A characterization of the fullness of continuous cores of type III_1 free product factors

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Abstract We prove that, for any type III_1 free product factor, its continuous core is full if and only if its τ-invariant is the usual topology on the real line. This trivially implies, as a particular case, the same result for free Araki–Woods factors. Moreover, our method shows the same result for full (generalized) Bernoulli crossed product factors of type III_1.

1. Introduction

Let M_1, M_2 be two nontrivial von Neumann algebras with separable preduals, and let φ_1, φ_2 be faithful normal states on them, respectively. Let $(M, \varphi) = (M_1, \varphi_1) \star (M_2, \varphi_2)$ be their free product (see, e.g., [25, Section 2.1]). Then M must be of the form $M = M_d \oplus M_c$ or M_c, where M_d is finite-dimensional (which can explicitly be determined) and M_c is diffuse. In what follows, we assume that $(\dim M_1, \dim M_2) \neq (2, 2)$; otherwise $M_c = L^\infty[0,1] \mathbin{\bar{\otimes}} M_2(\mathbb{C})$. Then M_c is a full factor of type II_1 (if both φ_i's are tracial), III_λ with $0 < \lambda < 1$ (if the modular actions σ^{φ_i} have a common (positive) period and the smallest one is $2\pi/|\log\lambda|$), or III_1 (otherwise). Hence, we call M_c a *free product factor* in what follows. Moreover, Connes's [4] τ-invariant $\tau(M_c)$ coincides with the weakest topology on $\mathbb{R}$ making the mapping $t \in \mathbb{R} \mapsto \sigma_t^\varphi \in \mathrm{Aut}(M)$ (equipped with the so-called u-topology; see, e.g., [4, Section III]) continuous. See [25, Theorem 4.1] and [26, Theorem 3.1] (with a trivial argument; see the proof of Theorem 1(2) $\Leftrightarrow$ Theorem 1(3) below) for these facts, respectively. In this way, almost all the basic invariants have been made clear for M (and M_c), but it still remains an open question when the continuous core of M_c becomes a full factor (if M_c is of type III_1). Here, for a given type III von Neumann algebra, we call the carrier algebra of its so-called *associated covariant system* (see [20, Definitions XII.1.3, XII.1.5]) its *continuous core*. In this article, we would like to report the following simple solution to the question.

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THEOREM 1
Assume that M_c is of type III_1. *Then the following conditions are equivalent.*

(1) *The continuous core* $\widetilde{M_c} := M_c \rtimes_{\sigma^{\varphi_c}} \mathbb{R}$ *with* $\varphi_c := \varphi\restriction_{M_c}$ *is full.*

(2) *The* τ*-invariant* $\tau(M_c)$, *that is, the weakest topology on* $\mathbb{R}$ *making the mapping* $t \in \mathbb{R} \mapsto \sigma_t^\varphi \in \mathrm{Aut}(M)$ *continuous in this particular case (see the above explanation), is the usual topology on* $\mathbb{R}$.

(3) *For any sequence* t_n *in* $\mathbb{R}$ *we have* $(\sigma_{t_n}^{\varphi_1}, \sigma_{t_n}^{\varphi_2}) \longrightarrow (\mathrm{Id}_{M_1}, \mathrm{Id}_{M_2})$ *in* $\mathrm{Aut}(M_1) \times \mathrm{Aut}(M_2)$ *as* $n \to \infty$ *implies* $t_n \longrightarrow 0$ *in the usual topology on* $\mathbb{R}$ *as* $n \to \infty$.

The above theorem completes the project to compute all the basic invariants for arbitrary free product von Neumann algebras. (Here we would like to mention that the triviality of the asymptotic bicentralizer of any type III_1 free product factor was confirmed by the second-named author by using only [7, Theorem 4, Corollary 8] and [25, Corollary 3.2, Theorem 4.1].) One of the important features of Theorem 1 is that the consequence is formulated in terms of modular automorphisms associated with given states rather than the τ-invariant itself; hence, it is suitable for practical use. Moreover, the next corollary is obtained as a particular case of the theorem (see Remark 10).

COROLLARY 2
Let $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ *be a free Araki–Woods factor of type* III_1 *(see [16]). Then the continuous core of* $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ *is full if and only if the weakest topology on* $\mathbb{R}$ *making* $t \mapsto U_t$ *(with respect to the strong operator topology) is the usual one.*

There are previously known cases where the continuous cores of free Araki–Woods factors become full (see Shlyakhtenko [17, Theorem 4.8], Houdayer [10, Theorem 1.2], and more recently Houdayer–Raum [11, Theorem B]; note that the second needs [14, Proposition 7] with $\mathcal{N}_0 = \mathcal{M}$ there). However, any kind of characterization such as the above corollary has never been known.

Another important class of full factors of type III_1 whose τ-invariants are already computed consists of *Bernoulli crossed products*. In fact, Vaes and Verraedt [30, Section 2.5] recently proved that any Bernoulli crossed product of nonamenable group must be a full factor and, moreover, computed its τ-invariant in terms of given data, generalizing Connes's [4] original work. In the appendix, we will explain that our method of proving Theorem 1 works well even for (generalized) Bernoulli crossed products (see Theorem 12 for the precise assertion).

