Trudinger's inequality and continuity for Riesz potentials of functions in grand Musielak–Orlicz–Morrey spaces over nondoubling metric measure spaces

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Abstract In this article we are concerned with Trudinger's inequality and continuity for Riesz potentials of functions in grand Musielak–Orlicz–Morrey spaces over nondoubling metric measure spaces.

1. Introduction

Grand Lebesgue spaces were introduced in [9] for the study of the Jacobian. They play important roles also in the theory of partial differential equations (see [5], [10], [28], etc.). The generalized grand Lebesgue spaces appeared in [7], where the existence and uniqueness of the nonhomogeneous N-harmonic equations were studied.

For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^N$ by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

The classical Trudinger's inequality for Riesz potentials of L^p-functions (see, e.g., [2, Theorem 3.1.4(c)]) has been also extended to various function spaces; see [19] and [20] for Morrey spaces of variable exponent, [6] for grand Morrey spaces of variable exponent, [24] for Musielak–Orlicz spaces, and [14] for Musielak–Orlicz–Morrey spaces. See also [26] and [27]. Recently, Trudinger's inequality has been extended to an inequality for Riesz potentials of functions in grand Musielak–Orlicz–Morrey spaces (see [15]).

We denote by (X, d, μ) a metric measure space, where X is a bounded set, d is a metric on X, and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at

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x with radius r and $d_X = \sup\{d(x,y) : x, y \in X\}$. We assume that $0 < d_X < \infty$,

$$\mu\big(\{x\}\big) = 0$$

for $x \in X$, and $\mu(B(x,r)) > 0$ for $x \in X$ and $r > 0$ for simplicity. In the present article, we do not postulate on μ, the so-called *doubling condition*. Recall that a Radon measure μ is said to be doubling if there exists a constant $C > 0$ such that $\mu(B(x,2r)) \leq C\mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)$ $(= X)$ and $r > 0$. Otherwise μ is said to be nondoubling.

For $\alpha > 0$ and $\tau > 0$, we define the Riesz potential of order α for a locally integrable function f on X by (e.g., see [8] and [22])

$$I_{\alpha,\tau} f(x) = \int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y).$$

Observe that this naturally extends the Riesz potential operator $I_\alpha f(x)$ when (X,d) is the N-dimensional Euclidean space and $\mu = dx$.

Our first aim in this article is to give a general version of Trudinger-type exponential integrability for Riesz potentials $I_{\alpha,\tau} f$ of functions in grand Musielak–Orlicz–Morrey spaces $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ over nondoubling metric measure spaces X (see, e.g., Corollary 5.5) as an extension of [15, Corollary 6.12] (see Sections 2 and 3 for the definitions of Φ, κ, η, ξ, and $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$). Since we discuss the Morrey version, our strategy is to find estimates of Riesz potentials $I_{\alpha,\tau} f$ by the use of other Riesz-type potentials $I_{\gamma,\tau} f$ of order γ $(< \alpha)$, which play the role of the maximal functions (see Section 4). What is new about this article is that we can pass our results to the nondoubling metric measure setting; the technique developed in [14] still works.

On the other hand, beginning with Sobolev's embedding theorem (see, e.g., [1], [2]), continuity properties of Riesz potentials or Sobolev functions have been studied by many authors. See [17] and [18] for generalized Morrey spaces $L^{1,\varphi}$, [21] for Orlicz–Morrey spaces, [21] for variable exponent Morrey spaces, and [19] for two variable exponent Morrey spaces.

Our second aim in this article is to give a general version of continuity for Riesz potentials $I_{\alpha,\tau} f$ of functions in grand Musielak–Orlicz–Morrey spaces over nondoubling metric measure spaces (see, e.g., Corollary 6.6), whose counterpart in the Euclidean setting was not considered in [15]. The result is new even for the Euclidean case.

2. Preliminaries

Throughout this article, let C denote various constants independent of the variables in question. In this article, we assume that X is a bounded set, that is, $d_X < \infty$. This implies that $\mu(X) < \infty$.

We consider a function

$$\Phi(x,t) = t\phi(x,t) : X \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi1)$–$(\Phi4)$:

($\Phi 1$) $\phi(\cdot,t)$ is measurable on X for each $t \geq 0$ and $\phi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;

($\Phi 2$) there exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \phi(x,1) \leq A_1 \quad \text{for all } x \in X;$$

($\Phi 3$) there exists a constant $\varepsilon_0 > 0$ such that $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing, namely, there exists a constant $A_2 \geq 1$ such that

$$t^{-\varepsilon_0}\phi(x,t) \leq A_2 s^{-\varepsilon_0}\phi(x,s)$$

for all $x \in X$ whenever $0 < t < s$;

($\Phi 4$) there exists a constant $A_3 \geq 1$ such that

$$\phi(x,2t) \leq A_3\phi(x,t) \quad \text{for all } x \in X \text{ and } t > 0.$$

Note that ($\Phi 3$) implies that

$$t^{-\varepsilon}\phi(x,t) \leq A_2 s^{-\varepsilon}\phi(x,s)$$

for all $x \in X$ and $0 < \varepsilon \leq \varepsilon_0$ whenever $0 < t < s$.

Also note that ($\Phi 2$), ($\Phi 3$), and ($\Phi 4$) imply

$$0 < \inf_{x \in X} \phi(x,t) \leq \sup_{x \in X} \phi(x,t) < \infty$$

for each $t > 0$ and there exists $\omega > 1$ such that

$$(A_1A_2)^{-1}t^{1+\varepsilon_0} \leq \Phi(x,t) \leq A_1A_2A_3t^{\omega} \tag{2.1}$$

for $t \geq 1$. In fact, we can take $\omega \geq 1 + \log A_3/\log 2$.

We shall also consider the following condition:

($\Phi 5$) for every $\gamma_1,\gamma_2 > 0$, there exists a constant $B_{\gamma_1,\gamma_2} \geq 1$ such that

$$\phi(x,t) \leq B_{\gamma_1,\gamma_2}\phi(y,t)$$

whenever $d(x,y) \leq \gamma_1 t^{-1/\gamma_2}$ and $t \geq 1$.

Let $\bar{\phi}(x,t) = \sup_{0 \leq s \leq t}\phi(x,s)$, and let

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in X$ and $t \geq 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leq \overline{\Phi}(x,t) \leq A_2\Phi(x,t)$$

for all $x \in X$ and $t \geq 0$.

EXAMPLE 2.1

Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\dots,k$, be measurable functions on X such that

$$1 < p^- := \inf_{x \in X} p(x) \leq \sup_{x \in X} p(x) =: p^+ < \infty$$

and

$$-\infty < q_j^- := \inf_{x\in X} q_j(x) \le \sup_{x\in X} q_j(x) =: q_j^+ < \infty, \quad j = 1,\dots,k.$$

Set $L(t) := \log(e+t)$, set $L^{(1)}(t) = L(t)$, and set $L^{(j)}(t) = L(L^{(j-1)}(t))$, $j = 2,\dots$. Then,

$$\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)} \prod_{j=1}^{k} \big(L^{(j)}(t)\big)^{q_j(x)}$$

satisfies ($\Phi1$), ($\Phi2$), ($\Phi3$) with $0 < \varepsilon_0 < p^- - 1$, and ($\Phi4$). For any $\omega > p^+$, (2.1) holds. Then, $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies ($\Phi5$) if $p(\cdot)$ is log-Hölder continuous, namely,

$$\big|p(x) - p(y)\big| \le \frac{C_p}{L(1/d(x,y))} \quad (x \ne y),$$

and $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely,

$$\big|q_j(x) - q_j(y)\big| \le \frac{C_{q_j}}{L^{(j+1)}(1/d(x,y))} \quad (x \ne y)$$

for $j = 1,\dots,k$ (cf. [13, Example 2.1]).

