Degenerate affine Grassmannians and loop quivers

Evgeny Feigin, Michael Finkelberg, and Markus Reineke

Abstract We study the connection between the affine degenerate Grassmannians in type A, quiver Grassmannians for one vertex loop quivers, and affine Schubert varieties. We give an explicit description of the degenerate affine Grassmannian of type GL_n and identify it with semi-infinite orbit closure of type A_{2n-1}. We show that principal quiver Grassmannians for the one vertex loop quiver provide finite-dimensional approximations of the degenerate affine Grassmannian. Finally, we give an explicit description of the degenerate affine Grassmannian of type $A_1^{(1)}$, propose a conjectural description in the symplectic case, and discuss the generalization to the case of the affine degenerate flag varieties.

Introduction

Let G be a simple Lie group, and let G/B be the flag variety attached to G. These varieties enjoy many nice properties; in particular, they are spherical, that is, the Borel subgroup acts on a flag variety with an open orbit. The varieties G/B can be degenerated in such a way that the action of the Borel subgroup degenerates (modulo torus) into an action of the abelian unipotent group of the same dimension, acting with an open orbit on the degenerate flag variety (see [6], [5], [8]). The construction is of Lie-theoretic nature and uses the theory of highest weight G-modules.

It was shown in [2] that in type A the degenerate flag varieties are closely related to the representation theory of the equioriented quivers of type A. More precisely, the degeneration of SL_n /B is isomorphic to a certain quiver Grassmannian of subrepresentations of a direct sum of injective and projective representations. This observation, on the one hand, allows one to use the representation theory of quivers in order to study the geometry of the degenerate flag varieties and, on the other hand, produces links between Lie theory and quiver Grassmannians.

Both the degenerate flag varieties and the quiver Grassmannians are known to be related to the theory of Schubert varieties. More precisely, one can identify certain quiver Grassmannians and degenerate flag varieties with certain Schubert varieties (see [3], [14], [21]). The main question we ask in this article is to what extent one can generalize the degeneration constructions above to the case of the affine Lie algebras and affine Lie groups? Is there a connection with the theory of quivers and with Schubert varieties in this case? We give partial answers to the questions above. In short, the affine story is rather complicated (even in type A); the correct replacement of the equioriented type A quiver is the (equioriented) cycle quiver.

In our article we concentrate on the case of the degenerate affine Grassmannians, although the general case is also discussed. From the point of view of representation theory this means that we restrict to the basic level one module of the affine Lie algebras. The general definition of the degenerate flag varieties goes through the theory of highest weight modules. In the affine case the same definition works perfectly. Unfortunately, we are not able to identify the resulting ind-variety even for affine $\mathfrak{sl}_n$. (However, we completely describe the A_1 case and put forward a conjecture for the symplectic Lie algebras.) The problems pop up even on the level of representations: we do not have a good enough description of the Poincaré–Birkhoff–Witt (PBW) graded level one basic representation L_0^a of the affine $\mathfrak{sl}_n$. In Section 2 we formulate a conjecture saying that L_0^a can be realized inside the semi-infinite wedge space. Unfortunately, we cannot prove the conjecture at the moment.

It turns out that there exists a similar object attached to the affine Lie algebra $\widehat{\mathfrak{gl}}_n$. More precisely, using the formalism of semi-infinite wedge spaces, we define the PBW degeneration $\mathrm{Gr}^a(\mathfrak{gl}_n)$ of the affine Grassmannians. The definition is very similar to the general Lie-theoretic one, and we are able to describe the degenerate object in linear algebra terms. Using this description, we make a connection to the theory of quiver Grassmannians for the one vertex loop quivers. More precisely, $\mathrm{Gr}^a(\mathfrak{gl}_n)$ can be realized as an inductive limit of quiver Grassmannians, which are analogues of Schubert varieties. We use representation-theoretic techniques to derive algebro-geometric properties of this finitization. In particular, we study the structure of orbits and construct desingularizations explicitly. We also identify them with certain classical affine Schubert varieties for larger groups.

Yet another approach to the study of the $\widehat{\mathfrak{gl}}_n$ degenerate Grassmannians comes from the identification with the closure of the semi-infinite orbit for the classical parabolic flag varieties for $\widehat{\mathfrak{sl}}_{2n}$. We identify the finitization as above with concrete subvarieties inside the closure. We thus realize $\mathrm{Gr}^a(\mathfrak{gl}_n)$ as a semi-infinite orbit closure inside the $\mathfrak{sl}_{2n}$ classical affine Grassmannian.

The article is organized as follows. In Section 1 we collect the basic objects and constructions on affine algebras and PBW degenerations. In Section 2 we define and study degenerate affine Grassmannians of type A. Section 3 is devoted to the quiver part of the story: the principal quiver Grassmannians for the one

vertex loop quivers are studied, and the identification with subvarieties in the degenerate flag varieties and with affine Schubert varieties is constructed. In Section 4 we discuss the degenerate affine Grassmannians of types $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sp}}_{2n}$.

1. The setup

1.1. Affine Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra, and let $\widehat{\mathfrak{g}}$ be the corresponding affine algebra. We have

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is central and $[d, x \otimes t^i] = -ix \otimes t^i$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the Cartan decomposition. Consider the decomposition for the affine algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^- \oplus \widehat{\mathfrak{g}}^0 \oplus \widehat{\mathfrak{g}}^+$, where

$$\widehat{\mathfrak{g}}^- = \mathfrak{n}^- \otimes 1 \oplus \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}], \qquad \widehat{\mathfrak{g}}^0 = \mathfrak{h} \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

$$\widehat{\mathfrak{g}}^+ = \mathfrak{n}^+ \otimes 1 \oplus \mathfrak{g} \otimes t\mathbb{C}[t].$$

Let θ be the highest root for $\mathfrak{g}$. For a dominant integral $\mathfrak{g}$-weight λ and a nonnegative integer k such that $(\lambda, \theta) \leq k$, let $L_{\lambda,k}$ be the corresponding irreducible integrable highest weight $\widehat{\mathfrak{g}}$-module with a highest weight vector $v_{\lambda,k}$. We have

$$\widehat{\mathfrak{g}}^+ v_{\lambda,k} = 0, \qquad \mathrm{U}(\widehat{\mathfrak{g}}^-)v_{\lambda,k} = L_{\lambda,k},$$

$$(h \otimes 1)v_{\lambda,k} = \lambda(h)v_{\lambda,k}, \qquad Kv_{\lambda,k} = kv_{\lambda,k}, \qquad dv_{\lambda,k} = 0.$$

Let G be a simple simply connected Lie group with a Borel subgroup B, let $\mathfrak{g} = \mathrm{Lie}(G)$, and let $\mathfrak{b} = \mathrm{Lie}(B)$. Let $\widehat{G}$ and I be the corresponding affine group and its Iwahori subgroup, respectively. For a parahoric subgroup $P \subset \widehat{G}$ such that $I \subset P \subset \widehat{G}$, let $\widehat{G}/P$ be the corresponding affine flag variety. These varieties are infinite-dimensional ind-varieties, that is, they are inductive limits of the finite-dimensional Schubert varieties. More precisely, let $T \subset \widehat{G}$ be the Cartan torus, and let p be a T-stable point in the affine flag variety. The corresponding Schubert variety is the closure of the I-orbit through p. If $P = P_0$ is the maximal parahoric subgroup corresponding to the affine simple root, then the corresponding flag variety is called the affine Grassmannian. Finally, we note that if P stabilizes the highest weight line $\mathbb{C}v_{\lambda,k}$ in $\mathbb{P}(L_{\lambda,k})$, then we have a natural embedding $\widehat{G}/P \subset \mathbb{P}(L_{\lambda,k})$.

1.2. Sato Grassmannians and flag varieties of type A_∞

We consider an infinite-dimensional vector space V with a basis v_i, $i \in \mathbb{Z}$. The semi-infinite wedge space $F = \Lambda^{\infty/2}$ is spanned by the elements

$$v_{i_0} \wedge v_{i_1} \wedge \cdots, \qquad i_0 > i_1 > \cdots, \qquad i_{k+1} = i_k - 1 \quad \text{for } k \text{ large enough.}$$

The charge of the wedge product $v_{i_0} \wedge v_{i_1} \wedge \cdots$ is defined as $(i_k + k)$ for k large enough. We have the decomposition $F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$, where $F^{(m)}$ is spanned by

the wedge products of charge m. We define the vacuum vectors

$$|m\rangle = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \cdots \in F^{(m)}.$$

The Lie algebra $\mathfrak{gl}_\infty$ is spanned by the matrix units $E_{i,j}$, that is, it consists of the infinite matrices with finite support. The natural action of $\mathfrak{gl}_\infty$ on the space V extends to an action on each space $F^{(m)}$. It is easy to see that each $F^{(m)}$ is an irreducible $\mathfrak{gl}_\infty$-module with highest weight vector $|m\rangle$.

The Sato Grassmannian SGr_m sits inside $\mathbb{P}(F^{(m)})$ via the Plücker embedding; it consists of subspaces $U \subset V$ with the property that there exists $N \in \mathbb{Z}$ such that

$$\mathrm{span}(v_N, v_{N-1}, \dots) \subset U \qquad \text{and} \qquad \dim U/\mathrm{span}(v_N, v_{N-1}, \dots) = m - N.$$

In particular, the line $\mathbb{C}|m\rangle$ belongs to (the image of) SGr_m. It is easy to see that all the varieties SGr_m are isomorphic (via shifts of the indices of the v_k's). Following [15] we define the flag variety $\mathcal{F}_\infty$ of type A_∞ as the subvariety of the product $\prod_{m \in \mathbb{Z}} \mathrm{SGr}_m$ consisting of collections $(U_m)_{m \in \mathbb{Z}}$, $U_m \in \mathrm{SGr}_m$, such that $U_m \subset U_{m+1}$.

1.3. Type $A_{n-1}^{(1)}$ case

Now let us consider an n-dimensional vector space W with a basis $w_1, \dots, w_n$. Let us identify the space $W \otimes \mathbb{C}[t, t^{-1}]$ with V by

$$(1.1) \qquad\qquad v_{nk+j} = w_j \otimes t^{-k-1}, \quad j = 1, \dots, n, k \in \mathbb{Z}.$$

In particular,

$$|0\rangle = (w_1 \otimes 1) \wedge \cdots \wedge (w_n \otimes 1) \wedge (w_1 \otimes t) \wedge \cdots \wedge (w_n \otimes t) \wedge \cdots .$$

This gives an embedding $\mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{gl}_\infty$ and induces an action of $\widehat{\mathfrak{sl}}_n$ on each $F^{(m)}$. The irreducible highest weight $\widehat{\mathfrak{sl}}_n$-modules can thus be realized in the semi-infinite picture. In particular, the level one module $L_{\omega_i, 1}$, $i = 0, \dots,$ $n - 1$, can be seen as the subspace of $F^{(i)}$ generated by $|i\rangle$. We also see that the affine Grassmannian $\widehat{\mathrm{SL}}_n / P_0$ of type A_n is naturally embedded into the Sato Grassmannian. The image of this embedding can be described explicitly:

$$(1.2) \qquad\qquad \widehat{\mathrm{SL}}_n / P_0 = \{U \in \mathrm{SGr}_0 : tU \subset U\}.$$

The complete flag variety for affine SL_n is formed by the collections $(U_i)_{i \in \mathbb{Z}}$, $U_i \in \mathrm{SGr}_i$, such that $U_i \subset U_{i+1}$, $tU_i \subset U_{i+n}$.

1.4. PBW filtration and degenerate flag varieties

We first collect some facts on the PBW degeneration in type A (see [10], [11], [6]). Given a highest weight representation V_λ of $\mathfrak{sl}_n$, we have an increasing filtration on V_λ induced by the PBW filtration on $U(\mathfrak{n}^-)$. The associated graded spaces V_λ^a are modules over the abelian algebra $(\mathfrak{n}^-)^a$ with the underlying vector space $\mathfrak{n}^-$. The degenerate flag variety $\mathcal{F}_\lambda^a(\mathfrak{g}) \subset \mathbb{P}(V_\lambda^a)$ is the closure of the orbit of the group $\exp((\mathfrak{n}^-)^a)$ through the line containing the highest weight vector v_λ.

