Triangulation extensions of self-homeomorphisms of the real line

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Abstract For every sense-preserving self-homeomorphism of the real axis, Hubbard constructed an extension that is a self-homeomorphism of the upper half-plane by triangulation. It is natural to ask if such extensions of quasisymmetric homeomorphisms of the real axis are all quasiconformal. Furthermore, for what sense-preserving self-homeomorphisms are such extensions David mappings? In this article, a sufficient and necessary condition for such extensions to be quasiconformal and a sufficient condition for such extensions to be David mappings are given.

1. Introduction

It is known that every quasiconformal self-homeomorphism f of the upper half-plane $\mathbb{H}$ can be extended to a self-homeomorphism $\tilde{f}$ of $\overline{\mathbb{H}}$. The restriction $h = \tilde{f}|_{\mathbb{R}}$ of $\tilde{f}$ to the real line $\mathbb{R}$ is a self-homeomorphism of $\mathbb{R}$, which is called the boundary value of f. The boundary value h of a quasiconformal self-homeomorphism f of $\mathbb{H}$ with $f(\infty) = \infty$ is quasisymmetric; that is, h is an increasing function of $\mathbb{R}$ onto itself and satisfies the M-condition

$$\frac{1}{M} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M, \quad \forall x \in \mathbb{R} \text{ and } t > 0,$$

for some constant $M \geq 1$. Conversely, every quasisymmetric self-homeomorphism of $\mathbb{R}$ can be extended to a quasiconformal homeomorphism of $\mathbb{H}$ by the well-known Beurling–Ahlfors extension (see [1]).

The quasiconformal extension of a quasisymmetric self-homeomorphism has been extensively studied. It is known that a quasisymmetric self-homeomorphism of $\mathbb{R}$ can be extended to a quasiconformal self-homeomorphism of $\mathbb{H}$ by the Beurling–Ahlfors extension, the Douady–Earle extension (see [5]), and the Carleson box extension (see [2]).

The boundary value problem has been recently studied for trans-quasiconformal mappings, which are also called $\mu(z)$-homeomorphisms (see [7], [3]). A homeomorphic solution in the Sobolev class $W^{1,1}_{\mathrm{loc}}$ of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}$$

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is called a *trans-quasiconformal mapping* if μ is a measurable function with $|\mu(z)|<1$ almost everywhere in $\mathbb{C}$ and $\|\mu\|_\infty=1$. Trans-quasiconformal mappings are quite different from quasiconformal mappings. Indeed, the homeomorphic solution of the Beltrami equation, for a measurable function μ with $|\mu|<1$ almost everywhere and $\|\mu\|_\infty=1$, does not always exist. In 1988, David [3] obtained a sufficient condition for the existence of the solution. These homeomorphic solutions are now called *David mappings*.

David mappings can be equivalently defined (see [9]) as functions $f\in W^{1,1}_{\mathbf{loc}}(U)$ ($U\subset\hat{\mathbb{C}}$) with

$$\iint_U \exp\big(pK_f(z)\big)\frac{1}{(1+x^2+y^2)^2}\,dx\,dy<\infty$$

for some $p>0$, where

$$K_f(z)=\frac{1+|\mu_f(z)|}{1-|\mu_f(z)|}.$$

Zakeri [9] and de Faria [4] studied when a sense-preserving self-homeomorphism of $\mathbb{R}$ can be extended to a David self-map of $\mathbb{H}$ by the Beurling–Ahlfors extension and Carleson box extension, respectively. Zakeri introduced the concept of scalewise distortion. Let $f(x)$ be the self-homeomorphism of the real axis $\mathbb{R}$, and let

$$\delta_f(x,t)=\max\left\{\frac{f(x+t)-f(x)}{f(x)-f(x-t)},\frac{f(x)-f(x-t)}{f(x+t)-f(x)}\right\}. \tag{1}$$

If

$$\rho_f(t)=\sup_{x\in\mathbb{R}}\delta_f(x,t) \tag{2}$$

exists, then $\rho_f(t)$ is called the *scalewise distortion* of $f(x)$. Then he proved the following theorem.

THEOREM A

Let $f(x)$ be a sense-preserving homeomorphism of the real axis $\mathbb{R}$, and let $f(x+1)=f(x)+1$. If $\rho_f(t)$ satisfies

$$\rho_f(t)=O\Big(\log\frac{1}{t}\Big),\quad t\to0^+,$$

then f can be extended to be a David mapping F of the upper half-plane $\mathbb{H}$.

Hubbard [6, pp. 172–175] proposed a method to extend a self-homeomorphism of $\mathbb{R}$ to $\mathbb{H}$ via triangulations of bands of $\mathbb{H}$ (see Section 2). Such extensions are called *triangulation extensions* here.

It is clear that every sense-preserving self-homeomorphism of $\mathbb{R}$ can be extended to a sense-preserving self-homeomorphism of $\mathbb{H}$ by triangulation extension. Since every quasisymmetric self-homeomorphism of $\mathbb{R}$ can be extended to a quasiconformal self-homeomorphism of $\mathbb{H}$ by either the Beurling–Ahlfors extension, Douady–Earle extension, or Carleson box extension, one may naturally ask whether the triangulation extension of a quasisymmetric self-homeomorphism of $\mathbb{R}$ is quasiconformal. Furthermore, one may ask when the triangulation extension of a sense-preserving self-homeomorphism of $\mathbb{R}$ can be extended to a David mapping as Zakeri [9] and de Faria [4] did for the Beurling–Ahlfors and Carleson box extensions.

Unfortunately, the answer to the first problem is not always true. Indeed, we prove the following theorem in this article.