This article uses the same standard notation as in [25] and [26] (except the appendix, where the notation follows [30, Section 2.5]). We will freely use (Ocneanu) ultraproducts and asymptotic centralizers (denoted by $N^\omega \supseteq N_\omega$, respectively, for given von Neumann algebras N), for which we refer to [25, Section 2.2] as a brief summary and to [1] as a detailed reference. Our discussion below is fairly simple, though it depends upon some previous works, namely,

[27], [28, Section 2.1] (based on [25, Theorem 4.1]), [30, Section 2.5], and the automorphism analysis due to Connes and Ocneanu.

2. Preliminary facts

Let us start with a general lemma on group actions on factors. Our intuition about it came from quite the recent work [12, Theorems 6.7, 6.8] about a characterization of the Rohlin property for flows on von Neumann algebras.

LEMMA 3

Let $\alpha\colon \Gamma \curvearrowright N$ be an action of a countable discrete abelian group on a factor with separable predual. Let $\alpha_\omega\colon \Gamma \curvearrowright N_\omega$ be the action on the asymptotic centralizer N_ω arising from α. Then, for every $p \in \mathrm{Ker}(\alpha_\omega)^\perp$ (in the dual $\widehat{\Gamma}$) there exists a unitary $u \in N_\omega$ such that $\alpha_{\omega,\gamma}(u) = \langle \gamma, p\rangle u$ holds for all $\gamma \in \Gamma$, where $\langle\cdot,\cdot\rangle$ is the dual pairing between Γ and $\widehat{\Gamma}$ and $\Lambda^\perp := \{p \in \widehat{\Gamma} \mid \langle \Lambda, p\rangle = 0\}$ for a subgroup Λ of Γ. Moreover, for every $p \in \mathrm{Ker}(\alpha_\omega)^\perp$, the dual action $\widehat{\alpha}_p$ is approximately inner, that is, it falls in the closure of $\mathrm{Int}(N \rtimes_\alpha \Gamma)$.

Proof

By [5, Proposition 2.1.2], the action α_ω induces a properly outer action of $\Gamma/\mathrm{Ker}(\alpha_\omega)$ on N_ω. Note that the dual of $\Gamma/\mathrm{Ker}(\alpha_\omega)$ is naturally identified with $\mathrm{Ker}(\alpha_\omega)^\perp$ in $\widehat{\Gamma}$. Let $p \in \mathrm{Ker}(\alpha_\omega)^\perp$ be arbitrarily chosen. We apply the so-called 1-*cohomology vanishing theorem* [13, Section 7.2] to the (rather simple) cocycle $\gamma \mapsto \langle \gamma, p\rangle 1 \in N_\omega$ with the above properly outer action, and we get the desired unitary $u \in N_\omega$.

Since $u \in N_\omega$ ($\subseteq N' \cap N^\omega$ trivially), one easily observes that $\widehat{\alpha}_p(x) = uxu^*$ holds inside $(N \rtimes_\alpha \Gamma)^\omega$ for every $x \in N \rtimes_\alpha \Gamma$. Thanks to [5, Proposition 1.1.3(b)] we can choose a representing sequence u_n of u in such a way that it consists of unitaries. Let ψ be a faithful normal state on N, and set $\widetilde{\psi} := \psi \circ E_N$ with the canonical conditional expectation $E_N\colon N \rtimes_\alpha \Gamma \to N$. For any $y, z \in N \rtimes_\alpha \Gamma$ one has

$$
\begin{aligned}
&\big|\big((y\widetilde{\psi}) \circ \widehat{\alpha}_p - (y\widetilde{\psi}) \circ \mathrm{Ad}\, u_n\big)(z)\big| \\
&\quad \le \big\|\widehat{\alpha}_p^{-1}(y) - u_n^* y u_n\big\|_{\widetilde{\psi}} \|z\|_\infty + \big|\widetilde{\psi}(u_n z u_n^* y) - \widetilde{\psi}(z u_n^* y u_n)\big| \\
&\quad \le \big\|\widehat{\alpha}_p^{-1}(y) - u_n^* y u_n\big\|_{\widetilde{\psi}} \|z\|_\infty + \|\psi u_n - u_n \psi\| \|y\|_\infty \|z\|_\infty
\end{aligned}
$$

so that

$$
\big\|(y\widetilde{\psi}) \circ \widehat{\alpha}_p - (y\widetilde{\psi}) \circ \mathrm{Ad}\, u_n\big\| \le \big\|\widehat{\alpha}_p^{-1}(y) - u_n^* y u_n\big\|_{\widetilde{\psi}} + \|\psi u_n - u_n \psi\| \|y\|_\infty \longrightarrow 0
$$

as $n \to \omega$. Therefore, $\widehat{\alpha}_p = \lim_{n\to\omega} \mathrm{Ad}\, u_n$ in $\mathrm{Aut}(N \rtimes_\alpha \Gamma)$, because the $y\widetilde{\psi}$'s form a dense subset of the predual. □

