega}(g)) \subseteq U$, but U is nilpotent, so

$$U = O_p(U) \times O_{p'}(U).$$

Also, $V \subseteq O_p(U)$ and $O_{p'}(U) \subseteq C(O_p(U))$, thus

$$O_{p'}(U) \subseteq C_{A_\Omega}(V).$$

So,

$$O_{p'}(U) \leq C_{S_\Omega}(V) = V \times Y_\Gamma.$$

As $U \leq A_\Omega$ and $V \leq A_\Omega$ $(p \neq 2)$, we obtain

$$O_{p'}(U) = O_{p'}(U) \cap A_\Omega \leq (V \times Y_\Gamma) \cap A_\Omega = V \times (Y_\Gamma \cap A_\Omega) = V \times Y_\Gamma^*.$$

Thus, $O_{p'} \leq Y_\Gamma^*$ as $p \mid |V|$ and, therefore,

$$U = O_p(U) \times O_{p'}(U) \leq C_{Y_A}(g) \times Y_\Gamma^*,$$

this implies that $U \leq C_{Y_A^*}(g) \times Y_\Gamma^* \leq Y_A^* \times Y_\Gamma^*$, as $p \neq 2$. Hence, by Lemma 2 we have

$$U = \big(U \cap C_{Y_A^*}(g)\big) \times (U \cap Y_\Gamma^*) = (U \cap Y_A^*) \times (U \cap Y_\Gamma^*). \qquad \square$$

Lemma 4. *Let Ω be a finite set and let U be a BG-injector of A_Ω, then there exists a partition $\Sigma = (A_1, A_2, \ldots, A_m)$ of Ω such that $U \leq Y_{A_1}^* \times Y_{A_2}^* \times \cdots \times Y_{A_m}^*$ and $U = (U \cap Y_{A_1}^*) \times \cdots \times (U \cap Y_{A_m}^*)$. Also, for $i = 1, 2, \ldots, m$, there exists a prime p_i such that $(U \cap Y_{A_i}^*)$ is a Sylow p_i-subgroup of $Y_{A_i}^*$.*

Proof. We consider two cases:

Case 1. U is a 2-group. Let Σ be the partition consisting of Ω alone, that is, $\Sigma = (\Omega)$. So, $Y_\Sigma^* = A_\Omega$ and $U = U \cap Y_\Sigma^*$. As U is a BG-injector of A_Ω, it is maximal nilpotent, and thus U is a Sylow 2-subgroup of A_Ω.

CASE 2. U is not a 2-group, so there exists a prime $p \neq 2$ such that $p \mid |U|$.

As U is nilpotent, it follows that there exists $z \in Z(U), o(z) = p$. Let A_1 be the set of nonfixed points of $Z = Z(U)$ and Γ be the set of fixed points of Z. By Lemma 3, we get

$$U \leq C_{A_\Omega}(z) = C_{Y^*_{A_1}}(z) \times Y^*_\Gamma \leq Y^*_{A_1} \times Y^*_\Gamma.$$

Also, by Lemma 3, we obtain

$$U = U \cap C_{Y^*_{A_1}}(z) \times (U \cap Y^*_\Gamma) = (U \cap Y^*_{A_1}) \times (U \cap Y^*_\Gamma),$$

$U \cap C_{Y^*_{A_1}}(z)$ is a BG-injector of $Y^*_{A_1}$, and $U \cap Y^*_\Gamma$ is a BG-injector of Y_Γ. As $U \cap C_{Y^*_{A_1}}(z)$ is a BG-injector of $C_{Y^*_{A_1}}(z)$ and $\Gamma^*(C_{Y^*_{A_1}}(z)) = O_p(C_{Y^*_{A_1}}(z))$, we get that $U \cap C_{Y^*_{A_1}}(z)$ is a Sylow p-subgroup of $Y^*_{A_1} \cong A_{A_1}$ and $U \cap Y^*_\Gamma$ is a BG-injector of $Y^*_\Gamma \cong A_\Gamma$. Repeating the argument for $U \cap Y_\Gamma$ and $Y^*_\Gamma \cong A_\Gamma$, the claim follows. $\square$

THEOREM 1. *Let K be a group isomorphic to $2A_\Omega$, where Ω is a finite set of size n. If B is a B-injector of K, then there exists a partition $\pi = (A_1, A_2, \ldots, A_m)$ of Ω such that:*

