The derivative and moment of the generalized Riesz–Nágy–Takács function

In-Soo Baek

Abstract We give the characterization of the differentiability and nondifferentiability points of a generalization of the Riesz–Nágy–Takács (RNT) singular function, namely, the generalized RNT (GRNT) singular function. A particular characterization generalizes recent multifractal and metric-number-theoretical results associated with the RNT singular function. Furthermore, we compute the moments of the GRNT singular function.

1. Introduction

Singular functions have been investigated by many authors (see [9], [13], [15], [10]). Recently, the characterization of the differentiability and nondifferentiability points of the Riesz–Nágy–Takács (RNT) singular function was studied (see [15]), and their dimensions were computed (see [3]). Their characterization and dimensions can be found if we use the distribution sets which give the information of the cylindrical local dimension sets for a self-similar set $[0,1]$ the attractor of an iteration function system (IFS), which has two similarity transformations. In this article, we consider the generalized Riesz–Nágy–Takács (GRNT) singular function, which is a generalized form of the RNT singular function. The characterization of the differentiability and nondifferentiability points of the GRNT singular function and their dimensions also can be found if we use the parameter distribution sets (see [4], [5]) which give the information of the cylindrical local dimension sets for a self-similar set $[0,1]$ the attractor of an IFS which has many similarity transformations. Furthermore, using the recurrence relation, we compute the moments of the GRNT singular function.

2. Preliminaries

For the probability vectors $(a_1,\ldots,a_N)\in(0,1)^N$ and $\mathbf{p}=(p_1,\ldots,p_N)\in(0,1)^N$ where $N\geq 2$ is a positive integer,

$$[0,1]=\bigcup_{k=1}^{N} S_k\bigl([0,1]\bigr),$$

Kyoto Journal of Mathematics, Vol. 57, No. 1 (2017), 185–195
DOI 10.1215/21562261-3759567,
Received October 17, 2012. Revised December 21, 2015. Accepted December 24, 2015.
2010 Mathematics Subject Classification: Primary 28A78; Secondary 11K55.
Baek's work supported by a research grant from Busan University of Foreign Studies in 2017.

where $S_k(x) = a_k x + \sum_{i=1}^{k-1} a_i$, and $\gamma_{\mathbf{p}}$ is the unique probability measure on $[0,1]$ such that

$$\gamma_{\mathbf{p}} = \sum_{i=1}^{N} p_i \gamma_{\mathbf{p}} \circ S_i^{-1}.$$

We define

$$f(x) = \gamma_{\mathbf{p}}\big([0,x]\big),$$

where $\gamma_{\mathbf{p}}$ is the self-similar measure on the self-similar set $[0,1]$. We call the function $f(x)$ the GRNT function.

In this paper, to avoid the degenerate case, we also assume that $\mathbf{p} = (p_1, \ldots, p_N) \neq (a_1, \ldots, a_N)$ (which happens if and only if $\frac{\log p_k}{\log a_k}$ is not the same for all $k = 1, \ldots, N$). Let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and the set of real numbers, respectively. The attractor K in the 1-dimensional Euclidean space $\mathbb{R}^1$ of the IFS $(S_1, \ldots, S_N)$ of contractions where $N \geq 2$ makes each point $q \in K$ have an infinite sequence $\omega = (m_1, m_2, \ldots) \in \Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ where

$$\{q\} = \bigcap_{n=1}^{\infty} K_{\omega|n}$$

for $K_{\omega|n} = K_{m_1,\ldots,m_n} = S_{m_1} \circ \cdots \circ S_{m_n}(K)$ (see [12]). In this case, we sometimes write $\pi(\omega)$ for such q using the natural projection $\pi : \Sigma \to K$ and write $c_n(q)$ for such $K_{\omega|n}$ and call it a fundamental interval. We note that, in our case, K is the unit interval $[0,1]$, which is the self-similar set satisfying the open set condition.