THEOREM 1

Let $f(x)$ be a sense-preserving homeomorphism of the real axis $\mathbb{R}$. If $f(x)$ is a quasisymmetric mapping, then $f(x)$ can be extended to be a quasiconformal mapping of the upper half-plane $\mathbb{H}$ by triangulation extension if and only if $f(x)$ satisfies

$$0 < \inf_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big) \le \sup_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big) < \infty.$$

For the second problem, we give the following sufficient condition for the triangulation extension of a sense-preserving self-homeomorphism of $\mathbb{R}$ to be a David self-homeomorphism of $\mathbb{H}$.

THEOREM 2

Let $f(x)$ be a sense-preserving homeomorphism of the real axis $\mathbb{R}$. If

$$0 < \inf_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big) \le \sup_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big) < \infty$$

and $\rho_f(t)$ satisfies

$$\rho_f(t) = O\Big(\log^{1/4}\Big(\frac{1}{t}\Big)\Big), \quad t \to 0^+,$$

then $f(x)$ can be extended to be a David mapping of the upper half-plane $\mathbb{H}$ by triangulation extension.

The article is arranged as follows. We give a detailed definition of the triangulation extension of a self-homeomorphism of $\mathbb{R}$ in Section 2 along with some analysis. Then we prove Theorems 1 and 2 in Section 3.

2. Triangulation extension

To give the triangulation extension to $\mathbb{H}$ of a sense-preserving self-homeomorphism f of $\mathbb{R}$, we divide the upper half-plane $\mathbb{H}$ into bands by horizontal lines $y = \frac{1}{2^n}$ $(n = 1, 2, \dots)$, denote the region between line $y = \frac{1}{2^n}$ and $y = \frac{1}{2^{n+1}}$ as band n, and denote the region $y > \frac{1}{2}$ as band 0. Then we triangulate each band as follows.

We first triangulate band 0. Draw lines on $\mathbb{H}$ with slope 1 from the points $k + \frac{j}{2}$ $(k \in \mathbb{Z}$ and $j = 0, 1)$ on the real line until they meet on the line $y = \frac{1}{2}$. Denote the intersection points by $A^0_{k,j}$. It is clear that $A^0_{k,1}$ is the middle point

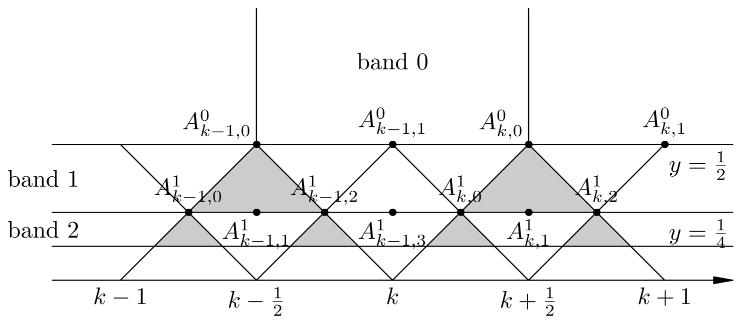

Figure 1

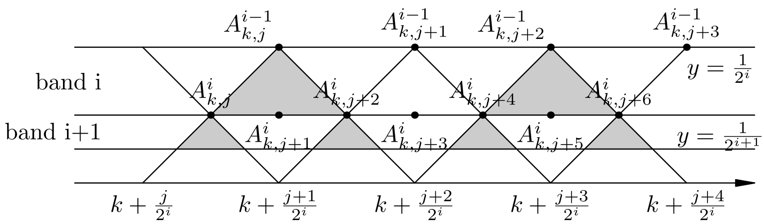

Figure 2

of segment $A^0_{k,0}A^0_{k+1,0}$. Draw vertical lines up from points $A^0_{k,0}$. These vertical lines and line segments $A^0_{k,0}A^0_{k+1,0}$ give a triangulation of band 0. Every triangle $\Delta A^0_{k,0}A^0_{k+1,0}\infty$ $(k\in\mathbb{Z})$ in the triangulation has a horizontal side and two vertical sides with a vertex at infinity.

Draw lines on $\mathbb{H}$ with slope 1 from points $k+\frac{j}{2^2}$ $(k\in\mathbb{Z},\ j=0,1,2,2^2-1)$ on the real line until they meet on the line $y=\frac{1}{2^2}$. Denote these intersection points by $A^1_{k,j}$. Then we get a triangulation of band 1 by using the points $A^0_{k,j}$ on the line $y=\frac{1}{2}$ and the points $A^1_{k,2j}$ on the line $y=\frac{1}{2^2}$ $(k\in\mathbb{Z}$ and $j=0,1)$ as the vertices of triangles of the triangulation (see Figure 1). The triangles can be listed as follows:

$$\Delta A^0_{k-1,1}A^1_{k,0}A^0_{k,0},\qquad \Delta A^0_{k,0}A^1_{k,0}A^1_{k,2},\qquad \Delta A^0_{k,0}A^1_{k,2}A^0_{k,1},\qquad \Delta A^0_{k,1}A^1_{k,2}A^1_{k+1,0},$$

where $k\in\mathbb{Z}$.

Generally, draw lines on $\mathbb{H}$ with slope 1 from points $k+\frac{j}{2^i}$ $(k\in\mathbb{Z},\ j=0,1,2,\dots,2^i-1)$ on the real line until they meet on the line $y=\frac{1}{2^i}$. Denote these intersection points by $A^{i-1}_{k,j}$. Then we get a triangulation of band i by using the points $A^{i-1}_{k,j}$ on the line $y=\frac{1}{2^i}$ and the points $A^i_{k,2j}$ on the line $y=\frac{1}{2^{i+1}}$ $(k\in\mathbb{Z}$ and $j=0,1,\dots,2^i-1)$ as the vertices of triangles of the triangulation (see Figure 2). The triangles can be listed as follows:

$$\Delta A^{i-1}_{k-1,2^i-1}A^i_{k,0}A^{i-1}_{k,0},$$
$$\Delta A^{i-1}_{k,0}A^i_{k,0}A^i_{k,2},\qquad \Delta A^{i-1}_{k,0}A^i_{k,2}A^{i-1}_{k,1},$$

$$\vdots$$

$$\Delta A^{i-1}_{k,j} A^{i}_{k,2j} A^{i}_{k,2(j+1)}, \qquad \Delta A^{i-1}_{k,j} A^{i}_{k,2(j+1)} A^{i-1}_{k,j+1},$$

$$\vdots$$

$$\Delta A^{i-1}_{k,2^i-2} A^{i}_{k,2(2^i-2)} A^{i}_{k,2(2^i-1)}, \qquad \Delta A^{i-1}_{k,2^i-2} A^{i}_{k,2(2^i-1)} A^{i-1}_{k,2^i-1},$$

$$\Delta A^{i-1}_{k,2^i-1} A^{i}_{k,2(2^i-1)} A^{i}_{k+1,0},$$

where $k \in \mathbb{Z}$.