(i) *For each $i, B \cap \hat{Y}^*_{A_i}$ is a B-injector of $\hat{Y}^*_{A_i}$.*
(ii) *Let $Z = Z(K)$ and $B_i = B \cap \hat{Y}^*_{A_i}$, then $B_i \cong Z \times O_{p_i}(B_i)$, for some prime $p_i \neq 2$.*
(iii) *$d_2(2A_{A_i}) = 2{p_i}^{n_i/p_i}$ and for any odd prime $p, p^{[n_i/p]} \leq {p_i}^{n_i/p_i}$.*
(iv) *There is at most one i with $p_i = 5$ and the union of the A_i's with $p_i = 3$ has size at most 6, and there are no i, j such that $p_i = 3$ and $p_j = 5$.*

Proof. (i) As B/Z is a B-injector of $K/Z \cong A_\Omega$, there exists by Lemma 4, a partition $\pi = (A_1, A_2, \ldots, A_m)$ of Ω such that

$$\hat{Y}^*_\pi = \hat{Y}^*_{A_1} \times \cdots \times \hat{Y}^*_{A_m}.$$

Let $B/Z = U$, then $U \leq Y^*_\pi$ and $U = (U \cap Y^*_{A_1}) \times \cdots \times (U \cap Y^*_{A_m})$. Thus,

$$B \leq \hat{Y}^*_\pi = \hat{Y}^*_{A_1} \circ \hat{Y}^*_{A_2} \circ \cdots \circ \hat{Y}^*_{A_m},$$

the central product of $\hat{Y}^*_{A_i}$, by Lemma 1. Hence,

$$B = (B \cap \hat{Y}^*_{A_1}) \times \cdots \times (B \cap \hat{Y}^*_{A_m})$$

and $B \cap \hat{Y}^*_{A_i}$ is a B-injector of $B \cap \hat{Y}^*_{A_i}$.

(ii) As $Z \leq B \cap \hat{Y}^*_{A_i} = B_i$ and $B_i/Z \cong U \cap Y^*_{A_i}$, then for any prime $p_i \neq 2$, we have $B_i = \prod_{p\text{-prime}} O_p(B_i)$ and $Z \leq O_2(B_i)$. So,

$$B_i/Z \cong O_2(B_i)/Z \times \prod_{p \neq 2} O_p(B_i).$$

As $B_i/Z \cong U \cap Y^*_{A_i}$ is a Sylow p_i-subgroup of $Y^*_{A_i}$ by Lemma 4, it follows that $B_i = Z \times O_{p_i}(B_i)$ and $U \cap Y^*_{A_i} \cong O_{p_i}(B_i)$, which is a Sylow p_i-subgroup of $Y^*_{A_i} \cong A_{A_i} = A_{n_i}$ where $|A_i| = n_i$.

(iii)

$$\begin{aligned} d_2(2A_{A_i}) &= d_2(\hat{Y}^*_{A_i}) = d_2(B_i) = d_2(Z)d_2(O_{p_i}(B_i)) \\ &= 2d_{2,p}(A_{A_i}) = 2{p_i}^{n_i/p}. \end{aligned}$$

Also, if p is a prime $\neq 2$, we have $d_{2,p}(2A_n) = 2d_{2,p}(A_n)$, because $2A_n$ and A_n have isomorphic Sylow p-subgroups. As $d_{2,p}(2A_n) \leq d_2(2A_n)$, we get

$$2p^{[n_i/p]} = 2d_{2,p}(A_{n_i}) = d_{2,p}(2A_{n_i}) \leq d_2(2A_{n_i}) = 2p_i^{n_i/p_i}$$

or $p^{[n_i/p]} \leq p_i^{n_i/p_i}$ for all odd primes.

(iv) Let $I \subseteq \{1, 2, \ldots, n\}$, so that p_i is an odd prime for all $i \in I$. Then for $A = \bigcup_{i \in I} A_i$, it follows that the central product

$$\prod_{\circ} \hat{Y}^*_{A_i} \leq \hat{Y}^*_A,$$

$B \cap \hat{Y}^*_A = \prod_{\circ}(B \cap \hat{Y}^*_{A_i})$ and $(B \cap \hat{Y}^*_A)$ is a B-injector in $\hat{Y}^*_A \cong 2A_A$.

Consider the following cases.

Case 1. Assume that there are disjoint A_i, A_j such that $p_i = p_j = 5$. So, $|A_i| = |A_j| = 5$. Set $A = A_i \cup A_j$. It follows that

$$B \cap \hat{Y}^*_A = (B \cap \hat{Y}^*_{A_i})(B \cap \hat{Y}^*_{A_j})$$

is a B-injector in $\hat{Y}^*_A$ of order $2 \cdot 5^2$: this is a contradiction, as $d_2(2A_{10}) \geq d_2(2A_8) \geq 2^6$.