We also consider a function $\kappa(x,r) : X \times (0,d_X) \to (0,\infty)$ satisfying the following conditions:

($\kappa1$) $\kappa(x,\cdot)$ is continuous on $(0,d_X)$ for each $x \in X$ and satisfies the uniform doubling condition: there is a constant $Q_1 \ge 1$ such that

$$Q_1^{-1}\kappa(x,r) \le \kappa(x,r') \le Q_1\kappa(x,r)$$

for all $x \in X$ whenever $0 < r \le r' \le 2r < d_X$;

($\kappa2$) $r \mapsto r^{-\delta}\kappa(x,r)$ is uniformly almost increasing for some $\delta > 0$, namely, there is a constant $Q_2 > 0$ such that

$$r^{-\delta}\kappa(x,r) \le Q_2 s^{-\delta}\kappa(x,s)$$

for all $x \in X$ whenever $0 < r < s < d_X$;

($\kappa3$) there are constants $Q > 0$ and $Q_3 \ge 1$ such that

$$Q_3^{-1}\min(1,r^Q) \le \kappa(x,r) \le Q_3$$

for all $x \in X$ and $0 < r < d_X$.

EXAMPLE 2.2

Let $\nu(\cdot)$ and $\beta(\cdot)$ be functions on X such that $\nu^- := \inf_{x\in X}\nu(x) > 0$, $\nu^+ := \sup_{x\in X}\nu(x) \le Q$, and $-c(Q-\nu(x)) \le \beta(x) \le c$ for all $x \in X$ and some constant $c > 0$. Then $\kappa(x,r) = r^{\nu(x)}(\log(e+1/r))^{\beta(x)}$ satisfies ($\kappa1$), ($\kappa2$), and ($\kappa3$); we can take any $0 < \delta < \nu^-$ for ($\kappa2$).

We say that f is a locally integrable function on X if f is an integrable function on all balls B in X. Given $\Phi(x,t)$ satisfying ($\Phi1$), ($\Phi2$), ($\Phi3$), and ($\Phi4$) and

$\kappa(x,r)$ satisfying (κ1), (κ2), and (κ3), we define the Musielak–Orlicz–Morrey space $L^{\Phi,\kappa}(X)$ by

$$L^{\Phi,\kappa}(X) = \Big\{ f \in L^1_{\rm loc}(X);\ \sup_{x\in X, 0<r<d_X} \frac{\kappa(x,r)}{\mu(B(x,r))} \int_{B(x,r)\cap X} \Phi\big(y, |f(y)|\big)\, d\mu(y) < \infty \Big\}.$$

It is a Banach space with respect to the norm (cf. [23])

$$\|f\|_{\Phi,\kappa;X} = \inf\Big\{ \lambda > 0;\ \sup_{x\in X, 0<r<d_X} \frac{\kappa(x,r)}{\mu(B(x,r))} \int_{B(x,r)\cap X} \overline{\Phi}\big(y, |f(y)|/\lambda\big)\, d\mu(y) \le 1 \Big\}.$$

3. Grand Musielak–Orlicz–Morrey space

For $\varepsilon \ge 0$, set $\Phi_\varepsilon(x,t) := t^{-\varepsilon}\Phi(x,t) = t^{1-\varepsilon}\phi(x,t)$. Then, $\Phi_\varepsilon(x,t)$ satisfies (Φ1), (Φ2) with the same A_1, and (Φ4) with the same A_3. If $\Phi(x,t)$ satisfies (Φ5), then so does $\Phi_\varepsilon(x,t)$ with the same $\{B_{\gamma_1,\gamma_2}\}_{\gamma_1,\gamma_2>0}$.

If $0 \le \varepsilon < \varepsilon_0$, then $\Phi_\varepsilon(x,t)$ satisfies (Φ3) with ε_0 replaced by $\varepsilon_0 - \varepsilon$ and the same A_2. It follows that

$$\frac{1}{2A_3}\Phi_\varepsilon(x,t) \le \overline{\Phi_\varepsilon}(x,t) \le A_2 \Phi_\varepsilon(x,t) \tag{3.1}$$

for all $x \in X$, $t \ge 0$, and $0 \le \varepsilon \le \varepsilon_0$.

Let

$$\tilde\sigma = \sup\big\{\sigma \ge 0 : r^{Q-\sigma}\kappa(x,r)^{-1} \text{ is bounded on } X \times \big(0, \min(1, d_X)\big)\big\}.$$

By (κ2) and (κ3), $0 \le \tilde\sigma \le Q$. If $\tilde\sigma = 0$, then let $\sigma_0 = 0$; otherwise fix any $\sigma_0 \in (0,\tilde\sigma)$. We also take δ_0 such that $0 < \delta_0 < \delta$ for δ in (κ2).

For $-\delta_0 \le \sigma \le \sigma_0$, set

$$\kappa_\sigma(x,r) = r^\sigma \kappa(x,r)$$

for $x \in X$ and $0 < r < d_X$. Then $\kappa_\sigma(x,r)$ satisfies (κ1), (κ2), and (κ3) with constants independent of σ.

LEMMA 3.1 ([15, PROPOSITION 3.2])

Assume that $\Phi(x,t)$ satisfies (Φ5). If $0 \le \varepsilon_1 \le \varepsilon_2 \le \varepsilon_0$, $-\delta_0 \le \sigma_j \le \sigma_0$, $j = 1,2$, and

$$\sigma_1 + \frac{\delta-\delta_0}{\omega}\varepsilon_1 \le \sigma_2 + \frac{\delta-\delta_0}{\omega}\varepsilon_2,$$

then $L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(X) \subset L^{\Phi_{\varepsilon_2},\kappa_{\sigma_2}}(X)$ and

$$\|f\|_{\Phi_{\varepsilon_2},\kappa_{\sigma_2};X} \le C\|f\|_{\Phi_{\varepsilon_1},\kappa_{\sigma_1};X}$$

for all $f \in L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(X)$ with $C > 0$ independent of ε_1, ε_2, σ_1, σ_2.

In particular,

$$L^{\Phi,\kappa}(X) \subset L^{\Phi_\varepsilon,\kappa_\sigma}(X)$$

if $0 \le \varepsilon \le \varepsilon_0$, $-\delta_0 \le \sigma \le \sigma_0$, *and* $\sigma + ((\delta - \delta_0)/\omega)\varepsilon \ge 0$.

