Boundary limits of monotone Sobolev functions in Musielak–Orlicz spaces on uniform domains in a metric space

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Dedicated to Professor Hiroaki Aikawa on the occasion of his sixtieth birthday

Abstract Our aim in this article is to deal with boundary limits of monotone Sobolev functions in Musielak–Orlicz spaces on uniform domains in a metric space.

1. Introduction

We denote by $B(x,r)$ the open ball centered at x with radius $r>0$ and set $\lambda B(x,r)=B(x,\lambda r)$ for $\lambda>0$. A continuous function u on an open set D in the n-dimensional Euclidean space $\mathbf{R}^n$ is called *monotone* in the sense of Lebesgue (see [13]) if the equalities

$$\max_{\overline{G}} u=\max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u=\min_{\partial G} u$$

hold whenever G is a domain with compact closure $\overline{G}\subset D$. If u is a monotone function on D satisfying

$$\int_D |\nabla u(z)|^p\,dz<\infty \quad \text{for some } p>n-1,$$

then

$$|u(x)-u(y)|\le C(n,p)r^{1-n/p}\Big(\int_{2B(x,r)}|\nabla u(z)|^p\,dz\Big)^{1/p} \tag{1.1}$$

whenever $y\in B(x,r)$ with $2B(x,r)\subset D$, where $C(n,p)$ is a positive constant depending only on n and p (see [17, Chapter 8], [20, Section 16]). By using this inequality (1.1), Lindelöf theorems for monotone Sobolev functions on the half-space of $\mathbf{R}^n$ were proved in [6], as an extension of [16, Theorem 2], [14], and [15]. Tangential boundary limits of monotone Sobolev functions with finite Dirichlet integral in the half-space were studied in [16]. For Orlicz spaces, see [3]. For related results, see [7], [12], [17], and [19].

Kyoto Journal of Mathematics, Vol. 57, No. 1 (2017), 147–164
DOI 10.1215/21562261-3759549,
Received July 31, 2015. Revised December 7, 2015. Accepted February 1, 2016.
2010 Mathematics Subject Classification: Primary 31B25; Secondary 46E35.
Ohno's work partially supported by Grant-in-Aid for Young Scientists (B), No. 16K17618, Japan Society for the Promotion of Science.
Shimomura's work partially supported by Grant-in-Aid Scientific Research (C), No. 15K04929, Japan Society for the Promotion of Science.

We denote by (X, d, μ) a metric measure space, where X is a set, d is a metric on X, and μ is a Borel measure on X which is positive and finite in every ball. We write $d(x,y) = |x-y|$ for simplicity. A domain D in X with $\partial D \neq \emptyset$ is a uniform domain if there exist constants $A_1 \geq 1$ and $A_2 \geq 1$ such that each pair of points $x, y \in D$ can be joined by a rectifiable curve γ in D for which

$$\ell(\gamma) \leq A_1 |x-y|, \tag{1.2}$$

$$\delta_D(z) \geq A_2 \min\{\ell(\gamma(x,z)), \ell(\gamma(y,z))\} \quad \text{for all } z \in \gamma, \tag{1.3}$$

where $\ell(\gamma)$, $\delta_D(z)$, and $\gamma(x,z)$ denote the length of γ, the distance from z to ∂D, and the subarc of γ connecting x and z, respectively. Roughly speaking, a domain D is a uniform domain if each pair of points in D can be joined by a cigar which is not too thin or too crooked. For example, a Lipschitz domain is a uniform domain (see [18]). Lindelöf theorems for monotone Sobolev functions on uniform domains were studied in [5].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with nonstandard growth conditions. For a survey, see [2] and [4]. Let $\mathbf{B}$ be the unit ball in $\mathbf{R}^n$. Lindelöf theorems for monotone Sobolev functions in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbf{B})$ were investigated in [9]. For the two variable exponents Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{B})$, see [10]. These spaces are special cases of so-called *Musielak–Orlicz spaces*. Futamura and the authors [8] studied Lindelöf theorems for monotone Sobolev functions in variable exponent Lebesgue spaces on uniform domains in a metric space.

Our main task in this article is to establish Lindelöf-type theorems for monotone Sobolev functions in Musielak–Orlicz spaces on uniform domains in a metric space (see Theorem 2.2) as an extension of the above results. What is new about this article is that we can pass our results to the Musielak–Orlicz spaces; the technique developed in [3, Section 2] still works. We shall also show tangential boundary limits of monotone Sobolev functions in our generalized setting (see Proposition 4.1). Theorem 2.2 and Proposition 4.1 are new even for a constant exponent case.

We state definitions and results in the next section. In Section 3, we prepare some lemmas to prove our results. We prove Theorem 2.2 and Proposition 4.1 in Section 4. Throughout this article, let C denote various constants independent of the variables in question.

2. Definitions and main results

In this article, for $p_0 > 1$, we are concerned with a positive continuous function $p(\cdot)$ on X satisfying the following conditions:

(p1) $p_0 \leq p^- \equiv \inf_{x \in X} p(x) \leq p^+ \equiv \sup_{x \in X} p(x) < \infty$,
(p2) $|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x-y|)}$ for all $x, y \in X$.

If $p(\cdot)$ satisfies (p2), we say that $p(\cdot)$ *satisfies* a log-Hölder condition.

Let φ be a positive function on $X \times (0, \infty)$ such that

(φ0) $0 < \inf_{x\in X} \varphi(x,1/2)$ and $\sup_{x\in X} \varphi(x,2) < \infty$;

(φ1) $\varphi(\cdot,t)$ is measurable for all $t > 0$ and $\varphi(x,\cdot)$ is uniformly quasi-increasing:

$$\varphi(x,s) \le C_1\varphi(x,t) \quad \text{for all } x \in X \text{ whenever } 0 < s < t.$$

We assume that φ is of log type; namely, there is a constant $C_2 > 0$ such that

(φ2) $\frac{1}{C_2} \le \frac{\varphi(x,t^2)}{\varphi(x,t)} \le C_2$ for all $x \in X$ and $t > 0$.

We further assume that φ satisfies the local log-Hölder-type condition:

(φ3) $\frac{1}{C_3} \le \frac{\varphi(x,r^{-1})}{\varphi(y,r^{-1})} \le C_3$ for all $x, y \in X$ with $|x-y| < r$ and $r \le 1$.

The constants C_1–C_3 are independent of $x, y \in X$ and $t, s, r > 0$.

We see that (φ0)–(φ2) imply the uniform doubling condition:

(φ2.1) $C^{-1} \le \frac{\varphi(x,t)}{\varphi(x,s)} \le C$ for all $x \in X$ and $2^{-1}s \le t \le 2s$.

Further,

(φ2.2) $t^{\varepsilon_0}\varphi(x,t)$ is uniformly quasi-increasing on $(0,\infty)$ for every $\varepsilon_0 > 0$;

(φ2.3) $t^{-\varepsilon_1}\varphi(x,t)$ is uniformly quasidecreasing on $(0,\infty)$ for every $\varepsilon_1 > 0$

(see, e.g., [17, Chapter 5, Lemma 3.1]). If $\varphi(x,t)$ is of log type, then $\varphi(x,t^{-1})$ is also of log type.