Case 2. Let J be the set of numbers j such that $p_j = 3$ and let $A = \bigcup_{j \in J} A_j$, then $(B \cap \hat{Y}^*_A)$ is a B-injector of $\hat{Y}^*_A$ and it is of the form $Z \times P$ for some Sylow 3-subgroup P of $\hat{Y}^*_A$. Hence, if $|A| = 3k$, then

$$d_2(2A_A) = d_2(\hat{Y}^*_A) = 2 \cdot 3^6.$$

So, $d_{2,2}(2A_A) \lneqq 2 \cdot 3^k$. By Corollary 3, we have $3k < 8$ or $3k = 15$, but $d_2(2A_{15}) \geq 2 \cdot \frac{d_{2,2}(2A_8)}{2} \cdot \frac{d_{2,2}(2A_4)}{2} \cdot \frac{d_{2,2}(2A_3)}{2} \geq 2 \cdot \frac{64}{2} \cdot \frac{8}{2} \cdot \frac{6}{2}$. Hence, $64 \cdot 24 \leq d_2(2A_{15}) = d_2(\hat{Y}^*_A) = 2 \cdot 3^5$ is a contradiction.

Case 3. Assume that there exist i, j such that $p_i = 5$ and $p_j = 3$, then $|A_i| = 5$ and $|A_j| = 3$ or 6. Set $A = A_i \cup A_j$, it follows that $(B \cap \hat{Y}^*_A)$ is a B-injector of $\hat{Y}^*_A \cong 2A_A$, and hence $|A_j| = 3$, thus $|A| = 8$ and $d_2(2A_8) = d_2(\hat{Y}^*_A) = 2 \cdot 3 \cdot 5 = 30$, a contradiction, as $64 \leq d_{2,2}(2A_8) \leq d_2(2A_8)$.

If $|A_j| = 6$, then $|A| = 11$ and $d_2(2A_{11}) = d_2(\hat{Y}^*_A) = 2 \cdot 5 \cdot 3^2 = 90$, a contradiction, as $d_2(2A_{11}) \geq 2 \cdot \frac{d_{2,2}(2A_8)}{2} \cdot d_{2,2}(2A_3) \geq 2 \cdot \frac{64}{2} \cdot \frac{6}{2} > 90$. □

Lemma 5. *Let Ω be a finite set of size n, and let P be a transitive p-subgroup of S_Ω of class ≤ 2. Then there exist integers $a \ge 0, b \ge 0$ such that $n = p^{a+b}$ and $|P| \le p^{a+b+ab}$.*

Proof. As P is transitive on Ω, $Z = Z(P)$ acts semiregularly on Ω that $Z_\alpha = 1$ $\forall \alpha \in \Omega$, because let $z \in Z_\alpha$, so $z \in Z(P)$, it follows that P leaves invariant the set of fixed points of Z, so $\mathrm{fix}(z) = \Omega$, and thus $z = 1$. As class $P \le 2$, it follows that $P' \le Z(P)$, and hence

$$(P_\alpha)' \le (P')_\alpha \le Z_\alpha = 1.$$

So, P_α is Abelian, and $M = \langle Z, P_\alpha \rangle = Z \times P_\alpha$ is an Abelian normal subgroup of P, as $P' \le Z \le M$ and $Z \cap Z_\alpha = Z_\alpha = 1$. Set $|P/M| = P^\alpha$ and $|Z| = p^b$, then there exist $t_1, t_2, \ldots, t_a \in P$ such that $P/M = \langle Mt_1, Mt_2, \ldots, Mt_a \rangle$. Next, consider the map $\sigma : P_\alpha \longrightarrow (P')^a$ defined by $\sigma(x) = ([x,t_1], \ldots, [x,t_a])$. As $\mathrm{class}(P) \le 2$, it follows that σ is a homomorphism. This can be seen as follows. In groups of class, at most two, we have the following relation:

$$[xy, t] = y^{-1}[x,t]y^t = [x,t]y^{-1}y^t$$

as $[x,t] \in P' \subseteq Z(P)$. So, $[xy,t] = [x,t][y,t]$, where $y^t = t^{-1}yt$ and $\ker \sigma = 1$, because let $x \in \ker \sigma$, it follows that $[x, t_i] = 1, i = 1, \ldots, a$, thus $t_1, \ldots, t_a$ are in $C_p(x)$. Furthermore, $x \in P_\alpha \subseteq M = Z \times P_\alpha$ and $M \subseteq C_p(x)$, as M is Abelian. Thus, $\langle M, t_1, \ldots, t_a \rangle \subseteq C_p(x)$. As $P/M = \langle Mt_1, \ldots, Mt_a \rangle$, it follows that $P = \langle M, t_1, \ldots, t_a \rangle \subseteq C_p(x)$, thus $x \in Z(P) \cap P_\alpha = (Z(P))_\alpha = 1$. Hence, $x = 1$. So, σ is injective. Therefore, $|P_\alpha| \le |P'|^a \le |Z(P)|^a = p^{ba}$ and $n = [P : P_\alpha] = [P : M][M : P_\alpha]$ as $P_\alpha \le M \le P$, it follows that

$$[P : P_\alpha] = p^a \frac{|M|}{|P_\alpha|} = p^a \frac{|Z||P_\alpha|}{|P_\alpha|} = p^a p^b = p^{a+b}$$

and $|P| = n|P_\alpha| \le np^{ab} = p^{a+b+ab}$. This completes the proof. $\square$

Corollary 1. *Let Ω be a finite set of size n and let P be a transitive p-subgroup of Ω of class ≤ 2, if $p \ne 2$, then $|P| \le p^{[n/p]}$, where equality holds if and only if for $n = p$ or $n = 9$ and $p = 3$.*

Proof. Since $p \ne 2$, by Lemma 4, there exist two integers $a \ge 0$, $b \ge 0$ such that $n = p^{a+b}$, $|P| \le p^{a+b+ab}$. As $p \ne 2$, it follows that $p^{a+b+ab} \le p^{n/p}$ if and only if $a + b + ab \le n/p = p^{a+b-1}$, where equality occurs if and only if $n = p$ or $n = 9$ and $p = 3$. $\square$

Lemma 6. *Let Ω be a finite set of size n and let P be a transitive p-subgroup of Ω of class ≤ 2, then*

(i) *If $p \ne 2, d_{2,p}(S_n) = d_{2,p}(A_n) = p^{[n/p]}$.*

(ii) *If $p = 2$, $d_{2,2}(S_n) = \varepsilon_n 8^{[n/4]}$ where*

$$\varepsilon_n = \begin{cases} 1, & n \equiv 0, 1 \ (mod\ 4), \\ 2, & n \equiv 2, 3 \ (mod\ 4). \end{cases}$$

and if $n>1$, $d_{2,2}(A_n)=\frac{1}{2}d_{2,2}(S_n)=\frac{1}{2}\varepsilon_n 8^{[n/4]}$.

Proof. S_n contains subgroups of order $p^{[n/p]}$ for any prime p. These groups are generated by $[n/p]$ cycles with disjoint support and $p^{[n/p]}\leq d_{2,p}(S_n)$. This can be explained as follows. Let $n=mp+r$, $0\leq r<p$, $m=[n/p]$, and let $\pi=(A_1,A_2,\dots,A_m,A)$ be a partition of Ω where $|A_i|=p, i=1,2,\dots,m$, and $|A|=r$. Let $t_i=(a_1a_2\cdots a_p)$ be a p-cycle in $A_i, i=1,2,\dots,m$. It follows that $\langle t_1,t_2,\dots,t_m\rangle$ is an elementary Abelian group of order $p^{[n/p]}$ and of class at most two. Also, S_n contains 2-subgroups of order $\varepsilon_n 8^{[n/4]}\leq d_{2,2}(S_n)$. This can be explained as follows.

Let $\pi=(A_1,A_2,\dots,A_m,A)$ be a partition of Ω where $|A_i|=4$, $i=1,2,\dots,m$ and $|A|=r$. Let $n=4m+r$, $0\leq r<4$. It follows that

$$H=Y_{A_1}\times Y_{A_2}\times\cdots\times Y_{A_m}\times Y_r\leq S_n,$$

where $Y_{A_i}\cong S_4$ and $Y_r\cong Z_{\varepsilon_n}$.

Hence, $H\cong S_4^m\times S_r$ contains $D_8^m\times Z_{\varepsilon_n}$ of class ≤ 2. It remains to show that for $p\neq 3$, these groups are exactly all possible p-subgroups of class $\leq$2 and order $d_{2,p}(S_n)$. Let $P\in a_{2,p}(S_n)$. Assume that P has orbits $A_1,A_2,\dots,A_m$, it follows that

$$P\leq Y_\Sigma=Y_{A_1}\times\cdots\times Y_{A_m},$$

where Y_{A_i} are the Young subgroups corresponding to the partition $\Sigma=(A_1,A_2,\dots,A_m)$.