Let $\eta(\varepsilon)$ be an increasing positive function on $(0,\infty)$ such that $\eta(0+) = 0$. Let $\xi(\varepsilon)$ be a function on $(0,\varepsilon_1]$ with some $\varepsilon_1 \in (0,\varepsilon_0/2]$ such that $-\delta_0 \le \xi(\varepsilon) \le \sigma_0$ for $0 < \varepsilon \le \varepsilon_1$, $\xi(0+) = 0$, and $\varepsilon \mapsto \xi(\varepsilon) + ((\delta-\delta_0)/\omega)\varepsilon$ is nondecreasing; in particular, $\xi(\varepsilon) + ((\delta - \delta_0)/\omega)\varepsilon \ge 0$ for $0 < \varepsilon \le \varepsilon_1$.

Given $\Phi(x,t)$, $\kappa(x,r)$, $\eta(\varepsilon)$, and $\xi(\varepsilon)$, the associated (generalized) grand Musielak–Orlicz–Morrey space is defined by (cf. [11] for generalized grand Morrey space)

$$\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X) = \Big\{ f \in \bigcap_{0<\varepsilon\le\varepsilon_1} L^{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)}}(X); \|f\|_{\Phi,\kappa;\eta,\xi;X} < \infty \Big\},$$

where

$$\|f\|_{\Phi,\kappa;\eta,\xi;X} = \sup_{0<\varepsilon\le\varepsilon_1} \eta(\varepsilon) \|f\|_{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)};X}.$$

Note that $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ is a Banach space with the norm $\|f\|_{\Phi,\kappa;\eta,\xi;X}$. Also note that, in view of Lemma 3.1, this space is determined independent of the choice of ε_1.

REMARK 3.2

If $\mu(B(x,r))$ satisfies $(\kappa1)$, $(\kappa2)$, and $(\kappa3)$, then the associated (generalized) grand Musielak–Orlicz–Morrey space $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ includes the following spaces:

- generalized grand Lebesgue spaces introduced in [3] if $\kappa(x,r) = \mu(B(x,r))$ and $\xi(\varepsilon) \equiv 0$;
- grand Orlicz spaces introduced in [12] (see also [4]) if $\kappa(x,r) = \mu(B(x,r))$, $\xi(\varepsilon) \equiv 0$, $\Phi(x,t) = \Phi(t)$, and

$$\sup_{0<\varepsilon\le\varepsilon_0} \eta(\varepsilon) \int_1^\infty t^{-N-\varepsilon}\Phi(t)\frac{dt}{t} < \infty;$$

- grand Morrey spaces introduced in [16] if $\xi(\varepsilon) \equiv 0$;
- grand grand Morrey spaces introduced in [25] and generalized grand Morrey spaces introduced in [11] if $\xi(\varepsilon)$ is an increasing positive function on $(0,\infty)$.

4. Lemmas

LEMMA 4.1 ([13, LEMMA 5.1])

Let $F(x,t)$ be a positive function on $X \times (0,\infty)$ satisfying the following conditions:

(F1) *$F(x,\cdot)$ is continuous on $(0,\infty)$ for each $x \in X$;*
(F2) *there exists a constant $K_1 \ge 1$ such that*

$$K_1^{-1} \le F(x,1) \le K_1 \quad \textit{for all } x \in X;$$

(F3) *$t \mapsto t^{-\varepsilon} F(x,t)$ is uniformly almost increasing for some $\varepsilon > 0$; namely, there exists a constant $K_2 \geq 1$ such that*

$$t^{-\varepsilon} F(x,t) \leq K_2 s^{-\varepsilon} F(x,s) \quad \text{for all } x \in X \text{ whenever } 0 < t < s.$$

Set

$$F^{-1}(x,s) = \sup\{t > 0; F(x,t) < s\}$$

for $x \in X$ and $s > 0$. Then:

(1) *$F^{-1}(x,\cdot)$ is nondecreasing.*

(2)

$$F^{-1}(x,\lambda t) \leq (K_2 \lambda)^{1/\varepsilon} F^{-1}(x,t) \tag{4.1}$$

for all $x \in X$, $t > 0$, and $\lambda \geq 1$.

(3)

$$F\big(x, F^{-1}(x,t)\big) = t \tag{4.2}$$

for all $x \in X$ and $t > 0$.

(4)

$$K_2^{-1/\varepsilon} t \leq F^{-1}\big(x, F(x,t)\big) \leq K_2^{2/\varepsilon} t$$

for all $x \in X$ and $t > 0$.

(5)

$$\min\Big\{1, \Big(\frac{s}{K_1 K_2}\Big)^{1/\varepsilon}\Big\} \leq F^{-1}(x,s) \leq \max\big\{1, (K_1 K_2 s)^{1/\varepsilon}\big\} \tag{4.3}$$

for all $x \in X$ and $s > 0$.

REMARK 4.2

We have that $F(x,t) = \Phi(x,t)$ satisfies (F1), (F2), and (F3) with $K_1 = A_1$, $K_2 = A_2$, and $\varepsilon = 1$.

By (κ3) and (4.3), we have the following result.

LEMMA 4.3

There exists a constant $C > 0$ such that

$$C^{-1} \leq \Phi^{-1}\big(x, \kappa(x,r)^{-1}\big) \leq C r^{-Q} \tag{4.4}$$

for all $x \in X$ and $0 < r \leq d_X$, where Q is a constant appearing in (κ3).

LEMMA 4.4 (CF. [15, LEMMA 3.1])

There exist constants $C \geq 1$ and $r_0 \in (0, \min(1, d_X))$ such that $\kappa_\sigma(x,r) \leq C r^{\delta - \delta_0}$ and

$$C^{-1} r^{-(\delta-\delta_0)/\omega} \leq \Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \leq C r^{-Q}$$

for all $x \in X$, $0 < r \leq r_0$, $-\delta_0 \leq \sigma \leq \sigma_0$, and $0 < \varepsilon \leq \varepsilon_0$, where Q is a constant appearing in (κ3).

Proof
In view of the proof of [15, Lemma 3.1], we only have to prove that there exists a constant $C \geq 1$ such that

$$\Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \leq Cr^{-Q}$$

for all $x \in X$, $0 < r \leq r_0$, $-\delta_0 \leq \sigma \leq \sigma_0$, and $0 < \varepsilon \leq \varepsilon_0$. First note from $(\Phi 3)$ that there exists a constant $C \geq 1$ such that

$$t^{-\varepsilon'}\Phi_\varepsilon(x,t) \leq Cs^{-\varepsilon'}\Phi_\varepsilon(x,s)$$

for all $x \in X$ and $0 < \varepsilon' \leq \varepsilon_0 - \varepsilon + 1$ whenever $0 < t < s$. By Lemma 4.1(5) with $\varepsilon' = 1$ and $(\kappa 3)$, we have

$$\Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \leq C\kappa_\sigma(x,r)^{-1} \leq Cr^{-Q}$$

for all $x \in X$, $0 < r \leq r_0$, $-\delta_0 \leq \sigma \leq \sigma_0$, and $0 < \varepsilon \leq \varepsilon_0$, as required. □

From now on, we assume:

(Ξ) $\xi(\varepsilon) \leq a\varepsilon$ for $0 < \varepsilon \leq \varepsilon_1$ with some $a \geq 0$.

Recall that $\xi(\varepsilon) \geq -((\delta - \delta_0)/\omega)\varepsilon$ by assumption. Let

$$\varepsilon(r) = \big(\log(e + 1/r)\big)^{-1}$$

for $r > 0$, and let $r_1 \in (0, \min(1, d_X))$ be such that $\varepsilon(r) \leq \varepsilon_1$ for $0 < r \leq r_1$.