EXAMPLE 2.1

Let $q_j(\cdot)$, $j = 1,\dots,k$, be measurable functions on X such that

(q1) $-\infty < q_j^- := \inf_{x\in X} q_j(x) \le \sup_{x\in X} q_j(x) =: q_j^+ < \infty$

for all $j = 1,\dots,k$.

Set $L^{(1)}(t) = \log(e+t)$ for $t \ge 0$ and $L^{(j+1)}(t) = L^{(1)}(L^{(j)}(t))$ inductively. Set

$$\varphi(x,t) = \prod_{j=1}^{k} \left(L^{(j)}(t)\right)^{q_j(x)}.$$

Then $\varphi(x,t)$ satisfies (φ2), and $\varphi(x,t)$ satisfies (φ1) if either

(i) $q_\ell^- > 0$ for some $1 \le \ell \le k$ and $q_j^- \ge 0$ for $j = 1,2,\dots,\ell-1$, or
(ii) $q_j^- \ge 0$ for all $j = 1,\dots,k$.

We see that $\varphi(x,t)$ satisfies (φ3) if

(q2) for each j, $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely,

$$\left|q_j(x) - q_j(y)\right| \le \frac{C_{q_j}}{L^{(j+1)}(1/|x-y|)}$$

for all $x, y \in X$ with constants $C_{q_j} > 0$.

For a function φ satisfying all the conditions (φ0)–(φ3), set

$$\Phi(x,t) = \begin{cases} t^{p(x)}\varphi(x,t) & t > 0, \\ 0 & t = 0. \end{cases}$$

We see from the assumption $p^- > 1$ in (p1) and (φ2.2) that

($\Phi 0$) $\lim_{t\to 0+} t^{-1}\Phi(x,t)=0$;
($\Phi 1$) $t\mapsto t^{-1}\Phi(x,t)$ is uniformly quasi-increasing on $(0,\infty)$.

Here note that if $\Phi(x,t)$ is convex for each $x\in X$, then ($\Phi 1$) holds; in fact, $t^{-1}\Phi(x,t)$ is nondecreasing for each $x\in X$.

Let D be a domain in X with $\partial D\neq\emptyset$. A continuous function u is called *monotone* in D (see [6]) if there exists a nonnegative function $g\in L^{p_0}_{\rm loc}(D)$ such that

$$|u(x)-u_B|\le Cr\Big(\frac{1}{\mu(\sigma B)}\int_{\sigma B} g(z)^{p_0}\,d\mu(z)\Big)^{1/p_0} \tag{2.1}$$

for every $x\in B$ with $\sigma B\subset D$, where $\sigma>1$, $B=B(y,r)$, p_0 is the constant appearing in (p1), and

$$u_B=\frac{1}{\mu(B)}\int_B u(z)\,d\mu(z).$$

In this article, following [5] and [7], we consider the boundary limits of functions u on a uniform domain D for which there exist a constant $\alpha\in\mathbf{R}$ and a nonnegative function $g\in L^{p_0}_{\rm loc}(D)$ such that

$$|u(x)-u(x')|\le Cr\Big(\frac{1}{\mu(\sigma B)}\int_{\sigma B} g(z)^{p_0}\,d\mu(z)\Big)^{1/p_0} \tag{2.2}$$

for every $x,x'\in B$ with $\sigma B\subset D$, where $\sigma>1$, $B=B(y,r)$, and

$$\int_D \Phi\big(z,g(z)\big)\delta_D(z)^\alpha\,d\mu(z)<1. \tag{2.3}$$

Note here that (2.1) implies (2.2). Let μ be a Borel measure on X satisfying the doubling condition

$$\mu(2B)\le c_d\mu(B)$$

for every ball $B\subset X$. We further assume that

$$\frac{\mu(B')}{\mu(B)}\ge C\Big(\frac{r'}{r}\Big)^s \tag{2.4}$$

for all balls $B'=B(x',r')$ and $B=B(x,r)$ with $x',x\in\overline{D}$ and $B'\subset B$, where $s>1$ (see, e.g., [11]). Here note that if μ satisfies the doubling condition, then

$$\frac{\mu(B')}{\mu(B)}\ge c_d^{-2}\Big(\frac{r'}{r}\Big)^{\log_2 c_d}$$

for all balls $B'=B(x',r')$ and $B=B(x,r)$ with $x',x\in\overline{D}$ and $B'\subset B$ (see, e.g., [1, Lemma 3.3]).

Let u be a function on D, and let $\xi\in\partial D$. For $\beta\ge 1$ and $c>0$, set

$$T_\beta(\xi;c)=\big\{x\in D: |x-\xi|^\beta\le c\delta_D(x)\big\}.$$

We say that u has a *tangential limit* of order β at ξ if the limit

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} u(x)$$

exists and is finite for every $c > 0$. In particular, a tangential limit of order 1 is called a *nontangential limit*.

Our main aim in this article is to establish the following result concerning the Lindelöf-type theorem.

THEOREM 2.2

Let u be a function on a uniform domain D with $g \ge 0$ satisfying (2.2) and (2.3), and let $\beta \ge 1$. Suppose $s+\alpha-1 < p^- \le p^+ < s+\alpha$, and set

$$E_\beta = \Big\{\xi \in \partial D : \limsup_{r\to 0} r^{\beta(p(\xi)-s-\alpha)+s}\varphi(\xi,r^{-1})^{-1}\mu\big(B(\xi,r)\big)^{-1} \times \int_{B(\xi,r)\cap D} \Phi\big(z,g(z)\big)\delta_D(z)^\alpha\, d\mu(z) > 0\Big\}.$$

If $\xi \in \partial D \setminus E_\beta$ and there exists a rectifiable curve γ in D tending to ξ along which u has a finite limit L, then u has a tangential limit L of order β at ξ.

REMARK 2.3

Let $\beta \ge 1$. Let $h_\beta(r;x) = r^{\beta(-p(x)+s+\alpha)-s}\varphi(x,r^{-1})\mu(B(x,r))$ for $x \in \partial D$ and $0 < r < \tilde r$, where $\tilde r > 0$. Assume that $h_\beta(\cdot;x)$ is nondecreasing on $(0,\tilde r)$ for each $x \in \partial D$. For $E \subset \partial D$ and $0 < r_0 < \tilde r$, let

$$H_{h_\beta}^{(r_0)}(E) = \inf\Big\{\sum_j h_\beta(r_j;x_j); E \subset \bigcup_j B(x_j,r_j), 0 < r_j \le r_0\Big\}.$$

Since $H_{h_\beta}^{(r_0)}(E)$ increases as r_0 decreases, we define the generalized Hausdorff measure with respect to h_β by

$$H_{h_\beta}(E) = \lim_{r_0\to+0} H_{h_\beta}^{(r_0)}(E).$$

Clearly, $H_{h_\beta}^{(r_0)}(E)$ and $H_{h_\beta}(E)$ are measures on X.

