The continuity of commutators on Herz-type Hardy spaces with variable exponent

Hongbin Wang

Abstract In this article, we study the continuity of some commutators generated by the Calderón–Zygmund singular integral operator, the fractional integral operator, the Marcinkiewicz integral operator, and Lipschitz functions on Herz-type Hardy spaces with variable exponent.

1. Introduction

The theory of function spaces with variable exponent has been extensively studied by researchers since the work of Kováčik and Rákosník [4]. Capone, Cruz-Uribe, and Fiorenza [1] and Wang, Fu, and Liu [8] studied the continuity of some integral operators on variable L^p-spaces. In addition, Wang and Liu [9] defined the Herz-type Hardy spaces with variable exponent and gave their atomic characterizations.

Motivated by [5] and [6], we will study the continuity of some commutators generated by the Calderón–Zygmund singular integral operator, the fractional integral operator, the Marcinkiewicz integral operator and Lipschitz functions on Herz-type Hardy spaces with variable exponent.

Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : \Omega \longrightarrow [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that, for some $\lambda > 0$,

$$\int_\Omega \Big(\frac{|f(x)|}{\lambda} \Big)^{p(x)} \, dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \Big\{ \lambda > 0 : \int_\Omega \Big(\frac{|f(x)|}{\lambda} \Big)^{p(x)} \, dx \le 1 \Big\}.$$

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These spaces are referred to as variable L^p-spaces, since they generalize the standard L^p-spaces: if $p(x)=p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$.

For all compact subsets $E\subset\Omega$, the space $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by $L^{p(\cdot)}_{\mathrm{loc}}(\Omega):=\{f: f\in L^{p(\cdot)}(E)\}$. Define $\mathcal{P}(\Omega)$ to be a set of $p(\cdot):\Omega\longrightarrow[1,\infty)$ such that

$$p^-=\operatorname{ess\,inf}\{p(x): x\in\Omega\}>1,\qquad p^+=\operatorname{ess\,sup}\{p(x): x\in\Omega\}<\infty.$$

Denote $p'(x)=p(x)/(p(x)-1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot)\in\mathcal{P}(\Omega)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

In variable L^p-spaces there are some important lemmas as follows.

LEMMA 1.1 ([4])
Let $p(\cdot)\in\mathcal{P}(\Omega)$. If $f\in L^{p(\cdot)}(\Omega)$ and $g\in L^{p'(\cdot)}(\Omega)$, then fg is integrable on Ω and

$$\int_\Omega \big|f(x)g(x)\big|\,dx\le r_p\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p=1+1/p^- - 1/p^+.$$

This inequality is called the generalized Hölder inequality with respect to the variable L^p-spaces.

LEMMA 1.2 ([2])
Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all balls B in $\mathbb{R}^n$ and all measurable subsets $S\subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\le C\frac{|B|}{|S|},\qquad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1},$$

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}\le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2},$$

where δ_1,δ_2 are constants with $0<\delta_1,\ \delta_2<1$ and $\chi_S,\ \chi_B$ are the characteristic functions of S, B, respectively.

Throughout this article δ_2 is as it is in Lemma 1.2.

LEMMA 1.3 ([2])
Suppose $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C>0$ such that, for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C.$$

For $0<\beta\le 1$, the Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is defined as

$$\mathrm{Lip}_\beta(\mathbb{R}^n)=\Big\{f:\|f\|_{\mathrm{Lip}_\beta}=\sup_{x,y\in\mathbb{R}^n;x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\beta}<\infty\Big\}.$$

Next we recall the definition of the Herz-type spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, and let $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and nonnegative integers, respectively, denote $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, denote $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$, and denote $\tilde{\chi}_0 = \chi_{B_0}$.