For a given type III_λ factor, we call the canonical type II_∞ factor $\mathcal{N}_0$ in [20, Theorem XII.2.1] the *discrete core* of the type III_λ factor. The discrete core is indeed uniquely determined from the given type III_λ factor. Moreover, it can explicitly be constructed based on the Takesaki duality (see, e.g., the proof of [20, Theorem XII.2.1] and also [28, Section 2.2]). Here is a small remark on this fact for the reader's convenience. Let Q be a type III_λ factor with separable predual ($0<\lambda<1$), and let χ be a faithful normal state on Q such that $\sigma_T^\chi = \mathrm{Id}_Q$ with $T := 2\pi/|\log\lambda|$ (the existence of such a state is well known; see, e.g., [6, Theorem 2.3], which fits the discussion here). Then we can view σ^χ as an action of $\mathbb{R}/T\mathbb{Z}$. Note that $Q \rtimes_{\sigma^\chi} (\mathbb{R}/T\mathbb{Z})$ must be properly infinite, since Q is of type III. It follows that $Q \rtimes_{\sigma^\chi} (\mathbb{R}/T\mathbb{Z}) \cong (Q \rtimes_{\sigma^\chi} (\mathbb{R}/T\mathbb{Z})) \bar{\otimes} B(\ell^2) \cong (Q \bar{\otimes} B(\ell^2))_{\sigma^\chi \bar{\otimes} \mathrm{Tr}} (\mathbb{R}/T\mathbb{Z})$ (with a faithful normal semifinite trace Tr on $B(\ell^2)$), which is confirmed to be the discrete core of Q in the proof of [20, Theorem XII.2.1].

PROPOSITION 4

Let $\lambda \in (0,1)$, and set $T := 2\pi/|\log\lambda|$. Let Q be a type III_λ factor with separable predual. Then Q is full if and only if so is its discrete core $\widehat{Q} := Q \rtimes_{\sigma^\chi} (\mathbb{R}/T\mathbb{Z})$ with a periodic state χ (i.e., a faithful normal state with $\sigma_T^\chi = \mathrm{Id}_Q$).

Proof

The 'if' part. Assume that $\widehat{Q}$ is full. It is known that $\widehat{Q}$ is stably isomorphic to Q_χ. Hence, Q_χ is also full. By [4, Proposition 2.3(2)] Q must be full.

The 'only if' part. Assume next that Q is full. Let us denote by $\theta\colon \mathbb{Z} \curvearrowright \widehat{Q}$ the dual action of $\sigma^\chi\colon \mathbb{R}/T\mathbb{Z} \curvearrowright Q$.

Suppose that θ_ω is a nontrivial action. By Lemma 3 there exists $\zeta \in \mathbb{T}\setminus\{1\}$ so that the dual action $\widehat{\theta}_\zeta$ falls in the closure of $\mathrm{Int}(\widehat{Q} \rtimes_\theta \mathbb{Z})$. Since $\widehat{Q} \rtimes_\theta \mathbb{Z} \cong Q$ is full, we conclude that $\widehat{\theta}_\zeta$ must be inner. However, $\widehat{\theta}$ is the bidual action of $\sigma^\chi\colon [0,T) = \mathbb{R}/T\mathbb{Z} = \mathbb{T} \curvearrowright Q$, and therefore, by [20, Theorem X.2.3(iv)] σ_t^χ is inner for some $0<t<T$, a contradiction. Hence, we have shown that θ_ω is indeed the trivial action.

Let $v \in \widehat{Q}_\omega$ be an arbitrary unitary. Since θ_ω is trivial, we observe that $x = vxv^*$ inside $(\widehat{Q} \rtimes_\theta \mathbb{Z})^\omega$ for every $x \in \widehat{Q} \rtimes_\theta \mathbb{Z}$. The same argument as that for getting $\widehat{\alpha}_p = \lim_{n\to\omega} \mathrm{Ad}\, u_n$ in the proof of Lemma 3 shows that $v \in (\widehat{Q} \rtimes_\theta \mathbb{Z})_\omega \cong Q_\omega = \mathbb{C}1$. □

The proof of Proposition 4 (especially, its only if part) actually works well, without any essential change, for showing the next proposition. Note that the discrete decomposition is well defined for any full type III_1 factor as long as it is possible (see [4] and also [28, Section 2.2] for its explicit construction based on the Takesaki duality).

PROPOSITION 5

The discrete core of any full type III_1 factor must be full (if it exists).

Our question is about the fullness of certain *continuous* crossed product factors *of type* II_∞, but the next lemma says that it is equivalent to that of certain *discrete* crossed product factors *of type* III_λ.

LEMMA 6

Let P be a type III_1 factor with separable predual, and let χ be a faithful normal state on it. Let $\lambda \in (0,1)$ be arbitrarily chosen, and set $T := 2\pi/|\log\lambda|$. Then the continuous core $\widetilde{P}$ is full if and only if the type III_λ factor $Q := P \rtimes_{\sigma^\chi_T} \mathbb{Z}$ is full.