We also note that the points in the unit interval which have two different infinite sequences $\omega = (m_1, m_2, \ldots)$, $\omega' = (m'_1, m'_2, \ldots) \in \Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ satisfying $\pi(\omega) = \pi(\omega')$ are only countable. Except for such countable points, each point in the unit interval has a unique infinite sequence $\omega = (m_1, m_2, \ldots) \in \Sigma = \{1, \ldots, N\}^{\mathbb{N}}$. We recall $A(\{x_n(\omega)\})$ (see [8], [4]) of the accumulation points in the $(N-1)$-simplex of probability vectors in $\mathbb{R}^N$ of the vector-valued sequence $\{x_n(\omega)\} = \{(u_1^{(n)}, \ldots, u_N^{(n)})\}$ of the probability vectors, where $u_k^{(n)}$ for $1 \leq k \leq N$ in the probability vector $(u_1^{(n)}, \ldots, u_N^{(n)})$ for each $n \in \mathbb{N}$ is defined by

$$u_k^{(n)} = \frac{|\{1 \leq l \leq n : m_l = k\}|}{n}.$$

LEMMA 2.1

Let $\mathbf{p} = (p_1, \ldots, p_N) \in (0,1)^N$ *with* $\sum_{k=1}^{N} p_k = 1$, *consider a self-similar measure* $\gamma_{\mathbf{p}}$ *on* K, *and let* $\mathbf{r} = (r_1, \ldots, r_N) \in [0,1]^N$ *with* $\sum_{k=1}^{N} r_k = 1$ *and*

$$g(\mathbf{r}, \mathbf{p}) = \frac{\sum_{k=1}^{N} r_k \log p_k}{\sum_{k=1}^{N} r_k \log a_k}.$$

Then we have the following.

(1) *The set* $\{\mathbf{r} : g(\mathbf{r}, \mathbf{p}) = \alpha\}$ *is a hyperplane of the* $(N-1)$*-simplex or empty.*

(2) *There are distinct integers* $1 \le i, j \le N$ *such that*

$$\alpha_{\min} \equiv \min_{1\le k\le N} \frac{\log p_k}{\log a_k} = \frac{\log p_i}{\log a_i} \le g(\mathbf{r},\mathbf{p}) \le \frac{\log p_j}{\log a_j} = \max_{1\le k\le N} \frac{\log p_k}{\log a_k} \equiv \alpha_{\max}.$$

Proof

Part (1) follows from the fact that $\{\mathbf{r} : g(\mathbf{r},\mathbf{p}) = \alpha\}$ is the intersection of two hyperplanes of $\mathbb{R}^N$. Part (2) follows from the fact that the two hyperplanes are not parallel because of our assumption that $(p_1,\dots,p_N) \ne (a_1,\dots,a_N)$. □

REMARK 2.2

From now on, without specific mention, we fix distinct i, j respectively determining the parameter axes (see [4]) satisfying

$$\frac{\log p_i}{\log a_i} = \min_{1\le k\le N} \frac{\log p_k}{\log a_k} < \max_{1\le k\le N} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j}.$$

Furthermore, we define $g(\mathbf{r},\mathbf{p})$ for $\mathbf{r} = (r_1,\dots,r_N) \in [0,1]^N$ and $\mathbf{p} = (p_1,\dots,p_N) \in (0,1)^N$ as in the above lemma.

DEFINITION 2.3

For $\mathbf{y}, \mathbf{z}$ such that $g(\mathbf{y},\mathbf{p}) = g(\mathbf{z},\mathbf{p})$ where $\mathbf{z} = (z_1,\dots,z_N)$ with $z_j = 1 - z_i$ and $z_k = 0$ if $k \ne i, j$, we define the natural projection $t(\mathbf{y})$ of $\mathbf{y}$ into the i-axis to be z_i (see [4]). We also define the parameter distribution sets (see [4]) with respect to the parameter axes

$$\underline{F}(t) \equiv \left\{\omega : \min_{\mathbf{y}\in A(\{x_n(\omega)\})} t(\mathbf{y}) = t\right\}$$

and

$$\overline{F}(t) \equiv \left\{\omega : \max_{\mathbf{y}\in A(\{x_n(\omega)\})} t(\mathbf{y}) = t\right\}.$$

We also write $F(t) \equiv \underline{F}(t) \cap \overline{F}(t)$ henceforth.