Furthermore, by Lemma 2, we have that

$$P=(P\cap Y_{A_1})\times\cdots\times(P\cap Y_{A_m})$$

and $P\cap Y_{A_i}\in a_{2,p}(Y_{A_i})$. As A_i is an orbit of P, $P\cap Y_{A_i}$ is a transitive subgroup of $Y_{A_i}\cong S_{A_i}$ of class $\leq$2.

Now we consider two cases.

Case 1. $p=2$. Let $|A_i|=n_i$, if $p\neq 2$, it follows that

$$p^{[n_i/p]}=p^{n_i/p}\leq d_{2,p}(S_{A_i})=|P\cap Y_{A_i}|.$$

By Corollary 1, $|P\cap Y_{A_i}|\leq p^{n_i/p}$. Therefore,

$$p^{n_i/p}=d_{2,p}(S_{A_i})=|P\cap Y_{A_i}|.$$

Also, by Corollary 1, it follows that $n_i=p$ or $n_i=9$ and $p=3$. If $p\neq 3$, then all orbits of P have lengths 1 or p. Thus, P is conjugate to the subgroup constructed above, and hence $d_{2,p}(S_n)=p^{[n/p]}$. As $p\neq 2$, it follows that

$$d_{2,p}(S_n)=d_{2,p}(A_n).$$

Case 2. $p=2$. Let $P\in a_{2,2}(S_n)$ and let $P\leq Y_\Sigma=Y_{A_1}\times\cdots\times Y_{A_m}$ where Y_{A_i}, $i=1,2,\dots,m$, be the Young subgroups corresponding to the partition $\Sigma=(A_1,A_2,\dots,A_m)$.

As above $P=(P\cap Y_{A_1})\times\cdots\times(P\cap Y_{A_m})$ where $P\cap Y_{A_i}\in a_{2,2}(Y_{A_i})$ and $P\cap Y_{A_i}$ is a transitive subgroup of Y_{A_i}. By Lemma 6, $|A_i|=1$ or 2 and $8^{n_i/4}\leq$

$d_2(S_{A_i}) = |P \cap Y_{A_i}| \leq 8^{n_i/4}$. This implies that $|P \cap Y_{A_i}| = 8^{n_i/4}$ and this occurs if and only if $n_i = 4$. Hence again, P is a group conjugate to the group constructed above. As $P \nleq A_n$, this implies that $d_{2,2}(A_n) = \frac{1}{2} d_{2,2}(S_n)$. □

LEMMA 7.

(i) *If p is a prime at least 7, then $p^k \lneqq 3^{[pk/3]}$ for all $k \geq 1$.*
(ii) $5^k \lneqq 3^{[5k/3]}$ *for $k \geq 2$.*

Proof. Easy. □

REMARK 3. By Theorem 1 and Lemma 7, we have $3^{[n_i/3]} \leq p_i^{n_i/p_i}$ and $5^{[n_i/5]} \leq p_i^{n_i/p_i}$ which implies that $p_i = 3$ or 5 and if $p_i = 5$, then $|A_i| = n_i = 5$. We need some information about $d_{2,2}(2A_n)$. This is a bit more complicated, as we cannot use our information about A_n directly, because if $X \leq 2A_n$, $Z \leq X$, then $X/Z \leq A_n$ and class$(X/Z) \leq$ class(X), but if $Y \leq A_n$ and it is a 2-group of class ≤ 2, then $\hat{Y}$ might have class equal to 3.

First, we know that in $S_n, n = 4m + r, 0 \leq r < 4, D_8^m \leq S_n$ and in $2A_n, n = 8m + r, 0 \leq r < 7$, we have the central product

$$X_1 \circ X_2 \circ \cdots \circ X_m \circ Y \leq 2A_m,$$

where $X_i \cong 2A_8$ and $Y \cong 2A_r$. In each X_i, we take a 2-group P_i of class ≤ 2 and in Y a 2-group Q of class ≤ 2, with $Z \leq P_i, Z \leq Q$, then it follows that

$$\langle P_1, \ldots, P_m, Q \rangle = P_1 \circ \cdots \circ P_m \circ Q$$

has class ≤ 2 and $|P_1 \circ \cdots \circ P_m \circ Q| = 2|P_1/Z||P_2/Z| \cdots |P_m/Z||Q/Z|$.