LEMMA 4.5 ([15, LEMMA 6.2])
There exists a constant $C \geq 1$ such that

$$C^{-1}\Phi^{-1}\big(x, \kappa(x,r)^{-1}\big) \leq \Phi_{\varepsilon(r)}^{-1}\big(x, \kappa_{\xi(\varepsilon(r))}(x,r)^{-1}\big) \leq C\Phi^{-1}\big(x, \kappa(x,r)^{-1}\big)$$

for all $x \in X$ and $0 < r \leq r_1$.

LEMMA 4.6
Assume that $\Phi(x,t)$ satisfies $(\Phi 5)$. Then there exists a constant $C > 0$ such that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)\cap X} f(y)\,d\mu(y) \leq C\Phi^{-1}\big(x, \kappa(x,r)^{-1}\big)\eta\Big(\big(\log(e+1/r)\big)^{-1}\Big)^{-1}$$

for all $x \in X$, $0 < r < d_X$, and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$.

Proof
Let f be a nonnegative function with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$. Then note from (3.1) that

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{\mu(B(x,r))}\int_{B(x,r)\cap X}\Phi_\varepsilon\big(y, \eta(\varepsilon)f(y)\big)\,d\mu(y) \leq 2A_3$$

for $x \in X$, $0 < r < d_X$, and $0 < \varepsilon < \varepsilon_1$, so that

$$\frac{\kappa_{\xi(\varepsilon(r))}(x,r)}{\mu(B(x,r))}\int_{B(x,r)\cap X}\Phi_{\varepsilon(r)}\big(y, \eta\big(\varepsilon(r)\big)f(y)\big)\,d\mu(y) \leq 2A_3$$

for $x \in X$ and $0 < r \le r_1$. Let $g_r(y) = \eta(\varepsilon(r)) f(y)$, and let

$$K(x,r) = \Phi^{-1}_{\varepsilon(r)}\big(x, \kappa_{\xi(\varepsilon(r))}(x,r)^{-1}\big).$$

Since there exist constants $C \ge 1$ and $r_0 \in (0, \min(1, d_X))$ such that

$$1 \le K(x,r) \le C r^{-Q}$$

for all $x \in X$ and $0 < r \le \min\{r_0, r_1\}$ by Lemma 4.4, we see from ($\Phi5$) and (4.2) that

$$\Phi_{\varepsilon(r)}\big(y, K(x,r)\big) \ge C\Phi_{\varepsilon(r)}\big(x, K(x,r)\big) = C\kappa_{\xi(\varepsilon(r))}(x,r)^{-1}$$

for all $y \in B(x,r)$ and $0 < r \le \min\{r_0, r_1\}$. Therefore, we have by ($\Phi3$)

$$\begin{aligned}
&\frac{1}{\mu(B(x,r))}\int_{B(x,r)\cap X} g_r(y)\,d\mu(y)\\
&\quad\le K(x,r) + \frac{A_2}{\mu(B(x,r))}\int_{B(x,r)\cap X} g_r(y)\frac{g_r(y)^{-1}\Phi_{\varepsilon(r)}(y, g_r(y))}{K(x,r)^{-1}\Phi_{\varepsilon(r)}(y, K(x,r))}\,d\mu(y)\\
&\quad\le CK(x,r)\Big\{1 + \frac{\kappa_{\xi(\varepsilon(r))}(x,r)}{\mu(B(x,r))}\int_{B(x,r)\cap X}\Phi_{\varepsilon(r)}\big(y, g_r(y)\big)\,d\mu(y)\Big\}\\
&\quad\le CK(x,r)
\end{aligned}$$

for $x \in X$ and $0 < r \le \min\{r_0, r_1\}$. Hence, we find by Lemma 4.5

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)\cap X} g_r(y)\,d\mu(y) \le C\Phi^{-1}\big(x, \kappa(x,r)^{-1}\big)$$

for all $x \in X$ and $0 < r \le \min\{r_0, r_1\}$.

When $\min\{r_0, r_1\} < r < d_X$, we have by (4.4)

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)\cap X} f(y)\,d\mu(y) \le C \le C\Phi^{-1}\big(x, \kappa(x,r)^{-1}\big)\eta\big((\log(e+1/r))^{-1}\big)^{-1},$$

as required. □

Set

$$\Gamma(x,s) = \int_{1/s}^{d_X} \rho^{\alpha}\Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big)\eta\big((\log(e+1/\rho))^{-1}\big)^{-1}\frac{d\rho}{\rho}$$

for $s \ge 2/d_X$ and $x \in X$. For $0 \le s < 2/d_X$ and $x \in X$, we set $\Gamma(x,s) = \Gamma(x, 2/d_X)(d_X/2)s$. Then note that $\Gamma(x,\cdot)$ is strictly increasing and continuous for each $x \in X$.

LEMMA 4.7 (CF. [14, LEMMA 3.5])

There exists a positive constant C' such that $\Gamma(x, 2/d_X) \ge C'$ for all $x \in X$.

LEMMA 4.8

Assume that $\Phi(x,t)$ satisfies ($\Phi5$). Let $\tau > 1$. Then there exists a constant $C > 0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \le C\Gamma\Big(x,\frac{1}{\delta}\Big)$$

for all $x\in X$, $0<\delta\le d_X/2$, *and nonnegative* $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ *with* $\|f\|_{\Phi,\kappa;\eta,\xi;X}\le 1$.

Proof

Let j_0 be the smallest positive integer such that $\tau^{j_0}\delta\ge d_X$. By Lemma 4.6, we have

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
&=\sum_{j=1}^{j_0}\int_{X\cap(B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta))}\frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
&\le \sum_{j=1}^{j_0}(\tau^j\delta)^\alpha\frac{1}{\mu(B(x,\tau^j\delta))}\int_{X\cap B(x,\tau^j\delta)} f(y)\,d\mu(y)\\
&\le C\Big(\sum_{j=1}^{j_0-1}(\tau^j\delta)^\alpha\Phi^{-1}\big(x,\kappa(x,\tau^j\delta)^{-1}\big)\eta\Big(\big(\log\big(e+1/(\tau^j\delta)\big)\big)^{-1}\Big)^{-1}\\
&\qquad + d_X^\alpha\Phi^{-1}\big(x,\kappa(x,d_X)^{-1}\big)\eta\Big(\big(\log(e+1/d_X)\big)^{-1}\Big)^{-1}\Big),
\end{aligned}$$

where we assume that $\sum_{j=1}^{0} a_j=0$ for $a_j\in\mathbf{R}$. By ($\kappa2$) and (4.1), we have