Let λ be a regular dominant weight for $\mathfrak{g} = \mathfrak{sl}_n$. Then the corresponding degenerate flag variety does not depend on λ and can be described explicitly as follows. Let $W = \mathbb{C}^n$ with a basis $w_1, \ldots, w_n$, and let pr_k be the projection along w_k to the span of the remaining basis vectors. Then the degenerate flag variety attached to a regular dominant weight consists of collections $(V_1, \ldots, V_{n-1})$, $V_k \in \mathrm{Gr}_k(W)$, such that $pr_{k+1} V_k \subset V_{k+1}$. We denote this variety by $\mathcal{F}^a(\mathfrak{sl}_n)$. Note that it was shown recently in [3, Theorem 1.2] that $\mathcal{F}^a(\mathfrak{sl}_n)$ is isomorphic to a Schubert variety for $\mathfrak{sl}_{2n-2}$. We also note that the degenerate Grassmann varieties for $\mathfrak{sl}_n$ coincide with their classical analogues.

The explicit realization of $\mathcal{F}^a(\mathfrak{sl}_n)$ can be naturally generalized to the A_∞ case. Recall the basis v_i of $V = \mathbb{C}^\infty$. We define the operators $pr_k : V \to V$ as projections along v_k to the span of v_i, $i \neq k$. The following definitions are degenerate versions of the classical analogues (see, e.g., [15]).

DEFINITION 1.1

The full degenerate flag variety $\mathcal{F}^a_\infty$ of type A_∞ is the subvariety of the product $\prod_{m \in \mathbb{Z}} \mathrm{SGr}_m$ consisting of all collections of subspaces $(U_m)_{m \in \mathbb{Z}}$, $U_m \in \mathrm{SGr}_m$, such that $pr_{m+1} U_m \subset U_{m+1}$.

We note that these varieties can be seen as infinite limits of the type A_n degenerate flag varieties.

REMARK 1.2

The degeneration procedure from $\mathcal{F}_\infty$ to $\mathcal{F}^a_\infty$ can be described as follows. Let $\mathbf{F}_\infty$ be the subvariety inside $\mathbb{A}^1 \times \prod_{m \in \mathbb{Z}} \mathrm{SGr}_m$, consisting of the points $(t, (U_m)_{m \in \mathbb{Z}})$ such that $pr_{m+1}(t) U_m \subset U_{m+1}$, where $pr_{m+1}(t) : V \to V$ is the map defined by $pr_{m+1}(t) v_k = v_k$, $k \neq m+1$, and $pr_{m+1}(t) v_{m+1} = t v_{m+1}$. Then we have the natural projection $\mathbf{F} \to \mathbb{A}^1$. The fiber over $t = 0$ is isomorphic to $\mathcal{F}^A_\infty$, and the general fiber is isomorphic to $\mathcal{F}_\infty$.

Let us now define the affine degenerate flag varieties of type $\mathfrak{gl}_n$. We denote by $pr_{w_i \otimes t^k} : W \otimes \mathbb{C}[t, t^{-1}] \to W \otimes \mathbb{C}[t, t^{-1}]$ the projections along $w_i \otimes t^k$ to the span of the remaining vectors of the form $w_j \otimes t^l$.

DEFINITION 1.3

The degenerate flag variety $\mathcal{F}^a(\mathfrak{gl}_n)$ of type $\mathfrak{gl}_n$ consists of collections $(U_i)_{i=0}^n$ such that $U_i \in \mathrm{SGr}_i$, $pr_{w_{i+1} \otimes t^{-1}} U_i \subset U_{i+1}$, and $U_n = t^{-1} U_0$.

REMARK 1.4

We note that if $(U_i)_{i=0}^n \in \mathcal{F}^a(\mathfrak{gl}_n)$, then

$$pr_{W \otimes 1} t U_0 = pr_{w_1 \otimes 1} \cdots pr_{w_n \otimes 1} t U_0 \subset U_0.$$

Indeed,

$$pr_{W \otimes t^{-1}} t U_n = pr_{w_1 \otimes t^{-1}} \cdots pr_{w_n \otimes t^{-1}} (U_0) \subset U_n.$$

Since $tU_n = U_0$, we obtain $pr_{W\otimes 1}tU_0 \subset U_0$.

Now let us consider the case of the affine Lie algebras. The standard PBW filtration $F_\bullet$ on the universal enveloping algebra $U(\widehat{\mathfrak{g}}^-)$ induces a filtration on the module $L_{\lambda,k}$. The associated graded space $L^a_{\lambda,k}$ is a representation of the abelian Lie algebra $\widehat{\mathfrak{g}}^{-,a}$ (with the underlying vector space $\widehat{\mathfrak{g}}^-$).

LEMMA 1.5

For any $x \in \widehat{\mathfrak{g}}^{-,a}$ there exists N such that $x^N v_{\lambda,k} = 0$ in $L^a_{\lambda,k}$.

Proof
Recall that the algebra $\widehat{\mathfrak{g}}^{-,a}$ is abelian. It suffices to prove the lemma for $x = r \otimes t^{-l}$ for some $r \in \mathfrak{g}$, $l > 0$. Since r can be included into an $\mathfrak{sl}_2$-triple, we may assume $\mathfrak{g} = \mathfrak{sl}_2$. Let e, h, f be the standard basis of $\mathfrak{sl}_2$. We know that for any i there exists M such that $(f \otimes t^{-i})^M v_{\lambda,k} = 0$. Also we have the adjoint action of $\mathfrak{sl}_2 = \mathfrak{sl}_2 \otimes 1$ on the symmetric algebra of $\mathfrak{sl}_2$, generating new relations. Hence, it suffices to prove that the algebra $\mathrm{Sym}(\mathfrak{sl}_2)/(U(\mathfrak{sl}_2)f^M)$ is finite-dimensional. But this algebra is known to be finite-dimensional by [12] (see also [7, Corollary 4.3]). $\square$

REMARK 1.6

We note that this lemma does not hold in the nondegenerate situation.

Let $\widehat{G}^{-,a} = \exp(\widehat{\mathfrak{g}}^{-,a})$ be the Lie group of the Lie algebra $\widehat{\mathfrak{g}}^{-,a}$. This group is isomorphic to the sum of (an infinite number of) copies of the groups $\mathbb{G}_a$ (one copy for each negative root of $\widehat{\mathfrak{g}}$). Let us now consider the projective space $\mathbb{P}(L^a_{\lambda,k})$. For a vector $v \in L^a_{\lambda,k}$ let $[v] \in \mathbb{P}(L^a_{\lambda,k})$ be the corresponding line in the projectivization.

DEFINITION 1.7

The degenerate affine flag variety $\mathcal{F}^a_{\lambda,k}$ is the closure of the orbit $\widehat{G}^{-,a}[v_{\lambda,k}]$ inside $\mathbb{P}(L^a_{\lambda,k})$.

REMARK 1.8

The orbit $\widehat{G}^{-,a}[v_{\lambda,k}]$ makes sense, since the action of the group $\widehat{G}^{-,a}$ on $\mathbb{P}(L^a_{\lambda,k})$ is well defined. In fact, thanks to Lemma 1.5, all the operators from the Lie algebra $\widehat{\mathfrak{g}}^{-,a}$ of the Lie group $\widehat{G}^{-,a}$ act as nilpotent operators and hence can be exponentiated, giving rise to the $\widehat{G}^{-,a}$-action.

REMARK 1.9

We do not describe the degeneration procedure in this article. Note, however, that for finite-dimensional Lie algebras of type A_n the closure of the orbit of the group $\widehat{G}^{-,a}$ (of its finite-dimensional analogue, to be precise) is indeed a (flat) degeneration of the classical flag variety (see [6, Proposition 5.8]).

2. Degenerate affine Grassmannians of type A

2.1. Main theorems

DEFINITION 2.1

The degenerate affine Grassmannian $\mathrm{Gr}^a(\mathfrak{gl}_n)$ of type $\mathfrak{gl}_n$ is the subvariety of the Sato Grassmannian SGr_0 consisting of subspaces U such that $pr_{W\otimes 1} t U \subset U$, where $pr_{W\otimes 1}$ is the projection along $W \otimes 1$ to the span of $W \otimes t^i$, $i \neq 0$.

We also define finite-dimensional approximations of the varieties $\mathrm{Gr}^a(\mathfrak{gl}_n)$.

DEFINITION 2.2

We define $\mathrm{Gr}^a_N(\mathfrak{gl}_n)$ as the subvariety of $\mathrm{Gr}^a(\mathfrak{gl}_n)$ consisting of U such that

$$W \otimes t^N \mathbb{C}[t] \subset U \subset W \otimes t^{-N} \mathbb{C}[t].$$

REMARK 2.3

Let $S_{N,n}$ be the quotient space $\frac{W \otimes t^{-N}\mathbb{C}[t]}{W \otimes t^N \mathbb{C}[t]}$. Then $\mathrm{Gr}^a_N(\mathfrak{gl}_n)$ is isomorphic to the variety of Nn-dimensional subspaces $U \subset S_{N,n}$ such that $pr_{W\otimes 1} t U \subset U$.

We note that $\mathrm{Gr}^a(\mathfrak{gl}_n)$ is naturally the inductive limit of its finite-dimensional pieces $\mathrm{Gr}^a_N(\mathfrak{gl}_n)$. We will prove the following theorems.

THEOREM 2.4

- $\mathrm{Gr}^a_N(\mathfrak{gl}_n)$ *is an irreducible projective variety of dimension* Nn^2.
- $\mathrm{Gr}^a_N(\mathfrak{gl}_n)$ *carries an action of an abelian unipotent group* $\mathbb{G}_a^{Nn^2}$ *with an open dense orbit.*
- $\mathrm{Gr}^a_N(\mathfrak{gl}_n)$ *is isomorphic to an affine Schubert variety for the group* $\widehat{\mathrm{SL}}_{2n}$.

THEOREM 2.5

- $\mathrm{Gr}^a(\mathfrak{gl}_n)$ *carries an action of an infinite-dimensional abelian unipotent group with an open dense orbit. The group is the inductive limit of the groups* $\mathbb{G}_a^{Nn^2}$ *from Theorem 2.4.*
- $\mathrm{Gr}^a(\mathfrak{gl}_n)$ *is isomorphic to a semi-infinite orbit closure inside the affine Grassmannian for the group* $\widehat{\mathrm{SL}}_{2n}$.

LEMMA 2.6

We have that $\mathrm{Gr}^a_N(\mathfrak{gl}_n)$ *is an irreducible projective variety of dimension* Nn^2.

Proof

We prove the claim by constructing a resolution of singularities. We consider the space $S_{N,n} = \frac{W \otimes t^{-N}\mathbb{C}[t]}{W \otimes t^N \mathbb{C}[t]}$. The operator $pr_{W\otimes 1} t$ naturally acts on $S_{N,n}$. In particular, $(pr_{W\otimes 1} t)^N = 0$. Let $S_{N,n}(k) = \mathrm{Im}(pr_{W\otimes 1})^k$ for $k = 0, 1, \ldots, N-1$. In particular, $\dim S_{N,n}(k) = n(2N - 2k)$.

Let $R_{N,n}$ be the variety of collections of vector spaces $(U_0, U_1, \ldots, U_N)$ such that the following holds:

(i) $U_i \subset S_{N,n}(k), \dim U_i = n(N - k)$,
(ii) $pr_{W \otimes 1} t U_i \subset U_{i+1}, i = 0, \ldots, N - 1$.