If g satisfies (2.3) and $p^- > s(1-1/\beta)+\alpha$, then $H_{h_\beta}(E_\beta) = 0$. In particular, if g satisfies (2.3) and $p^- > \alpha$, then $H_{h_1}(E_1) = 0$.

COROLLARY 2.4

Let $q = q_1$ be as in Example 2.1. Let u be a monotone Sobolev function on a uniform domain D in $\mathbf{R}^n$ satisfying

$$\int_D |\nabla u(z)|^{p(z)}\big(\log(e+|\nabla u(z)|)\big)^{q(z)}\delta_D(z)^\alpha\, dz < \infty. \tag{2.5}$$

Suppose $\max\{n-1, n+\alpha-1\} < p^- \le p^+ < n+\alpha$*. Set*

$$E'_\beta = \Big\{\xi \in \partial D : \limsup_{r\to 0} r^{\beta(p(\xi)-n-\alpha)}\big(\log(e+r^{-1})\big)^{-q(\xi)} \times \int_{B(\xi,r)\cap D} |\nabla u(z)|^{p(z)}\big(\log(e+|\nabla u(z)|)\big)^{q(z)}\delta_D(z)^\alpha\, dz > 0\Big\}.$$

If $\xi \in \partial D \setminus E'_\beta$ and there exists a rectifiable curve γ in D tending to ξ along which u has a finite limit L, then u has a tangential limit L of order β at ξ.

3. Preliminary lemmas

Let us begin with the following result borrowed from [9, Lemma 3].

LEMMA 3.1

Let $\{p_j\}$ be a sequence such that $p_* = \inf p_j > 1$ and $p^* = \sup p_j < \infty$. Then

$$\sum |a_j b_j| \le 2\Big(\sum |a_j|^{p_j}\Big)^{1/q}\Big(\sum |b_j|^{p_j'}\Big)^{1/q'},$$

where $1/p_j + 1/p_j' = 1$, $q = p_*$ if $\sum |a_j|^{p_j} \ge \sum |b_j|^{p_j'}$, and $q = p^*$ if $\sum |a_j|^{p_j} \le \sum |b_j|^{p_j'}$.

LEMMA 3.2 (CF. [5, LEMMA 1])

Let D be a uniform domain in X. Then for each $\xi \in \partial D$ there exists a rectifiable curve γ_ξ in D ending at ξ such that

$$\delta_D(z) \ge A_3 \ell\big(\gamma_\xi(\xi, z)\big) \tag{3.1}$$

for all $z \in \gamma_\xi$, where A_3 is a constant depending only on A_1 and A_2.

Fix $\xi \in \partial D$. For $x \in D$, set

$$r(x) = |\xi - x|.$$

Now, we give the estimate of

$$F_u(x,y) = \min\big\{\big|u(x) - u(y)\big|^{p^-}, \big|u(x) - u(y)\big|^{p^+}\big\}$$

whenever x and y can be joined by a rectifiable curve γ in D such that

$$\delta_D(z) \ge A_0 \ell\big(\gamma(x,z)\big) \qquad \text{and} \qquad \sigma B(z) \subset B\big(\xi, c_0 r(x)\big) \tag{3.2}$$

for all $z \in \gamma$, where A_0 and c_0 are positive constants, σ is the constant appearing in (2.2), and $B(z) = B(z, \delta_D(z)/(2\sigma))$.

REMARK 3.3

Let D be a uniform domain. Suppose that $x, y \in D$ satisfy

$$Q^{-1} r(x) \le r(y) \le Q r(x)$$

for some $Q \ge 1$. Here let γ be a rectifiable curve in D joining x and y and satisfying (1.2) and (1.3). Take $\zeta \in \gamma$ such that $\ell(\gamma(x,\zeta)) = \ell(\gamma(y,\zeta))$, and set $\gamma_1 = \gamma(x,\zeta)$ and $\gamma_2 = \gamma(y,\zeta)$. Then each γ_i satisfies (3.2) with $A_0 = A_2$ and $c_0 = 3(A_1(Q+1)+1)/2$.

In fact, we have by (1.3)

$$\delta_D(z) \ge A_2 \min\big\{\ell\big(\gamma(x,z)\big), \ell\big(\gamma(z,y)\big)\big\} = A_2 \ell\big(\gamma_1(x,z)\big)$$

for $z \in \gamma_1$. Take $w \in \sigma B(z)$ for $z \in \gamma_1$. Then note that

$$|w - \xi| \le |w - z| + |z - \xi| \le \frac{3}{2}|z - \xi| \le \frac{3}{2}\big(r(x) + \ell(\gamma)\big) \le \frac{3(A_1(Q+1)+1)}{2} r(x)$$

since we have by (1.2)

$$\ell(\gamma) \le A_1|x-y| \le A_1(Q+1)r(x).$$

Similarly, we have

$$\delta_D(z) \ge A_2\ell\big(\gamma_2(y,z)\big)$$

and $\sigma B(z) \subset B(\xi, c_o r(y))$ for $z \in \gamma_2$.

LEMMA 3.4 (CF. [3, LEMMA 2.2])

Let $\lambda \in \mathbf{R}$, and let $x, y \in D$. Let u be a function on D with $g \ge 0$ satisfying (2.2) and (2.3). Suppose that points x and y are joined by a rectifiable curve γ in D satisfying (3.2). Let $0 < \varepsilon < 1$.

(1) *If $p^+ < s-\lambda$, $x \in T_\beta(\xi;c)$ for some $c>0$, and $r(x) < \min\{1/c_0, A_0/c_0, 1\}$, then*

$$F_u(x,y) \le C\Big\{r(x)^{\beta(p(\xi)-s+\lambda)+s}\varphi\big(\xi, r(x)^{-1}\big)^{-1}\mu\big(B\big(\xi, r(x)\big)\big)^{-1} \times \int_{B(\xi,c_0 r(x))\cap D} \Phi\big(z, g(z)\big)\delta_D(z)^{-\lambda}\, d\mu(z) + r(x)^{p^-(1-\varepsilon)}\Big\},$$

where C may depend on ε.

(2) *If $p^- > s-\lambda$, $x \in D$, and $r(x) < \min\{1/c_0, A_0/c_0, 1\}$, then*

$$F_u(x,y) \le C\Big\{r(x)^{p(\xi)+\lambda}\varphi\big(\xi, r(x)^{-1}\big)^{-1}\mu\big(B\big(\xi, r(x)\big)\big)^{-1} \times \int_{B(\xi,c_0 r(x))\cap D} \Phi\big(z, g(z)\big)\delta_D(z)^{-\lambda}\, d\mu(z) + r(x)^{p^-(1-\varepsilon)}\Big\},$$

where C may depend on ε.