DEFINITION 1.1 ([2])

Let $\alpha \in \mathbb{R}$, let $0 < p \le \infty$, and let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \big\{ f \in L^{q(\cdot)}_{\mathrm{loc}}\big(\mathbb{R}^n \setminus \{0\}\big) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty \big\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \Big\}^{1/p}.$$

The nonhomogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \big\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty \big\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \Big\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \Big\}^{1/p}.$$

Wang and Liu [9] gave the definition of Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and the atomic decomposition characterizations. Here, $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N(f)(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N(f)(x) = \sup_{\phi \in \mathcal{A}_N} \big|\phi^*_\nabla(f)(x)\big|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \le N} |x^\alpha D^\beta \phi(x)| \le 1\}$, $N > n+1$, and ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} \big|\phi_t * f(y)\big|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

DEFINITION 1.2 ([9])

Let $\alpha \in \mathbb{R}$, let $0 < p < \infty$, let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and let $N > n+1$.

(i) The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \big\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \big\}$$

and $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii) The nonhomogeneous Herz-type Hardy space with variable exponent $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

For $x \in \mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x.

DEFINITION 1.3 ([9])
Let $n\delta_2 \le \alpha < \infty$, let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and let nonnegative integer $s \ge [\alpha - n\delta_2]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom if it satisfies:
(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$;
(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n}$;
(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \le s$.
(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type if it satisfies conditions (2) and (3) above and
$(1)'$ $\operatorname{supp} a \subset B(0,r)$, $r \ge 1$.

If $r = 2^k$ for some $k \in \mathbb{Z}$ in Definition 1.3, then the corresponding central $(\alpha, q(\cdot))$-atom is called a dyadic central $(\alpha, q(\cdot))$-atom.

LEMMA 1.4 ([9])
Let $n\delta_2 \le \alpha < \infty$, let $0 < p < \infty$, and let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) if and only if

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \Big(\text{or} \sum_{k=0}^{\infty} \lambda_k a_k\Big), \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha, q(\cdot))$-atom (or central $(\alpha, q(\cdot))$-atom of restricted type) with support contained in B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ (or $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$). Moreover,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf\Big(\sum_{k=-\infty}^{\infty} |\lambda_k|^p\Big)^{1/p} \Big(\text{or } \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf\Big(\sum_{k=0}^{\infty} |\lambda_k|^p\Big)^{1/p}\Big),$$

where the infimum is taken over all the above decompositions of f.

2. Commutator of the Calderón–Zygmund singular integral operator

Let $b \in \operatorname{Lip}_\beta(\mathbb{R}^n)$, and let T be a Calderón–Zygmund singular integral operator; that is,

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy,$$

p.v. or (principal value integrals) where $\Omega \in C^2(S^{n-1})$ is homogeneous of degree zero and has mean value zero on the unit sphere. The commutator $[b, T]$ generated by b and T is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

Noting that $|[b,T]f(x)| \le C\|b\|_{\mathrm{Lip}_\beta} I_\beta(|f|)(x)$, where I_β is the fractional integral operator with $0<\beta<n$. Capone, Cruz-Uribe, and Fiorenza [1] proved that I_β is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$ for $1/q_1(x) - 1/q_2(x) = \beta/n$ and $q_1(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ with $q_1^+ < n/\beta$. Izuki [3] generalized the result to the case of Herz–Morrey spaces with variable exponent. So we can easily obtain the following theorem.

THEOREM 2.1

Suppose that $b\in\mathrm{Lip}_\beta(\mathbb{R}^n)$, $0<\beta\le 1$. If $q_1(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ with $q_1^+<n/\beta$, $1/q_1(x) - 1/q_2(x) = \beta/n$, $0<p_1\le p_2<\infty$, and $0<\alpha<n\delta_2$, then $[b,T]$ maps $\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$) continuously into $\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$).

Furthermore, we can obtain the following theorem when $\alpha\ge n\delta_2$.

THEOREM 2.2

Suppose that $b\in\mathrm{Lip}_\beta(\mathbb{R}^n)$, $0<\beta\le 1$. If $q_1(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ with $q_1^+<n/\beta$, $1/q_1(x) - 1/q_2(x) = \beta/n$, $0<p_1\le p_2<\infty$, and $n\delta_2\le\alpha<n\delta_2+\beta$, then $[b,T]$ maps $H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$) continuously into $\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$).