One easily sees that $R_{N,n}$ is an N-floor tower of fibrations over a point, each fibration having fiber $\mathrm{Gr}(n, 2n)$. In addition, $R_{N,n}$ surjects onto $\mathrm{Gr}_N^a(\mathfrak{gl}_n)$ (forgetting all the U_i's with $i > 0$), and this surjection is generically one-to-one. In fact, let $R_{N,n}^0 \subset R_{N,n}$ be the subvariety cut out by the condition $\dim(pr_{W \otimes 1})^k t U_0 = n(N - k)$. By the definition of $R_{N,n}$, the restriction of the surjection $R_{N,n} \to \mathrm{Gr}_N^a(\mathfrak{gl}_n)$ to $R_{N,n}^0$ is one-to-one. Now one easily sees that $R_{N,n}^0$ is an open dense part of $R_{N,n}$. This proves the lemma. $\square$

REMARK 2.7
We generalize the claim of Lemma 2.6 in Example 3.2 and Corollary 3.7.

In order to prove the remaining statements of Theorems 2.4 and 2.5 we construct a projective embedding of $\mathrm{Gr}^a(\mathfrak{gl}_n)$ and hence of its finite-dimensional pieces $\mathrm{Gr}_N^a(\mathfrak{gl}_n)$.

2.2. Semi-infinite abelianization
Let us decompose the Lie algebra $\mathfrak{gl}_\infty$ into four blocks

$$\mathfrak{gl}_\infty = \mathfrak{gl}_\infty^{-,-} \oplus \mathfrak{gl}_\infty^{-,+} \oplus \mathfrak{gl}_\infty^{+,-} \oplus \mathfrak{gl}_\infty^{+,+},$$

where $\mathfrak{gl}_\infty^{-,+}$ is spanned by the matrix units $E_{i,j}$ with $i \le 0$, $j > 0$, $\mathfrak{gl}_\infty^{+,-}$ is spanned by the matrix units $E_{i,j}$ with $i > 0$, $j \le 0$, and the two other summands are similarly spanned. For example, $\mathfrak{gl}_\infty^{+,-}$ is spanned by the operators mapping $v_{\le 0}$ to $v_{>0}$.

REMARK 2.8
The summand $\mathfrak{gl}_\infty^{+,-}$ is abelian.

Let $p : \mathfrak{gl}_\infty \to \mathfrak{gl}_\infty^{+,-}$ be the projection along the three other summands. Recall the embedding $\mathfrak{gl}_n \otimes \mathbb{C}[t, t^{-1}] \subset \mathfrak{gl}_\infty$. Combining this embedding with p we obtain a map

$$\mathfrak{gl}_n \otimes t^{-1} \mathbb{C}[t^{-1}] \to \mathfrak{gl}_\infty^{+,-}.$$

We define the space

$$L_0(\mathfrak{gl}_n^a) = U\big(p\big(\mathfrak{gl}_n \otimes t^{-1}\mathbb{C}[t^{-1}]\big)\big)|0\rangle.$$

Since $\mathfrak{gl}_\infty^{+,-}$ is abelian, $L_0(\mathfrak{gl}_n^a)$ is a cyclic representation of the abelian Lie algebra $\mathfrak{gl}_n^a \otimes t^{-1}\mathbb{C}[t^{-1}]$.

EXAMPLE 2.9
Let us consider the Lie algebra $\widehat{\mathfrak{gl}_1}$. This is nothing but the Heisenberg algebra. Let $F^{(0)}$ be the Fock module. Let us denote by h_i, $i \in \mathbb{Z}$, the basis of $\widehat{\mathfrak{gl}_1}$. Then the elements h_i, $i < 0$, are represented by the formula $h_i = \sum_{k \in \mathbb{Z}} E_{k-i,k}$.

The projection p to $\mathfrak{gl}_\infty^{+,-}$ defines a representation of the abelian Lie algebra spanned by h_i, $i < 0$, on $F^{(0)}$. This representation is no longer irreducible. In particular, the subspace generated from the highest weight vector is defined by the relation $(h_{-1} + zh_{-2} + \cdots)^2 = 0$. Here z is a variable and the relation above means that the coefficients of z^k vanish for all $k \geq 0$.

Let us consider the affine Lie algebra $\widehat{\mathfrak{sl}}_{2n}$ and its vacuum representation $L_0(\mathfrak{sl}_{2n})$ with the highest weight vector l_0. Let us embed $\mathfrak{gl}_n^a$ into $\mathfrak{sl}_{2n}$ as the unipotent radical corresponding to the weight ω_n. Explicitly, we consider the abelian subalgebra $\mathfrak{a}$ in $\mathfrak{sl}_{2n}$, spanned by the matrix units $E_{i,j}$, $n + 1 \leq i \leq 2n$, $1 \leq j \leq n$. Clearly, $\mathfrak{a} \simeq \mathfrak{gl}_n^a$.

LEMMA 2.10

The identification $\mathfrak{a} \simeq \mathfrak{gl}_n^a$ induces an isomorphism between $L_0(\mathfrak{gl}_n^a)$ and the subspace $U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}])l_0$ of $L_0(\mathfrak{sl}_{2n})$.

Proof
Consider the space R with basis $r_1, \ldots, r_{2n}$. We write $R = R^< \oplus R^>$, where $R^<$ is spanned by $r_1, \ldots, r_n$ and $R^>$ is spanned by $r_{n+1}, \ldots, r_{2n}$. Then $L_0(\mathfrak{sl}_{2n})$ sits inside $\Lambda^{\infty/2}(R \otimes \mathbb{C}[t, t^{-1}])$ with l_0 being the wedge product

$$l_0 = \bigwedge_{i \geq 0} (r_1 \otimes t^i \wedge \cdots \wedge r_{2n} \otimes t^i).$$

Let us write $R \otimes \mathbb{C}[t, t^{-1}]$ as a direct sum of four subspaces $R^{<,+}$, $R^{<,-}$, $R^{>,+}$, and $R^{>,-}$, where

$$R^{<,+} = R^< \otimes \mathbb{C}[t], \qquad R^{<,-} = R^< \otimes t^{-1}\mathbb{C}[t^{-1}],$$

and similar statements hold for $R^{>,+}$ and $R^{>,-}$. For example, l_0 is the wedge product of the "top" wedge powers of $R^{<,+}$ and of $R^{>,+}$.

Now let us look at the subspace $U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}])l_0$. We take $a \in \mathfrak{a}$ and consider the vector $(a \otimes t^i)l_0$. Clearly, the only nontrivial terms showing up come from the action of $a \otimes t^i$ sending $R^{<,+}$ to $R^{>,-}$. Recall the space W with the basis $w_1, \ldots, w_n$ used to construct the wedge representation for $\widehat{\mathfrak{gl}}_n$. Let us embed $W \otimes \mathbb{C}[t, t^{-1}]$ into $R \otimes \mathbb{C}[t, t^{-1}]$ as follows:

$$w_j \otimes t^i \mapsto r_j \otimes t^i, \quad i \geq 0, \qquad w_j \otimes t^i \mapsto r_{j+n} \otimes t^i, \quad i < 0.$$

Thus, the image of this embedding coincides with $R^{<,+} \oplus R^{>,-}$. Now it remains to note that the operators of the form $p(\mathfrak{gl}_n^a)$ map $W \otimes \mathbb{C}[t]$ to $W \otimes t^{-1}\mathbb{C}[t^{-1}]$. $\square$

Let us consider the Lie group $\mathrm{GL}_n^a(t^{-1}\mathbb{C}[t^{-1}]) = \exp(\mathfrak{gl}_n^a \otimes t^{-1}\mathbb{C}[t^{-1}])$. The group $\mathrm{GL}_n^a(t^{-1}\mathbb{C}[t^{-1}])$ is isomorphic to the direct sum of an infinite number of copies of the additive group $\mathbb{G}_a$ of the base field. Consider the action of $\mathrm{GL}_n^a(t^{-1}\mathbb{C}[t^{-1}])$ on the projectivization $\mathbb{P}(L_0(\mathfrak{gl}_n^a))$, and (temporarily) denote by $G(n)$ the closure of the orbit through the line containing the highest weight vector. Our goal is to identify $G(n)$ with $\mathrm{Gr}^a(\mathfrak{gl}_n)$. Let us define a finite-dimensional approximation of

$G(n)$ as

$$G_N(n) = \overline{\exp\Big(\bigoplus_{i=1}^{N} \mathfrak{gl}_n^a \otimes t^{-i}\Big) \cdot \mathbb{C}|0\rangle}.$$

LEMMA 2.11

Let $U \in G_N(n)$. Then $pr_{W \otimes 1}(tU) \subset U$ and $W \otimes t^N \mathbb{C}[t] \subset U \subset W \otimes t^{-N}\mathbb{C}[t]$.

Proof
Take an element $g = \exp(\sum_{i=1}^{N} x_i \otimes t^{-i})$, $x_i \in \mathfrak{gl}_n$. Then the space corresponding to the line $g \cdot \mathbb{C}|0\rangle$ is spanned by the vectors

$$w_j \otimes t^k + x_i w_j \otimes t^{k-i}, \quad j = 1, \dots, n, i = 1, \dots, N, k = 0, \dots, N-1, k-i < 0.$$

Now the claim is clear.   □

PROPOSITION 2.12

We have $G_N(n) \simeq \mathrm{Gr}_N^a(\mathfrak{gl}_n)$.

Proof
We know that both $G_N(n)$ and $\mathrm{Gr}_N^a(\mathfrak{gl}_n)$ are irreducible. According to Lemma 2.11, $G_N(n)$ sits inside $\mathrm{Gr}^a(\mathfrak{gl}_n)$. Also note that the open orbit

$$\exp\Big(\bigoplus_{i=1}^{N} \mathfrak{gl}_n^a \otimes t^{-i}\Big) \cdot \mathbb{C}|0\rangle$$

consists of all $U \in \frac{W \otimes t^{-N}\mathbb{C}[t]}{W \otimes t^N \mathbb{C}[t]}$ such that the Plücker coordinate of U, corresponding to the set of vectors $w_j \otimes t^i$, $i \leq 0$, does not vanish. Hence, the open parts of $G_N(n)$ and $\mathrm{Gr}^a(\mathfrak{gl}_n)$ coincide.   □

Proof of Theorem 2.4
The first claim follows from Lemma 2.6, and the second claim follows from Proposition 2.12. The proof of the last claim is postponed until the end of the next section (see Corollary 3.16).   □

Proof of Theorem 2.5
Since $G_N(n) \simeq \mathrm{Gr}_N^a(\mathfrak{gl}_n)$, we obtain $G(n) \simeq \mathrm{Gr}^a(\mathfrak{gl}_n)$. By Lemma 2.10 we obtain that $\mathrm{Gr}^a(\mathfrak{gl}_n)$ is naturally embedded into the classical affine Grassmannian for the group $\widehat{\mathrm{SL}}_{2n}$. The image of this embedding is the closure of the orbit of the abelian unipotent group $\exp(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}])$ through the highest weight line.   □

2.3. $\mathfrak{sl}_n$ case
Let $L_0(\mathfrak{sl}_n)$ be the basic level one $\mathfrak{sl}_n$-module with highest weight vector l_0. Recall the PBW graded version $L_0^a(\mathfrak{sl}_n)$; in particular, $L_0^a(\mathfrak{sl}_n) = U(\mathfrak{sl}_n^a \otimes t^{-1}\mathbb{C}[t^{-1}])l_0$. We define

$$\tilde{L}_0(\mathfrak{sl}_n) = U\big(p\big(\mathfrak{sl}_n \otimes t^{-1}\mathbb{C}[t^{-1}]\big)\big)|0\rangle.$$

CONJECTURE 2.13

The $\mathfrak{n}^{-,a} \otimes t^{-1}\mathbb{C}[t^{-1}]$-modules $\tilde{L}_0(\mathfrak{sl}_n)$ and $L_0^a(\mathfrak{sl}_n)$ are isomorphic.

We prove the following lemma.

LEMMA 2.14

There exists a surjection $L_0^a(\mathfrak{sl}_n) \to \tilde{L}_0(\mathfrak{sl}_n)$ of $\mathfrak{sl}_n^a \otimes t^{-1}\mathbb{C}[t^{-1}]$-modules.