Proof

We can take a finite chain of balls $B_0, B_1, \ldots, B_N$ such that

(i) $B_j = B(x_j)$, $x_j \in \gamma$, $x_0 = x$, and $y \in B_N$;
(ii) $\ell(\gamma(x_j, x_{j+1})) \ge \delta_D(x_j)/(2\sigma)$ and $\ell(\gamma(x, x_{j+1})) > \ell(\gamma(x, x_j))$;
(iii) $B_j \cap B_k \ne \emptyset$ if and only if $|j-k| \le 1$;
(iv) $c_1\delta_D(x) \le \delta_D(x_j) \le c_0 r(x)$, where c_1 is a positive constant depending only on A_0 and σ;
(v) for each $t>0$, the number of x_j's such that $t < \delta_D(x_j) \le 2t$ is less than c_2, where c_2 is a positive constant depending only on A_0 and σ;
(vi) $\sum_{j=0}^N \chi_{B_j}(z) \le c_3$, where χ_E denotes the characteristic function of E and c_3 is a positive constant depending only on the doubling constant of μ and σ.

See [7, Lemmas 2.1 and 2.2] and [8, Lemma 2.3].

Consider the function $p_*(x_j) = \inf_{z\in\sigma B_j} p(z)$. Since $p_*(x_j) \ge p_0$, we see that

$$\big|u(\zeta_1) - u(\zeta_2)\big| \le C\delta_D(x_j)\Big(\frac{1}{\mu(\sigma B_j)}\int_{\sigma B_j} g(z)^{p_*(x_j)}\, d\mu(z)\Big)^{1/p_*(x_j)}$$

for every $\zeta_1, \zeta_2 \in B_j$. Set $G_j = \{z \in \sigma B_j : g(z) \geq \delta_D(x_j)^{-\varepsilon}\}$ for $0 < \varepsilon < 1$. Then

$$\begin{aligned}
&\int_{\sigma B_j} g(z)^{p_*(x_j)}\, d\mu(z) \\
&\quad = \int_{G_j} g(z)^{p(z)} g(z)^{p_*(x_j)-p(z)}\, d\mu(z) \\
&\qquad + \int_{\sigma B_j \setminus G_j} g(z)^{p_*(x_j)}\, d\mu(z) \\
&\quad \leq \int_{G_j} g(z)^{p(z)}\, d\mu(z) + \mu(\sigma B_j)\delta_D(x_j)^{-\varepsilon p_*(x_j)}
\end{aligned}$$

since $\delta_D(x_j) \leq c_0 r(x) < 1$ by (iv). By (φ1), (φ2), and (φ3), we have

$$\begin{aligned}
\varphi\big(z, g(z)\big)^{-1} &\leq C\varphi\big(z, \delta_D(x_j)^{-\varepsilon}\big)^{-1} \leq C(\varepsilon)\varphi\big(z, \delta_D(x_j)^{-1}\big)^{-1} \\
&\leq C(\varepsilon)\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1}
\end{aligned}$$

since $|z - x_j| \leq \delta_D(x_j)/2 \leq c_0 r(x)/2 < 1/2$ when $z \in G_j$. Hence, we obtain

$$\begin{aligned}
&\big|u(\zeta_1) - u(\zeta_2)\big| \\
&\quad \leq C\Big\{\delta_D(x_j)\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p_*(x_j)}\mu(\sigma B_j)^{-1/p_*(x_j)} \\
&\qquad \times \Big(\int_{\sigma B_j} \Phi\big(z, g(z)\big)\, d\mu(z)\Big)^{1/p_*(x_j)} + \delta_D(x_j)^{1-\varepsilon}\Big\}.
\end{aligned}$$

Here note from (2.4) that

$$\begin{aligned}
&\mu(\sigma B_j)^{-1/p_*(x_j)} \\
&\quad = \mu(\sigma B_j)^{-1/p(x_j)}\mu(\sigma B_j)^{-(p(x_j)-p_*(x_j))/(p(x_j)p_*(x_j))} \\
&\quad \leq \mu(\sigma B_j)^{-1/p(x_j)}\Big\{C\mu\big(B(\xi, c_0)\big)\Big(\frac{\delta_D(x_j)}{2c_0}\Big)^s\Big\}^{-(p(x_j)-p_*(x_j))/(p(x_j)p_*(x_j))} \\
&\quad \leq C\mu(\sigma B_j)^{-1/p(x_j)}\delta_D(x_j)^{-C/\log(1/\delta_D(x_j))} \\
&\quad \leq C\mu(\sigma B_j)^{-1/p(x_j)}
\end{aligned}$$

since $\delta_D(x_j) \leq c_0$ by (iv) and $\sigma B_j \subset B(\xi, c_0 r(x)) \subset B(\xi, c_0)$. Similarly, we have

$$C^{-1}\delta_D(x_j)^{1/p_*(x_j)} \leq \delta_D(x_j)^{1/p(x_j)} \leq C\delta_D(x_j)^{1/p_*(x_j)},$$

and by (φ2.2),

$$\begin{aligned}
&\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p_*(x_j)} \\
&\quad = \varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{1/p(x_j)-1/p_*(x_j)}\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p(x_j)} \\
&\quad \leq C\big(\delta_D(x_j)^{\varepsilon_0}\big)^{-C/\log(1/\delta_D(x_j))}\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p(x_j)} \\
&\quad \leq C\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p(x_j)}.
\end{aligned}$$

Therefore, for $\lambda \in \mathbf{R}$, we find by (2.3),

$$\begin{aligned}
&\left|u(\zeta_1) - u(\zeta_2)\right| \\
&\le C\Big\{\delta_D(x_j)\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p(x_j)}\mu(\sigma B_j)^{-1/p(x_j)} \\
&\quad \times \Big(\int_{\sigma B_j} \Phi\big(z, g(z)\big)\, d\mu(z)\Big)^{1/p(x_j)} + \delta_D(x_j)^{1-\varepsilon}\Big\} \\
&\le C\Big\{\delta_D(x_j)^{1+\lambda/p(x_j)}\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p(x_j)}\mu(\sigma B_j)^{-1/p(x_j)} \\
&\quad \times \Big(\int_{\sigma B_j} \Phi\big(z, g(z)\big)\delta_D(z)^{-\lambda}\, d\mu(z)\Big)^{1/p(x_j)} + \delta_D(x_j)^{1-\varepsilon}\Big\}
\end{aligned}$$

since $\delta_D(x_j)/2 \le \delta_D(z) \le 3\delta_D(x_j)/2$ for $z \in \sigma B_j$.