Proof

Although many proofs seem to be available for this fact (see [21, Lemma XVIII.4.17(i)] and its proof), we would like to give a proof, which we believe to be elementary, for the reader's convenience. Let $E\colon P \rtimes_{\sigma^\chi_T} \mathbb{Z} \to P$ be the canonical conditional expectation. By, for example, [15, Proposition 2.1] we have $(P \rtimes_{\sigma^\chi_T} \mathbb{Z}) \rtimes_{\sigma^{\chi\circ E}} \mathbb{R} \cong (P \rtimes_{\sigma^\chi} \mathbb{R}) \rtimes_{\sigma^\chi_T \bar\otimes \mathrm{Id}} \mathbb{Z} \cong (P \rtimes_{\sigma^\chi} \mathbb{R}) \bar\otimes L(\mathbb{Z})$, which sends the generators $x \in P$, $\lambda^{\sigma^\chi_T}(n)$ $(n \in \mathbb{Z})$, and $\lambda^{\sigma^{\chi\circ E}}(t)$ $(t \in \mathbb{R})$ in the leftmost algebra to $x \otimes 1$, $(\lambda^{\sigma^\chi}(T) \otimes u)^n$, and $\lambda^{\sigma^\chi}(t) \otimes 1$ in the rightmost algebra with the canonical generator u of $L(\mathbb{Z})$. In particular, the center of the leftmost algebra is generated by $v := \lambda^{\sigma^\chi_T}(1)\lambda^{\sigma^{\chi\circ E}}(T)^*$, since we know that $P \rtimes_{\sigma^\chi} \mathbb{R}$ is a factor. (Note that P is a factor of type III_1.) Then the dual action θ of $\sigma^{\chi\circ E}$ satisfies $\theta_s(v) = e^{isT}v$, $s \in \mathbb{R}$. Therefore, the (smooth) flow of weights of $Q = P \rtimes_{\sigma^\chi_T} \mathbb{Z}$ is a transitive flow of period $-\log\lambda$ so that Q must be a type III_λ factor. Choose a faithful normal state ψ on Q with $\sigma^\psi_T = \mathrm{Id}_Q$ (see the explanation before Proposition 4). Then we see that $\widetilde{P} \bar\otimes L(\mathbb{Z}) \cong Q \rtimes_{\sigma^{\chi\circ E}} \mathbb{R} \cong Q \rtimes_{\sigma^\psi} \mathbb{R} \cong (Q \rtimes_{\sigma^\psi} (\mathbb{R}/T\mathbb{Z})) \bar\otimes L(\mathbb{Z})$, where the last isomorphism follows from [9, Proposition 5.6]. (Note that its proof uses only $\sigma^\chi_T = \mathrm{Id}_Q$.) By the uniqueness of the central decomposition we obtain $\widetilde{P} \cong Q \rtimes_{\sigma^\psi} (\mathbb{R}/T\mathbb{Z})$. Thus, the desired assertion immediately follows from Proposition 4. $\square$

We remark that the use of central decomposition can be replaced with taking the fixed-point algebra under the canonical extension of the dual action of σ^χ_T to the continuous core of Q.

3. Proof of Theorem 1

Our main concern is to prove Theorem 1(2) $\Rightarrow$ Theorem 1(1). If both M_1, M_2 are (possibly infinite) direct sums of type I factors, then both φ_1, φ_2 are almost periodic and so is the positive linear functional φ_c (see [26, Theorem 2.1]); hence, $\tau(M_c)$ never becomes the usual topology on $\mathbb{R}$. Therefore, we may and do assume that M_1 has a diffuse direct summand. Note here that the τ-invariant is a von Neumann algebraic invariant. Hence, by the trick explained at the beginning of [28, Section 2.1] we may and do further assume that M_1 is either (a) a diffuse von Neumann algebra with no type III_1 factor direct summands or (b) a type III_1 factor. In each case, $M = M_c$ holds thanks to [25, Theorem 4.1]. In what follows, we fix $\lambda \in (0,1)$ and set $T := 2\pi/|\log\lambda|$, and it suffices, thanks to Lemma 6, to prove that $M \rtimes_{\sigma^\varphi_T} \mathbb{Z}$ is full under Theorem 1(2). We need two technical lemmas.

LEMMA 7

With the conditional expectation $E_{\varphi_1} := (\varphi_1 \bar{\otimes} \mathrm{Id}) \restriction_{M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}} \colon M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z} \to \mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$, one can find a faithful normal state ψ on $\mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$ so that for each natural number $n \geq 2$ there exists a unitary $u_n \in (M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})_{\psi\circ E_{\varphi_1}}$ such that $E_{\varphi_1}(u_n^k) = 0$ as long as $1 \leq k \leq n-1$.

Proof

We first treat case (a). It is easy to see that $(M_1)_{\varphi_1}$ is diffuse (see the proof of [25, Theorem 3.4] with standard facts; see, e.g., [23, Lemmas 11, 12]). Let $E_{M_1} : M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z} \to M_1$ be the canonical conditional expectation. One can easily confirm $\tau_{\mathbb{Z}} \circ E_{\varphi_1} = \varphi_1 \circ E_{M_1}$ with the canonical tracial state $\tau_{\mathbb{Z}}$ on $L(\mathbb{Z}) = \mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$ naturally, and thus, $(M_1)_{\varphi_1} \bar{\otimes} L(\mathbb{Z})$ naturally sits in $(M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})_{\tau_{\mathbb{Z}}\circ E_{\varphi_1}}$. Let $v \in (M_1)_{\varphi_1}$ be a Haar unitary with respect to φ_1 (see, e.g., the proof of [25, Theorem 3.7] for its existence), and $\psi := \tau_{\mathbb{Z}}$ and $u_n := v \otimes 1$ (for every n) are the desired ones.