REMARK 4. Let $\pi = (A_1, \ldots, A_m)$ is a partition of Ω and $Y_\pi^* = Y_{A_1}^* \times \cdots \times Y_{A_m}^*$. Assume that in each $Y_{A_1}^*$, a nilpotent subgroup X_i of class ≤ 2 such that its preimage $\hat{X}_i$ has also class ≤ 2, then the group $\langle \hat{X}_1, \ldots, \hat{X}_m \rangle$ is a central product of the $\hat{X}_i$'s of class ≤ 2 and of order

$$2|\hat{X}_1/Z||\hat{X}_2/Z| \cdots |\hat{X}_m/Z| = 2|X_1||X_2| \cdots |X_m|.$$

To get an estimation for $d_{2,2}(2A_n)$, we prove the following lemma.

LEMMA 8. $d_{2,2}(2A_8) = 2^6$.

Proof. As $d_{2,2}(A_8) = \frac{1}{2} d_{2,2}(S_8) = \frac{1}{2} 8^2 = 2^5$ (use Lemma 6), it follows that

$$d_{2,2}(2A_8) \leq 2 d_{2,2}(A_8) = 2^6.$$

Furthermore,

$$d_{2,2}(2A_n) \leq 2 d_{2,2}(A_n),$$

because, if $P \leq 2A_n$ a 2-group of class ≤ 2 with $Z \leq P$, then we have class $(P/Z) \leq 2$, and this implies that $|P/Z| \leq d_{2,2}(A_n)$, and hence

$$\frac{|P|}{2} \leq d_{2,2}(A_n).$$

□

Lemma 9. *Let $H \cong 2^{1+4}$ be the extra special group of $A_8 \cong GL(4,2)$, then the preimage $\hat{H}$ of H has class at most 2.*

Proof. Let

$$H_1 = \left\{ \begin{bmatrix} 1 & & & \\ * & 1 & & \\ * & 0 & 1 & \\ * & 0 & 0 & 1 \end{bmatrix} \right\} \quad \text{and} \quad H_2 = \left\{ \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ * & * & * & 1 \end{bmatrix} \right\}.$$

It is clear that $H = H_1 H_2$ where $H_1 \cong H_2 \cong {Z_2}^3$ and $\hat{H} = Z_2$.

Also, $[H_1, H_2] = H_1 \cap H_2 = Z(H) \cong Z_2$, where

$$H_1 \cap H_2 = \left\langle \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 1 & 0 & 0 & 1 \end{bmatrix} \right\rangle.$$

All nonidentity elements of $H_1 \cup H_2$ are transfection, and in particular are conjugate elements in $H_1 \cup H_2 \setminus \{1\}$. From this, it follows that the preimages $\hat{H}_1, \hat{H}_2$ are elementary Abelian. This can be proved as follows.

Let $x, y \in \hat{H}_1 \backslash Z(K), K = 2A_8$, such that $x, y \notin Z(k)$, then we have

$$xZ(K) \sim yZ(K) \sim xyZ(K),$$

and hence

$$x^2 = y^2 = (xy)^2 = z \in Z(K).$$

So,

$$z = xyxy = xy^2y^{-1}xy = xy^2x^y = xzx^y.$$

This implies $x = zx^{-1}$ and $x^y = xz$. As $|H_1| = 8$, there exists $a, b, c \in \hat{H}_1$, where $a, b, c, ab, ac, bc \notin Z(K)$. So,

$$z = (abc)^2 = zb^a c^a bc = z^3(bc)^2 = z^4 = 1.$$

Hence, $o(z) = 1$ or 2, so $z = 1$ and $a^2 = b^2 = c^2 = 1 = [a, b]$. Therefore, $\hat{H}_1, \hat{H}_2$ are elementary Abelian groups. So,

$$\begin{aligned} \hat{H} &= \hat{H}_1\hat{H}_2, \hat{H}_1 \leq \hat{H}, \hat{H}' = (\hat{H}_1\hat{H}_2)' \\ &= (\hat{H}_1)'[\hat{H}_1, \hat{H}_2](\hat{H}_2)' = [\hat{H}_1, \hat{H}_2] \subseteq \hat{H}_1 \cap \hat{H}_2. \end{aligned}$$

As $\hat{H}_1, \hat{H}_2$ are elementary Abelian, it follows that $\hat{H}_1 \cap \hat{H}_2 \subseteq Z(\hat{H})$. Hence, $\hat{H}' \subseteq Z(\hat{H})$ and class $\hat{H} \leq 2$. □