Set $p_j = p(x_j)$, and pick $z_j \in B_{j-1} \cap B_j$ for $1 \le j \le N$; set $z_0 = x$ and $z_{N+1} = y$. By the above inequality, we see that

$$\begin{aligned}
&\left|u(x) - u(y)\right| \\
&\le \sum_{j=0}^{N}\left|u(z_{j+1}) - u(z_j)\right| \\
&\le C\Big\{\sum_{j=0}^{N}\delta_D(x_j)^{1+\lambda/p_j}\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-1/p_j}\mu(\sigma B_j)^{-1/p_j} \\
&\quad \times \Big(\int_{\sigma B_j} \Phi\big(z, g(z)\big)\delta_D(z)^{-\lambda}\, d\mu(z)\Big)^{1/p_j} + \sum_{j=0}^{N}\delta_D(x_j)^{1-\varepsilon}\Big\}.
\end{aligned}$$

Taking integers k_0 and k_1 such that $2^{-k_0-1} \le c_0 r(x) < 2^{-k_0}$ and $2^{-k_1-1} \le c_1\delta_D(x) < 2^{-k_1}$, we see from (iv) and (v) that

$$\begin{aligned}
\sum_{j=0}^{N}\delta_D(x_j)^{1-\varepsilon} &\le \sum_{k=k_0}^{k_1}\Big(\sum_{2^{-k-1} \le \delta_D(x_j) < 2^{-k}}\delta_D(x_j)^{1-\varepsilon}\Big) \\
&\le c_2\sum_{k=k_0}^{k_1}(2^{-k})^{1-\varepsilon} \le C(2^{-k_0})^{1-\varepsilon} \le Cr(x)^{1-\varepsilon}.
\end{aligned}$$

Hence, we have by Lemma 3.1

$$\begin{aligned}
&\left|u(x) - u(y)\right| \\
&\le C\Big\{\Big(\sum_{j=0}^{N}\delta_D(x_j)^{p_j'(1+\lambda/p_j)}\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-p_j'/p_j}\mu(\sigma B_j)^{-p_j'/p_j}\Big)^{1/q'} \\
&\quad \times \Big(\sum_{j=0}^{N}\int_{\sigma B_j} \Phi\big(z, g(z)\big)\delta_D(z)^{-\lambda}\, d\mu(z)\Big)^{1/q} + r(x)^{1-\varepsilon}\Big\} \\
&\le C\Big\{\Big(I^{q-1}\int_{\cup\sigma B_j} \Phi\big(z, g(z)\big)\delta_D(z)^{-\lambda}\, d\mu(z)\Big)^{1/q} + r(x)^{1-\varepsilon}\Big\},
\end{aligned}$$

where q is a number in $\{\min p_j, \max p_j\}$ and

$$I = \sum_{j=0}^{N} \delta_D(x_j)^{p'_j(1+\lambda/p_j)} \varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-p'_j/p_j} \mu(\sigma B_j)^{-p'_j/p_j}.$$

Since

$$\delta_D(x_j) \ge A_0 \ell\big(\gamma(x, x_j)\big) \ge A_0 |x - x_j|$$

by (3.2), we have

$$\begin{aligned} \Big|\frac{p_j+\lambda}{p_j-1} - \frac{p(x)+\lambda}{p(x)-1}\Big| &= \Big|\frac{(\lambda+1)(p(x)-p_j)}{(p(x)-1)(p_j-1)}\Big| \\ &\le C\big|p(x)-p_j\big| \le \frac{C}{\log(1/|x-x_j|)} \le \frac{C}{\log(1/\delta_D(x_j))} \end{aligned}$$

and

$$\Big|\frac{p'_j}{p_j} - \frac{p'(x)}{p(x)}\Big| = \Big|\frac{p(x)-p_j}{(p(x)-1)(p_j-1)}\Big| \le C\big|p(x)-p_j\big| \le \frac{C}{\log(1/\delta_D(x_j))},$$

where $1/p(x) + 1/p'(x) = 1$. Therefore, we have

$$\begin{aligned} \delta_D(x_j)^{p'_j(1+\lambda/p_j)} &= \delta_D(x_j)^{\frac{p(x)+\lambda}{p(x)-1}} \delta_D(x_j)^{\frac{p_j+\lambda}{p_j-1} - \frac{p(x)+\lambda}{p(x)-1}} \\ &\le \delta_D(x_j)^{\frac{p(x)+\lambda}{p(x)-1}} \delta_D(x_j)^{-C/\log(1/\delta_D(x_j))} \\ &\le C\delta_D(x_j)^{\frac{p(x)+\lambda}{p(x)-1}}, \end{aligned}$$

since $\delta_D(x_j) \le c_0 r(x) < 1$ by (iv). Here note from ($\varphi0$), ($\varphi1$), and ($\varphi2.3$) that

$$\begin{aligned} &\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-p'_j/p_j} \\ &\quad = \varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{p'(x)/p(x) - p'_j/p_j} \varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-p'(x)/p(x)} \\ &\quad \le C\big(\delta_D(x_j)^{-\varepsilon_1}\big)^{C/\log(1/\delta_D(x_j))} \varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-p'(x)/p(x)} \\ &\quad \le C\varphi\big(x_j, \delta_D(x_j)^{-1}\big)^{-p'(x)/p(x)} \\ &\quad \le C\varphi\big(x, \delta_D(x_j)^{-1}\big)^{-p'(x)/p(x)} \end{aligned}$$

since $|x - x_j| \le \delta_D(x_j)/A_0 \le c_0 r(x)/A_0 < 1$. Further, we note from (2.4) that

$$\begin{aligned} \mu(\sigma B_j)^{-p'_j/p_j} &= \mu(\sigma B_j)^{-p'(x)/p(x)} \mu(\sigma B_j)^{-(p'_j/p_j - p'(x)/p(x))} \\ &\le \mu(\sigma B_j)^{-p'(x)/p(x)} \Big\{C\mu\big(B(\xi, c_0)\big)\Big(\frac{\delta_D(x_j)}{2c_0}\Big)^s\Big\}^{-C/\log(1/\delta_D(x_j))} \\ &\le C\mu(\sigma B_j)^{-p'(x)/p(x)} \delta_D(x_j)^{-C/\log(1/\delta_D(x_j))} \\ &\le C\mu(\sigma B_j)^{-p'(x)/p(x)} \end{aligned}$$

since $\delta_D(x_j) \le c_0$ by (iv) and $\sigma B_j \subset B(\xi, c_0 r(x)) \subset B(\xi, c_0)$. Hence, we obtain by ($\varphi$2.1),

$$
\begin{aligned}
I &\le C \sum_{j=0}^{N} \delta_D(x_j)^{(p(x)+\lambda)/(p(x)-1)} \varphi\big(x, \delta_D(x_j)^{-1}\big)^{-p'(x)/p(x)} \mu(\sigma B_j)^{-p'(x)/p(x)} \\
&\le C \sum_{j=0}^{N} \delta_D(x_j)^{(p(x)+\lambda)/(p(x)-1)} \mu\big(B\big(\xi, r(x)\big)\big)^{-p'(x)/p(x)} r(x)^{sp'(x)/p(x)} \\
&\quad \times \delta_D(x_j)^{-sp'(x)/p(x)} \varphi\big(x, \delta_D(x_j)^{-1}\big)^{-p'(x)/p(x)} \\
&= C \mu\big(B\big(\xi, r(x)\big)\big)^{-p'(x)/p(x)} r(x)^{sp'(x)/p(x)} \\
&\quad \times \sum_{j=0}^{N} \delta_D(x_j)^{(p(x)+\lambda-s)/(p(x)-1)} \varphi\big(x, \delta_D(x_j)^{-1}\big)^{-p'(x)/p(x)} \\
&\le C \Big(\mu\big(B\big(\xi, r(x)\big)\big)^{-1} r(x)^s\Big)^{\frac{1}{p(x)-1}} \int_{c_1 \delta_D(x)/2}^{2c_0 r(x)} t^{\frac{p(x)-s+\lambda}{p(x)-1}} \varphi(x, t^{-1})^{-1/(p(x)-1)} \frac{dt}{t}.
\end{aligned}
$$