Proof
We need to show that any relation which holds in $L_0^a(\mathfrak{sl}_n)$ is true in $\tilde{L}_0(\mathfrak{sl}_n)$. We consider a grading on the semi-infinite wedge space $\Lambda^{\infty/2}(V)$. The sth graded component is spanned by vectors $v_{i_1} \wedge v_{i_2} \wedge \cdots$ such that the number of positive indices among the i_l's is equal to s. For example, the zeroth component is spanned by $|0\rangle$. Then any $x \in \mathfrak{gl}_\infty^{+,-}$ increases this grading by one. The operators from the three remaining summands either decrease the grading or preserve it.

Now assume that we have a relation in $L_0^a(\mathfrak{sl}_n)$. To make it explicit, let us fix a basis $x_1, x_2, \ldots$ in $\mathfrak{n}^- \otimes t^{-1}\mathbb{C}[t^{-1}]$. Then any relation can be represented by a homogeneous total degree N polynomial $p(x_1, x_2, \ldots)$, which vanishes in L_0^a. This means that we have an equality in L_0:

$$p(x_1, x_2, \ldots)|0\rangle = q(x_1, x_2, \ldots)|0\rangle \text{ for some } q \text{ of degree less than } N.$$

This equality implies that $p(x_1, x_2, \ldots)|0\rangle$ vanishes in $\tilde{L}_0(\mathfrak{sl}_n)$, since this is exactly the degree N component of $p(x_1, x_2, \ldots)|0\rangle$ considered as a vector in $L_0(\mathfrak{sl}_n)$. □

REMARK 2.15

To complete the proof of Conjecture 2.13 it thus suffices to show that the character of $L_0^a(\mathfrak{sl}_n)$ is equal to the character of $\tilde{L}_0(\mathfrak{sl}_n)$. Unfortunately, we are not able to do this at the moment.

2.4. Flatness

In this section we discuss the infinite-dimensional analogue of the flatness of the degeneration of the $\mathfrak{sl}_n$ flag varieties. We consider the space $W \otimes \mathbb{C}[t, t^{-1}]$, $W = \mathbb{C}^n$. Let $U \in \mathrm{SGr}_0$ be a subspace with $tU \subset U$. We start with the open cell containing the base point $U_0 = W \otimes \mathbb{C}[t]$. The coordinates in this cell are given by the collection of linear mappings $(A_{k,i})$, $k \geq 1$, $i \geq 0$, $A_{k,i} \in \mathrm{End}(W)$, such that for a fixed i the operators $A_{k,i}$ vanish for k large enough. The subspace U corresponding to a collection $(A_{k,i})$ is defined as the linear span of the vectors

(2.1) $$w \otimes t^i + A_{1,i}w \otimes t^{-1} + A_{2,i}w \otimes t^{-2} + \cdots.$$

Now we introduce the operator $\alpha_\hbar \in (\mathrm{End}\, W \otimes \mathbb{C}[t, t^{-1}])$ defined by

$$\alpha_\hbar(w \otimes t^i) = \begin{cases} w \otimes t^i & \text{if } i \neq 0, \\ \hbar w \otimes t^0 & \text{if } i = 0. \end{cases}$$

We consider the ind-subscheme $\mathbf{Gr} \subset \mathrm{SGr}_0 \times \mathbb{A}^1$ in the product of the Sato Grassmannian and the affine line with coordinate $\hbar$ cut out by the equations $\alpha_\hbar t U \subset U$. We note that if $\hbar \neq 0$, then the fiber of $\mathbf{Gr}$ over $\hbar$ is isomorphic to the affine Grassmannian for $\mathfrak{gl}_n$. The special fiber $\mathbf{Gr}_0$ is the degenerate affine Grassmannian.

PROPOSITION 2.16

The morphism $\mathbf{Gr} \to \mathbb{A}^1$ is flat.

Proof

For each connected component $\mathbf{Gr}^n, n \in \mathbb{Z}$, of $\mathbf{Gr}$, it suffices to produce a dense open affine subscheme $S^n \subset \mathbf{Gr}^n$ such that $\hbar \in \mathbb{C}[S^n]$ is not a zero divisor. We will exhibit S^0; the other connected components are taken care of similarly. We define S^0 as the intersection of $\mathbf{Gr}^0$ with the open cell in SGr_0 formed by the subspaces transversal to $W \otimes t^{-1}\mathbb{C}[t^{-1}]$.

LEMMA 2.17

Let $U \in \mathrm{SGr}_0$ be defined by the collection of operators $(A_{k,i})$. Then $(U, \hbar) \in \mathbf{Gr}$ if and only if

$$(2.2) \qquad A_{k,i} = A_{k-1,i+1} + \hbar A_{k-1,0} A_{1,i}$$

for all $k, i \geq 1$.

Proof
We need

$$(2.3) \qquad w \otimes t^{i+1} + \hbar A_{1,i} w \otimes t^0 + A_{2,i} w \otimes t^{-1} + \cdots \in U.$$

This means that the vector (2.3) is a linear combination of the vectors (2.1) starting with $w \otimes t^{i+1}$ and $w \otimes t^0$ with coefficients 1 and $\hbar A_{1,i}$. This gives the desired system of equations. □

REMARK 2.18

If $\hbar = 0$, then (2.2) reduces to $A_{k,i} = A_{k-1,i+1}$ (block Hankel matrices).

Now the ind-scheme S^0 is a union of the finite-type subschemes $S^0(N)$ cut out by the conditions $A_{k,i} = 0$ if $k > N$ or $i \geq N$. The relations (2.2) allow one to express any operator $A_{k,i}$ with $k > 0$ in terms of $A_{1,i}$. Hence, the coordinate ring $\mathbb{C}[S^0]$ is a certain completion of the polynomial ring in $\hbar$ and the matrix coefficients of $A_{1,0}, A_{1,1}, A_{1,2}, \ldots$. The relations (2.2) are homogeneous if we set $\deg \hbar = 0, \deg A_{k,i} = k + i$. If we had a relation $\hbar f = 0$ for $0 \neq f \in \mathbb{C}[S^0]$, then it would imply that $\hbar f^{(n)} = 0$ for each homogeneous component $f^{(n)}$ of f. Since each $f^{(n)}$ is an element of the polynomial ring in $\hbar$ and the matrix coefficients of $A_{1,0}, A_{1,1}, A_{1,2}, \ldots$, we conclude that $f^{(n)} = 0$. Hence, $f = 0$. The proposition is proved. □

REMARK 2.19

The special fiber $\mathbf{Gr}_0$ is a union of reduced finite-type schemes of growing dimensions. For $\hbar \neq 0$, in the case of $\mathfrak{gl}_1$, the fiber $\mathbf{Gr}_\hbar$ is a union of 0-dimensional schemes with nilpotents. So the flatness of $\mathbf{Gr}$ over $\mathbb{A}^1$ holds only at the level of the inductive limit and looks somewhat counterintuitive.

REMARK 2.20

Again let us consider the case of $\mathfrak{gl}_1$. Then for $\hbar \neq 0$ the fiber lives in the projectivization of the Fock module $\mathbb{C}[h_{-1}, h_{-2}, \dots]$. (The Fock module can be naturally identified with the space of sections of the tautological line bundle.) However, in the degenerate situation the space of sections of the tautological line bundle is smaller. (More precisely, $\mathbb{C}[h_{-1}, h_{-2}, \dots]/(h(z)^2)$, that is, the ideal we quotient out is generated by all the coefficients of the series $(\sum_{i>0} z^{i-1} h_{-i})^2$ (see Example 2.9).)

3. Quiver Grassmannians for (truncated) loop quivers

3.1. Basics

For $N \geq 1$, let $A_N = \mathbb{C}[t]/(t^N)$ be the truncated polynomial ring, which we view as a finite-dimensional and self-injective algebra over $\mathbb{C}$ (A_N being an indecomposable projective and injective module over itself).

For a finitely generated (possibly noncommutative) algebra A, a finite-dimensional A-module M, and an integer $k \leq \dim M$, we denote by $\mathrm{Gr}_k^A(M)$ the Grassmannian of k-dimensional subrepresentations U of M. This is a projective variety, admitting a closed embedding into the Grassmannian $\mathrm{Gr}_k(M)$ of k-dimensional $\mathbb{C}$-linear subspaces of M.

Let W be an m-dimensional vector space, and consider $W \otimes A_N$ as an A_N-module, which is thus projective and injective of dimension mN. Our aim is to study the following varieties and to relate them to degenerate affine Grassmannians and to affine Schubert varieties.

DEFINITION 3.1

With notation as above, define
$$X_{k,m}^{(N)} = \mathrm{Gr}_k^{A_N}(W \otimes A_N)$$
as the Grassmannian of k-dimensional A_N-subrepresentations of $W \otimes A_N$.

EXAMPLE 3.2

We have $X_{Nn,2n}^{(N)} \simeq \mathrm{Gr}_N^a(\mathfrak{gl}_n)$ (see Remark 2.3).

Let us interpret this definition in linear algebra terms. Consider the operator φ on $W \otimes A_N$ which is given by $\varphi = \mathrm{id}_W \otimes t$, where t means multiplication by t on A_N (which is a regular nilpotent operator on A_N). Thus, φ is nilpotent, and its Jordan canonical form consists of m nilpotent Jordan blocks, each of size N.

The variety $X_{k,m}^{(N)}$ parameterizes φ-invariant k-dimensional subspaces of $W \otimes A_N$; thus, it is naturally a closed subvariety of $\mathrm{Gr}_k(W \otimes A_N)$.

Note that the group $\mathrm{GL}_m(A_N) \simeq \mathrm{GL}(W \otimes A_N)$ acts on $X_{k,m}^{(N)}$. This group is of dimension mN, with reductive part $\mathrm{GL}_m(\mathbb{C})$ and unipotent radical $1 + \mathrm{Mat}_m(tA_N)$.

Our aim in the next section is to realize $X_{k,m}^{(N)}$ as a geometric quotient of a well-known variety by a (free) group action. We recall the prototype for such quotient realizations. Let V be a k-dimensional vector space. Then the linear Grassmannian $\mathrm{Gr}_k(W \otimes A_N)$ can be viewed as the quotient of the set $\mathrm{Hom}^0(V, W \otimes A_N)$ of injective linear maps from V to $W \otimes A_N$ modulo the action of $\mathrm{GL}(V)$. Thus,

$$\mathrm{Gr}_k(W \otimes A_N) \simeq \mathrm{Hom}^0(V, W \otimes A_N)/\mathrm{GL}(V).$$

3.2. Interpretation as framed moduli

We recall a result of classical invariant theory (see [13, Section 5.1]). For two vector spaces V and W, consider the action of $\mathrm{GL}(V)$ on $\mathrm{End}(V) \times \mathrm{Hom}(V, W)$ by $g \cdot (\varphi, f) = (g\varphi g^{-1}, fg^{-1})$. We call a pair (φ, f) *stable* if $\bigcap_{i \geq 0} \mathrm{Ker}(f\varphi^i) = 0$. This is equivalent to the map

$$\bigoplus_{i=0}^{\dim V - 1} f\varphi^i : V \to W^{\dim V}$$

being injective. Denote by $(\mathrm{End}(V) \times \mathrm{Hom}(V, W))^{\mathrm{st}}$ the open subset of stable points; the action of $\mathrm{GL}(V)$ is free on this stable locus.

THEOREM 3.3

The set $(\mathrm{End}(V) \times \mathrm{Hom}(V, W))^{\mathrm{st}}$ *admits a geometric quotient by* $\mathrm{GL}(V)$. *It embeds into the Grassmannian* $\mathrm{Gr}_{\dim V}(W^{\dim V})$ *of subspaces of* $W^{\dim V}$ *of dimension* $\dim V$ *by mapping the class of* (φ, f) *to the subspace* $\mathrm{Im}(\bigoplus_{i=0}^{\dim V - 1} f\varphi^i)$.