It is not difficult to verify that $A^{i-1}_{k,2j}$ ($i \in \mathbb{N}$, $k \in \mathbb{Z}$, and $j = 0, 1, 2, \ldots, 2^{i-1}-1$) is the intersection point of the line with slope 1 from point $k + \frac{2j}{2^i}$ and the line with slope -1 from point $k + \frac{2j+2}{2^i}$, that $A^{i-1}_{k,2j+1}$ ($i \in \mathbb{N}$, $k \in \mathbb{Z}$, and $j = 0, 1, 2, \ldots, 2^{i-1}-2$) is the middle point of segment $A^{i-1}_{k,2j} A^{i-1}_{k,2(j+1)}$, and that $A^{i-1}_{k,2^i-1}$ is the middle point of segment $A^{i-1}_{k,2^i-2} A^{i-1}_{k+1,0}$.

To give the triangulation extension of f, we also divide $\mathbb{H}$ into distorted bands with triangulations corresponding to the above bands and their triangulations.

For every $k \in \mathbb{Z}$, draw a line in $\mathbb{H}$ with slope 1 from point $f(k)$ on the real line until it meets the line with slope -1 from point $f(k+1)$. Let $B^0_{k,0}$ be such an intersection point, and denote the middle point of segment $B^0_{k,0} B^0_{k+1,0}$ by $B^0_{k,1}$ for $k \in \mathbb{Z}$. The domain over the zigzag line $\cdots B^0_{k,0} B^0_{k,1} B^0_{k+1,0} \cdots$ is the distorted band 0. By drawing vertical lines from points $B^0_{k,0}$, we get a triangulation of the distorted band 0 with triangles $\Delta B^0_{k,0} B^0_{k+1,0} \infty$ (see Figure 3).

Then draw lines in $\mathbb{H}$ with slopes ± 1 from points $f(k + \frac{1}{2})$ on the real line until they meet the line with slope 1 from point $f(k)$ and the line with slope -1 from point $f(k+1)$, and denote by $B^1_{k,0}$ and $B^1_{k,2}$ the intersection points, respectively. Connect adjacent points by line segments, and denote by $B^1_{k,1}$ the middle point of the segment $B^1_{k,0} B^1_{k,2}$. Then the domain between the zigzag line $\cdots B^1_{k,0} B^1_{k,1} B^1_{k,2} \cdots$ and the zigzag line $\cdots B^0_{k,0} B^0_{k,1} B^0_{k+1,0} \cdots$ is the distorted

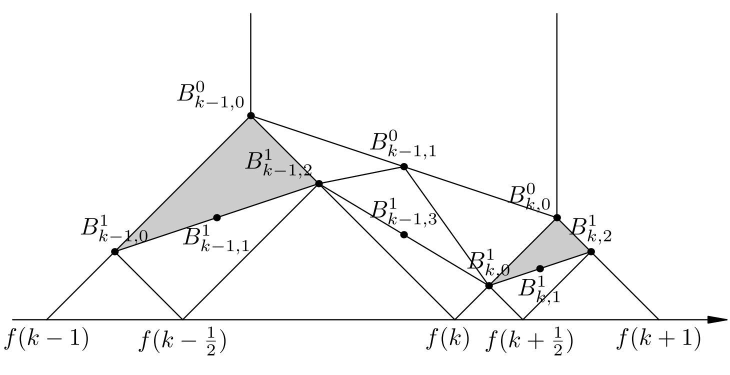

Figure 3

band 1. The distorted band 1 is triangulated by triangles

$$\Delta B^0_{k-1,1}B^1_{k,0}B^0_{k,0}, \qquad \Delta B^0_{k,0}B^1_{k,0}B^1_{k,2}, \qquad \Delta B^0_{k,0}B^1_{k,2}B^0_{k,1}, \qquad \Delta B^0_{k,1}B^1_{k,2}B^1_{k+1,0},$$

where $k \in \mathbb{Z}$ (see Figure 3).

Generally, let $B^{i-1}_{k,2j}$ ($i \in \mathbb{N}$, $k \in \mathbb{Z}$, and $j = 0, 1, 2, \ldots, 2^{i-1}-1$) be the intersection point of the line with slope 1 from point $f(k + \frac{2j}{2^i})$ and the line with slope -1 from point $f(k + \frac{2j+2}{2^i})$, let $B^{i-1}_{k,2j+1}$ ($i \in \mathbb{N}$, $k \in \mathbb{Z}$, and $j = 0, 1, 2, \ldots, 2^{i-1}-2$) be the middle point of segment $B^{i-1}_{k,2j}B^{i-1}_{k,2(j+1)}$, and let $B^{i-1}_{k,2^i-1}$ be the middle point of segment $B^{i-1}_{k,2^i-2}B^{i-1}_{k+1,0}$. Then the distorted band i is the domain between the zigzag lines $\cdots B^{i-1}_{k,2j}B^{i-1}_{k,2j+1}B^{i-1}_{k,2j+2}\cdots$ and $\cdots B^{i}_{k,2j}B^{i}_{k,2j+1}B^{i}_{k,2j+2}\cdots$. The distorted band i is triangulated by triangles