First consider the case $p^+ < s - \lambda$ and $x \in T_\beta(\xi; c)$. Since $r(x)^\beta \le c\delta_D(x)$ and $|x - x_j| \le (1 + c_0) r(x)$, we see that

$$
\begin{aligned}
&\Big| \frac{(p(x) - s + \lambda)(q-1)}{p(x) - 1} - \big(p(\xi) - s + \lambda\big) \Big| \\
&\quad = \Big| \frac{(p(x) - s + \lambda)(q - p(x))}{p(x) - 1} + \big(p(x) - p(\xi)\big) \Big| \\
&\quad \le C\big|q - p(x)\big| + \big|p(x) - p(\xi)\big| \le \frac{C}{\log(1/r(x))} \le \frac{C}{\log(1/\delta_D(x))}
\end{aligned}
$$

and

$$
\Big| \frac{q-1}{p(x)-1} - 1 \Big| \le C\big|q - p(x)\big| \le \frac{C}{\log(1/r(x))} \le \frac{C}{\log(1/\delta_D(x))}.
$$

Then we have by (p2) and (φ3)

$$
\begin{aligned}
I^{q-1} &\le C \Big(\mu\big(B\big(\xi, r(x)\big)\big)^{-1} r(x)^s\Big)^{\frac{q-1}{p(x)-1}} \delta_D(x)^{(p(x)-s+\lambda)(q-1)/(p(x)-1)} \\
&\quad \times \varphi\big(x, \delta_D(x)^{-1}\big)^{-(q-1)/(p(x)-1)} \\
&\le C \mu\big(B\big(\xi, r(x)\big)\big)^{-1} r(x)^s \delta_D(x)^{p(\xi)-s+\lambda} \varphi\big(\xi, \delta_D(x)^{-1}\big)^{-1}
\end{aligned}
$$

since

$$
\Big(\frac{\mu(B(\xi, r(x)))}{\mu(B(\xi, 1))}\Big)^{-C|q-p(x)|} \le C r(x)^{-C|q-p(x)|} \le C
$$

by (2.4). Hence, we obtain by (vi) and $c^{-1} r(x)^\beta \le \delta_D(x) \le r(x)$

$$
\begin{aligned}
F_u(x, y) &\le \big|u(x) - u(y)\big|^q \\
&\le C \Big\{ \mu\big(B\big(\xi, r(x)\big)\big)^{-1} r(x)^s \delta_D(x)^{p(\xi)-s+\lambda} \varphi\big(\xi, \delta_D(x)^{-1}\big)^{-1}
\end{aligned}
$$

$$\times \int_{\cup \sigma B_j} \Phi\big(z, g(z)\big) \delta_D(z)^{-\lambda}\, d\mu(z) + r(x)^{q(1-\varepsilon)}\Big\}$$
$$\le C\Big\{\mu\big(B\big(\xi, r(x)\big)\big)^{-1} r(x)^{\beta(p(\xi)-s+\lambda)+s} \varphi\big(\xi, r(x)^{-1}\big)^{-1}$$
$$\times \int_{B(\xi, c_0 r(x)) \cap D} \Phi\big(z, g(z)\big) \delta_D(z)^{-\lambda}\, d\mu(z) + r(x)^{p^-(1-\varepsilon)}\Big\}.$$

Next consider the case $p^- > s - \lambda$. Noting that

$$\Big| \frac{(p(x)-s+\lambda)(q-1)}{p(x)-1} - \big(p(\xi)-s+\lambda\big)\Big| \le \frac{C}{\log(1/r(x))},$$

we have

$$I^{q-1} \le C\Big(\mu\big(B\big(\xi, r(x)\big)\big)^{-1} r(x)^s\Big)^{\frac{q-1}{p(x)-1}} r(x)^{(p(x)-s+\lambda)(q-1)/(p(x)-1)}$$
$$\times \varphi\big(\xi, r(x)^{-1}\big)^{-(q-1)/(p(x)-1)}$$
$$\le C\mu\big(B\big(\xi, r(x)\big)\big)^{-1} r(x)^{p(\xi)+\lambda} \varphi\big(\xi, r(x)^{-1}\big)^{-1}.$$

Thus, we can show the second part in the same manner as the first part. □

REMARK 3.5

Let $\lambda \in \mathbf{R}$, and let $x, y, w \in D$. Let u be a function on D with $g \ge 0$ satisfying (2.2) and (2.3). Let γ_1 be a rectifiable curve in D joining x and w satisfying (3.2), and let γ_2 be a rectifiable curve in D joining y and w satisfying (3.2). Let $0 < \varepsilon < 1$.

(1) If $p^+ < s - \lambda$, $x, y \in T_\beta(\xi; c)$ for some $c > 0$, and $r(x) = r(y) < \min\{1/c_0, A_0/c_0, 1\}$, then

$$F_u(x,y)$$
$$\le C\Big\{r(x)^{\beta(p(\xi)-s+\lambda)+s} \varphi\big(\xi, r(x)^{-1}\big)^{-1} \mu\big(B\big(\xi, r(x)\big)\big)^{-1}$$
$$\times \int_{B(\xi, c_0 r(x)) \cap D} \Phi\big(z, g(z)\big) \delta_D(z)^{-\lambda}\, d\mu(z) + r(x)^{p^-(1-\varepsilon)}\Big\}.$$

(2) If $p^- > s - \lambda$, $x, y \in D$, and $r(x) = r(y) < \min\{1/c_0, A_0/c_0, 1\}$, then

$$F_u(x,y)$$
$$\le C\Big\{r(x)^{p(\xi)+\lambda} \varphi\big(\xi, r(x)^{-1}\big)^{-1} \mu\big(B\big(\xi, r(x)\big)\big)^{-1}$$
$$\times \int_{B(\xi, c_0 r(x)) \cap D} \Phi\big(z, g(z)\big) \delta_D(z)^{-\lambda}\, d\mu(z) + r(x)^{p^-(1-\varepsilon)}\Big\}.$$

REMARK 3.6

In Lemma 3.4, we can replace

$$\int_{B(\xi, c_0 r(x)) \cap D} \Phi\big(z, g(z)\big) \delta_D(z)^{-\lambda}\, d\mu(z)$$

by

$$\int_{B(\xi,c_0r(x))\cap D}\Phi\big(z,g(z)\big)\delta_D(z)^\alpha\big|r(x)-|z-\xi|\big|^{-\lambda-\alpha}\,d\mu(z)$$

if $\alpha+\lambda>0$ (see [8, Remark 2.5]). Here note that

$$\begin{aligned}\big|r(x)-|z-\xi|\big| &\le |x-z|\le|x-x_j|+|x_j-z|\le\ell\big(\gamma(x,x_j)\big)+\frac{\delta_D(x_j)}{2}\\&\le\Big(A_0+\frac{1}{2}\Big)\delta_D(x_j)\end{aligned}$$

and $\delta_D(x_j)\le 2\delta_D(z)$ for $z\in\sigma B_j$.