Let $\mathcal{N}(V)$ be the closed subvariety of nilpotent operators in $\mathrm{End}(V)$, and let $\mathcal{N}^{(N)}(V)$ be the set of operators φ such that $\varphi^N = 0$, for $N \geq 1$.

As restrictions of (geometric) quotients to invariant closed subvarieties are again (geometric) quotients, we can consider the $\mathrm{GL}(V)$-invariant subset $(\mathcal{N}^{(N)}(V) \times \mathrm{Hom}(V, W))^{\mathrm{st}}$ of $(\mathrm{End}(V) \times \mathrm{Hom}(V, W))^{\mathrm{st}}$ and get, after some reindexing and using N-nilpotency, the following result.

COROLLARY 3.4

The set $(\mathcal{N}^{(N)}(V) \times \mathrm{Hom}(V, W))^{\mathrm{st}}$ *admits a geometric quotient by* $\mathrm{GL}(V)$. *This quotient embeds into the Grassmannian* $\mathrm{Gr}_{\dim V}(W \otimes A_N)$ *of* $\dim V$-*dimensional subspaces of* $W[[t]]/(t^N) \simeq W \otimes A_N$ *by mapping the class of* (φ, f) *to the subspace* $\mathrm{Im}(\sum_{i=0}^{N-1} f\varphi^{N-1-i}t^i)$.

The main technical result of this section is the following (cf. [18, Section 2]).

LEMMA 3.5

The image of the above embedding coincides with the subvariety $X^{(N)}_{\dim V, \dim W}$ of $\mathrm{Gr}_{\dim V}(W \otimes A_N)$.

Proof

For φ and f such that $\varphi^N = 0$, the image of $\sum_i f\varphi^{N-1-i}t^i$ is t-invariant, namely,

$$t \sum_{i=0}^{N-1} f\varphi^{N-1-i}t^i = \sum_{i=0}^{N-1} f\varphi^{N-1-i}t^{i+1} = \sum_{i=0}^{N-1} f\varphi^{N-i}t^i = \Big(\sum_{i=0}^{N-1} f\varphi^{N-1-i}t^i \Big)\varphi,$$

and thus,

$$t \cdot \mathrm{Im}\Big(\sum_i f\varphi^{N-1-i}t^i \Big) = \mathrm{Im}\Big(t \sum_i f\varphi^{N-1-i}t^i \Big)$$

$$= \mathrm{Im}\Big(\Big(\sum_i f\varphi^{N-1-i}t^i \Big)\varphi \Big) \subset \mathrm{Im}\Big(\sum_i f\varphi^{N-1-i}t^i \Big).$$

Conversely, let the image of an injective map $\sum_{i=0}^{N-1} f_i t^i$ be t-invariant. This means that there exists an endomorphism φ such that

$$\sum_{i=1}^{N} f_{i-1}t^i = t \sum_{i=0}^{N-1} f_i t^i = \Big(\sum_{i=0}^{N-1} f_i t^i \Big)\varphi = \sum_{i=0}^{N-1} f_i\varphi t^i.$$

By comparing coefficients, this is equivalent to

$$f_0\varphi = 0, \qquad f_0 = f_1\varphi, \qquad f_1 = f_2\varphi, \qquad \dots, \qquad f_{N-2} = f_{N-1}\varphi;$$

thus, $f_i = f_{N-1}\varphi^{N-1-i}$ for all $i = 0, \dots, N-1$, and $f_{N-1}\varphi^N = 0$. The latter condition means that $\mathrm{Im}(\varphi^{i+1}) \subset \mathrm{Ker}(f_{N-1}\varphi^{N-1-i})$ for all i; thus,

$$\mathrm{Im}(\varphi^N) \subset \bigcap_{i=0}^{N-1} \mathrm{Ker}(f_{N-1}\varphi^{N-1-i}) = \bigcap_{i=0}^{N-1} \mathrm{Ker}(f_i) = 0$$

by the injectivity of $\sum_i f_i t^i$, which means $\varphi^N = 0$. Thus, the pair (φ, f_{N-1}) maps to the image of $\sum_i f_i t^i$. $\qquad \square$

We have thus proved the following result.

COROLLARY 3.6

The variety $X^{(N)}_{\dim V, \dim W}$ is isomorphic to the quotient

$$\big(\mathcal{N}^{(N)}(V) \times \mathrm{Hom}(V, W) \big)^{\mathrm{st}} / \mathrm{GL}(V).$$

3.3. Geometric consequences

The above corollary allows us to easily derive various geometric properties of the varieties $X^{(N)}_{k,m}$.

COROLLARY 3.7

The variety $X_{k,m}^{(N)}$ is irreducible, normal, and Cohen–Macaulay with rational singularities. It has dimension $\dim \mathcal{N}^{(N)}(\mathbb{C}^k) + k(m-k)$. In particular, it has dimension $k(k-1) + k(m-k) = k(m-1)$ for $N \geq k$.

Proof

Every variety $\mathcal{N}^{(N)}(\mathbb{C}^k)$ is irreducible, normal, and Cohen–Macaulay with rational singularities by [16, Theorem 0.1], since it is the closure of a single conjugacy class. These properties are preserved under passing to open subsets and geometric $\mathrm{GL}_k(\mathbb{C})$-quotients. Namely, for irreducibility this is clear, whereas normality and rational singularities are preserved under arbitrary quotients (the latter by Boutot's theorem; see [1, Corollaire]). For the Cohen–Macaulay property, we use the fact that a geometric $\mathrm{GL}_k(\mathbb{C})$-quotient is a principal bundle under a special group and, thus, Zariski locally trivial (see [22, Théorème 2]). Finally, the dimension formula follows by a direct calculation.   □

The natural sequence of embeddings of the varieties $\mathcal{N}^{(N)}(V)$, stabilizing in $\mathcal{N}^{(\dim V)}(V) = \mathcal{N}(V)$, induces a chain of embeddings

$$X_{k,m}^{(1)} \subset X_{k,m}^{(2)} \subset \cdots \subset X_{k,m}^{(k)} = X_{k,m}^{(k+1)} = \cdots,$$

whose limit we define as $X_{k,m} = X_{k,m}^{(N)}$ for $N \geq k$, which is the quotient of stable pairs (φ, f) for φ an arbitrary nilpotent operator by the action of $\mathrm{GL}_k(\mathbb{C})$.

3.4. Examples

We give some examples of the varieties $X_{k,m}$. First, it is easy to see that $X_{1,m} \simeq \mathbf{P}^{m-1}$. Second, let us consider the case $k = 2$. By the above, we thus consider the set of $2m \times 2$ matrices of rank 2 of the form

$$\begin{bmatrix} AB \\ A \end{bmatrix}$$

for an $(m \times 2)$-matrix A and a nilpotent 2×2 matrix B, up to the GL_2-action on columns. We can embed this variety into projective space via the Plücker embedding. Namely, we choose homogeneous coordinates $x_{i,j}$ for $1 \leq i < j \leq 2m$ and map the above matrix to the collection of its (2×2)-minors $T_{i,j}$. A priori these obey the Plücker relations

$$T_{i,k}T_{j,l} = T_{i,j}T_{k,l} + T_{i,l}T_{j,k}$$

for all $1 \leq i < j < k < l \leq 2m$. Since $\det(B) = 0$, we have $T_{i,j} = 0$ for $1 \leq i < j \leq m$. Since $\mathrm{tr}(B) = 0$, we have $T_{i,j+m} = T_{j,i+m}$ for all $1 \leq i, j \leq m$. We conclude that $X_{2,m}$ can be realized as the set of points in projective space with coordinates $x_{i,j}$ for $1 \leq i < j \leq 2m$ subject to the relations

$$x_{i,k}x_{j,l} = x_{i,j}x_{k,l} + x_{i,l}x_{j,k} \quad \text{for } 1 \leq i < j < k < l \leq 2m,$$

$$x_{i,j} = 0 \quad \text{for } 1 \leq i < j \leq m,$$

$$x_{i,j+m} = x_{j,i+m} \quad \text{for } 1 \leq i, j \leq m.$$

For example, in the case in which $m = 2$ we can eliminate the variables $x_{1,2}$ and $x_{2,3}$ and realize $X_{2,2}$ as the singular surface in $\mathbf{P}^3$ with coordinates $x_{1,3}, x_{1,4}, x_{2,4}$, $x_{3,4}$ and defining equation $x_{1,3}x_{2,4} = x_{1,4}^2$. (Then $(0:0:0:1)$ is an isolated singularity.)

For general m, we can eliminate the variables $x_{i,j}$ for $1 \leq i < j \leq m$ and $x_{j,i+m}$ for $1 \leq i < j \leq m$. We can rename the remaining variables as $v_{i,j} = x_{i+m,j+m}$ for $1 \leq i < j \leq m$ and $w_{i,j} = x_{i,j+m}$ for $1 \leq i, j \leq m$ and rewrite the above relations in these terms. Computer experiments with $m \leq 5$ suggest the following.

CONJECTURE 3.8

The variety $X_{2,m}$ is isomorphic to the closed subvariety of projective space with coordinates $v_{i,j}$ for $1 \leq i < j \leq m$ and $w_{i,j}$ for $1 \leq i \leq j \leq m$ given by the following equations:

(i)   *the symmetric matrix $W = (w_{i,j})$ has rank 1,*
(ii) *$WZ = 0$ for the matrix Z with m rows, with columns indexed by tuples (j, k, l) for $1 \leq j < k < l \leq m$, and with entries*

$$Z_{i,(j,k,l)} = \begin{cases} v_{k,l} & i = j, \\ -v_{j,l} & i = k, \\ v_{j,k} & i = l, \\ 0 & i \neq j, k, l, \end{cases}$$

(iii) *the Plücker relations for the $v_{i,j}$'s.*

3.5. Orbit structure

Now we consider the action of $\mathrm{GL}_m(A_N)$ on $X_{k,m}^{(N)}$ and determine the orbit structure. The group $\mathrm{GL}_m(A_N)$ embeds into the group $\mathrm{GL}(W \otimes A_N)$ as the subgroup of automorphisms commuting with $\mathrm{id}_W \otimes t$; every such automorphism can be written uniquely as

$$\sum_{i=0}^{N-1} \psi_i t^i$$

for $\psi_0 \in \mathrm{GL}(W)$, $\psi_i \in \mathrm{End}(W)$ for $i = 1, \ldots, N-1$, where each summand acts on $W \otimes A_N$ by applying the endomorphism ψ_i in the W-component and multiplying by t^i in the A_N-component. In particular, such an element acts on a linear map $\sum_i f_i t^i$ from V to $W \otimes A_N$ by

$$\left(\sum_i \psi_i t^i \right) \left(\sum_i f_i t^i \right) = \sum_i \left(\sum_{i'+i''=i} \psi_{i'} f_{i''} \right) t^i.$$

This allows us to conclude the following.

LEMMA 3.9

Under the isomorphism

$$X^{(N)}_{\dim V, \dim W} \simeq \left(\mathcal{N}^{(N)}(V) \times \mathrm{Hom}(V, W) \right)^{\mathrm{st}} / \mathrm{GL}(V),$$

the action of $\mathrm{GL}_{\dim W}(A_N)$ *on* $X^{(N)}_{\dim V, \dim W}$ *translates to*

$$\left(\sum_i \psi_i t^i \right) \cdot \overline{(\varphi, f)} = \overline{\left(\varphi, \sum_i \psi_i f \varphi^i \right)}$$

for $\sum_i \psi_i t^i \in \mathrm{GL}_{\dim W}(A_N)$, $\varphi \in \mathcal{N}^{(N)}(V)$, *and* $f \in \mathrm{Hom}(V, W)$.

Proof
Compute the action of $\sum_i \psi_i t^i$ on $\sum_i f \varphi^{N-1-i} t^i$ as above, and compare the t^{N-1}-coefficients. $\qquad \square$

To parameterize the orbits of $\mathrm{GL}_m(A_N)$ in $X^{(N)}_{k,m}$, we use representation theory of the algebra A_N and some of the methods developed in [2]. More precisely, we will use the following facts.