$$\begin{aligned}
&\Delta B^{i-1}_{k-1,2^i-1}B^i_{k,0}B^{i-1}_{k,0},\\
&\Delta B^{i-1}_{k,0}B^i_{k,0}B^i_{k,2}, \qquad \Delta B^{i-1}_{k,0}B^i_{k,2}B^{i-1}_{k,1},\\
&\vdots\\
&\Delta B^{i-1}_{k,j}B^i_{k,2j}B^i_{k,2(j+1)}, \qquad \Delta B^{i-1}_{k,j}B^i_{k,2(j+1)}B^{i-1}_{k,j+1},\\
&\vdots\\
&\Delta B^{i-1}_{k,2^i-2}B^i_{k,2(2^i-2)}B^i_{k,2(2^i-1)}, \qquad \Delta B^{i-1}_{k,2^i-2}B^i_{k,2(2^i-1)}B^{i-1}_{k,2^i-1},\\
&\Delta B^{i-1}_{k,2^i-1}B^i_{k,2(2^i-1)}B^i_{k+1,0},
\end{aligned}$$

where $k \in \mathbb{Z}$.

The triangulation extension F of f is defined as follows. The extension F on $\Delta A^0_{k,0}A^0_{k+1,0}\infty$ is defined as a mapping from $\Delta A^0_{k,0}A^0_{k+1,0}\infty$ onto $\Delta B^0_{k,0}B^0_{k+1,0}\infty$ which is piecewise linear and an isometry on vertical lines. This completes the extension of f to the band 0. The extension F on band $i \in \mathbb{N}$ can be realized by the barycentric coordinate mappings

$$\begin{aligned}
&\Delta A^{i-1}_{k-1,2^i-1}A^i_{k,0}A^{i-1}_{k,0} \to \Delta B^{i-1}_{k-1,2^i-1}B^i_{k,0}B^{i-1}_{k,0},\\
&\Delta A^{i-1}_{k,j}A^i_{k,2j}A^i_{k,2(j+1)} \to \Delta B^{i-1}_{k,j}B^i_{k,2j}B^i_{k,2(j+1)},\\
&\Delta A^{i-1}_{k,j}A^i_{k,2(j+1)}A^{i-1}_{k,j+1} \to \Delta B^{i-1}_{k,j}B^i_{k,2(j+1)}B^{i-1}_{k,j+1},\\
&\Delta A^{i-1}_{k,2^i-1}A^i_{k,2(2^i-1)}A^i_{k+1,0} \to \Delta B^{i-1}_{k,2^i-1}B^i_{k,2(2^i-1)}B^i_{k+1,0},
\end{aligned}$$

where $k \in \mathbb{Z}$ and $j = 0, 1, \ldots, 2^i - 2$.

3. Two lemmas

In order to prove Theorems 1 and 2, we give two lemmas here.

LEMMA 1

Let f be a sense-preserving self-homeomorphism of the real axis $\mathbb{R}$, and let F be the triangulation extension of f to the upper half-plane $\mathbb{H}$. Then, for all $z \in$

$\Delta A^0_{k-1,0}A^0_{k,0}\infty$,

$$\left|\mu_F(z)\right|^2 = \frac{(f(k+1)-f(k-1)-2)^2 + (f(k+1)-2f(k)+f(k-1))^2}{(f(k+1)-f(k-1)+2)^2 + (f(k+1)-2f(k)+f(k-1))^2},$$

where μ_F is the complex dilatation of F and $k \in \mathbb{Z}$.

Proof

For every $k \in \mathbb{Z}$, $A^0_{k,0}$ and $B^0_{k,0}$ can be expressed, respectively, by the complex numbers

$$z_k = k + \frac{1}{2} + \mathrm{i}\frac{1}{2} \qquad \text{and} \qquad w_k = \frac{f(k)+f(k+1)}{2} + \mathrm{i}\frac{f(k+1)-f(k)}{2}.$$

Then one can verify that the restriction of F to $\Delta A^0_{k-1,0}A^0_{k,0}\infty$ is

$$F|_{\Delta A^0_{k-1,0}A^0_{k,0}\infty}(z) = \frac{1}{2}\Big(\frac{w_k - w_{k-1}}{z_k - z_{k-1}} + 1\Big)z + \frac{1}{2}\Big(\frac{w_k - w_{k-1}}{z_k - z_{k-1}} - 1\Big)\overline{z} + C,$$

where $C = (z_k w_{k-1} - z_{k-1}w_k)/(z_k - z_{k-1})$. The lemma follows easily. □

LEMMA 2

Let f be a sense-preserving self-homeomorphism of the real axis $\mathbb{R}$, and let F be the triangulation extension of f to the upper half-plane $\mathbb{H}$. Then the maximal dilatation K_i of $F|_{\mathrm{band}\,i}$ satisfies

$$K_i \le \rho_f^4\Big(\frac{1}{2^i}\Big), \quad \forall i \in \mathbb{N}^+.$$

Proof

It is easy to see that the complex dilatation of the barycentric coordinate mapping g between triangles $\Delta z_1 z_2 z_3$ and $\Delta w_1 w_2 w_3$ is

$$\mu_g = \begin{vmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{vmatrix} \begin{vmatrix} w_1 & \overline{z}_1 & 1 \\ w_2 & \overline{z}_2 & 1 \\ w_3 & \overline{z}_3 & 1 \end{vmatrix}^{-1}.$$

Especially, if $(z_1 - z_2)\mathrm{i} = z_2 - z_3$, we have

$$|\mu_g| = \frac{|(w_1 - w_2) + i(w_2 - w_3)|}{|(w_1 - w_2) - i(w_2 - w_3)|}. \tag{3}$$

In order to prove Lemma 2, it is sufficient to consider the following four cases for a given band i ($i \in \mathbb{N}^+$):