DEFINITION 2.4

Let t_0 be the real number satisfying

$$\frac{t_0 \log p_i + (1-t_0)\log p_j}{t_0 \log a_i + (1-t_0)\log a_j} = g(\mathbf{r}_0,\mathbf{p})$$

for $\mathbf{r}_0 = (a_1,\dots,a_N)$ with $\sum_{k=1}^N a_k = 1$. That is, t_0 is the natural projection $t(\mathbf{r}_0)$ of $\mathbf{r}_0$ into the i-axis satisfying the above condition. Let t_1 be the real number satisfying

$$\frac{t_1 \log p_i + (1-t_1)\log p_j}{t_1 \log a_i + (1-t_1)\log a_j} = 1.$$

REMARK 2.5

We note that there is q satisfying $\beta'(q) = -1$, where $\sum_{k=1}^N p_k^q a_k^{\beta(q)} = 1$. Then

$\mathbf{r}_1 = (p_1^q a_1^{\beta(q)}, \ldots, p_N^q a_N^{\beta(q)})$ satisfies $g(\mathbf{r}_1, \mathbf{p}) = 1$ (see [4, Example 2]). That is, t_1 is the natural projection $t(\mathbf{r}_1)$ of $\mathbf{r}_1$ into the i-axis satisfying the above condition. We will use t_0 and t_1 as above henceforth in this article.

3. The differentiability points and the nondifferentiability points

Let $\omega \in \Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ be given. Then $A(\{x_n(\omega)\})$, which is a subset of the $(N-1)$-simplex, is determined. Then $t(A(\{x_n(\omega)\}))$ is a nonempty subinterval of the unit interval since t is a continuous map and $A(\{x_n(\omega)\})$ is a continuum (see [4], [14]).

THEOREM 3.1

If $(a_1, \ldots, a_N) \neq (p_1, \ldots, p_N)$, *then* $\frac{\log p_i}{\log a_i} < 1 < \frac{\log p_j}{\log a_j}$.

Proof

We note that $\frac{\log p_i}{\log a_i} = \min_{1 \le k \le N} \frac{\log p_k}{\log a_k} \le \max_{1 \le k \le N} \frac{\log p_k}{\log a_k} = \frac{\log p_j}{\log a_j}$ for some $1 \le i, j \le N$. If $(a_1, \ldots, a_N) \neq (p_1, \ldots, p_N)$, then there are $a_k \neq p_k$ for some $1 \le k \le N$, say, $a_k > p_k$. Since $(a_1, \ldots, a_N)$ and $(p_1, \ldots, p_N)$ are probability vectors, there are also $a_l < p_l$ for some $1 \le l (\neq k) \le N$. Therefore, $\frac{\log p_i}{\log a_i} \le \frac{\log p_k}{\log a_k} < 1 < \frac{\log p_l}{\log a_l} \le \frac{\log p_j}{\log a_j}$. $\square$

THEOREM 3.2

If $p_k \neq a_k$ *for all* $k = 1, \ldots, N$, *then we have*

$$\{x \in (0,1] : 0 < f'(x) < \infty\} = \phi.$$

Proof

We note that

$$0 \le \lim_{n \to \infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} = f'(x) \le \infty,$$

where $\gamma_{\mathbf{p}}$ is a self-similar probability measure on $[0,1]$ if $f'(x) \in [0, \infty]$ exists. Since

$$\{x \in (0,1] : 0 < f'(x) < \infty\} \subset \Big\{x \in (0,1] : 0 < \lim_{n \to \infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} < \infty\Big\},$$

we only need to show that

$$\Big\{x \in (0,1] : 0 < \lim_{n \to \infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} < \infty\Big\} = \phi.$$

Noting that $0 < \min_k a_k \le \frac{|c_{n+1}(x)|}{|c_n(x)|} \le 1$, we see that if there is an $x \in (0,1]$ such that $0 < \lim_{n \to \infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} = l < \infty$, then

$$\lim_{n \to \infty} \left| \frac{\gamma_{\mathbf{p}}(c_{n+1}(x))}{\gamma_{\mathbf{p}}(c_n(x))} - \frac{l|c_{n+1}(x)|}{l|c_n(x)|} \right| = 0.$$

However,

$$\left|\frac{\gamma_{\mathbf{p}}(c_{n+1}(x))}{\gamma_{\mathbf{p}}(c_n(x))} - \frac{l|c_{n+1}(x)|}{l|c_n(x)|}\right| = |p_k - a_k| \neq 0,$$

since $p_k \neq a_k$ for all $k = 1, \ldots, N$. $\square$

PROPOSITION 3.1 ([4])