- The indecomposable representations of A_N are (up to isomorphism) $U_i = \mathbb{C}[t]/(t^i)$ for $i = 1, \ldots, N$. The representation U_i has dimension i and socle U_1. We have $\dim \mathrm{Hom}_{A_N}(U_i, U_j) = \min(i, j)$. In particular, A_N admits only finitely many isomorphism classes of representations of fixed dimension.
- Two subrepresentations R, R' of an injective representation I of a finite-dimensional algebra A are conjugate under $\mathrm{Aut}(I)$ if and only if they are isomorphic.
- A representation R embeds into an injective representation I of a finite-dimensional algebra A if and only if the socles embed, that is, $\mathrm{soc}(R)$ embeds into $\mathrm{soc}(I)$.
- If an A-representation M admits only finitely many isomorphism classes of subrepresentations of a given dimension k, then the Grassmannian of subrepresentations $\mathrm{Gr}^A_k(M)$ is stratified into locally closed subsets $\mathcal{S}_{[R]}$ consisting of subrepresentations in a fixed isomorphism class $[R]$; we have $\dim \mathcal{S}_{[R]} = \dim \mathrm{Hom}(R, M) - \dim \mathrm{End}(R)$.

Combining the above statements, we have the following result.

PROPOSITION 3.10

Suppose that A is an algebra of finite representation type, and let I be an injective representation of A. Then the $\mathrm{Aut}(I)$-orbits $\mathcal{O}_{[R]}$ in $\mathrm{Gr}_k(I)$ are parameterized by the isomorphism classes $[R]$ such that R is k-dimensional and $\mathrm{soc}(R) \subset \mathrm{soc}(I)$. The orbit $\mathcal{O}_{[R]}$ has dimension $\dim \mathrm{Hom}(R, I) - \dim \mathrm{End}(R)$.

This proposition applies to our setting since $A = A_N$ is of finite representation type and $I = U_N^m$ is injective. Every U_i has simple socle; thus, a representation embeds into U_N^m if and only if it has at most m indecomposable direct summands.

A k-dimensional such representation can be written as $U_\lambda = U_{\lambda_1} \oplus \cdots \oplus U_{\lambda_m}$ for a partition $N \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$ with $\sum_i \lambda_i = k$. (Note that the parts are allowed to be zero.)

COROLLARY 3.11

The $\mathrm{GL}_m(A_N)$-orbits $\mathcal{O}_\lambda$ in $X_{k,m}^{(N)}$ are parameterized by partitions λ of k of length m with at most N parts. We have

$$\dim \mathcal{O}_\lambda = mk - \sum_i (\lambda_i')^2,$$

where λ' denotes the conjugate partition of λ.

Proof
We know that the dimension of $\dim \mathcal{O}_\lambda$ is equal to

$$\dim \mathrm{Hom}(U_\lambda, U_N^m) - \dim \mathrm{End}(U_\lambda).$$

We note that $\dim \mathrm{Hom}(U_{\lambda_i}, U_N) = \lambda_i$ and $\dim \mathrm{Hom}(U_{\lambda_i}, U_{\lambda_j}) = \lambda_{\max(i,j)}$. Hence, $\dim \mathcal{O}_\lambda = (m+1)k - 2\sum_{i=1}^m i\lambda_i$. Now using [19, Chapter I, (1.6)], we arrive at the desired formula.   □

Let us determine the closure relation of these orbits. We recall some facts on orderings on partitions from, for example, [19].

  · We write $\lambda \geq \mu$ (the so-called dominance ordering on partitions) if $\sum_{j \leq i} \lambda_j \geq \sum_{j \leq i} \mu_j$ for all i.
  · We have $\lambda > \mu$ minimally if and only if there exist entries $i < j$ such that $\mu_i = \lambda_i - 1$, $\mu_j = \lambda_j + 1$, and $\mu_k = \lambda_k$ for all $k \neq i, j$.
  · We have $\lambda \geq \mu$ if and only if for the conjugate partitions one has $\mu' \geq \lambda'$.

  Consider partitions λ, μ as in the corollary.

THEOREM 3.12

The closure of $\mathcal{O}_\lambda$ contains $\mathcal{O}_\mu$ if and only if $\lambda \geq \mu$.

Proof
Suppose $\lambda \geq \mu$. To prove that $\mathcal{O}_\mu$ is contained in the closure of $\mathcal{O}_\lambda$, it suffices to do this in the case where $\lambda \geq \mu$ minimally; thus, μ differs from λ only in two positions $i < j$ as above. We can then reduce to the case $m = 2$. Thus, we want to prove that the closure of the orbit corresponding to the partition (λ_1, λ_2) contains the orbit corresponding to $(\lambda_1 - 1, \lambda_2 + 1)$. (In particular, $\lambda_1 \geq \lambda_2 + 2$.) We consider the following family U_z for $z \in \mathbb{C}$ of t-invariant subspaces of A_N^2: the subspace U_z is generated by $(zt^{N-\lambda_1}, t^{N-\lambda_1+1})$ and $(t^{N-\lambda_2-1}, zt^{N-\lambda_2})$. Then U_z belongs to $\mathcal{O}_{(\lambda_1, \lambda_2)}$ for $z \neq 0, 1, -1$ and to $\mathcal{O}_{(\lambda_1-1, \lambda_2+1)}$ for $z = 0$. (The $z = 0$ case is clear; if $z \neq 0$, then $U_z \simeq U_{\lambda_1} \oplus U_{\lambda_2}$ if and only if the vectors $t^{\lambda_1-\lambda_2-1}(zt^{N-\lambda_1}, t^{N-\lambda_1+1})$ and $(t^{N-\lambda_2-1}, zt^{N-\lambda_2})$ are linearly independent, which is equivalent to $z \neq \pm 1$.)

To prove the converse, we define for a sequence $k_* = (k_1, \ldots, k_N)$ the subset

$$C(k_*) = \{U \in X_{k,m}^{(N)} : \dim t^i U \le k_{i+1} \text{ for } i = 0, \ldots, N-1\}.$$

This is a closed subset since the dimension inequalities can be interpreted as rank conditions. For $U \in \mathcal{O}_\lambda$, we have

$$\dim t^i U = \sum_{j:\lambda_j \ge i} (\lambda_j - i).$$

Thus, defining $k_*(\lambda)$ by

$$k_i(\lambda) = \sum_{j:\lambda_j \ge i-1} (\lambda_j - i + 1) = \sum_{j \ge i} \lambda_j',$$

we have $\mathcal{O}_\lambda \subset C(k_*(\lambda))$. This immediately implies $\overline{\mathcal{O}_\lambda} \subset C(k_*(\lambda))$. Since we already know that $\overline{\mathcal{O}_\lambda}$ contains all $\mathcal{O}_\mu$'s for $\lambda \ge \mu$, the claim follows once we prove that $C(k_*(\lambda))$ is contained in the union of the $\mathcal{O}_\mu$'s for $\lambda \ge \mu$. So let us assume that $\mathcal{O}_\mu$ is contained in $C(k_*(\lambda))$. Then we know, by the definitions, that

$$\sum_{j:\mu_j \ge i} (\mu_j - i) \le \sum_{j:\lambda_j \ge i} (\lambda_j - i)$$

for all $i = 0, \ldots, N-1$. This can be rewritten as $\mu' \ge \lambda'$; thus, $\lambda \ge \mu$. The theorem is proved. $\qquad\square$

In particular, we can determine the open orbit of $\mathrm{GL}_m(A_N)$ in $X_{k,m}^{(N)}$; it corresponds to the partition $(\underbrace{N, \ldots, N}_{s}, r, 0, \ldots, 0)$, where $k = sN + r$ for $0 \le r < N$.

3.6. Desingularization

For N, k, m as before and a sequence $(k_1, \ldots, k_N)$ as in the proof of Theorem 3.12 (in particular, $k_1 = k$), we consider the variety $Y(k_*)$ of tuples

$$(U_1, \ldots, U_N) \in \prod_{i=0}^{N-1} \mathrm{Gr}_{k_{i+1}}(W \otimes t^i A_N)$$

such that

(i) $U_1 \supset U_2 \supset \cdots \supset U_N$,
(ii) $t(U_i) \subset U_{i+1}$ for $i = 1, \ldots, N-1$.

(As before, we abbreviate the map $\mathrm{id}_W \otimes t$ simply by t.) This is a closed subvariety of the product $\prod_{i=0}^{N-1} \mathrm{Gr}_{k_{i+1}}(W \otimes t^i A_N)$ and thus projective.

REMARK 3.13

We can view the variety $Y(k_*)$ as a quiver Grassmannian as follows. Consider the quiver Γ_N with N vertices $v_1, \ldots, v_N$ and $2(N-1)$ arrows $\alpha_i : v_i \to v_{i+1}$ for $i = 1, \ldots, N-1$ and $\beta_i : v_{i+1} \to v_i$ for $i = 1, \ldots, N-1$. We consider the admissible ideal I in $\mathbb{C}\Gamma_N$ generated by the elements

$$\alpha_i \beta_i - \beta_{i+1} \alpha_{i+1}, \quad i = 1, \ldots, N-1,$$

where, for $i = N - 1$, the relation has to be read as $\alpha_{N-1}\beta_{N-1} = 0$; that is, we formally define α_N and β_N as zero. The algebra $\mathbb{C}\Gamma_N/I$ is then isomorphic to the Auslander algebra of A_N.

We construct a specific representation $\mathbf{W}$ of $\mathbb{C}\Gamma_N/I$: we define $\mathbf{W}_{v_i} = W \otimes t^i A_N$, $\mathbf{W}_{\alpha_i} = t$ (multiplication by t in the second component), and $\mathbf{W}_{\beta_i} = \iota$, the inclusion map. We see that this representation of Γ_N indeed satisfies the relations in I. Thus, we can interpret $Y(k_*)$ as $\mathrm{Gr}_{k_*}^{\mathbb{C}\Gamma_N/I}(\mathbf{W})$.

PROPOSITION 3.14

For every partition λ as above, projection to $\mathrm{Gr}_k(W \otimes A_N)$ induces a desingularization map

$$\pi_\lambda : Y(k_*(\lambda)) \to \overline{\mathcal{O}_\lambda}.$$

Proof
We first verify that the image of $Y(k_*)$ under the projection

$$\pi : \prod_i \mathrm{Gr}_{k_{i+1}}(W \otimes t^i A_N) \to \mathrm{Gr}_k(W \otimes A_N)$$

given by $(U_1, \ldots, U_N) \mapsto U_1$ is contained in $X_{k,m}^{(N)}$ and even in $C(k_*)$. Indeed, the defining relations of $Y(k_*)$ imply that

$$t(U_1) \subset U_2 \subset U_1,$$

which shows that π projects to $X_{k,m}^{(N)}$. More generally, for all $i = 0, \ldots, N-1$, iterating the defining relations shows that

$$t^i(U_1) \subset U_{i+1},$$

which is thus a subspace of dimension at most k_{i+1}. But these are precisely the defining conditions of $C(k_*)$. Applying this argument to the special case $k_* = k_*(\lambda)$ for a partition λ as before, we see that the image $\pi(Y(k_*(\lambda)))$ is contained in $C(k_*(\lambda))$, which by the proof of Theorem 3.12 equals the closure of the orbit $\mathcal{O}_\lambda$.

Again by the proof of Theorem 3.12, we already know that, for a point $U \in \mathcal{O}_\lambda$, we have $\dim t^i(U) = k_{i+1}(\lambda)$ for $i = 0, \ldots, N-1$. Thus, the fiber of π over such a point U consists of the single point $(U, tU, \ldots, t^{N-1}U)$ for dimension reasons. First, this proves that $\mathcal{O}_\lambda$ is contained in the image of the map $\pi : Y(k_*(\lambda)) \to \overline{\mathcal{O}_\lambda}$. But since π is proper and thus has closed image, even $\overline{\mathcal{O}_\lambda}$ is contained in the image. Thus, by what has already been proven, π maps onto $\overline{\mathcal{O}_\lambda}$. Second, the above argument proves that π is generically one-to-one.