We then treat case (b). Let us denote by $u \in \mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$ the canonical unitary generator. By a standard fact (see, e.g., [20, Theorem X.1.17]) together with the identity $\tau_{\mathbb{Z}} \circ E_{\varphi_1} = \varphi_1 \circ E_{M_1}$ we observe that $\sigma_T^{\tau_{\mathbb{Z}}\circ E_\varphi} = \mathrm{Ad}\, u$. One can choose a positive invertible $h \in \mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$ so that $u^* = h^{iT}$. Set $\psi := \tau_{\mathbb{Z}}(h)^{-1}\tau_{\mathbb{Z}}(h-)$, and one has $\sigma_T^{\psi\circ E_{\varphi_1}} = \mathrm{Id}_{M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}}$. By [3, Theorem 4.2.6], $(M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})_{\psi\circ E_\varphi}$ must be a type II_1 factor and contain $\mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$. Since $\mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$ is diffuse, for each natural number $n \geq 2$ there exist n orthogonal $e_0, \dots, e_{n-1} \in (\mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})^p$, all of which are equivalent in $(M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})_{\psi\circ E_\varphi}$, and $\sum_{i=0}^{n-1} e_i = 1$. Then one can construct a unitary $u_n \in (M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})_{\psi\circ E_\varphi}$ in such a way that $u_n e_0 = e_1 u_n$, $u_n e_1 = e_2 u_n$, $\dots, u_n e_{n-1} = e_0 u_n$. Since $\mathbb{C}1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}$ is commutative, one has $E_{\varphi_1}(u_n^k) = 0$ for every $1 \leq k \leq n-1$. $\square$

LEMMA 8

We have $(M \rtimes_{\sigma_T^\varphi} \mathbb{Z})_\omega = (M \rtimes_{\sigma_T^\varphi} \mathbb{Z})' \cap (M \rtimes_{\sigma_T^\varphi} \mathbb{Z})^\omega = M' \cap (\mathbb{C}1 \rtimes_{\sigma_T^\varphi} \mathbb{Z})^\omega$, where M canonically sits in $M \rtimes_{\sigma_T^\varphi} \mathbb{Z}$.

Proof

Similar to [22, Theorem 5.1], we have

$$(M \rtimes_{\sigma_T^\varphi} \mathbb{Z}, E_\varphi) = (M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z}, E_{\varphi_1}) \star_{\mathbb{C}1 \rtimes_{\sigma_T^\varphi} \mathbb{Z}} (M_2 \rtimes_{\sigma_T^{\varphi_2}} \mathbb{Z}, E_{\varphi_2})$$

naturally, to which [27, Proposition 3.5] is applicable thanks to Lemma 7. Since M_2 is nontrivial, one can choose an invertible $y \in \mathrm{Ker}(\varphi_2)$ so that $E_{\varphi_2}(y^*y) = \varphi_2(y^*y)1 \neq 0$. Therefore, [27, Proposition 3.5] actually says that $(M \rtimes_{\sigma_T^\varphi} \mathbb{Z})_\omega \subseteq (M \rtimes_{\sigma_T^\varphi} \mathbb{Z})' \cap (M \rtimes_{\sigma_T^\varphi} \mathbb{Z})^\omega \subseteq (M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})^\omega$. For any $x \in (M_2 \rtimes_{\sigma_T^{\varphi_2}} \mathbb{Z})' \cap (M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})^\omega$ one has $y(x - E_\varphi^\omega(x)) + yE_\varphi^\omega(x) = yx = xy = E_\varphi^\omega(x)y + (x - E_\varphi^\omega(x))y$, and the free independence of $(M_1 \rtimes_{\sigma_T^{\varphi_1}} \mathbb{Z})^\omega$ and $(M_2 \rtimes_{\sigma_T^{\varphi_2}} \mathbb{Z})^\omega$ in $((M \rtimes_{\sigma_T^\varphi} \mathbb{Z})^\omega, E_\varphi^\omega)$ (see [24, Proposition 4]) forces at least $y(x - E_1^\omega(x)) = 0$. This implies $x = E_\varphi^\omega(x) \in$

$(\mathbb{C}1 \rtimes_{\sigma^\varphi_T} \mathbb{Z})^\omega$ thanks to the invertibility of y. Consequently, $(M \rtimes_{\sigma^\varphi_T} \mathbb{Z})_\omega \subseteq (\mathbb{C}1 \rtimes_{\sigma^\varphi_T} \mathbb{Z})^\omega$, from which the desired assertion immediately follows. $\square$

We are ready to prove the desired assertion.

Proof of Theorem 1(2) ⇒ Theorem 1(1)
We prove its contraposition. Namely, assume that $\widetilde{M}$ is not full. Lemma 6 together with [20, Theorem XIV.3.8, Theorem XIV.4.7] says that $(M \rtimes_{\sigma^\varphi_T} \mathbb{Z})_\omega \neq \mathbb{C}1$.

CLAIM 9
There exists a sequence k_l in $\mathbb{Z}\setminus\{0\}$ such that $\sigma^\varphi_{k_l T} \longrightarrow \mathrm{Id}_M$ in $\mathrm{Aut}(M)$ as $l \to \infty$ or, equivalently, that $\|x - \sigma^\varphi_{k_l T}(x)\|_\varphi \longrightarrow 0$ as $l \to \infty$ for every $x \in M$ (since σ^φ_t preserves φ; see [20, Theorem IX.1.15, Proposition IX.1.17]).