For $t \in [0,1]$ satisfying $\frac{t\log p_i + (1-t)\log p_j}{t\log a_i + (1-t)\log a_j} = \alpha$, we have the following. For

$$\overline{E}_{\alpha}^{(\mathbf{p})} = \Big\{x \in [0,1] : \limsup_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} = \alpha\Big\},$$

$$\underline{E}_{\alpha}^{(\mathbf{p})} = \Big\{x \in [0,1] : \liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} = \alpha\Big\},$$

(1)

$$\overline{E}_{\alpha}^{(\mathbf{p})} = \pi\big(\underline{F}(t)\big),$$

(2)

$$\underline{E}_{\alpha}^{(\mathbf{p})} = \pi\big(\overline{F}(t)\big).$$

Proof

It follows from [4, Theorems 3.4, 3.5] with $s = 1$. $\square$

REMARK 3.2

From now on, we only consider $x = \pi(\omega)$, which is not an endpoint of the fundamental interval for simplicity. We also note that the endpoints of the fundamental intervals are countable.

THEOREM 3.3

We have the following.

(1) *If $t(A(\{x_n(\omega)\})) \subset [0, t_1)$, then $f'(x) = 0$ when $f'(x)$ exists for $x = \pi(\omega)$.*

(2) *If $t(A(\{x_n(\omega)\})) \subset (t_1, 1]$, then $f'(x) = \infty$ when $f'(x)$ exists for $x = \pi(\omega)$.*

(3) *If $t_1 \in [t(A(\{x_n(\omega)\}))]^o$, then $f'(x)$ does not exist for $x = \pi(\omega)$.*

Proof

For (1), if $t(A(\{x_n(\omega)\})) \subset [0, t_1)$, then

$$\liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} > 1$$

for $x = \pi(\omega)$, where $\omega \in \bigcup_{0\le t<t_1} \overline{F}(t)$ from Proposition 3.1(2). Therefore, for some $\epsilon > 0$, $\liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} \geq 1 + \epsilon$, which gives

$$0 \le \limsup_{n\to\infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} \le \limsup_{n\to\infty} \big|c_n(x)\big|^{\epsilon} = 0.$$

Using Proposition 3.1(1) instead of Proposition 3.1(2), we have (2) from arguments similar to those given above.

For (3), if $t_1 \in [t(A(\{x_n(\omega)\}))]^o$, then

$$\liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} < 1 < \limsup_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|}$$

for $x = \pi(\omega)$, where

$$\omega \in \Big[\bigcup_{t_1 < t \le 1} \overline{F}(t)\Big] \cap \Big[\bigcup_{0 \le t < t_1} \underline{F}(t)\Big]$$

from Propositions 3.1(1) and 3.1(2). This gives

$$0 = \liminf_{n\to\infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} < \limsup_{n\to\infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} = \infty$$

from arguments similar to those given above. □

THEOREM 3.4

We have the following.

(1) If $f'(x) = 0$ for $x = \pi(\omega)$, then $t(A(\{x_n(\omega)\})) \subset [0, t_1]$.

(2) If $f'(x) = \infty$ for $x = \pi(\omega)$, then $t(A(\{x_n(\omega)\})) \subset [t_1, 1]$.

(3) If $0 < f'(x) < \infty$ for $x = \pi(\omega)$, then $t(A(\{x_n(\omega)\})) = \{t_1\}$.

Proof

For (1), assume that $t(A(\{x_n(\omega)\})) \not\subset [0, t_1]$. Noting that $t(A(\{x_n(\omega)\}))$ is connected, we have $t(A(\{x_n(\omega)\})) \subset (t_1, 1]$ or $t_1 \in [t(A(\{x_n(\omega)\}))]^o$ or $t(A(\{x_n(\omega)\})) = [t_1, t_2]$ where $t_1 < t_2 \le 1$. If $t(A(\{x_n(\omega)\})) \subset (t_1, 1]$ or $t_1 \in [t(A(\{x_n(\omega)\}))]^o$, from Theorems 3.3(2) and 3.3(3), we have $f'(x) = \infty$ for $x = \pi(\omega)$ or $f'(x)$ does not exist for $x = \pi(\omega)$. For $t(A(\{x_n(\omega)\})) = [t_1, t_2]$, we have $\limsup_{n\to\infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} = \infty$ from the fact that $\liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} < 1$. Similarly, (2) follows.