Finally, we prove that $Y(k_*)$ is always smooth by realizing it as a tower of Grassmann bundles. To do this, we consider truncated versions of the variety $Y(k_*)$, namely, for $i = 1, \ldots, N$, we consider the subvariety $Y_i(k_*)$ of $\prod_{j=i-1}^{N-1} \mathrm{Gr}_{k_{j+1}}(W \otimes t^j A_N)$ given by the same conditions as before, that is, the subvariety consisting of tuples $(U_i, \ldots, U_N)$ such that $tU_j \subset U_{j+1}$ for all relevant

j's and $U_i \supset \cdots \supset U_N$. We have a sequence of projections

$$Y(k_*) = Y_1(k_*) \to Y_2(k_*) \to \cdots \to Y_N(k_*).$$

Obviously, we have

$$Y_N(k_*) \simeq \mathrm{Gr}_{k_N}(\underbrace{W \otimes t^{N-1} A_N}_{\simeq W}).$$

Now we consider the projection $Y_{N-1}(k_*) \to Y_N(k_*)$, which is equivariant for the action of $\mathrm{GL}(W \otimes t^{N-1} A_N)$, the latter being transitive on the target. Its fiber over a point U_N consists of all U_{N-1}'s such that

$$U_N \subset U_{N-1} \subset t^{-1} U_N.$$

Note that $U_N \subset t^{-1} U_N$ since $t U_N = 0 \subset U_N$. Thus, the fiber over U_N is isomorphic to

$$\mathrm{Gr}_{k_{N-1} - k_N}(\underbrace{t^{-1} U_N / U_N}_{\simeq W}).$$

We continue inductively. The fiber of a projection $Y_{i-1}(k_*) \to Y_i(k_*)$ over a point U_i consists of all U_{i-1}'s such that $U_i \subset U_{i-1} \subset t^{-1} U_1$; we note that $t U_i \subset U_{i+1} \subset U_i$ and thus $U_i \subset t^{-1} U_i$. Thus, the fiber over U_i is isomorphic to

$$\mathrm{Gr}_{k_{i-1} - k_i}(\underbrace{t^{-1} U_i / U_i}_{\simeq W}).$$

Thus, we see that $Y(k_*)$ is a tower of Grassmann bundles with fibers isomorphic to that $\mathrm{Gr}_{k_i - k_{i+1}}(W)$ for $i = 1, \ldots, N$ (formally defining $k_{N+1} = 0$), and in particular, $Y(k_*)$ is irreducible, smooth, and of dimension

$$mk_1 - \sum_i (k_i - k_{i+1})^2.$$

In the special case $k_* = k_*(\lambda)$, this dimension can be written as $mk - \sum_i (\lambda_i')^2$ as expected. $\qquad \square$

3.7. All $X_{k,m}^{(N)}$'s are affine Schubert varieties

Let P_i, $i = 0, \ldots, m$, be the maximal parahoric subgroup of $\widehat{\mathrm{SL}}_m$ corresponding to the simple root α_i of the affine algebra $\widehat{\mathfrak{sl}}_m$. We prove the following proposition.

PROPOSITION 3.15
The variety $X_{k,m}^{(N)}$ is isomorphic to a Schubert variety in $\widehat{\mathrm{SL}}_m / P_{k \bmod m}$.

Proof
Recall the Iwahori group $I \subset \widehat{\mathrm{SL}}_m$. Consider the T-fixed point $x_{k,m}^{(N)} \in X_{k,m}^{(N)}$ such that $I x_{k,m}^{(N)}$ is the open part of $X_{k,m}^{(N)}$.

Namely, fixing the standard basis $e_1, \ldots, e_m$ of $\mathbb{C}^m$ let us write $k = Nr + s$, $0 \leq j < N$. Then $x_{k,m}^{(N)}$ is the subspace of $\mathbb{C}^m \otimes \mathbb{C}[t]/t^N$ spanned by the vectors $e_i \otimes t^j$, $i = 1, \ldots, r-1$, $j = 0, \ldots, N-1$, and $e_r \otimes t^j$, $j = N-s, \ldots, N-1$.

Our goal is to prove the existence of a T-fixed point $y_{k,m}^{(N)} \in \widehat{\mathrm{SL}}_m/P_{k \bmod m}$ such that the closure of $Iy_{k,m}^{(N)}$ (the corresponding Schubert variety) is isomorphic to the closure of $Ix_{k,m}^{(N)}$. To this end we construct an embedding of the I-module $\Lambda^k(\mathbb{C}^m \otimes \mathbb{C}[t]/t^N)$ into the semi-infinite wedge space with the image contained in the level one representation $L_{k \bmod m}$ corresponding to the $\mathfrak{sl}_m$ fundamental weight $\omega_{k \bmod m}$.

Let us write $k = ma + b$, where $b = k \bmod m$. Then $X_{k,m}^{(N)}$ contains the span of

$$\mathbb{C}^m \otimes t^j, \quad j = t^{N-1}, \ldots, t^{N-a}, \qquad e_1 \otimes t^{N-a-1}, \ldots, e_b \otimes t^{N-a-1}.$$

This point is obviously I-fixed. We want to identify it with the highest weight line in L_b. This line inside $F^{(b)}$ is spanned by $\mathbb{C}^m \otimes t^j$, $j \geq 0$, and $w_1 \otimes t^{-1}, \ldots, w_b \otimes t^{-1}$. Hence, the map $e_i \otimes t^j \mapsto w_i \otimes t^{j-N+a}$ induces the injective map of I-modules from $\Lambda^k(\mathbb{C}^m \otimes \mathbb{C}[t]/t^N)$ to L_b, sending $X_{k,m}^{(N)}$ to the Schubert variety $\overline{Iy_{k,m}^{(N)}} \subset \widehat{\mathrm{SL}}_m/P_b$. $\qquad\qquad\square$

COROLLARY 3.16
We have that $\mathrm{Gr}_N^a(\mathfrak{gl}_n)$ is an affine Schubert variety for $\widehat{\mathrm{SL}}_{2n}$.

REMARK 3.17
We note that Proposition 3.15 implies that $X_{k,m}^{(N)}$ is irreducible, normal, and Cohen–Macaulay with rational singularities (see [20, Théorèmes 2.Σ, 3], [17, Theorems 2.16, 2.23]). In particular, this re-proves the first part of Corollary 3.7.

4. Affine Lie algebras $\widehat{\mathfrak{sl}}_2$ and $\widehat{\mathfrak{sp}}_{2n}$

4.1. Type A_1
In this section we restrict to the case $\mathfrak{g} = \mathfrak{sl}_2$. We prove Conjecture 2.13 and give an explicit realization of the degenerate affine Grassmannian inside the Sato Grassmannian SGr_0.

Recall the identification $V \simeq W \otimes \mathbb{C}[t, t^{-1}]$, $\dim W = 2$. Let pr be the projection operator along $W \otimes 1$ to the span of the basis vectors $w_i \otimes t^j$, $j \neq 1$. We will also need a skew-symmetric form on V defined by

$$(w_1 \otimes t^i, w_2 \otimes t^j) = \delta_{i+j,-1}, \qquad (w_1 \otimes t^i, w_1 \otimes t^j) = (w_2 \otimes t^i, w_2 \otimes t^j) = 0.$$

THEOREM 4.1
The degenerate affine Grassmannian $\mathrm{Gr}^a(\mathfrak{sl}_2)$ sits inside SGr_0 as the subvariety of subspaces U satisfying the following conditions:

  (i) $\mathrm{pr}(tU) \subset U$,
  (ii) U *is isotropic with respect to the above symplectic form.*

We first show the existence of the embedding $\mathrm{Gr}^a(\mathfrak{sl}_2) \subset \mathrm{SGr}_0$ by proving Conjecture 2.13. To do this we prove that there exists a basis of $L_0(\mathfrak{sl}_2)$ such that

its vectors are linearly independent when considered in $\tilde{L}_0(\mathfrak{sl}_2)$ (see Section 2.3). Let e, h, f be the standard basis of $\mathfrak{sl}_2$. For an element $x \in \mathfrak{sl}_2$ we set $x_i = x \otimes t^i$. Recall the construction of the ehf-basis of L_0 from [4, Theorem 4]. A monomial of the form

$$(4.1) \qquad \cdots f_{-n}^{a_n} h_{-n}^{b_n} e_{-n}^{c_n} \cdots f_{-1}^{a_1} h_{-1}^{b_1} e_{-1}^{c_1}$$

is called an ehf-monomial if it satisfies the following conditions:

   (a) $a_i + a_{i+1} + b_{i+1} \leq 1$ for $i > 0$,
   (b) $a_i + b_{i+1} + c_{i+1} \leq 1$ for $i > 0$,
   (c) $a_i + b_i + c_{i+1} \leq 1$ for $i > 0$,
   (d) $b_i + c_i + c_{i+1} \leq 1$ for $i > 0$.

Then applying the ehf-monomials to a highest weight vector of L_0 one gets a basis. The following picture from [4] illustrates the set of ehf-monomials:

$$c_1 \quad c_2 \ \ldots$$

$$a_1 \quad a_2 \ \ldots$$
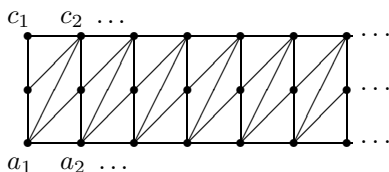
Namely, one considers the set of monomials (4.1) such that the sum of exponents over the ends of any segment is less than or equal to 1.

LEMMA 4.2
The monomials (4.1) subject to the conditions (a)–(d) form a basis of $\tilde{L}_0$.

Proof
Recall the identification $V \simeq W \otimes \mathbb{C}[t, t^{-1}]$ given by $v_{2i+1} \mapsto w_1 \otimes t^{-i-1}$, $v_{2i} \mapsto w_2 \otimes t^{-i}$ (not to be confused with $x_k = x \otimes t^k$ for $x \in \mathfrak{sl}_2$). We note that

$$h_k v_{2i+1} = v_{2i+1+2k}, \qquad h_k v_{2i} = -v_{2i+2k},$$

$$e_k v_{2i+1} = 0, \qquad e_k v_{2i} = v_{2i+2k-1},$$

$$f_k v_{2i+1} = v_{2i+2k+1}, \qquad f_k v_{2i} = 0.$$

Now assume that we are given a monomial m of the form (4.1) subject to the conditions (a)–(d). Then $m|0\rangle$ is decomposed as a sum of several semi-infinite wedge products of vectors v_i. We attach to m one wedge product $w(m)$ from this decomposition. We then show that, given a linear combination of ehf-monomials, we can find a monomial m in it such that $w(m)$ does not show up in any other ehf-monomial.

   We note that f_{-1} shifts an index by 1, h_{-1} by 2, e_{-1} by 3, f_{-2} by 3, h_{-2} by 4, e_{-2} by 5, and so on. Now we can see that if m is an ehf-monomial, then all powers a_i, b_i, and c_i are zeros or ones, and moreover, if x_i and y_j show up in m, then the difference of their shifts is at least 2. Hence, $m|0\rangle$ contains a

semi-infinite wedge product of the form

$$E_{i_1,j_1} \cdots E_{i_k,j_k}(v_0 \wedge v_1 \wedge v_2 \wedge \cdots), \quad j_k > j_{k-1} > \cdots > j_1 > i_1 > \cdots > i_k,$$

where the $E_{i,j}$'s are matrix units. Now it is easy to see that given a linear combination of ehf-monomials we can find a wedge product as above, contained in a single ehf-monomial. □

Now let us consider the degenerate affine Grassmannian $\mathrm{Gr}^a(\mathfrak{sl}_2)$; since affine Grassmannians are special cases of affine flag varieties, the general Definition 1.7 applies (with $k = 1$ and $\lambda = 0$).