Proof
On the contrary, suppose that there exist $\varepsilon > 0$ and a finite subset $\mathfrak{F}$ of M such that $\sum_{y\in\mathfrak{F}} \|y - \sigma^\varphi_{mT}(y)\|_\varphi^2 \geq \varepsilon$ as long as $m \neq 0$. Let $x \in (M \rtimes_{\sigma^\varphi_T} \mathbb{Z})_\omega$ be arbitrarily chosen with a representing sequence x_n. Lemma 8 shows that x falls in $(\mathbb{C}1 \rtimes_{\sigma^\varphi_T} \mathbb{Z})^\omega$ so that we can approximate each x_n in the σ-strong topology by a bounded net consisting of finite linear combinations of the form $\sum_m c_m u^m$ with scalars c_m (see the proof of Lemma 7 for the symbol u). We have

$$\begin{aligned}
&\Big\|\Big(\sum_m c_m u^m\Big) - \tau_{\mathbb{Z}}\Big(\sum_m c_m u^m\Big)1\Big\|_{\tau_{\mathbb{Z}}}^2 \\
&\quad= \sum_{m\neq 0} |c_m|^2 \\
&\quad\leq \varepsilon^{-1} \sum_{m\neq 0} |c_m|^2 \sum_{y\in\mathfrak{F}} \big\|y - \sigma^\varphi_{mT}(y)\big\|_\varphi^2 \\
&\quad\leq \varepsilon^{-1} \sum_{y\in\mathfrak{F}} \Big\|\sum_m c_m \big(y - \sigma^\varphi_{mT}(y)\big) u^m\Big\|_{\varphi\circ E}^2 \\
&\quad= \varepsilon^{-1} \sum_{y\in\mathfrak{F}} \Big\|y\Big(\sum_m c_m u^m\Big) - \Big(\sum_m c_m u^m\Big)y\Big\|_{\varphi\circ E}^2.
\end{aligned}$$

It follows that $\|x_n - \tau_{\mathbb{Z}}(x_n)1\|_{\tau_{\mathbb{Z}}}^2 \leq \varepsilon^{-1} \sum_{y\in\mathfrak{F}} \|yx_n - x_n y\|_{\varphi\circ E}^2$ for every n. Taking the limit of this inequality as $n \to \omega$ we get

$$0 \leq \big\|x - \tau^\omega_{\mathbb{Z}}(x)1\big\|_{\tau^\omega_{\mathbb{Z}}} \leq \varepsilon^{-1} \sum_{y\in\mathfrak{F}} \|yx - xy\|_{(\varphi\circ E)^\omega}^2 = 0.$$

This implies that $x = \tau^\omega_{\mathbb{Z}}(x)1$, a contradiction to $(M \rtimes_{\sigma^\varphi_T} \mathbb{Z})_\omega \neq \mathbb{C}1$. Hence, we have proved the claim. $\square$

Since $|k_l T| \geq T > 0$ for all l, the sequence $k_l T$ in the claim never converges to 0 in the usual topology on $\mathbb{R}$. Nevertheless, $\sigma^{\varphi}_{k_l T} \longrightarrow \mathrm{Id}_M$ in $\mathrm{Aut}(M)$ as $l \to \infty$. These clearly contradict Theorem 1(2). Hence, we are done. $\square$

Here are quick proofs of Theorem 1(1) $\Rightarrow$ Theorem 1(2) and Theorem 1(2) $\Leftrightarrow$ Theorem 1(3) for the sake of completeness. We remark that the former can also be derived as a consequence of the more general [18, Corollary 3.4].

Proof of Theorem 1(1) $\Rightarrow$ Theorem 1(3)
Suppose, on the contrary, that there exists a sequence t_n of real numbers so that $\sigma^{\varphi_c}_{t_n} \longrightarrow \mathrm{Id}_{M_c}$ in $\mathrm{Aut}(M_c)$ as $n \to \infty$ but t_n does not converge to 0 in the usual topology as $n \to \infty$. Let $\lambda^{\varphi_c} \colon \mathbb{R} \to \widetilde{M_c} = M_c \rtimes_{\sigma^{\varphi_c}} \mathbb{R}$ be the canonical unitary representation. Passing to a subsequence, we may assume that there is a positive constant $\varepsilon > 0$ so that $|t_n| \geq 3\varepsilon$ for all n. Then the regular representation $\lambda \colon \mathbb{R} \curvearrowright L^2(\mathbb{R})$ enjoys that $\|\lambda(t_n)\chi_{[-\varepsilon,\varepsilon]} - \zeta\chi_{[-\varepsilon,\varepsilon]}\|_2^2 \geq 2\varepsilon$ for all n and all $\zeta \in \mathbb{C}$. It follows that the sequence $\lambda^{\varphi_c}(t_n)$ never defines a scalar in $(\widetilde{M_c})^\omega$. Set $E_{\varphi_c} := (\varphi_c \bar{\otimes} \mathrm{Id})\restriction_{\widetilde{M_c}}$, a positive scalar multiple of faithful normal conditional expectation onto $\mathbb{C}1_{M_c} \rtimes_{\sigma^{\varphi_c}} \mathbb{R}$. With a faithful normal state ψ on $\mathbb{C}1_{M_c} \rtimes_{\sigma^{\varphi_c}} \mathbb{R}$ we have $\|\lambda^{\varphi_c}(t_n)x - x\lambda^{\varphi_c}(t_n)\|_{\psi\circ E_{\varphi_c}} = \|\sigma^{\varphi_c}_{t_n}(x) - x\|_{\varphi_c} \longrightarrow 0$ as $n \to \infty$ for every $x \in M_c$ so that the sequence $\lambda^{\varphi_c}(t_n)$ defines a nonscalar element of $(\widetilde{M_c})' \cap (\widetilde{M_c})^\omega$, a contradiction. $\square$