For (3), assume that $t(A(\{x_n(\omega)\})) \ne \{t_1\}$. Then $\liminf_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} < 1$ or $\limsup_{n\to\infty} \frac{\log \gamma_{\mathbf{p}}(c_n(x))}{\log |c_n(x)|} > 1$. This gives

$$\limsup_{n\to\infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} = \infty$$

or

$$\liminf_{n\to\infty} \frac{\gamma_{\mathbf{p}}(c_n(x))}{|c_n(x)|} = 0.$$ □

From now on, M denotes the nondifferentiability points, D_0 denotes the null-differentiability points, D_∞ denotes the ∞-differentiability points, and D denotes the set $\{x \in [0,1] : 0 < f'(x) < \infty\}$.

COROLLARY 3.3

We have

$$\pi\Big(\Big[\bigcup_{t_1<t\le 1}\overline{F}(t)\Big]\cap\Big[\bigcup_{0\le t<t_1}\underline{F}(t)\Big]\Big)\subset M,$$

$$\bigcup_{0\le t<t_1}\pi\big(\overline{F}(t)\big)\subset D_0\cup M,$$

$$\bigcup_{t_1<t\le 1}\pi\big(\underline{F}(t)\big)\subset D_\infty\cup M;$$

furthermore,

$$D_0\subset\bigcup_{0\le t\le t_1}\pi\big(\overline{F}(t)\big),$$

$$D_\infty\subset\bigcup_{t_1\le t\le 1}\pi\big(\underline{F}(t)\big),$$

and

$$D\subset\pi\big(F(t_1)\big).$$

Proof

It follows from Theorems 3.3 and 3.4. □

From now on, $\dim(E)$ denotes the Hausdorff dimension of E and $\mathrm{Dim}(E)$ denotes the packing dimension of E [12]. We note that $\dim(E)\le\mathrm{Dim}(E)$ for every set E [12].

COROLLARY 3.4

The GRNT function f which is not the identity function is a singular function, and $0<t_0<t_1<1$ for t_0,t_1 in Definition 2.4.

Proof

The GRNT function f is an increasing function, so its derivative is zero or finite almost everywhere (see [16]). We note that $D\subset\pi(F(t_1))$ from Corollary 3.3 and $\dim(D)\le\dim(\pi(F(t_1)))=g(\mathbf{r}_1,\mathbf{r}_1)$ (see the proofs of [4, Theorems 4.2, 4.3]). Further, $\alpha q+\beta(q)=g(\mathbf{r}_1,\mathbf{r}_1)<g(\mathbf{r}_0,\mathbf{r}_0)=1$ where $\alpha=1$ and $\beta'(q)=-1$ from [12, (11.37)]. Hence, the GRNT function has null derivative almost everywhere.

Since β is strictly convex (see [12]) where $\sum_{k=1}^{N}p_k^q a_k^{\beta(q)}=1$,

$$\frac{t_0\log p_i+(1-t_0)\log p_j}{t_0\log a_i+(1-t_0)\log a_j}=g(\mathbf{r}_0,\mathbf{p})=-\beta'(0)<\frac{\log p_j}{\log a_j}=-\lim_{q\to-\infty}\beta'(q).$$

This gives $t_0>0$.

We note that $t_0\ne t_1$ from Gibbs's inequality since $(p_1,\dots,p_N)\ne(a_1,\dots,a_N)$. Assuming $t_1=1$, we have $\frac{\log p_i}{\log a_i}=1$, which is a contradiction by Theorem 3.1. Therefore, $t_1<1$.

Assume that $t_1 < t_0$. Noting that $\Gamma \subset F(t_0)$, where $\pi(\Gamma)$ is the set of the normal points (see [8], [7], [11], [14]), that is, $\Gamma = \{\omega : A(\{x_n(\omega)\}) = \{(a_1, \dots, a_N)\}\}$, we have $(D_0 \cup D) \cap \pi(F(t_0)) = \phi$ since $D_0 \cup D \subset \bigcup_{0 \le t \le t_1} \pi(\overline{F}(t))$ from Corollary 3.3. Then a contradiction arises from the fact that $D_0 \cup D$ is of full Lebesgue measure and $\pi(\Gamma)$ is also of full Lebesgue measure. This and $t_0 \ne t_1$ give $t_0 < t_1$. □