Theorem 2. *If Ω is a set of size n, and $\pi = (A_1, A_2, \ldots, A_m)$ is a partition of Ω with $|A_i| = n_i$, then*

$$d_{2,2}(2A_\Omega) \geq 2 \cdot \frac{d_{2,2}(2A_{A_1})}{2} \cdot \frac{d_{2,2}(2A_{A_2})}{2} \cdot \ldots \cdot \frac{d_{2,2}(2A_{A_m})}{2}.$$

Proof. Consider the Young subgroup $Y_\pi^* = Y_{A_1}^* \times \cdots \times Y_{A_m}^*$. The preimage

$$\hat{Y}_\pi^* = \hat{Y}_{A_1}^* \circ \hat{Y}_{A_2}^* \circ \cdots \circ \hat{Y}_{A_m}^*,$$

is the central product of $\hat{Y}_{A_i}^* \cong 2A_{A_i}, i = 1, 2, \ldots, m$. By Lemma 8 and Remark 4, we have in each $\hat{Y}_{A_i}^*$, there exists a 2-group of class ≤ 2 and of order $d_{2,2}(2A_{A_i})$. These groups generate a subgroup of $2A_\Omega$ of class at most 2 and of order $2 \cdot \frac{d_{2,2}(2A_{A_1})}{2} \cdot \frac{d_{2,2}(2A_{A_2})}{2} \cdot \cdots \cdot \frac{d_{2,2}(2A_{A_m})}{2}$. □

Corollary 2. *Let $n = 8.k + r, 0 \leq r < 8$, then*

$$d_{2,2}(2A_n) \geq 2 \cdot (32)^k \frac{d_{2,2}(2A_r)}{2}.$$

Corollary 3. *If $n \geq 8, n \neq 15$, then*

$$d_{2,2}(2A_n) \gneqq 2.d_{2,3}(2A_n).$$

Proof. Use the inequality

$$d_{2,2}(2A_n) \geq 2 \cdot (32)^k \frac{d_{2,2}(2A_r)}{2}$$

if $n = 8.k + r, 0 \leq r < 8$, and Table 1. □

Corollary 4. *Let $8 \mid |\Omega|$, then*

$$d_{2,2}(2A_\Omega) \geq 2 \cdot 32^{n/8}.$$

Proof. As $8 \mid |\Omega|$, there exists a partition $\pi = (A_1, A_2, \ldots, A_m)$ of Ω such that $|A_i| = 8$. By Theorem 2, it follows that

$$d_{2,2}(2A_8) \geq 2 \cdot \frac{d_{2,2}(\hat{Y}_{A_1}^*)}{2} \cdots \frac{d_{2,2}(\hat{Y}_{A_m}^*)}{2} \geq 2 \cdot (32)^m$$

as $d_{2,2}(\hat{Y}_{A_i}^*) = d_{2,2}(2A_8) \geq 64$. □

Table 1.

n	$d_{2,2}(2A_n)$
0	1
1	1
2	2
3	2
4	8 $2A_4 \cong SL(2,3)$
5	8 $2A_5 \cong SL(2,5)$
6	8 $2A_6 \cong SL(2,9)$
7	8 $2A_6$ and $2A_7$ have isomorphic Sylow 2-groups

COROLLARY 5. *If $X \in a_{22}(2A_\Omega)$, then $Z = Z(2A_\Omega) \subseteq X$, and any orbit of X/Z in Ω has length ≤ 8, or $|\Omega| \leq 7$; also if A is an orbit of length 8, then*

$$C_X(A) \in a_{22}(\hat{Y}^*_{\Omega\setminus A}).$$

Proof. Let A be an orbit of X/Z of length ≥ 8, and let $\Gamma = \Omega \setminus A$.

The partition $\pi = (A, \Gamma)$ implies $2d_{2,2}(\hat{Y}^*_A)d_{2,2}(\hat{Y}^*_\Gamma) \leq d_{2,2}(2A_\Omega) = |X|$. So,

$$C_X(A) = \{x \in X, x \text{ fixes all points in } A \leq \hat{Y}^*_\Gamma \text{ and it is of class} \leq 2\}.$$

So, $|C_X(A)| \leq d_{2,2}(\hat{Y}^*_\Gamma)$, also $X/C_X(A)$ is a transitive subgroup of S_A of class $\leq$2. Furthermore,

$$2\frac{d_{2,2}(\hat{Y}^*_A)}{2}\frac{d_{2,2}(\hat{Y}^*_\Gamma)}{2} \leq |X| = |X/C_X(A)| \cdot |C_X(x)| \leq |X/C_X(A)| \cdot d_{2,2}(\hat{Y}^*_\Gamma).$$