Proof of Theorem 1(2) $\Leftrightarrow$ Theorem 1(3)
This follows from the equivalence between $\sigma^{\varphi}_{t_n} \longrightarrow \mathrm{Id}_M$ in $\mathrm{Aut}(M)$ as $n \to \infty$ and $(\sigma^{\varphi_1}_{t_n}, \sigma^{\varphi_2}_{t_n}) \longrightarrow (\mathrm{Id}_{M_1}, \mathrm{Id}_{M_2})$ in $\mathrm{Aut}(M_1) \times \mathrm{Aut}(M_2)$ as $n \to \infty$. First, the linear span of the identity 1 and all alternating words in $\mathrm{Ker}(\varphi_k)$, $k = 1, 2$ forms a dense subspace of the standard form $L^2(M)$, which can be understood as the completion of M with respect to the norm $\|\cdot\|_\varphi$. Second, the free independence of M_1, M_2 with respect to φ together with the formula $\sigma^\varphi_t = \sigma^{\varphi_1}_t \star \sigma^{\varphi_2}_t$ (see [2], [8]) enables us to see that $\|\sigma^{\varphi}_{t_n}(x) - x\|_\varphi \leq (\max_{1\leq i\leq l}\|x_i\|_\infty)^{l-1}\sum_{i=1}^{l}\|\sigma^{\varphi_{k_i}}_{t_n}(x_i) - x_i\|_{\varphi_{k_i}}$ for every alternating word $x = x_1 \cdots x_l$ with $x_i \in \mathrm{Ker}(\varphi_{k_i})$. The equivalence is immediate from these facts (thanks to [20, Theorem IX.1.15, Proposition IX.1.17]). $\square$

In closing this section we give a simple remark explaining Corollary 2.

REMARK 10
The corollary is indeed a particular case of Theorem 1, since *any free Araki–Woods factor $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ with its distinguished state φ_U can be written as a free product of two nontrivial von Neumann algebras* (see the proof of [29, Theorem 2.7] for a related claim). This should be known by experts, but we do give an explanation about this for the reader's convenience.

Let $U_t = \exp(\sqrt{-1}tA)$ with $A = \int_{-\infty}^{+\infty} sE_A(ds)$ be the Stone representation on the complexification $\mathcal{H}_\mathbb{R} + \sqrt{-1}\mathcal{H}_\mathbb{R}$. The unitary conjugation $J \colon \xi + \sqrt{-1}\eta \mapsto \xi - \sqrt{-1}\eta$ for $\xi + \sqrt{-1}\eta \in \mathcal{H}_\mathbb{R} + \sqrt{-1}\mathcal{H}_\mathbb{R}$ enjoys the property that $JE_A(B)J =$

$E_A(-B)$ for each Borel subset B of $\mathbb{R}$. This shows that if $\sharp(\mathrm{Sp}(A)\cap(0,+\infty))\le 1$, then $\mathrm{Sp}(A)$ must be either $\{0\}$, $\{-s,s\}$, or $\{-s,0,s\}$ with $s>0$. Hence, the desired free product decomposition is obtained in the proof of [16, Theorem 6.1]. If $\sharp(\mathrm{Sp}(A)\cap(0,+\infty))\ge 2$, then $\mathrm{Sp}(A)\cap(0,+\infty)$ is decomposed into two nontrivial Borel subsets B_1,B_2. Set $P_1:=E_A(-B_1\cup\{0\}\cup B_1)$, $P_2:=E_A(-B_2\cup B_2)$, both of which commute with U_t and J. Thus, we have $(\mathcal{H}_\mathbb{R},U_t)=(P_1\mathcal{H}_\mathbb{R},U_t\restriction_{P_1\mathcal{H}_\mathbb{R}})\oplus(P_2\mathcal{H}_\mathbb{R},U_t\restriction_{P_2\mathcal{H}_\mathbb{R}})$ so that $\Gamma(\mathcal{H}_\mathbb{R},U_t)''$ becomes a free product of two free Araki–Woods factors (see [16, Theorem 2.11]).

In this way, almost all general results on free Araki–Woods factors follow as particular cases from the corresponding ones on free product von Neumann algebras (see [25], [26], [28]). Only two nontrivial facts (i.e., [16, Theorem 5.4] and [17, Theorem 4.8]), both of which heavily depend upon matricial models, have not been re-proved in the general framework of free product von Neumann algebras. These lacks seem to be related to the question from [25, Section 5.4].

Appendix. Bernoulli crossed products

Throughout this section, we follow the notation from [30, Section 2.5], which is different from that in the other sections. Let Λ be a countable group acting on a countable set I such that $\Lambda\curvearrowright I$ *has no invariant mean*, and let (P,ϕ) be a nontrivial von Neumann algebra equipped with a faithful normal state ϕ. Let $(P,\phi)^I\rtimes\Lambda$ (or $P^I\rtimes\Lambda$ for short) be the (generalized) Bernoulli crossed product (see, e.g., [30, Section 2.5]). Set $\varphi:=\phi^I\circ E_{P^I}$ with the canonical conditional expectation $E_{P^I}: P^I\rtimes\Lambda\to P^I$.

LEMMA 11

For every countable subgroup G of $\mathbb{R}$, any central sequence (see [21, Definition XIV.3.2]) in $(P^I\rtimes\Lambda)\rtimes_{\sigma^\varphi}G$ is equivalent to an (operator norm-)bounded one in $(\mathbb{C}1\rtimes\Lambda)\rtimes_{\sigma^\varphi}G=\mathbb{C}1\mathbin{\bar\otimes}L(\Lambda\times G)$.