1. $\Delta A^{i-1}_{k,2j}A^i_{k,4j}A^i_{k,2+4j} \to \Delta B^{i-1}_{k,2j}B^i_{k,4j}B^i_{k,2+4j}$,
2. $\Delta A^{i-1}_{k,2j}A^i_{k,2+4j}A^{i-1}_{k,1+2j} \to \Delta B^{i-1}_{k,2j}B^i_{k,2+4j}B^{i-1}_{k,1+2j}$,
3. $\Delta A^{i-1}_{k,1+2j}A^i_{k,2+4j}A^i_{k,4+4j} \to \Delta B^{i-1}_{k,1+2j}B^i_{k,2+4j}B^i_{k,4+4j}$,
4. $\Delta A^{i-1}_{k,1+2j}A^i_{k,4+4j}A^{i-1}_{k,2+2j} \to \Delta B^{i-1}_{k,1+2j}B^i_{k,4+4j}B^{i-1}_{k,2+2j}$,

where $k \in \mathbb{Z}$ and $j = 0, 1, \ldots, 2^{i-1} - 1$. When $j = 2^{i-1} - 1$, we identify

$$A^{i-1}_{k,2+2j} = A^{i-1}_{k+1,0}, \qquad A^{i}_{k,4+4j} = A^{i}_{k+1,0},$$
$$B^{i-1}_{k,2+2j} = B^{i-1}_{k+1,0}, \qquad \text{and} \qquad B^{i}_{k,4+4j} = B^{i}_{k+1,0}.$$

This is another classification of the triangles in the same band, which is different from the classification in Section 2. The second classification is based on the relationship between vertices of triangles and the boundary value of $f(x)$. Consequently, it is more convenient for computing the dilatation.

We use the following notation to simplify the computations. Let

$$x_\alpha = k + \frac{2j+\alpha}{2^i} \qquad \text{and} \qquad y_\alpha = f(x_{\alpha+1}) - f(x_\alpha) \quad (\alpha = 0,1,2,3,4).$$

Therefore, the vertices $B^{i-1}_{k,2j}$, $B^{i-1}_{k,2j+1}$, $B^{i}_{k,4j}$, $B^{i}_{k,4j+2}$, and $B^{i}_{k,4j+4}$ satisfy

$$B^{i-1}_{k,2j} = \frac{f(x_0)+f(x_2)}{2} + \mathrm{i}\frac{f(x_2)-f(x_0)}{2}, \tag{4}$$

$$B^{i-1}_{k,2j+1} = \frac{f(x_4)+2f(x_2)+f(x_0)}{4} + \mathrm{i}\frac{f(x_4)-f(x_0)}{4}, \tag{5}$$

$$B^{i}_{k,4j} = \frac{f(x_0)+f(x_1)}{2} + \mathrm{i}\frac{f(x_1)-f(x_0)}{2}, \tag{6}$$

$$B^{i}_{k,4j+2} = \frac{f(x_1)+f(x_2)}{2} + \mathrm{i}\frac{f(x_2)-f(x_1)}{2}, \tag{7}$$

$$B^{i}_{k,4j+4} = \frac{f(x_3)+f(x_2)}{2} + \mathrm{i}\frac{f(x_3)-f(x_2)}{2}. \tag{8}$$

Case 1: The estimate of the complex dilatation on $\Delta A^{i-1}_{k,2j}A^{i}_{k,4j}A^{i}_{k,2+4j}$. Let $\mu^{k,j}_1$ be the complex dilatation of $F|_{\Delta A^{i-1}_{k,2j}A^{i}_{k,4j}A^{i}_{k,2+4j}}$. As $F|_{\Delta A^{i-1}_{k,2j}A^{i}_{k,4j}A^{i}_{k,2+4j}}$ is a barycentric coordinate mapping between triangles $\Delta A^{i-1}_{k,2j}A^{i}_{k,4j}A^{i}_{k,2+4j}$ and $\Delta B^{i-1}_{k,2j}B^{i}_{k,4j}B^{i}_{k,2+4j}$ (see Figure 4 for the image triangle), by (3) for $w_1 = B^{i}_{k,4j+2}$, $w_2 = B^{i-1}_{k,2j}$, and $w_3 = B^{i}_{k,4j}$, we have

$$|\mu^{k,j}_1| = \Bigg|\frac{1 - \frac{f(x_2)-f(x_1)}{f(x_1)-f(x_0)}}{1 + \frac{f(x_2)-f(x_1)}{f(x_1)-f(x_0)}}\Bigg|.$$

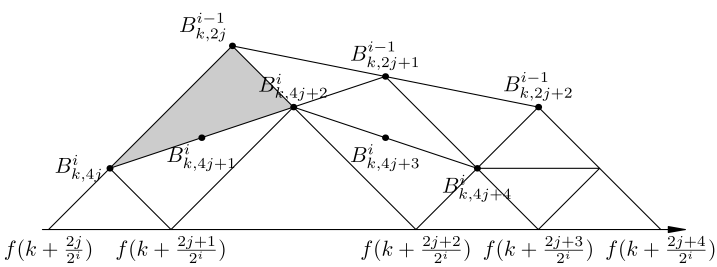

Figure 4

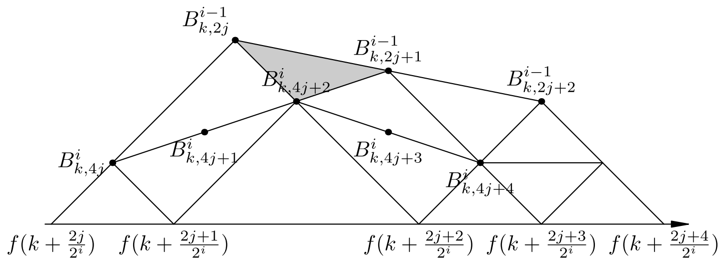

Figure 5

Consequently, by (1) and (2) we have

$$|\mu_1^{k,j}| \le \frac{\delta_f(x_1, \frac{1}{2^i}) - 1}{\delta_f(x_1, \frac{1}{2^i}) + 1} \le \frac{\rho_f(\frac{1}{2^i}) - 1}{\rho_f(\frac{1}{2^i}) + 1}.$$