REMARK 3.7

The number of balls $B_0,B_1,\ldots,B_N$ in Lemma 3.4 is less than (see [8, Remark 2.6])

$$c_2\Big(\log_2\frac{c_0r(x)}{c_1\delta_D(x)}+2\Big).$$

In fact,

$$\begin{aligned}N+1&=\sum_{k=k_0}^{k_1}\#\big\{j:2^{-k-1}\le\delta_D(x_j)<2^{-k}\big\}\\&\le\sum_{k=k_0}^{k_1}c_2=c_2(k_1-k_0+1)\le c_2\Big(\log_2\frac{c_0r(x)}{c_1\delta_D(x)}+2\Big),\end{aligned}$$

where we take k_0 and k_1 as in the proof of Lemma 3.4.

LEMMA 3.8 (CF. [8, LEMMA 2.7])

Let u be a function on a uniform domain D with $g\ge0$ satisfying (2.2) and (2.3). If $\xi\in\partial D\setminus E_1$ and there exist a rectifiable curve γ_ξ in D ending at ξ satisfying (3.1) and a sequence $\{y_j\}$ such that $y_j\in\gamma_\xi$, $2^{-j-1}\le|\xi-y_j|<2^{-j}$, and $u(y_j)$ has a finite limit L, then u has a nontangential limit L at ξ.

Proof

Fix $\xi\in\partial D\setminus E_1$. Take $x_j\in T_1(\xi;c)$ with $2^{-j-1}\le|x_j-\xi|<2^{-j}$. Let γ be a rectifiable curve in D joining x_j and y_j satisfying (1.2) and (1.3). Take $y\in\gamma$ such that $\ell(\gamma(x_j,y))=\ell(\gamma(y_j,y))$, and set $\gamma_1=\gamma(x_j,y)$ and $\gamma_2=\gamma(y_j,y)$. Then each γ_i satisfies (3.2) with $A_0=A_2$ and $c_0=3(3A_1+1)/2$ by Remark 3.3.

Then, for γ_i, we can take a finite chain of balls $B_0^i,B_1^i,\ldots,B_{N_i}^i$ with $B_k^i=B(w_k^i)$ as in the proof of Lemma 3.4. By Remark 3.7, we note that N_i is less than a positive constant C_1, since

$$\frac{r(x_j)}{\delta_D(x_j)}\le\frac{cr(x_j)}{|x_j-\xi|}=c$$

and

$$\frac{r(y_j)}{\delta_D(y_j)} \le \frac{r(y_j)}{A_3|\xi - y_j|} = \frac{1}{A_3}$$

by (3.1). Furthermore, we note from the proof of [8, Lemma 2.7] that

$$C^{-1}2^{-j} \le \delta_D(w_k^i) \le C2^{-j}$$

and

$$C^{-1}|w_k^i - \xi| \le \delta_D(w_k^i) \le |w_k^i - \xi|.$$

Hence, we obtain by (3.3) and (vi) in the proof of Lemma 3.4 that

$$\begin{aligned}
&\big|u(x_j) - u(y_j)\big| \\
&\quad \le \big|u(x_j) - u(y)\big| + \big|u(y_j) - u(y)\big| \\
&\quad \le C\Big\{\sum_{i=1}^{2}\sum_{k=0}^{N_i} \delta_D(w_k^i)^{1-\alpha/p(w_k^i)}\varphi\big(w_k^i, \delta_D(w_k^i)^{-1}\big)^{-1/p(w_k^i)}\mu(\sigma B_k^i)^{-1/p(w_k^i)} \\
&\qquad \times \Big(\int_{\sigma B_k^i} \Phi\big(z, g(z)\big)\delta_D(z)^{\alpha}\, d\mu(z)\Big)^{1/p(w_k^i)} + \sum_{i=1}^{2}\sum_{k=0}^{N_i}\delta_D(w_k^i)\Big\} \\
&\quad \le C\Big\{\sum_{i=1}^{2}\sum_{k=0}^{N_i}\Big(\delta_D(w_k^i)^{p(\xi)-\alpha}\mu(\sigma B_k^i)^{-1}\varphi\big(\xi, \delta_D(w_k^i)^{-1}\big)^{-1} \\
&\qquad \times \int_{\sigma B_k^i} \Phi\big(z, g(z)\big)\delta_D(z)^{\alpha}\, d\mu(z)\Big)^{1/p(w_k^i)} + \sum_{i=1}^{2}\sum_{k=0}^{N_i}\delta_D(w_k^i)\Big\} \\
&\quad \le C\Big\{2^{-j} + \big(2^{-j(p(\xi)-\alpha)}\mu\big(B(\xi, 2^{-j})\big)^{-1}\varphi(\xi, 2^{j})^{-1} \\
&\qquad \times \int_{B(\xi, c_0 2^{-j})} \Phi\big(z, g(z)\big)\delta_D(z)^{\alpha}\, d\mu(z)\big)^{1/p^+}\Big\}.
\end{aligned}$$

Since $\xi \in D \setminus E_1$ and $\lim_{j\to\infty} u(y_j) = L$, u has a nontangential limit L at ξ. □

4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. First, we show the following proposition as an extension of [16, Theorem 4], [8, Theorem 1.1], [10, Theorem 1.1], and [3, Remark 3.1].

PROPOSITION 4.1

Let u be a function on a uniform domain D with $g \ge 0$ satisfying (2.2) and (2.3), and let $\beta \ge 1$. Suppose $p^+ < s + \alpha$. If $\xi \in \partial D \setminus E_\beta$ and there exists a rectifiable curve γ in $T_\beta(\xi; \tilde{c})$ tending to ξ along which u has a finite limit L for some $\tilde{c} > 0$, then u has a tangential limit L of order β at ξ.