Proof

The idea used in the proof of [30, Lemma 2.7] works for proving this lemma. Let x_n be a central sequence in $(P^I\rtimes\Lambda)\rtimes_{\sigma^\varphi}G$. Consider those x_n's as vectors in the standard Hilbert space $L^2((P^I\rtimes\Lambda)\rtimes_{\sigma^\varphi}G)\cong[L^2((P,\phi)^I\ominus\mathbb{C})\mathbin{\bar\otimes}\ell^2(\Lambda)\mathbin{\bar\otimes}\ell^2(G)]\oplus[\ell^2(\Lambda)\mathbin{\bar\otimes}\ell^2(G)]$. We remark that this Hilbert space decomposition is given by the conditional expectation from $(P^I\rtimes\Lambda)\rtimes_{\sigma^\varphi}G$ onto $(\mathbb{C}1\rtimes\Lambda)\rtimes_{\sigma^\psi}G$ defined to be the restriction of $\phi^I\mathbin{\bar\otimes}\mathrm{Id}\mathbin{\bar\otimes}\mathrm{Id}$ to $(P^I\rtimes\Lambda)\rtimes_{\sigma^\varphi}G$ and also that the Bernoulli action commutes with the modular action associated with ϕ^I. Hence, as in the proof of [30, Lemma 2.7] (note that one of the keys there is that any tensor product representation of nonamenable one with arbitrary one must be nonamenable again; see, e.g., [19, Proposition 2.7]), we see that x_n is equivalent to $(\phi^I\mathbin{\bar\otimes}\mathrm{Id}\mathbin{\bar\otimes}\mathrm{Id})(x_n)$ in $(\mathbb{C}1\rtimes\Lambda)\rtimes_{\sigma^\psi}G=\mathbb{C}1\mathbin{\bar\otimes}L(\Lambda\times G)$. Hence, we are done. $\square$

With the above lemma, one can prove the next proposition in the essentially same way as in the proof of Theorem 1(1) $\Leftrightarrow$ Theorem 1(2).

THEOREM 12

If $P^I \rtimes \Lambda$ is a full factor of type III_1*, then the following conditions are equivalent.*

(1) *The continuous core $(P^I \rtimes \Lambda) \rtimes_{\sigma^\varphi} \mathbb{R}$ of $P^I \rtimes \Lambda$ is a full factor.*

(2) *The τ-invariant $\tau(P^I \rtimes \Lambda)$, that is, the weakest topology on $\mathbb{R}$ making the mapping $t \in \mathbb{R} \mapsto \sigma_t^\phi \in \mathrm{Aut}(P)$ continuous in this particular case (see [30, Lemma 2.7]), is the usual topology on $\mathbb{R}$.*

Proof

That $\sigma_{t_n}^\varphi \longrightarrow \mathrm{Id}$ in $\mathrm{Aut}(P^I)$ is easily seen to be equivalent to that $\sigma_{t_n}^\phi \longrightarrow \mathrm{Id}$ in $\mathrm{Aut}(P)$. Hence, it suffices to prove Theorem 12(2) $\Rightarrow$ Theorem 12(1). In fact, the proof of Theorem 1(1) $\Rightarrow$ Theorem 1(2) works by replacing φ_c there with ϕ^I. We will explain how to modify the proof of Theorem 1(2) $\Rightarrow$ Theorem 1(1).

We prove its contraposition. Namely, by Lemma 6 we assume that there exists a nontrivial strongly central sequence x_n in $(P^I \rtimes \Lambda) \rtimes_{\sigma_T^\varphi} \mathbb{Z}$ for some $T > 0$ (see [21, Definition XIV.3.2] for the notion of strongly central sequences). By Lemma 11 we may and do assume that all the x_n's fall in $(\mathbb{C}1 \rtimes \Lambda) \rtimes_{\sigma_T^\varphi} \mathbb{Z} = \mathbf{C}1 \mathbin{\bar{\otimes}} L(\Lambda \times \mathbb{Z})$. As in the proof of Theorem 1(2) $\Rightarrow$ Theorem 1(1), it suffices to prove that there exists a sequence k_l in $\mathbb{Z} \setminus \{0\}$ such that $\|x - \sigma_{k_l T}^\varphi(x)\|_\varphi \longrightarrow 0$ as $l \to \infty$ for every $x \in P^I \rtimes \Lambda$. Suppose, on the contrary, that this is not the case. Since $\mathbb{C}1 \rtimes \Lambda \subseteq (P^I \rtimes \Lambda)_\varphi$, the fixed-point algebra of the modular action σ^φ, the same argument as in Claim in the proof of Theorem 1(2) $\Rightarrow$ Theorem 1(1) actually works for proving that $\|x_n - (\mathrm{Id} \mathbin{\bar{\otimes}} \mathrm{Id} \mathbin{\bar{\otimes}} \tau_\mathbb{Z})(x_n)\|_2 \longrightarrow 0$ as $n \to \infty$. Set $y_n := (\mathrm{Id} \mathbin{\bar{\otimes}} \mathrm{Id} \mathbin{\bar{\otimes}} \tau_\mathbb{Z})(x_n) \in \mathbb{C}1 \rtimes \Lambda \subseteq P^I \rtimes \Lambda$. Since we have assumed that $P^I \rtimes \Lambda$ is full, by [20, Theorem XIV.3.8] the y_n's (and hence the x_n's) must be trivial, a contradiction. $\square$

So far, we have established, for every explicit example of full type III_1 factor whose τ-invariant is already computed, that the τ-invariant is the usual topology if and only if the continuous core is full. Therefore, one may conjecture that this is true even for any full type III_1 factor. Actually, this question seems important from the theoretical point of view, and we are still working on this general question.

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