$$\begin{aligned}
&\int_{\tau^{j-1}\delta}^{\tau^j\delta} t^\alpha\Phi^{-1}\big(x,\kappa(x,t)^{-1}\big)\eta\Big(\big(\log(e+1/t)\big)^{-1}\Big)^{-1}\frac{dt}{t}\\
&\ge(\tau^{j-1}\delta)^\alpha\Phi^{-1}\big(x,Q_2^{-1}\kappa(x,\tau^j\delta)^{-1}\big)\eta\Big(\big(\log\big(e+1/(\tau^j\delta)\big)\big)^{-1}\Big)^{-1}\log\tau\\
&\ge\frac{(\tau^j\delta)^\alpha\log\tau}{\tau^\alpha A_2Q_2}\Phi^{-1}\big(x,\kappa(x,\tau^j\delta)^{-1}\big)\eta\Big(\big(\log\big(e+1/(\tau^j\delta)\big)\big)^{-1}\Big)^{-1}\\
&=C(\tau^j\delta)^\alpha\log\tau\,\Phi^{-1}\big(x,\kappa(x,\tau^j\delta)^{-1}\big)\eta\Big(\big(\log\big(e+1/(\tau^j\delta)\big)\big)^{-1}\Big)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{d_X/2}^{d_X} t^\alpha\Phi^{-1}\big(x,\kappa(x,t)^{-1}\big)\eta\Big(\big(\log(e+1/t)\big)^{-1}\Big)^{-1}\frac{dt}{t}\\
&\ge\frac{d_X^\alpha\log 2}{2^\alpha A_2Q_2}\Phi^{-1}\big(x,\kappa(x,d_X)^{-1}\big)\eta\Big(\big(\log(e+1/d_X)\big)^{-1}\Big)^{-1}\\
&=Cd_X^\alpha\Phi^{-1}\big(x,\kappa(x,d_X)^{-1}\big)\eta\Big(\big(\log(e+1/d_X)\big)^{-1}\Big)^{-1}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
&\le\frac{C}{\log\tau}\Big(\sum_{j=1}^{j_0-1}\int_{\tau^{j-1}\delta}^{\tau^j\delta} t^\alpha\Phi^{-1}\big(x,\kappa(x,t)^{-1}\big)\eta\Big(\big(\log(e+1/t)\big)^{-1}\Big)^{-1}\frac{dt}{t}
\end{aligned}$$

$$+\int_{d_X/2}^{d_X} t^{\alpha}\Phi^{-1}\big(x,\kappa(x,t)^{-1}\big)\eta\Big(\big(\log(e+1/t)\big)^{-1}\Big)^{-1}\frac{dt}{t}\Big)$$
$$\le \frac{C}{\log\tau}\Gamma\Big(x,\frac{1}{\delta}\Big),$$

as required. □

LEMMA 4.9

Assume that $\Phi(x,t)$ satisfies $(\Phi 5)$. Let $\tau > 2$ and $\vartheta > 1$ be such that $\tau > (\vartheta+1)/(\vartheta-1)$. Let $\gamma > 0$, and define

$$\lambda_\gamma(z,r) = \frac{1}{1+\int_r^{d_X}\rho^\gamma\Phi^{-1}(z,\kappa(z,\rho)^{-1})\eta((\log(e+1/\rho))^{-1})^{-1}\frac{d\rho}{\rho}}$$

for $z\in X$ and $0<r<d_X$. Then there exists a constant $C_{I,\gamma}>0$ such that

$$\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))}\int_{X\cap B(z,r)} I_{\gamma,\tau}f(x)\,d\mu(x) \le C_{I,\gamma}$$

for all $z\in X$, $0<r<d_X$, and nonnegative $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X}\le 1$.

Proof

Let $z\in X$, and let $0<r<d_X$. Write

$$\begin{aligned} I_{\gamma,\tau}f(x) &= \int_{X\cap B(z,\vartheta r)} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &\quad + \int_{X\setminus B(z,\vartheta r)} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \\ &= I_1(x)+I_2(x) \end{aligned}$$

for $x\in B(z,r)$. By Fubini's theorem,

$$\begin{aligned} &\int_{X\cap B(z,r)} I_1(x)\,d\mu(x) \\ &\quad = \int_{X\cap B(z,\vartheta r)}\Big(\int_{X\cap B(z,r)} \frac{d(x,y)^\gamma}{\mu(B(x,\tau d(x,y)))}\,d\mu(x)\Big)f(y)\,d\mu(y) \\ &\quad \le \int_{X\cap B(z,\vartheta r)}\Big(\int_{X\cap B(y,(\vartheta+1)r)} \frac{d(x,y)^\gamma}{\mu(B(x,\tau d(x,y)))}\,d\mu(x)\Big)f(y)\,d\mu(y). \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{X\cap B(z,r)} I_1(x)\,d\mu(x) \\ &\le \int_{X\cap B(z,\vartheta r)}\Big(\sum_{j=1}^{\infty}\int_{X\cap(B(y,R_j)\setminus B(y,R_{j+1}))} \frac{d(x,y)^\gamma}{\mu(B(x,\tau d(x,y)))}\,d\mu(x)\Big)f(y)\,d\mu(y) \\ &\le \int_{X\cap B(z,\vartheta r)}\Big(\sum_{j=1}^{\infty}\int_{X\cap(B(y,R_j)\setminus B(y,R_{j+1}))} \frac{R_j^\gamma}{\mu(B(x,\tau R_{j+1}))}\,d\mu(x)\Big)f(y)\,d\mu(y) \end{aligned}$$

$$\le \int_{X\cap B(z,\vartheta r)} \Big(\sum_{j=1}^{\infty} \int_{X\cap (B(y,R_j)\setminus B(y,R_{j+1}))} \frac{R_j^\gamma}{\mu(B(y,R_j))}\,d\mu(x)\Big) f(y)\,d\mu(y)$$

$$\le \int_{X\cap B(z,\vartheta r)} \Big(\sum_{j=1}^{\infty} R_j^\gamma\Big) f(y)\,d\mu(y)$$

$$= \frac{(\vartheta+1)^\gamma(\tau/2)^\gamma}{(\tau/2)^\gamma-1} r^\gamma \int_{X\cap B(z,\vartheta r)} f(y)\,d\mu(y),$$

where $R_j = (\vartheta+1)(\tau/2)^{-j+1}r$. Now, by Lemma 4.6, ($\kappa$2), and (4.1), we have

$$\begin{aligned} r^\gamma & \int_{X\cap B(z,\vartheta r)} f(y)\,d\mu(y) \\ & \le C r^\gamma \mu\big(B(z,\vartheta r)\big)\Phi^{-1}\big(z,\kappa(z,\vartheta r)^{-1}\big)\eta\Big(\big(\log(e+1/(\vartheta r))\big)^{-1}\Big)^{-1} \\ & \le \frac{C}{\log\vartheta}\mu\big(B(z,\vartheta r)\big)\int_r^{\vartheta r} \rho^\gamma \Phi^{-1}\big(z,\kappa(z,\rho)^{-1}\big)\eta\Big(\big(\log(e+1/\rho)\big)^{-1}\Big)^{-1}\frac{d\rho}{\rho} \end{aligned}$$

if $0<r\le d_X/\vartheta$, and by Lemma 4.6 and (4.4), we have

$$\begin{aligned} r^\gamma & \int_{X\cap B(z,\vartheta r)} f(y)\,d\mu(y) \\ & = r^\gamma \int_{B(z,d_X)} f(y)\,d\mu(y) \\ & \le C {d_X}^\gamma \mu\big(B(z,d_X)\big)\Phi^{-1}\big(z,\kappa(z,d_X)^{-1}\big)\eta\Big(\big(\log(e+1/d_X)\big)^{-1}\Big)^{-1} \\ & \le C\mu\big(B(z,\vartheta r)\big) \end{aligned}$$

if $d_X/\vartheta < r < d_X$. Therefore,

$$\int_{X\cap B(z,r)} I_1(x)\,d\mu(x) \le \frac{C}{((\tau/2)^\gamma-1)\log\vartheta}\frac{\mu(B(z,\vartheta r))}{\lambda_\gamma(z,r)}$$

for all $0<r<d_X$.