Proof

We only prove the homogeneous case. The nonhomogeneous case can be proved in the same way. Let $f\in H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$, and let $b\in\mathrm{Lip}_\beta(\mathbb{R}^n)$. By Lemma 1.4, we get $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)}\approx\inf(\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over the above decompositions of f), and a_j is a dyadic central $(\alpha,q_1(\cdot))$-atom with support B_j. Note that $p_1\le p_2$. We have

$$\begin{aligned}
\big\|[b,T](f)\big\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &= \Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_2}\big\|[b,T](f)\chi_k\big\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big\}^{p_1/p_2}\\
&\le \sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\big\|[b,T](f)\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|\big\|[b,T](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1} \qquad (2.1)\\
&\quad + C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\big\|[b,T](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\\
&=: E_1+E_2.
\end{aligned}$$

Now we estimate E_1. For each $k\in\mathbb{Z}$, $j\le k-2$, and almost every $x\in A_k$, by the vanishing moments of a_j and the generalized Hölder inequality, we have

$$\begin{aligned}\big|[b,T](a_j)(x)\big| &\le \Big|\big(b(x)-b(0)\big)\int_{B_j}\Big(\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\Big)a_j(y)\,dy\Big| \\ &\quad+\Big|\int_{B_j}\frac{\Omega(x-y)}{|x-y|^n}\big(b(y)-b(0)\big)a_j(y)\,dy\Big| \\ &\le C\|b\|_{\mathrm{Lip}_\beta}\Big(|x|^\beta\int_{B_j}\frac{|y|}{|x|^{n+1}}\big|a_j(y)\big|\,dy+\int_{B_j}\frac{|y|^\beta}{|x-y|^n}\big|a_j(y)\big|\,dy\Big) \\ &\le C\|b\|_{\mathrm{Lip}_\beta}\big(|x|^{\beta-n-1}2^j\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\ &\quad+|x|^{-n}2^{j\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\big) \\ &\le C2^{j\beta-kn}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Since

$$I_\beta(\chi_{B_k})(x)\ge\int_{B_k}\frac{dy}{|x-y|^{n-\beta}}\chi_{B_k}(x)\ge C2^{k\beta}\chi_{B_k}(x), \tag{2.2}$$

by Lemmas 1.2 and 1.3 we have

$$\begin{aligned}&\big\|[b,T](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{j\beta-kn}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{j\beta-k(n+\beta)}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\big\|I_\beta(\chi_{B_k})\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{j\beta-k(n+\beta)}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{(j-k)\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \\ &\le C2^{-j\alpha+(j-k)(\beta+n\delta_2)}\|b\|_{\mathrm{Lip}_\beta}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}E_1&\le C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{-j\alpha+(j-k)(\beta+n\delta_2)}\Big)^{p_1} \\ &=C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)(\beta+n\delta_2-\alpha)}\Big)^{p_1}.\end{aligned}$$

For $1<p_1<\infty$, take $1/p_1+1/p_1'=1$. Since $\beta+n\delta_2-\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}E_1\le C\|b\|^{p_1}_{\mathrm{Lip}_\beta}&\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big) \\ &\times\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2}\Big)^{p_1/p_1'}\end{aligned}$$

$$\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big) \tag{2.3}$$

$$= C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\Big(\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big)$$

$$\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.$$

If $0 < p_1 \leq 1$, then we have

$$\begin{aligned} E_1 &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\Big) \\ &= C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\Big(\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\Big) \\ &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}. \end{aligned} \tag{2.4}$$

Let us now estimate E_2. By $|[b,T]f(x)| \leq C\|b\|_{\mathrm{Lip}_\beta}I_\beta(|f|)(x)$ and the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$-boundedness of I_β we have

$$\begin{aligned} E_2 &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1} \\ &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|2^{(k-j)\alpha}\Big)^{p_1}. \end{aligned}$$

If $0 < p_1 \leq 1$, then we have

$$\begin{aligned} E_2 &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\sum_{k=-\infty}^{j+1}2^{(k-j)\alpha p_1} \\ &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}. \end{aligned} \tag{2.5}$$

If $1 < p_1 < \infty$, then by the Hölder inequality we have

$$\begin{aligned} E_2 &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-1}^{\infty}|\lambda_j|^{p_1}2^{(k-j)\alpha p_1/2}\Big)\Big(\sum_{j=k-1}^{\infty}2^{(k-j)\alpha p_1'/2}\Big)^{p_1/p_1'} \\ &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}. \end{aligned} \tag{2.6}$$