Proof
It is sufficient to prove

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} u(x) = L$$

for every $c \geq \tilde{c}$. Let $c \geq \tilde{c}$. We may assume that, for each $x \in T_\beta(\xi;c)$, there exists a point $y(x) \in \gamma$ such that $r(x) = r(y(x)) < \min\{1/c_0, A_0/c_0, 1\}$ since $T_\beta(\xi;\tilde{c}) \subset T_\beta(\xi;c)$. As in the proof of Lemma 3.8, let γ_0 be a rectifiable curve in D joining x and $y(x)$ satisfying (1.2) and (1.3). Take $w \in \gamma_0$ such that $\ell(\gamma_0(x,w)) = \ell(\gamma_0(y(x),w))$, and set $\gamma_1 = \gamma_0(x,w)$ and $\gamma_2 = \gamma_0(y(x),w)$. Here note that γ_1 and γ_2 satisfy (3.2). Since $\xi \notin E_\beta$, we have by Lemma 3.4(1) with $\lambda = -\alpha$ and Remark 3.5

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} F_u\big(x, y(x)\big) = 0,$$

so that

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} \big|u(x) - u\big(y(x)\big)\big| = 0.$$

Since $\lim_{x\to\xi} u(y(x)) = L$ by our assumption,

$$\lim_{T_\beta(\xi;c)\ni x\to\xi} u(x) = L,$$

as required. □

COROLLARY 4.2
Let $q = q_1$ be as in Example 2.1. Let u be a monotone Sobolev function on a uniform domain D in $\mathbf{R}^n$ satisfying (2.5). Suppose $n-1 < p^- \leq p^+ < n+\alpha$. If $\xi \in \partial D \setminus E'_\beta$ and there exists a rectifiable curve γ in $T_\beta(\xi;\tilde{c})$ tending to ξ along which u has a finite limit L for some $\tilde{c} > 0$, then u has a tangential limit L of order β at ξ.

Next we give the following result concerning the Lindelöf-type theorem as an extension of [3], [5], [6], [14]–[16] in the constant exponent case and the authors [9, Theorem], [10, Theorem 1.2], and [8, Theorem 1.2] in the variable exponent case.

PROPOSITION 4.3
Let u be a function on a uniform domain D with $g \geq 0$ satisfying (2.2) and (2.3). Suppose $p^- > s+\alpha-1$. If $\xi \in \partial D \setminus E_1$ and there exists a rectifiable curve γ in D tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

Proof
Take $\lambda \in \mathbf{R}$ such that $\max\{s+\alpha-p^-, 0\} < \lambda+\alpha < 1$. Let γ_ξ be as in Lemma 3.2. For $r > 0$ sufficiently small, take $x(r) \in \gamma \cap \partial B(\xi,r)$ and $y(r) \in \gamma_\xi \cap \partial B(\xi,r)$. Let γ_0 be a rectifiable curve in D joining $x(r)$ and $y(r)$ satisfying (1.2) and (1.3).

Take $w \in \gamma_0$ such that $\ell(\gamma_0(x(r),w)) = \ell(\gamma_0(y(r),w))$, and set $\gamma_1 = \gamma_0(x(r),w)$ and $\gamma_2 = \gamma_0(y(r),w)$. Here note that γ_1 and γ_2 satisfy (3.2). By Lemma 3.4(2) and Remarks 3.5 and 3.6, we have

$$
\begin{aligned}
&F_u\big(x(r),y(r)\big)\\
&\quad\le C\Big\{r^{p(\xi)+\lambda}\varphi(\xi,r^{-1})^{-1}\mu\big(B(\xi,r)\big)^{-1}\\
&\qquad\times\int_{B(\xi,c_0r)\cap D}\Phi\big(z,g(z)\big)\delta_D(z)^{\alpha}\big|r-|z-\xi|\big|^{-\lambda-\alpha}\,d\mu(z)+r^{p^-(1-\varepsilon)}\Big\}.
\end{aligned}
$$

Moreover, since $0<\lambda+\alpha<1$, we see that

$$
\int_{2^{-j-1}}^{2^{-j}}\big|r-|z-\xi|\big|^{-\lambda-\alpha}\,dr\le C2^{-j(1-\lambda-\alpha)}.
$$

Hence, it follows that

$$
\begin{aligned}
&\inf_{2^{-j-1}\le r<2^{-j}}F_u\big(x(r),y(r)\big)\\
&\quad\le C\Big\{\int_{2^{-j-1}}^{2^{-j}}r^{p(\xi)+\lambda}\varphi(\xi,r^{-1})^{-1}\mu\big(B(\xi,r)\big)^{-1}\\
&\qquad\times\Big(\int_{B(\xi,c_0r)\cap D}\Phi\big(z,g(z)\big)\delta_D(z)^{\alpha}\big|r-|z-\xi|\big|^{-\lambda-\alpha}\,d\mu(z)\Big)\frac{dr}{r}\\
&\qquad+(2^{-j})^{p^-(1-\varepsilon)}\Big\}\\
&\quad\le C\Big\{2^{-j\{p(\xi)+\lambda-1\}}\varphi(\xi,2^{j})^{-1}\mu\big(B(\xi,2^{-j})\big)^{-1}\\
&\qquad\times\int_{B(\xi,c_02^{-j})\cap D}\Phi\big(z,g(z)\big)\delta_D(z)^{\alpha}\Big(\int_{2^{-j-1}}^{2^{-j}}\big|r-|z-\xi|\big|^{-\lambda-\alpha}\,dr\Big)\,d\mu(z)\\
&\qquad+(2^{-j})^{p^-(1-\varepsilon)}\Big\}\\
&\quad\le C\Big\{2^{-j\{p(\xi)-\alpha\}}\varphi(\xi,2^{j})^{-1}\mu\big(B(\xi,2^{-j})\big)^{-1}\\
&\qquad\times\int_{B(\xi,c_02^{-j})\cap D}\Phi\big(z,g(z)\big)\delta_D(z)^{\alpha}\,d\mu(z)+(2^{-j})^{p^-(1-\varepsilon)}\Big\}.
\end{aligned}
$$

Since $\xi\notin E_1$, we see that

$$
\lim_{j\to\infty}\inf_{2^{-j-1}\le r<2^{-j}}F_u\big(x(r),y(r)\big)=0.
$$

Hence, we find a sequence $\{r_j\}$ such that $2^{-j-1}\le r_j<2^{-j}$ and

$$
\lim_{j\to\infty}F_u\big(x(r_j),y(r_j)\big)=0.
$$

Since u has a finite limit L at ξ along γ, we have

$$
\lim_{j\to\infty}u\big(y(r_j)\big)=\lim_{j\to\infty}u\big(x(r_j)\big)=L.
$$

Thus, u has a nontangential limit L at ξ by Lemma 3.8. □

COROLLARY 4.4

Let $q = q_1$ be as in Example 2.1. Let u be a monotone Sobolev function on a uniform domain D in $\mathbf{R}^n$ satisfying (2.5). Suppose $p^- > \max\{n-1, n+\alpha-1\}$. If $\xi \in \partial D \setminus E_1'$ and there exists a rectifiable curve γ in D tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

REMARK 4.5

In Proposition 4.1, unlike Theorem 2.2, it is necessary for the rectifiable curve γ to be included in $T_\beta(\xi; \tilde{c})$. On the other hand, in Proposition 4.3, we only show that u has a nontangential limit L at ξ.

Proof of Theorem 2.2

Since $T_1(\xi; c) \cap B(\xi, 1) \subset T_\beta(\xi; c) \cap B(\xi, 1)$ and $E_1 \subset E_\beta$ for all $\beta \geq 1$ and $c > 0$, we obtain the required result by Propositions 4.1 and 4.3. □

Acknowledgment

We would like to express our thanks to the referee for his/her kind comments and suggestions.

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