THEOREM 3.5

For the vector $\mathbf{r}_1$ in Remark 2.5, we have

$$0 < g(\mathbf{r}_1, \mathbf{r}_1) \le \dim(M) \le \mathrm{Dim}(M) = 1.$$

Furthermore,

$$\dim(D_\infty) \le \mathrm{Dim}(D_\infty) \le g(\mathbf{r}_1, \mathbf{r}_1) < 1.$$

Proof

We note that $0 < t_0 < t_1 < 1$. We also note that

$$g(\mathbf{r}, \mathbf{r}) = \frac{\sum_{k=1}^N r_k \log r_k}{\sum_{k=1}^N r_k \log a_k} > 0$$

if $r_k > 0$ for all $k = 1, \dots, N$ with $\sum_{k=1}^N r_k = 1$. This gives $g(\mathbf{r}_1, \mathbf{r}_1) > 0$. Moreover, $g(\mathbf{r}_1, \mathbf{r}_1) < 1$ from the proof of Corollary 3.4. Noting that $\pi(\underline{F}(0) \cap \overline{F}(1)) \subset M$ from Corollary 3.3 and

$$\{\omega : A(\{x_n(\omega)\}) = C\} \subset \underline{F}(0) \cap \overline{F}(1),$$

where C is the $(N-1)$-simplex, we have $\mathrm{Dim}(M) = 1$ from [8, Theorem 2]. Then

$$g(\mathbf{r}_1, \mathbf{r}_1) \le \dim(M)$$

follows from $\pi([\bigcup_{t_1 < t \le 1} \overline{F}(t)] \cap [\bigcup_{0 \le t < t_1} \underline{F}(t)]) \subset M$ and

$$\sup_{t_1 < t_2 \le 1, 0 \le t_3 < t_1} \dim\big(\pi\big(\overline{F}(t_2) \cap \underline{F}(t_3)\big)\big) = g(\mathbf{r}_1, \mathbf{r}_1),$$

which follows from arguments similar to those of the proofs of [4, Theorems 4.2 and 4.3].

From [5, Corollary 3.5(1)], we see that

$$\mathrm{Dim}\Big(\bigcup_{t_1 \le t \le 1} \pi\big(\underline{F}(t)\big)\Big) = g\big(\mathbf{r}(t_1), \mathbf{r}(t_1)\big) = g(\mathbf{r}_1, \mathbf{r}_1).$$

Then $\mathrm{Dim}(D_\infty) \le g(\mathbf{r}_1, \mathbf{r}_1)$ follows from $D_\infty \subset \bigcup_{t_1 \le t \le 1} \pi(\underline{F}(t))$ by Corollary 3.3. □

CONJECTURE 3.5

For the vector $\mathbf{r}_1$ in Remark 2.5, we positively conjecture

$$0 < \dim(D_\infty) = \mathrm{Dim}(D_\infty) = g(\mathbf{r}_1, \mathbf{r}_1) < 1$$

for the integer $N \ge 3$ as we have the same result (see [6]) for $N = 2$.

4. Moments of the GRNT function

We have the recurrence relation of the GRNT function by using the arguments of [1] and [2].

THEOREM 4.1

The GRNT function f satisfies the $N-1$ equations

$$f\Big(\sum_{k=1}^{l-1} a_k + a_l x\Big) = \sum_{k=1}^{l-1} p_k + p_l f(x),$$

where $l = 1, \ldots, N$.

Proof

It follows from the definition of the GRNT function f. □

COROLLARY 4.2

For the continuous function G and the GRNT function f, we have

$$\int_{\sum_{j=1}^{k-1} a_j}^{\sum_{j=1}^{k} a_j} G(x)\, df(x) = p_k \int_0^1 G\Big(\sum_{j=1}^{k-1} a_j + a_k x\Big)\, df(x),$$

where $k = 1, \ldots, N$.

Proof

It follows from Theorem 4.1. □

COROLLARY 4.3

For the nth moment $c_n = \int_0^1 x^n\, df(x)$, we have

$$c_n = \frac{\sum_{k=2}^{N} p_k (\sum_{j=0}^{n-1} \binom{n}{j} (\sum_{i=1}^{k-1} a_i)^{n-j} a_k^j c_j)}{1 - p_1 a_1^n - \sum_{k=1}^{N} p_k a_k^n},$$

where $n = 0, 1, \ldots$.