Therefore,

$$K(F|_{\Delta A^{i-1}_{k,2j} A^{i}_{k,4j} A^{i}_{k,2+4j}}) \le \rho_f\Big(\frac{1}{2^i}\Big) \le \rho_f^4\Big(\frac{1}{2^i}\Big). \tag{9}$$

Case 2: The estimate of the complex dilatation on $\Delta A^{i-1}_{k,2j} A^{i}_{k,2+4j} A^{i-1}_{k,1+2j}$. Let $\mu_2^{k,j}$ be the complex dilatation of $F|_{\Delta A^{i-1}_{k,2j} A^{i}_{k,2+4j} A^{i-1}_{k,1+2j}}$. Since F is a barycentric coordinate mapping between triangles $\Delta A^{i-1}_{k,2j} A^{i}_{k,2+4j} A^{i-1}_{k,1+2j}$ and $\Delta B^{i-1}_{k,2j} B^{i}_{k,4j+2} B^{i-1}_{k,1+2j}$ (see Figure 5 for the image triangle), by using (3), (4), (5), and (7), we have

$$\begin{aligned}|\mu_2^{k,j}| &= \Big|\frac{f(x_0) - 2f(x_2) + f(x_4) + \mathrm{i}(4f(x_1) - 3f(x_0) - f(x_4))}{3f(x_0) - 4f(x_1) + 2f(x_2) - f(x_4) + \mathrm{i}(f(x_4) - f(x_0))}\Big| \\ &= \Big|\frac{y_2 + y_3 - y_0 - y_1 + \mathrm{i}(3y_0 - y_1 - y_2 - y_3)}{y_1 - 3y_0 - y_2 - y_3 + \mathrm{i}(y_0 + y_1 + y_2 + y_3)}\Big|.\end{aligned}$$

This implies

$$|\mu_2^{k,j}|^2 = 1 - \frac{16y_0(y_2 + y_3)}{(3y_0 - y_1 + y_2 + y_3)^2 + (y_0 + y_1 + y_2 + y_3)^2}.$$

Noting that

$$\begin{aligned}&\rho_f^{-1}\Big(\frac{1}{2^i}\Big) \le \frac{y_1}{y_0} \le \rho_f\Big(\frac{1}{2^i}\Big), \qquad \rho_f^{-1}\Big(\frac{1}{2^i}\Big) \le \frac{y_2}{y_1} \le \rho_f\Big(\frac{1}{2^i}\Big), \\ &\rho_f^{-1}\Big(\frac{1}{2^i}\Big) \le \frac{y_3}{y_2} \le \rho_f\Big(\frac{1}{2^i}\Big),\end{aligned} \tag{10}$$

one can prove

$$|\mu_2^{k,j}|^2 \le \frac{(\rho_f^4(2^{-i}) - 1)^2}{(\rho_f^4(2^{-i}) + 1)^2} \tag{11}$$

by elementary methods. We prove (11) using Maple. The details are given in the Appendix.

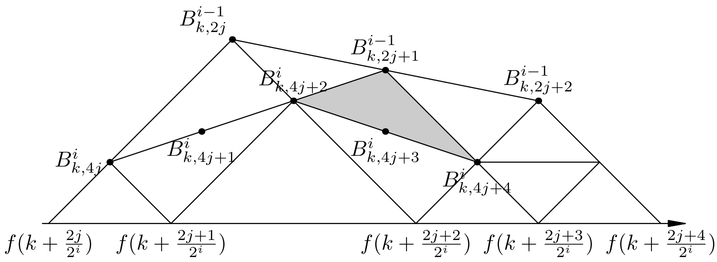

Figure 6

Therefore, by (11) we have

$$K(F|_{\Delta A^{i-1}_{k,2j}A^{i}_{k,2+4j}A^{i-1}_{k,1+2j}}) \le \rho_f^4\Big(\frac{1}{2^i}\Big).$$

Case 3: The estimate of the complex dilatation on $\Delta A^{i-1}_{k,1+2j}A^{i}_{k,2+4j}A^{i}_{k,4+4j}$. Let $\mu_3^{k,j}$ be the complex dilatation of $F|_{\Delta A^{i-1}_{k,1+2j}A^{i}_{k,2+4j}A^{i}_{k,4+4j}}$. Since F is a barycentric coordinate mapping of $\Delta A^{i-1}_{k,1+2j}A^{i}_{k,2+4j}A^{i}_{k,4+4j}$ onto $\Delta B^{i-1}_{k,1+2j}B^{i}_{k,2+4j}B^{i}_{k,4+4j}$ (see Figure 6 for the image triangle), by using (1), (2), (3), (5), (7), and (8), we have

$$\begin{aligned}|\mu_3^{k,j}| &= \Big|\frac{f(x_2)-f(x_1)+f(x_3)-f(x_4)+\mathrm{i}(f(x_0)-f(x_1)+f(d)-f(x_2))}{f(x_1)-f(x_0)-f(x_2)+f(x_3)+\mathrm{i}(f(x_1)-f(x_2)+f(x_3)-f(x_4))}\Big|\\ &= \Big|\frac{y_1-y_3+\mathrm{i}(y_2-y_0)}{y_0+y_2-\mathrm{i}(y_1+y_3)}\Big|.\end{aligned}$$

This implies

$$|\mu_3^{k,j}|^2 = 1-\frac{4(y_0y_2+y_1y_3)}{(y_0+y_2)^2+(y_1+y_3)^2}.$$

As in Case 2, we can prove

$$|\mu_2^{k,j}|^2 \le \frac{(\rho_f^4(2^{-i})-1)^2}{(\rho_f^4(2^{-i})+1)^2} \tag{12}$$

by using Maple, which implies

$$K(F|_{\Delta A^{i-1}_{k,1+2j}A^{i}_{k,2+4j}A^{i}_{k,4+4j}}) \le \rho_f^4\Big(\frac{1}{2^i}\Big).$$

The details of the proof of (12) are given in the Appendix.