Therefore, by (2.1) and (2.3)–(2.6) we complete the proof of Theorem 2.2. □

3. Commutator of the fractional integral operator

Let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, and let I_σ denote the fractional integral operator with $0 < \sigma < n$. The commutator of fractional integral operator $[b, I_\sigma]$ is defined by

$$[b, I_\sigma]f(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))}{|x-y|^{n-\sigma}} f(y)\,dy.$$

Wang, Fu, and Liu [8] proved that the commutator $[b, I_\sigma]$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$ for $1/q_1(x) - 1/q_2(x) = (\sigma+\beta)/n$ and $q_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $q_1^+ < n/(\sigma+\beta)$. In this section, we will give the corresponding result about the commutator $[b, I_\sigma]$ on Herz-type Hardy spaces with variable exponent.

THEOREM 3.1

Suppose that $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ with $0 < \beta \le 1$ and $0 < \sigma < n - \beta$. If $q_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $q_1^+ < n/(\sigma+\beta)$, $1/q_1(x) - 1/q_2(x) = (\sigma+\beta)/n$, $0 < p_1 \le p_2 < \infty$, and $n\delta_2 \le \alpha < n\delta_2 + \beta$, then $[b, I_\sigma]$ maps $H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ (or $HK^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$) continuously into $\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ (or $K^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$).

Proof

Similar to Theorem 2.2, it suffices to prove the homogeneous case. Let $f \in H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$, and let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$. By Lemma 1.4 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over the above decompositions of f), and a_j is a dyadic central $(\alpha, q_1(\cdot))$-atom with support B_j. Note that $p_1 \le p_2$. We have

$$\begin{aligned}
\big\|[b, I_\sigma](f)\big\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &= \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \big\|[b, I_\sigma](f)\chi_k\big\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \Big\}^{p_1/p_2} \\
&\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \big\|[b, I_\sigma](f)\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| \big\|[b, I_\sigma](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \Big)^{p_1} \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \big\|[b, I_\sigma](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \Big)^{p_1} \\
&=: F_1 + F_2.
\end{aligned} \tag{3.1}$$

We first estimate F_1. For each $k \in \mathbb{Z}$, $j \le k-2$, and almost every $x \in A_k$, using the vanishing moments of a_j and the generalized Hölder inequality we have

$$\begin{aligned}
\big|[b, I_\sigma](a_j)(x)\big| &\le \int_{B_j} \big|b(x) - b(y)\big| \frac{|a_j(y)||y|^\beta}{|x-y|^{n-\sigma+\beta}}\,dy \\
&\le C 2^{-k(n-\sigma+\beta)+j\beta} \int_{B_j} \big|b(x) - b(y)\big|\big|a_j(y)\big|\,dy
\end{aligned}$$

$$
\begin{aligned}
&\le C2^{-k(n-\sigma+\beta)+j\beta}\Big(\big|b(x)-b(0)\big|\int_{B_j}\big|a_j(y)\big|\,dy \\
&\quad+\int_{B_j}\big|a_j(y)\big|\big|b(y)-b(0)\big|\,dy\Big) \\
&\le C2^{-k(n-\sigma+\beta)+j\beta}\|b\|_{\mathrm{Lip}_\beta}\Big(|x|^\beta\int_{B_j}\big|a_j(y)\big|\,dy+\int_{B_j}|y|^\beta\big|a_j(y)\big|\,dy\Big) \\
&\le C2^{-k(n-\sigma+\beta)+j\beta}\|b\|_{\mathrm{Lip}_\beta}\big(|x|^\beta\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\
&\quad+2^{j\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\big) \\
&\le C2^{-k(n-\sigma)+j\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