Set $c=(\tau(\vartheta-1)-1)/\vartheta>1$. For I_2, first note that $I_2(x)=0$ if $x\in X$ and $r\ge d_X/\vartheta$. Let $0<r<d_X/\vartheta$. Let j_0 be the smallest positive integer such that $\vartheta c^{j_0} r\ge d_X$. Here we claim that $x\in B(z,r)$ and $y\in X\setminus B(z,\vartheta r)$ imply that

$$d(y,z)\le \frac{\vartheta}{\vartheta-1} d(x,y) \tag{4.5}$$

and

$$B\big(z,c\,d(z,y)\big)\subset B\big(x,\tau d(x,y)\big). \tag{4.6}$$

Indeed, we have $d(x,z)<r$ and $d(y,z)\ge\vartheta r$. Hence, it follows that

$$d(y,z)\le d(x,y)+d(x,z)\le d(x,y)+\frac{1}{\vartheta}d(y,z),$$

which yields (4.5). Also observe that, when $w\in B(z,c\,d(z,y))$, we have by (4.5)

$$\begin{aligned} d(w,x) &\le d(z,x)+d(w,z) \\ &\le \frac{1}{\vartheta} d(z,y)+c\,d(z,y) \\ &\le \Big(c+\frac{1}{\vartheta}\Big)\frac{\vartheta}{\vartheta-1} d(x,y)=\tau d(x,y), \end{aligned}$$

which yields (4.6).

Consequently it follows from (4.6) that

$$I_2(x) \le C\int_{X\setminus B(z,\vartheta r)} \frac{d(z,y)^\gamma f(y)}{\mu(B(z,c\,d(z,y)))}\,d\mu(y) \quad \text{for } x\in X\cap B(z,r).$$

By Lemma 4.6, we have

$$\begin{aligned} I_2(x) &\le C\sum_{j=1}^{j_0}\int_{B(z,\vartheta c^j r)\setminus B(z,\vartheta c^{j-1} r)} \frac{d(z,y)^\gamma}{\mu(B(z,c\,d(z,y)))} f(y)\,d\mu(y) \\ &\le C\sum_{j=1}^{j_0}(\vartheta c^j r)^\gamma \frac{1}{\mu(B(z,\vartheta c^j r))}\int_{X\cap B(z,\vartheta c^j r)} f(y)\,d\mu(y) \\ &\le C\Big(\sum_{j=1}^{j_0-1}(\vartheta c^j r)^\gamma \Phi^{-1}\big(x,\kappa(x,\vartheta c^j r)^{-1}\big)\eta\Big(\big(\log\big(e+1/(\vartheta c^j r)\big)\big)^{-1}\Big)^{-1} \\ &\qquad + d_X^\gamma \Phi^{-1}\big(x,\kappa(x,d_X)^{-1}\big)\eta\Big(\big(\log(e+1/d_X)\big)^{-1}\Big)^{-1}\Big), \end{aligned}$$

where we assume that $\sum_{j=1}^{0} a_j = 0$ for $a_j\in\mathbf{R}$. As in the proof of Lemma 4.8, we obtain

$$\begin{aligned} I_2(x) &\le \frac{C}{\log c}\Big(\sum_{j=1}^{j_0-1}\int_{\vartheta c^{j-1} r}^{\vartheta c^j r} \rho^\gamma\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\Big(\big(\log(e+1/\rho)\big)^{-1}\Big)^{-1}\frac{d\rho}{\rho} \\ &\qquad + \int_{d_X/2}^{d_X} \rho^\gamma\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\Big(\big(\log(e+1/\rho)\big)^{-1}\Big)^{-1}\frac{d\rho}{\rho}\Big) \\ &\le C\int_r^{d_X} \rho^\gamma\Phi^{-1}\big(z,\kappa(z,\rho)^{-1}\big)\eta\Big(\big(\log(e+1/\rho)\big)^{-1}\Big)^{-1}\frac{d\rho}{\rho} \\ &\le \frac{C}{\log c}\frac{1}{\lambda_\gamma(z,r)} \end{aligned}$$

for all $x\in X\cap B(z,r)$. Hence,

$$\int_{X\cap B(z,r)} I_2(x)\,d\mu(x) \le \frac{C}{\log c}\frac{\mu(B(z,r))}{\lambda_\gamma(z,r)} \le \frac{C}{\log c}\frac{\mu(B(z,\vartheta r))}{\lambda_\gamma(z,r)}.$$

Thus, this lemma is proved. □

5. Trudinger's inequality for grand Musielak–Orlicz–Morrey spaces

In this section, we deal with the case in which $\Gamma(x,t)$ satisfies the uniform log-type condition:

($\Gamma_{\log}$) there exists a constant $c_\Gamma > 0$ such that

$$\Gamma(x,t^2) \le c_\Gamma \Gamma(x,t)$$

for all $x \in X$ and $t \ge 1$.

By ($\Gamma_{\log}$), together with Lemma 4.7, we see that $\Gamma(x,t)$ satisfies the uniform doubling condition in t.

LEMMA 5.1 (CF. [14, LEMMA 4.2])

Suppose $\Gamma(x,t)$ satisfies ($\Gamma_{\log}$). For every $a > 1$, there exists $b > 0$ such that $\Gamma(x,at) \le b\Gamma(x,t)$ for all $x \in X$ and $t > 0$.

THEOREM 5.2

Assume that $\Phi(x,t)$ satisfies ($\Phi5$) and $\Gamma(x,t)$ satisfies ($\Gamma_{\log}$). For each $x \in X$, let $\gamma(x) = \sup_{s>0} \Gamma(x,s)$. Suppose $\Psi(x,t) : X \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:

($\Psi1$) *$\Psi(\cdot,t)$ is measurable on X for each $t \in [0,\infty)$ and $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;*

($\Psi2$) *there is a constant $A_1' \ge 1$ such that $\Psi(x,t) \le \Psi(x,A_1's)$ for all $x \in X$ whenever $0 < t < s$;*

($\Psi3$) *$\Psi(x,\Gamma(x,t)/A_2') \le A_3't$ for all $x \in X$ and $t > 0$ with constants $A_2', A_3' \ge 1$ independent of x.*

Let $\tau > 2$ and $\vartheta > 1$ be such that $\tau > (\vartheta+1)/(\vartheta-1)$. Then, for $0 < \gamma < \alpha$, there exists a constant $C^* > 0$ such that $I_{\alpha,\tau}f(x)/C^* < \gamma(x)$ for almost every $x \in X$ and

$$\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))} \int_{X\cap B(z,r)} \Psi\Big(x, \frac{I_{\alpha,\tau}f(x)}{C^*}\Big)\, d\mu(x) \le 1$$

for all $z \in X$, $0 < r < d_X$, and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$.