Proof

It follows from Theorem 4.1. See also [1]. □

COROLLARY 4.4

For the fundamental interval of the form $[\gamma, \gamma + a_1^{k_1} \cdots a_N^{k_N}]$, we have

$$\int_{\gamma}^{\gamma + a_1^{k_1} \cdots a_N^{k_N}} x^i\, df(x) = p_1^{k_1} \cdots p_N^{k_N} \sum_{j=0}^{i} \binom{i}{j} [a_1^{k_1} \cdots a_N^{k_N}]^j \gamma^{i-j} c_j.$$

Proof
It follows from the formula

$$\int_{\gamma}^{\gamma+\delta} G(x)\, df(x) = p(\delta) \int_{0}^{1} G(\gamma+\delta x)\, df(x)$$

with $G(x) = x^i$ and the fundamental interval of the form $[\gamma, \gamma+\delta]$ with $p(\delta) = f(\gamma+\delta) - f(\gamma)$. $\square$

Acknowledgment. The author thanks the referee for revising this article.

References

[1] I. S. Baek, *A note on the moments of the Riesz-Nágy-Takács distribution*, J. Math. Anal. Appl. **348** (2008), 165–168. MR 2449335. DOI 10.1016/j.jmaa.2008.07.014.

[2] ———, *The moments of the Riesz-Nágy-Takács distribution over a general interval*, Bull. Korean Math. Soc. **47** (2010), 187–193. MR 2604244. DOI 10.4134/BKMS.2010.47.1.187.

[3] ———, *Derivative of the Riesz-Nágy-Takács function*, Bull. Korean Math. Soc. **48** (2011), 261–275. MR 2809129. DOI 10.4134/BKMS.2011.48.2.261.

[4] ———, *The parameter distribution set for a self-similar measure*, Bull. Korean Math. Soc. **49** (2012), 1041–1055. MR 3012971. DOI 10.4134/BKMS.2012.49.5.1041.

[5] ———, *Spectral classes and the parameter distribution set*, Commun. Korean Math. Soc. **30** (2015), 221–226. MR 3383630. DOI 10.4134/CKMS.2015.30.3.221.

[6] ———, *Sufficient condition for the differentiability points of RNT functions*, preprint.

[7] I. S. Baek and L. Olsen, *Baire category and extremely non-normal points of invariant sets of IFS's*, Discrete Contin. Dyn. Syst. **27** (2010), 935–943. MR 2629566. DOI 10.3934/dcds.2010.27.935.

[8] I. S. Baek, L. Olsen, and N. Snigireva, *Divergence points of self-similar measures and packing dimension*, Adv. Math. **214** (2007), 267–287. MR 2348031. DOI 10.1016/j.aim.2007.02.003.

[9] L. Berg and M. Krüppel, *De Rham's singular function and related functions*, Z. Anal. Anwend. **19** (2000), 227–237. MR 1748045. DOI 10.4171/ZAA/947.

[10] G. de Rham, *Sur quelques courbes definies par des équations fonctionelles*, Rend. Semin. Mat. Univ. Politec. Torino. **16** (1956/1957), 101–113. MR 0095227.

[11] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, Chichester, 1990. MR 1102677.

[12] ———, *Techniques in Fractal Geometry*, Wiley, Chichester, 1997. MR 1449135.

[13] K. Kawamura, *On the set of points where Lebesgue's singular function has the derivative zero*, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), 162–166. MR 2863359.

[14] L. Olsen and S. Winter, *Normal and non-normal points of self-similar sets and divergence points of self-similar measures*, J. Lond. Math. Soc. (2) **67** (2003), 103–122. MR 1942414. DOI 10.1112/S0024610702003630.

[15] J. Paradís, P. Viader, and L. Bibiloni, *Riesz-Nágy singular functions revisited*, J. Math. Anal. Appl. **329** (2007), 592–602. MR 2306825. DOI 10.1016/j.jmaa.2006.06.082.

[16] H. L. Royden, *Real Analysis*, 3rd ed., Macmillan, New York, 1988. MR 1013117.

Department of Mathematics, Busan University of Foreign Studies, Busan, Korea;
email: isbaek@bufs.ac.kr