Case 4: The estimate of the complex dilatation on $\Delta A^{i-1}_{k,2j+1}A^{i}_{k,4j+4}A^{i-1}_{k,4j+2}$. Since the image triangle $\Delta B^{i-1}_{k,2j+1}B^{i}_{k,4j+4}B^{i-1}_{k,4j+2}$ (see Figure 7) is determined by $f(x_0)$, $f(x_2)$, $f(x_3)$, and $f(x_4)$, similarly to Case 2, we also have

$$K^i \le \max\Big\{\rho_f\Big(\frac{1}{2^n}\Big), \rho_f^4\Big(\frac{1}{2^n}\Big)\Big\} = \rho_f^4\Big(\frac{1}{2^n}\Big) \tag{13}$$

by using (9), (11), and (12). □

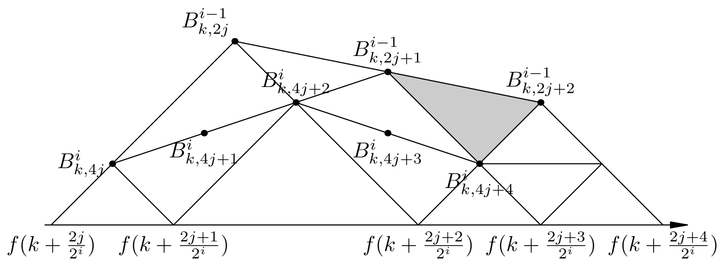

Figure 7

4. Proof of the main theorems

Now we give the proof of Theorems 1 and 2. Assume that $f(x)$ is a sense-preserving self-homeomorphism of the real axis $\mathbb{R}$. Let $F(z)$ be the triangulation extension to the upper half-plane $\mathbb{H}$ of $f(x)$, and let $K(z)$ be the dilatation of $F(z)$.

Proof of Theorem 1
By Lemma 1, we have

$$1-|\mu_{n,0}|^2=\frac{8(f(n+1)-f(n-1))}{(f(n+1)-f(n-1)+2)^2+(f(n+1)-2f(n)+f(n-1))^2},$$

where $\mu_{n,0}=\mu|_{\Delta A^0_{n-1,0}A^0_{n,0}\infty}$. Since

$$0<\inf_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big)\le\sup_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big)<\infty,$$

there exist positive numbers c_1 and c_2 such that

$$c_1\le\inf_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big)\le\sup_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big)\le c_2.$$

So

$$1-|\mu_{0,n}|^2\ge\frac{16c_1}{(2c_2+2)^2+(2c_2)^2}.$$

Therefore, it is easy to see that there exists a positive number $0<c_3<1$ such that $K|_{\text{band}0}\le c_3$.

By Lemma 2, we have $K|_{\text{band}n}\le\rho_f^4(1/2^n)$. Since $f(x)$ is quasisymmetric, there exists a positive number $0<c_4<1$ such that $\rho_f^4(1/2^n)\le c_4$ for every $n\in\mathbb{N}$. So $K\le\max\{c_3,c_4\}$; that is, $F(z)$ is a quasiconformal mapping.

Conversely, assume that

$$\inf_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big)=0\qquad\text{or}\qquad\sup_{n\in\mathbb{Z}}\big(f(n+1)-f(n)\big)=\infty.$$

Since $f(x)$ is a quasisymmetric mapping, there exists a positive constant ρ_f such that

$$\rho_f^{-1}<\frac{f(n+1)-f(n)}{f(n)-f(n-1)}<\rho_f.$$

Let $a_n = f(n+1) - f(n)$ $(n \in \mathbb{Z})$ and $b_n = a_{n-1}/a_n$. Then $\rho_f^{-1} < b_n < \rho_f$ for every $n \in \mathbb{Z}$.

If $\inf_{n\in\mathbb{Z}}(f(n+1) - f(n)) = 0$, then there exists a subsequence $\{a_{n_k}\}$ $(k \in \mathbb{Z})$ such that $\lim_{k\to+\infty} a_{n_k} = 0$ or $\lim_{k\to-\infty} a_{n_k} = 0$. Without loss of generality, we may only consider $k \to +\infty$. By Lemma 1, we have

$$\lim_{k\to+\infty}\left(1 - |\mu_{n,0}|^2\right) = \lim_{k\to+\infty} \frac{8(a_{n_k} + b_{n_k}a_{n_k})}{(a_{n_k} + b_{n_k}a_{n_k} + 2)^2 + (a_{n_k} + b_{n_k}a_{n_k})^2} = 0.$$

This is a contradiction.

If $\sup_{n\in\mathbb{Z}}(f(n+1) - f(n)) = \infty$, then there exists a subsequence $\{a_{n_l}\}$ $(l \in \mathbb{Z})$ such that $\lim_{l\to+\infty} a_{n_l} = \infty$ or $\lim_{l\to-\infty} a_{n_l} = \infty$. Without loss of generality, we may only consider $l \to +\infty$. By Lemma 1, we have

$$\begin{aligned}
&\lim_{l\to+\infty}\left(1 - |\mu_{0,n}|^2\right)\\
&= \lim_{l\to+\infty} \frac{8(a_{n_l} + b_{n_l}a_{n_l})}{(a_{n_l} + b_{n_l}a_{n_l} + 2)^2 + (a_{n_l} + b_{n_l}a_{n_l})^2}\\
&\le \lim_{l\to+\infty} \frac{8(a_{n_l} + b_{n_l}a_{n_l} + 2)}{(a_{n_l} + b_{n_l}a_{n_l} + 2)^2} = 0.
\end{aligned}$$

Then it is easy to see that $\lim_{l\to+\infty} |\mu_{0,n}| = 1$. This is also a contradiction. □

Now we give a counterexample, which is a quasisymmetric mapping that cannot be extended to be a quasiconformal mapping by triangulation extension. Take $g(x) = x^3$. Since it is a quasisymmetric mapping and $\sup_{n\in\mathbb{Z}}\{(n+1)^3 - n^3\} = \infty$, we see that $g(x) = x^3$ cannot be extended to be a quasiconformal mapping by triangulation extension.