Proof

Let $f \ge 0$ and $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$. Fix $x \in X$. For $0 < \delta \le d_X/2$, Lemma 4.8 implies

$$\begin{aligned} I_{\alpha,\tau}f(x) &\le \int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) + C\Gamma\Big(x,\frac{1}{\delta}\Big) \\ &= \int_{X\cap B(x,\delta)} d(x,y)^{\alpha-\gamma} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) + C\Gamma\Big(x,\frac{1}{\delta}\Big) \\ &\le C\Big\{\delta^{\alpha-\gamma} I_{\gamma,\tau}f(x) + \Gamma\Big(x,\frac{1}{\delta}\Big)\Big\} \end{aligned}$$

with constants $C > 0$ independent of x.

If $I_{\gamma,\tau}f(x) \le 2/d_X$, then we take $\delta = d_X/2$. Then, by Lemma 4.7

$$I_{\alpha,\tau}f(x) \le C\Gamma\Big(x,\frac{2}{d_X}\Big).$$

By Lemma 5.1, there exists $C_1^* > 0$ independent of x such that

$$I_{\alpha,\tau} f(x) \le C_1^* \Gamma\Big(x, \frac{1}{2A_3'}\Big) \quad \text{if } I_{\gamma,\tau} f(x) \le 2/d_X. \tag{5.1}$$

Next, suppose $2/d_X < I_{\gamma,\tau} f(x) < \infty$. Let $m = \sup_{s \ge 2/d_X, x \in X} \Gamma(x,s)/s$. By $(\Gamma_{\log})$, $m < \infty$. Define δ by

$$\delta^{\alpha-\gamma} = \frac{(d_X/2)^{\alpha-\gamma}}{m} \Gamma\big(x, I_{\gamma,\tau} f(x)\big) \big(I_{\gamma,\tau} f(x)\big)^{-1}.$$

Since $\Gamma(x, I_{\gamma,\tau} f(x))(I_{\gamma,\tau} f(x))^{-1} \le m$, $0 < \delta \le d_X/2$. Then by Lemma 4.7

$$\begin{aligned}
\frac{1}{\delta} &\le C\Gamma\big(x, I_{\gamma,\tau} f(x)\big)^{-1/(\alpha-\gamma)} \big(I_{\gamma,\tau} f(x)\big)^{1/(\alpha-\gamma)} \\
&\le C\Gamma(x, 2/d_X)^{-1/(\alpha-\gamma)} \big(I_{\gamma,\tau} f(x)\big)^{1/(\alpha-\gamma)} \\
&\le C\big(I_{\gamma,\tau} f(x)\big)^{1/(\alpha-\gamma)}.
\end{aligned}$$

Hence, using $(\Gamma_{\log})$ and Lemma 5.1, we obtain

$$\Gamma\Big(x, \frac{1}{\delta}\Big) \le \Gamma\Big(x, C\big(I_{\gamma,\tau} f(x)\big)^{1/(\alpha-\gamma)}\Big) \le C\Gamma\big(x, I_{\gamma,\tau} f(x)\big).$$

By Lemma 5.1 again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$I_{\alpha,\tau} f(x) \le C_2^* \Gamma\Big(x, \frac{1}{2C_{I,\gamma} A_3'} I_{\gamma,\tau} f(x)\Big) \quad \text{if } 2/d_X < I_{\gamma,\tau} f(x) < \infty, \tag{5.2}$$

where $C_{I,\gamma}$ is the constant given in Lemma 4.9.

Now, let $C^* = A_1' A_2' \max(C_1^*, C_2^*)$. Then, by (5.1) and (5.2),

$$\frac{I_{\alpha,\tau} f(x)}{C^*} \le \frac{1}{A_1' A_2'} \max\Big\{\Gamma\Big(x, \frac{1}{2A_3'}\Big), \Gamma\Big(x, \frac{1}{2C_{I,\gamma} A_3'} I_{\gamma,\tau} f(x)\Big)\Big\}$$

whenever $I_{\gamma,\tau} f(x) < \infty$. Since $I_{\gamma,\tau} f(x) < \infty$ for almost every $x \in X$ by Lemma 4.9, $I_{\alpha,\tau} f(x)/C^* < \gamma(x)$ for almost every $x \in X$, and by $(\Psi 2)$ and $(\Psi 3)$, we have

$$\begin{aligned}
&\Psi\Big(x, \frac{I_{\alpha,\tau} f(x)}{C^*}\Big) \\
&\le \max\Big\{\Psi\Big(x, \Gamma\Big(x, \frac{1}{2A_3'}\Big)/A_2'\Big), \Psi\Big(x, \Gamma\Big(x, \frac{1}{2C_{I,\gamma} A_3'} I_{\gamma,\tau} f(x)\Big)/A_2'\Big)\Big\} \\
&\le \frac{1}{2} + \frac{1}{2C_{I,\gamma}} I_{\gamma,\tau} f(x)
\end{aligned}$$

for almost every $x \in X$. Thus, noting that $\lambda_\gamma(z,r) \le 1$ and using Lemma 4.9, we have

$$\begin{aligned}
&\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))} \int_{X \cap B(z,r)} \Psi\Big(x, \frac{I_{\alpha,\tau} f(x)}{C^*}\Big)\, d\mu(x) \\
&\quad \le \frac{1}{2}\lambda_\gamma(z,r) + \frac{1}{2C_{I,\gamma}} \frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))} \int_{X \cap B(z,r)} I_{\gamma,\tau} f(x)\, d\mu(x)
\end{aligned}$$

$$\leq \frac{1}{2} + \frac{1}{2}$$
$$= 1$$

for all $z \in X$ and $0 < r < d_X$. □

REMARK 5.3

If $\Gamma(x,s)$ is bounded, that is,

$$\sup_{x \in X} \int_0^{d_X} \rho^\alpha \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) \eta\Big(\big(\log(e+1/\rho)\big)^{-1}\Big)^{-1} d\rho < \infty,$$

then by Lemma 4.8 we see that $I_{\alpha,\tau}|f|$ is bounded for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$.

REMARK 5.4

We cannot take $\gamma = \alpha$ in Theorem 5.2. For details, see [17, Remark 2.8].

As in the proof of [14, Corollary 4.6], we obtain the following corollary by applying Theorem 5.2 to the special Φ and κ given in Examples 2.1 and 2.2.

COROLLARY 5.5

Let κ be as in Example 2.2, and let $p(x)$ and $q(x) = q_1(x)$ be as in Example 2.1. Let $\tau > 2$ and $\vartheta > 1$ be such that $\tau > (\vartheta+1)/(\vartheta-1)$. Set $\eta(t) = t^\theta$ for $\theta > 0$ and $\Phi(x,t) = t^{p(x)}(\log(e+t))^{q(x)}$.

Assume that

$$\alpha - \nu(x)/p(x) = 0 \quad \textit{for all } x \in X.$$

(1) *Suppose*

$$\inf_{x \in X} \big(-q(x)/p(x) - \beta(x)/p(x) + \theta + 1\big) > 0.$$

Then for $0 < \gamma < \alpha$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\gamma}}{\mu(B(z,\vartheta r))} \int_{B(z,r)\cap X} \exp\bigg(\Big(\frac{I_{\alpha,\tau}f(x)}{C^*}\Big)^{p(x)/(p(x)+\theta p(x)-\beta(x)-q(x))}\bigg) d\mu(x) \leq C^{**}$$

for all $z \in X$, $0 < r \leq d_X$, and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$.