COROLLARY 4.3

The $\widehat{\mathfrak{sl}}_2$ degenerate Grassmannian is contained in the Sato Grassmannian SGr_0.

LEMMA 4.4

Let $U \in \mathrm{Gr}^a(\mathfrak{sl}_2)$. Then U satisfies all the conditions of Theorem 4.1.

Proof
We use the standard basis w_1, w_2 of the 2-dimensional vector representation of $\mathfrak{sl}_2$. In particular, $fw_1 = w_2$, $ew_2 = w_1$, $hw_1 = w_1$, and $hw_2 = -w_2$. Consider an element $g = \exp(\sum_{i<0} x_i e_{-i} + y_i h_{-i} + z_i f_{-i})$, $g \in \widehat{G}^{-,a}$. Then $g|0\rangle$ is spanned by vectors of the form

$$w_1 \otimes 1 + y_1 w_1 \otimes t^{-1} + z_1 w_2 \otimes t^{-1} + y_2 w_1 \otimes t^{-2} + z_2 w_2 \otimes t^{-2} + \cdots,$$

$$w_2 \otimes 1 + x_1 w_1 \otimes t^{-1} - y_1 w_2 \otimes t^{-1} + x_2 w_1 \otimes t^{-2} - y_2 w_2 \otimes t^{-2} + \cdots,$$

$$w_1 \otimes t + y_2 w_1 \otimes t^{-1} + z_2 w_2 \otimes t^{-1} + y_3 w_1 \otimes t^{-2} + z_3 w_2 \otimes t^{-2} + \cdots,$$

$$w_2 \otimes t + x_2 w_1 \otimes t^{-1} - y_2 w_2 \otimes t^{-1} + x_3 w_1 \otimes t^{-2} - y_3 w_2 \otimes t^{-2} + \cdots.$$

Now it is easy to check that the linear span of these vectors satisfies all the conditions of Theorem 4.1. Since $\mathrm{Gr}(\mathfrak{sl}_2)$ is the closure of the $\widehat{G}^{-,a}$-orbit, the lemma follows. □

COROLLARY 4.5

Theorem 4.1 holds.

Proof
One sees from the proof of Lemma 4.4 that the $\widehat{G}^{-,a}$-orbit of the line $\mathbb{C}|0\rangle$ consists of the subspaces U satisfying the conditions of Theorem 4.1 and such that U has a nontrivial Plücker coordinate corresponding to the subspace $W \otimes \mathbb{C}[t]$. Since this is an open condition, Theorem 4.1 follows. □

Finally, let us compute the torus acting on the degenerate affine Grassmannian. Assume that we have a torus scaling the basis vectors as $w_1 \otimes t^i \to q_i w_1 \otimes t^i$, $w_2 \otimes t^j \to p_j w_2 \otimes t^j$ for some numbers p_i, q_j. We want this torus to act on

the open cell. In particular, each vector in the list of vectors from the proof of Lemma 4.4 has to be invariant (up to scaling) with respect to the torus action. This gives the following set of relations, labeled by positive numbers k:

$$\frac{p_0}{q_{-k}} = \frac{p_1}{q_{-k+1}} = \cdots = \frac{p_{k-1}}{q_{-1}},$$

$$\frac{q_0}{q_{-k}} = \frac{q_1}{q_{-k+1}} = \cdots = \frac{q_{k-1}}{q_{-1}} = \frac{p_0}{p_{-k}} = \frac{p_1}{p_{-k+1}} = \cdots = \frac{p_{k-1}}{p_{-1}},$$

$$\frac{q_0}{p_{-k}} = \frac{q_1}{p_{-k+1}} = \cdots = \frac{q_{k-1}}{p_{-1}}.$$

REMARK 4.6

The values $p_k = p_1 r^{k-1}$, $q_k = q_1 r^{k-1}$ for arbitrary p_1, q_1, r solve the equations above. This is a torus (effectively 2-dimensional, since it contains the 1-dimensional torus $r = 1$, $p_1 = q_1$, scaling all the vectors by the same number), generated by the loop rotation and the Cartan torus of SL_2.

The system above is equivalent to the statement that for any $a \geq 0$, $b < 0$ the quantities $\frac{p_a}{q_b}$, $\frac{q_a}{p_b}$, $\frac{p_a}{p_b}$, $\frac{q_a}{q_b}$ depend only on the difference $a - b$ and $\frac{p_a}{p_b} = \frac{q_a}{q_b}$. This means that there exists a complex number r such that

$$q_a = q_0 r^a, \qquad p_a = p_0 r^a, \quad a \geq 0, \qquad q_b = q_{-1} r^{b+1}, \qquad p_b = p_{-1} r^{b+1}, \quad b < 0,$$

with the additional condition $\frac{q_0}{q_{-1}} = \frac{p_0}{p_{-1}}$.

COROLLARY 4.7

The torus acting on the degenerate Grassmannian is 3-dimensional. It is generated by the loop rotation, the 1-dimensional Cartan torus of SL_2, and an additional 1-dimensional torus with $p_{\geq 0} = q_{\geq 0} = 1$, $p_{<0} = q_{<0} = \mathrm{const}$.

4.2. Symplectic degenerate affine Grassmannian

It turns out that the construction of $\mathrm{Gr}^a(\mathfrak{sl}_2)$ has a natural (though conjectural) generalization to the case of symplectic algebras.

Let W be a $2n$-dimensional vector space endowed with a nondegenerate skew-symmetric form $(\cdot, \cdot)$. We fix a basis $w_1, \ldots, w_{2m}$ of W such that $(w_i, w_{2n+1-i}) = 1$ for $i = 1, \ldots, n$. Consider the space $W \otimes \mathbb{C}[t, t^{-1}]$ and the corresponding sector $F^{(0)}$ of the semi-infinite wedge power.

Now consider the Lie algebra $\mathfrak{sp}_{2n}^a$, which is an abelian Lie algebra with the underlying vector space $\mathfrak{sp}_{2n}$. Define the action of $\mathfrak{sp}_{2n}^a \otimes t^{-1}\mathbb{C}[t^{-1}]$ on $W \otimes \mathbb{C}[t, t^{-1}]$ as follows. For $x \in \mathfrak{sp}_{2n}^a$, $w \in W$ we define

$$(x \otimes t^i)(w \otimes t^j) = \begin{cases} xw \otimes t^{i+j} & \text{if } j \geq 0, i+j < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $L_0(\mathfrak{sp}_{2n})$ be the basic level one module of the affine Lie algebra $\widehat{\mathfrak{sp}}_{2n}$, and let L_0^a be its degenerate analogue. Let $l_0 \in L_0$ be the highest weight vector. In particular, $L_0(\mathfrak{sp}_{2n}) = \mathrm{U}(\mathfrak{sp}_{2n} \otimes t^{-1}\mathbb{C}[t^{-1}])l_0$.

CONJECTURE 4.8

Let $\tilde{L}_0(\mathfrak{sp}_{2n}) = \mathrm{U}(\mathfrak{sp}_{2n}^a \otimes t^{-1}\mathbb{C}[t^{-1}])|0\rangle$. Then we have an isomorphism of $\mathfrak{sp}_{2n}^a \otimes t^{-1}\mathbb{C}[t^{-1}]$-modules

$$\tilde{L}_0(\mathfrak{sp}_{2n}) \simeq L_0^a, \qquad |0\rangle \mapsto l_0.$$

The action of the abelian Lie algebra $\mathfrak{sp}_{2n}^a \otimes t^{-1}\mathbb{C}[t^{-1}]$ as above induces an action of the corresponding infinite-dimensional Lie group $\exp(\mathfrak{sp}_{2n}^a \otimes t^{-1}\mathbb{C}[t^{-1}])$. Because of Conjecture 4.8 the natural candidate for the degenerate symplectic affine Grassmannian $\mathrm{Gr}^a(\widehat{\mathfrak{sp}}_{2n})$ is the closure of the orbit of this group through the highest weight vector inside $\mathbb{P}(\tilde{L}_0(\mathfrak{sp}_{2n}))$. We denote this closure by $\mathrm{G}^a(\widehat{\mathfrak{sp}}_{2n})$.

Define a symplectic form on the infinite-dimensional space $W \otimes \mathbb{C}[t, t^{-1}]$:
$\langle v \otimes t^i, w \otimes t^j \rangle = (v, w)\delta_{i+j,-1}$.

THEOREM 4.9

The variety $\mathrm{G}^a(\widehat{\mathfrak{sp}}_{2n})$ consists of points U of the Sato Grassmannian SGr_0 such that U is isotropic with respect to the form $\langle \cdot, \cdot \rangle$ and such that $pr(tU) \subset U$.

CONJECTURE 4.10

We have $\mathrm{Gr}^a(\widehat{\mathfrak{sp}}_{2n}) \simeq \mathrm{G}^a(\widehat{\mathfrak{sp}}_{2n})$.

We sketch the proof of Theorem 4.9. Let $\mathrm{O} \subset \mathrm{SGr}_0$ be the subvariety of spaces that intersect trivially with $W \otimes t^{-1}\mathbb{C}[t^{-1}]$ (i.e., the Plücker coordinate corresponding to $W \otimes \mathbb{C}[t]$ does not vanish).

PROPOSITION 4.11

The $\exp(\mathfrak{sp}_{2n} \otimes t^{-1}\mathbb{C}[t^{-1}]) \cdot \mathbb{C}|0\rangle$-orbit coincides with $\mathrm{O} \cap \overline{\mathrm{G}^a(\widehat{\mathfrak{sp}}_{2n})}$.

To prove Theorem 4.9 we consider the finitization of $\mathrm{G}^a(\widehat{\mathfrak{sp}}_{2n})$, thus making explicit the ind-variety structure.

DEFINITION 4.12

For $N \geq 0$, let $\mathrm{G}_N^a(\widehat{\mathfrak{sp}}_{2n})$ be the finite-dimensional subvariety of $\mathrm{G}^a(\widehat{\mathfrak{sp}}_{2n})$ consisting of subspaces U such that

(i) $W \otimes t^N\mathbb{C}[t] \subset U \subset W \otimes t^{-N}\mathbb{C}[t]$,
(ii) U is isotropic,
(iii) $pr(tU) \subset U$.

LEMMA 4.13

We have that $\mathrm{G}_N^a(\widehat{\mathfrak{sp}}_{2n})$ coincides with the closure of the orbit of the group $\exp(\mathfrak{sp}_{2n}^a \otimes \mathrm{span}(t^{-1}, \ldots, t^{-N}))$ through the line spanned by $|0\rangle$. In particular, $\mathrm{G}_N^a(\widehat{\mathfrak{sp}}_{2n})$'s are irreducible and of dimension $N \dim \mathfrak{sp}_{2n}$.

LEMMA 4.14

We have $\mathrm{G}_N^a(\widehat{\mathfrak{sp}}_{2n}) \cap O = \exp(\mathfrak{sp}_{2n}^a \otimes \mathrm{span}(t^{-1}, \ldots, t^{-N})) \cdot \mathbb{C}|0\rangle \cap O.$

Now it remains to prove the irreducibility of the varieties $\mathrm{G}_N^a(\widehat{\mathfrak{sp}}_{2n})$. This is achieved by constructing explicitly the desingularization via the same procedure as in Lemma 2.6 and Section 3.6 (see also [9, Definition 5.1]).

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Feigin: Department of Mathematics, National Research University Higher School of Economics, Moscow, 117312, Russia
Skolkovo Institute of Science and Technology, Skolkovo Innovation Center, Moscow, 143026, Russia; evgfeig@gmail.com

Finkelberg: National Research University Higher School of Economics, Moscow, 117312, Russia
Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow, 127051, Russia; fnklberg@gmail.com

Reineke: Faculty of Mathematics, Ruhr-Universitaet Bochum, D-44780 Bochum, Germany; markus.reineke@ruhr-uni-bochum.de