Proof of Theorem 2

Since $\rho_f(t) = O(\log^{1/4}(\frac{1}{t}))$, $t \to 0^+$, there exists a positive number $C > 0$ such that

$$\rho_f(t) \le C \log^{1/4}\left(\frac{1}{t}\right)$$

when t is small enough. Let $t = 1/2^n$. Then there exists $N \in \mathbb{N}$ such that

$$\rho_f\left(\frac{1}{2^n}\right) \le C \log^{1/4}(2^n), \quad \forall n > N.$$

Take $p = \frac{1}{2C^4}$. By Lemma 2, we have

$$\exp(pK_n) \le \sqrt{2^n} \quad (n \ge N).$$

By Lemma 1, it is clear that there exists a positive number M_1 such that

$$\iint_{y\ge1} \frac{\exp(pK_0(z))}{(1 + x^2 + y^2)^2}\, dx\, dy < M_1.$$

So

$$\begin{aligned}
&\iint_{\mathbb{H}} \exp\bigl(pK(z)\bigr)\frac{1}{(1+x^2+y^2)^2}\,dx\,dy \\
&\quad= \iint_{y\geq 1}\frac{\exp(pK_0(z))}{(1+x^2+y^2)^2}\,dx\,dy + \iint_{0\leq y\leq 1}\frac{\exp(pK(z))}{(1+x^2+y^2)^2}\,dx\,dy \\
&\quad\leq M_1 + \sum_{n=0}^{\infty}\iint_{S_n}\frac{\exp(K_n(z))}{(1+x^2+y^2)^2}\,dx\,dy \\
&\quad= M_1 + \Bigl(\sum_{n=0}^{N} + \sum_{n=N+1}^{\infty}\Bigr)\iint_{S_n}\frac{\exp(K_n(z))}{(1+x^2+y^2)^2}\,dx\,dy \\
&\quad= M_1 + M_2 + \frac{\pi}{2}\sum_{n=N}^{\infty}\sqrt{2^n}\,\frac{1}{2^n}\Bigl(\frac{1}{\sqrt{1+4^{-n}}} - \frac{1}{\sqrt{4+4^{-n}}}\Bigr) \\
&\quad\leq M_1 + M_2 + \sum_{n=N}^{\infty}\frac{1}{(\sqrt{2})^n}\pi \\
&\quad< \infty,
\end{aligned}$$

where M_2 is a positive number and S_n is the domain of band n ($n \in \mathbb{N}^+$). So, the triangulation extension $F(z)$ of $f(x)$ is a David mapping of the upper half-plane $\mathbb{H}$. □

Appendix

The inequalities (10) and (12) can be proved by elementary methods, but it is very complicated. Here we use the inequality package BOTTEMA (see [8]) of Maple to prove the inequalities (10) and (12).

Proof of (10)
Let

$$z_i = y_i/y_0 \quad (i=1,2,3), \qquad w = z_2 + z_3, \qquad \text{and} \qquad p = \rho_f(1/2^i).$$

Then it is easy to see that inequality (11) holds if the following inequality holds:

$$\begin{aligned}
&(p^4+1)^2\bigl[(3-z_1+w)^2 + (1+z_1+w)^2 - 16w\bigr] \\
&\quad - (p^4-1)^2\bigl[(3-z_1+w)^2 + (1+z_1+w)^2\bigr] \leq 0,
\end{aligned}$$

where

$$p^{-1} \leq z_1 \leq p \qquad \text{and} \qquad p^{-2}+p^{-3} \leq w \leq p^2 + p^3.$$

Simplify the left-hand term by using Maple. It is equivalent to

$$5p^4 - 2p^8 w + p^4 z_1^2 + p^4 w^2 - 2w - 2p^4 z_1 \leq 0.$$

Use the following orders in Maple:

```
xprove(5*p^4-2*p^8*w+p^4*z1^2+p^4*w^2-2*w-2*p^4*z1<=0,
[1/p<=z1,z1<=p,(1/p^2+1/p^3)<=w,w<=(p^2+p^3)]);
```

We know that the inequality (10) holds. $\square$

Proof of (12)
Inequality (12) is equivalent to the following polynomial inequality:

$$\begin{aligned}(p^4+1)^2\left[\left((y_0+y_2)^2+(y_1+y_3)^2\right)-(4y_0y_2+y_1y_3)\right]\\ -(p^4-1)^2\left[(y_0+y_2)^2+(y_1+y_3)^2\right]\le 0.\end{aligned}$$

Simplify the left-hand term by using Maple. It is equivalent to

$$-4y_1y_3-4y_0y_2+4y_2^2p^4-4p^8y_0y_2+4y_1^2p^4-4p^8y_1y_3+4p^4y_0^2+4p^4y_3^2\le 0.$$

Use the following orders in Maple:

```
xprove(-4*y1*y3-4*y0*y2+4*y2^2*p^4-4*p^8*y0*y2+4*y1^2*p^4-
4*p^8*y1*y3+4*p^4*y0^2+4*p^4*y3^2<=0,[y0<=y1*p,y1<=p*y0,
y0<=y2*p^2,y2<=y0*p^2,y0<=y3*p^3,y3<=y0*p^3,p>=1]);
```

We know that the inequality (12) holds. $\square$

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