This implies that $|X/C_X(A)| \geq 32^{|A|/8}$. By Lemma 5, there exist integers a, b such that $|A| = 2^{a+b}$ and $|X/C_A(A)| \leq 2^{a+b+ab}$. So,

$$2^{a+b+ab} \geq |X/C_X(A)| \geq 32^{|A|/8},$$

then it follows that $a + b + ab \geq 5|A|/8 = 5.2^{a+b-3}$. Hence, $|A| = 8$. We also see that in all estimations equality must hold. Thus, $C_X(A) \in a_{22}(\hat{Y}^*_{\Omega\setminus A})$. $\square$

COROLLARY 6. *If $|\Omega|$ is even, then*

$$d_{2,2}(2A_\Omega) = \begin{cases} 2 \cdot 32^{[n/8]}, & \text{if } |\Omega| \equiv 0, 2 \text{ mod } 8, \\ 2 \cdot 4 \cdot 32^{[32/8]}, & \text{if } |\Omega| \equiv 4, 6 \text{ mod } 8. \end{cases}$$

COROLLARY 7. *Let $|\Omega| = n$. The B-injectors in $2A_\Omega$ are as follows:*

- *$n \equiv 0, 1, 4$ mod 8, the B-injectors are Sylow 2-subgroups.*
- *$n \equiv 3, 7$ mod 8, the B-injectors correspond to the partition $\pi = (A, \Gamma)$, $|A| = 3$. So, the B-injectors are $Z_3 \times T_2$, where T_2 is a Sylow 2-subgroup in $\hat{Y}^*_\Gamma$.*
- *$n \equiv 6, 2$ mod 8, the B-injectors correspond to the partition $\pi = (A, \Gamma)$, $|A| = 6$. So, the B-injectors are $Z_3 \times Z_3 \times T_2$, where T_2 is a Sylow 2-subgroup in $\hat{Y}^*_\Gamma$.*
- *$n \equiv 5$ mod 8, the B-injectors correspond to the partition $\pi = (A, \Gamma)$, $|A| = 5$. Hence, the B-injectors are $Z_5 \times T_2$, where T_2 is a Sylow 2-subgroup in $\hat{Y}^*_\Gamma$.*

THEOREM 3. *B-injectors in $3A_6$ are the Sylow 3-subgroups.*

Proof. As 3-subgroups of $3A_6$ have order 3^3, and hence have class ≤ 2. It suffices to show that there are no nilpotent subgroups of class at most 2 and of order > 27. So, let X be a nilpotent subgroup of $3A_6$. If $5 \mid |X|$, it follows that $X \leq C(z)$ for some element z of order 5. As elements of order 5 in A_6 are self centralizing, it follows that $|X| \leq 3 \cdot 5 = 15$. If $2 \mid |X|$, then $X \leq C(z)$ for some involution $z \in 3A_6$. As centralizers of involutions in A_6 have order 8, it follows that $|X| \leq 3 \cdot 8 = 24 < 27$. So, the claim follows. $\square$

Theorem 4. *B-injectors in $3A_7$ are the groups of order 36, and are the preimages in $3A_7$ of subgroups $Z_2^2 \times Z_3$ of Young subgroups $A_4 \times A_3 \leq A_7$.*

Proof. As elements of order 5 or 7 are self-centralizing in A_7, it follows that nilpotent subgroups of $3A_7$ which are divisible by 5 or 7 can have orders at most 15 or 21, respectively. As Sylow 3-subgroups of $3A_7$ have order $27 < 36$, then any nilpotent subgroup of $3A_7$ of class ≤ 2 and order ≥ 36 must be contained in a centralizer of an involution. As centralizers of involutions in A_7 have order 24 and are not nilpotent, the claim follows. □

Theorem 5. *B-injectors in $6A_6$ are the groups $Z.T_3$, where Z is the center and T_3 is a Sylow 3-subgroup of order 54.*

Proof. As element of order 5 in A_6 are self-centralizing, it follows that nilpotent subgroups in $6A_6$, whose order is divisible by 5 can have at most order $30 < 54$. As centralizers of involutions in A_6 have order 8. It follows that nilpotent subgroups of whose Sylow 2-subgroups are not contained in the center of $6A_6$ can have order at most $48 < 54$. So, the claim follows. □

Theorem 6. *B-injectors in $6A_7$ are groups of order 72 corresponding to subgroups $Z_2^2 \times Z_3$ in Young subgroups $A_4 \times A_3 \leq A_7$.*

Proof. Similar as above. □

Acknowledgments. The authors' thanks are due to Mr. H. Jorg for his discussion, and to the referee for his remarks.

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