So by (2.2) and Lemmas 1.2 and 1.3 we have

$$
\begin{aligned}
&\big\|[b,I_\sigma](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad\le C2^{-k(n-\sigma)+j\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\
&\quad\le C2^{-kn+(j-k)\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\big\|I_{\beta+\sigma}(\chi_{B_k})\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad\le C2^{-kn+(j-k)\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\quad\le C2^{(j-k)\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \\
&\quad\le C2^{-j\alpha+(j-k)(\beta+n\delta_2)}\|b\|_{\mathrm{Lip}_\beta}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
F_1&\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{-j\alpha+(j-k)(\beta+n\delta_2)}\Big)^{p_1} \\
&=C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)(\beta+n\delta_2-\alpha)}\Big)^{p_1}.
\end{aligned}
$$

When $1<p_1<\infty$, take $1/p_1+1/p_1'=1$. Since $\beta+n\delta_2-\alpha>0$, by the Hölder inequality we have

$$
\begin{aligned}
F_1&\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big) \\
&\quad\times\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2}\Big)^{p_1/p_1'} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big) \qquad (3.2)
\end{aligned}
$$

$$= C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \Big(\sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \Big)$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

When $0 < p_1 \le 1$, we have

$$\begin{aligned} F_1 &\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \Big) \\ &= C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \Big(\sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \Big) \\ &\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{3.3}$$

Next we estimate F_2; by the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, I_\sigma]$ we have

$$\begin{aligned} F_2 &\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \Big)^{p_1} \\ &\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \Big)^{p_1} \\ &\le C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{3.4}$$

Thus, by (3.1)–(3.4) we complete the proof of Theorem 3.1. □

4. Commutator of the Marcinkiewicz integral operator

Suppose that S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ $(n \ge 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in \mathrm{Lip}_\nu(\mathrm{S}^{n-1})$ for $0 < \nu \le 1$ be a homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \ne 0$. In 1958, Stein [7] introduced the Marcinkiewicz integral operator, which is related to the Littlewood–Paley g-function on $\mathbb{R}^n$ as

$$\mu(f)(x) = \Big(\int_0^\infty \big|F_t(f)(x)\big|^2 \frac{dt}{t^3} \Big)^{1/2},$$

where

$$F_t(f)(x) = \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy.$$

Let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$. The commutator generated by the Marcinkiewicz integral operator μ and b is defined by

$$[b,\mu](f)(x) = \Big(\int_0^\infty \Big| \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \big[b(x)-b(y)\big] f(y)\, dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2}.$$

Wang, Fu, and Liu [8] proved that the commutator $[b,\mu]$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$. In this section, we will generalize the result to the case of Herz-type Hardy spaces with variable exponent.

THEOREM 4.1

Suppose that $\Omega \in \mathrm{Lip}_\nu(S^{n-1})$ $(0<\nu\le 1)$ *and* $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, $0<\beta\le \nu/2$. *If* $q_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *with* $q_1^+ < n/\sigma$, $1/q_1(x) - 1/q_2(x) = \beta/n$, $0<p_1\le p_2<\infty$, *and* $n\delta_2 \le \alpha < n\delta_2 + \beta$, *then* $[b,\mu]$ *maps* $H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ *(or* $HK^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$*) continuously into* $\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ *(or* $K^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$*).*

Proof

Similar to Theorem 2.2, it suffices to prove the homogeneous case. Let $f \in H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$, and let $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$. By Lemma 1.4 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over the above decompositions of f), and a_j is a dyadic central $(\alpha, q_1(\cdot))$-atom with support B_j. Note that $p_1 \le p_2$. We have

$$\begin{aligned}
\big\|[b,\mu](f)\big\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &= \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \big\|[b,\mu](f)\chi_k\big\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \Big\}^{p_1/p_2} \\
&\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \big\|[b,\mu](f)\chi_k\big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \Big(\sum_{j=-\infty}^{k-2} |\lambda_j| \big\|[b,\mu](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \Big)^{p_1} \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \Big(\sum_{j=k-1}^{\infty} |\lambda_j| \big\|[b,\mu](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \Big)^{p_1} \\
&=: G_1 + G_2.
\end{aligned} \tag{4.1}$$

We first estimate G_1. Note that

$$\begin{aligned}
&\big|[b,\mu](a_j)(x)\big| \\
&\quad\le \Big(\int_0^{|x|+2^{j+1}} \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \big[b(x)-b(y)\big] a_j(y)\, dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2} \\
&\qquad + \Big(\int_{|x|+2^{j+1}}^\infty \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \big[b(x)-b(y)\big] a_j(y)\, dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2} \\
&\quad =: I_1 + I_2.
\end{aligned}$$