(2) *If*

$$\sup_{x \in X} \big(-q(x)/p(x) - \beta(x)/p(x) + \theta + 1\big) \leq 0,$$

then for $0 < \gamma < \alpha$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\gamma}}{\mu(B(z,\vartheta r))} \int_{B(z,r)\cap X} \exp\bigg(\exp\Big(\frac{I_{\alpha,\tau}f(x)}{C^*}\Big)\bigg) d\mu(x) \leq C^{**}$$

for all $z \in X$, $0 < r < d_X$, and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$.

6. Continuity for grand Musielak–Orlicz–Morrey spaces

In this section, we discuss the continuity of Riesz potentials $I_{\alpha,\tau}f$ of functions in grand Musielak–Orlicz–Morrey spaces under the condition that there are constants $\theta>0$, $\iota>1$, and $C_0>0$ such that

$$\left|\frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))}-\frac{d(z,y)^\alpha}{\mu(B(z,\tau d(z,y)))}\right|\le C_0\left(\frac{d(x,z)}{d(x,y)}\right)^\theta\frac{d(x,y)^\alpha}{\mu(B(x,\iota d(x,y)))} \tag{6.1}$$

whenever $d(x,z)\le d(x,y)/2$.

We consider the functions

$$\omega(x,r)=\int_0^r \rho^\alpha\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\big(\big(\log(e+1/\rho)\big)^{-1}\big)^{-1}\frac{d\rho}{\rho}$$

and

$$\omega_\theta(x,r)=r^\theta\int_r^{d_X} \rho^{\alpha-\theta}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\big(\big(\log(e+1/\rho)\big)^{-1}\big)^{-1}\frac{d\rho}{\rho}$$

for $\theta>0$ and $0<r\le d_X$.

LEMMA 6.1 (CF. [14, LEMMA 5.1])
Let $E\subset X$. If $\omega(x,r)\to 0$ as $r\to 0+$ uniformly in $x\in E$, then $\omega_\theta(x,r)\to 0$ as $r\to 0+$ uniformly in $x\in E$.

LEMMA 6.2 (CF. [14, LEMMA 5.2])
There exists a constant $C>0$ such that

$$\omega(x,2r)\le C\omega(x,r)$$

for all $x\in X$ and $0<r\le d_X/2$.

THEOREM 6.3
Assume that $\Phi(x,t)$ satisfies $(\Phi5)$. Let $\tau>1$. Then there exists a constant $C>0$ such that

$$\big|I_{\alpha,\tau}f(x)-I_{\alpha,\tau}f(z)\big|\le C\big\{\omega\big(x,d(x,z)\big)+\omega\big(z,d(x,z)\big)+\omega_\theta\big(x,d(x,z)\big)\big\}$$

for all $x,z\in X$ with $d(x,z)\le d_X/4$ and nonnegative $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X}\le 1$.

Before giving a proof of Theorem 6.3, we prepare two more lemmas.

LEMMA 6.4
Assume that $\Phi(x,t)$ satisfies $(\Phi5)$. Let $\tau>1$, and let f be a nonnegative function on X such that $\|f\|_{\Phi,\kappa;\eta,\xi;X}\le 1$. Then there exists a constant $C>0$ such that

$$\int_{X\cap B(x,\delta)}\frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\le C\omega(x,\delta)$$

for all $x\in X$ and $0<\delta\le d_X$.

Proof
Let f be a nonnegative function on X with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$. As usual we start by decomposing $B(x,\delta)$ dyadically:

$$\begin{aligned}
&\int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
&\quad= \sum_{j=1}^{\infty} \int_{X\cap(B(x,\tau^{-j+1}\delta)\setminus B(x,\tau^{-j}\delta))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
&\quad\le \sum_{j=1}^{\infty} (\tau^{-j+1}\delta)^\alpha \frac{1}{\mu(B(x,\tau^{-j+1}\delta))} \int_{B(x,\tau^{-j+1}\delta)} f(y)\,d\mu(y).
\end{aligned}$$

By Lemma 4.6, we have

$$\begin{aligned}
&\int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
&\le C\sum_{j=1}^{\infty} (\tau^{-j+1}\delta)^\alpha \Phi^{-1}\big(x,\kappa(x,\tau^{-j+1}\delta)^{-1}\big)\eta\Big(\big(\log\big(e+1/(\tau^{-j+1}\delta)\big)\big)^{-1}\Big)^{-1}\\
&\le \frac{C}{\log\tau}\int_0^\delta \rho^\alpha \Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\Big(\big(\log(e+1/\rho)\big)^{-1}\Big)^{-1}\frac{d\rho}{\rho}\\
&= C\omega(x,\delta).
\end{aligned}$$

□

The following lemma can be proved in the same manner as Lemma 4.8.

LEMMA 6.5
Assume that $\Phi(x,t)$ satisfies $(\Phi 5)$. Let $\theta \in \mathbf{R}$, and let $\tau > 1$. Let f be a nonnegative function on X such that $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$. Then there exists a constant $C > 0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \le C\delta^{-\theta}\omega_\theta(x,\delta)$$

for all $x \in X$ and $0 < \delta \le d_X/2$.

Proof of Theorem 6.3
Let f be a nonnegative function on X with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$, and let $x, z \in X$ with $d(x,z) \le d_X/4$. Write

$$\begin{aligned}
&I_{\alpha,\tau}f(x) - I_{\alpha,\tau}f(z)\\
&\quad= \int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
&\qquad- \int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\,d\mu(y)\\
&\qquad+ \int_{X\setminus B(x,2d(x,z))} \Big(\frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,\tau d(z,y)))}\Big) f(y)\,d\mu(y).
\end{aligned}$$

Using Lemmas 6.2 and 6.4, we have

$$\int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y) \le C\omega\big(x,2d(x,z)\big) \le C\omega\big(x,d(x,z)\big)$$

and

$$\begin{aligned}\int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\,d\mu(y) &\le \int_{X\cap B(z,3d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\,d\mu(y)\\ &\le C\omega\big(z,3d(x,z)\big) \le C\omega\big(z,d(x,z)\big).\end{aligned}$$

On the other hand, by (6.1) and Lemma 6.5, we have

$$\begin{aligned}&\int_{X\setminus B(x,2d(x,z))} \left|\frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,\tau d(z,y)))}\right| f(y)\,d\mu(y)\\ &\le Cd(x,z)^\theta \int_{X\setminus B(x,2d(x,z))} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\iota d(x,y)))}\,d\mu(y)\\ &\le C\omega_\theta\big(x,2d(x,z)\big) \le C\omega_\theta\big(x,d(x,z)\big).\end{aligned}$$

Then we have the conclusion. □

In view of Lemma 6.1, we obtain the following corollary.

COROLLARY 6.6

Assume that $\Phi(x,t)$ satisfies $(\Phi5)$. Let $\tau > 1$.

(a) *Let $x_0 \in X$, and suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in X \cap B(x_0,\delta)$ for some $\delta > 0$. Then $I_{\alpha,\tau} f$ is continuous at x_0 for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$.*

(b) *Suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in X$. Then $I_{\alpha,\tau} f$ is uniformly continuous on X for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$.*

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