When $x \in A_k$ and $|x-y|<t$ with $t<|x|+2^{j+1}$, it follows from $j \le k-2$ that $|x-y| \sim |x| \sim |x|+2^{j+1}$. Then by the Minkowski inequality, $\mathrm{Lip}_\nu(S^{n-1}) \subset L^\infty(S^{n-1})$, and the generalized Hölder inequality, we have

$$\begin{aligned}
I_1 &\le C\int_{\mathbb{R}^n}\Big(\int_{|x-y|}^{|x|+2^{j+1}}\frac{dt}{t^3}\Big)^{1/2}\frac{|b(x)-b(y)||a_j(y)|}{|x-y|^{n-1}}\,dy\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\int_{\mathbb{R}^n}\frac{|x|^\beta|a_j(y)|}{|x|^{n-1}}\frac{|y|^{1/2}}{|x|^{3/2}}\,dy\\
&\le C\|b\|_{\mathrm{Lip}_\beta}|x|^{\beta-n-1/2}\int_{B_j}|a_j(y)||y|^{1/2}\,dy\\
&\le C\|b\|_{\mathrm{Lip}_\beta}|x|^{\beta-n-1/2}2^{j/2}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\\
&= C\|b\|_{\mathrm{Lip}_\beta}|x|^{-n}|x|^{\beta-1/2}2^{j(1/2-\beta)}2^{j\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn+j\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Notice that, from $t \ge |x|+2^{j+1}$ and $y \in B_j$, it follows that $t \ge |x|+|y| \ge |x-y|$. By the vanishing moments of a_j and the generalized Hölder inequality we have

$$\begin{aligned}
I_2 &\le \Big(\int_{|x|+2^{j+1}}^\infty\Big|\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-1}}\big[b(x)-b(y)\big]a_j(y)\,dy\Big|^2\frac{dt}{t^3}\Big)^{1/2}\\
&= \Big|\int_{B_j}\frac{\Omega(x-y)}{|x-y|^{n-1}}\big[b(x)-b(y)\big]a_j(y)\,dy\Big|\Big(\int_{|x|+2^{j+1}}^\infty\frac{dt}{t^3}\Big)^{1/2}\\
&\le C\Big|\int_{B_j}\frac{\Omega(x-y)}{|x-y|^{n-1}}\big[b(x)-b(y)\big]a_j(y)\,dy\Big|\frac{1}{|x|+2^{j+1}}\\
&\le C\Big|\big[b(x)-b(0)\big]\int_{B_j}\Big(\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x)}{|x|^{n-1}}\Big)a_j(y)\,dy\Big|\frac{1}{|x|+2^{j+1}}\\
&\quad +C\Big|\int_{B_j}\frac{\Omega(x-y)}{|x-y|^{n-1}}\big[b(y)-b(0)\big]a_j(y)\,dy\Big|\frac{1}{|x|+2^{j+1}}\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\Big(|x|^\beta\int_{B_j}\frac{|y|^\nu|a_j(y)|}{|x|^{n-1+\nu}}\,dy+\int_{B_j}\frac{|y|^\beta|a_j(y)|}{|x-y|^{n-1}}\,dy\Big)\frac{1}{|x|+2^{j+1}}\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\Big(|x|^\beta\int_{B_j}\frac{|y|^\nu|a_j(y)|}{|x|^{n+\nu}}\,dy+\int_{B_j}\frac{|y|^\beta|a_j(y)|}{|x-y|^{n}}\,dy\Big)\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\big(|x|^{\beta-n-\nu}2^{j\nu}+|x|^{-n}2^{j\beta}\big)\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn+j\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

So by (2.2) and Lemmas 1.2 and 1.3 we have

$$\begin{aligned}
&\big\|[b,\mu](a_j)\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\quad\le C2^{-kn+j\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq C2^{-kn+(j-k)\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\big\|I_\beta(\chi_{B_k})\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$
$$\leq C2^{-kn+(j-k)\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$
$$\leq C2^{(j-k)\beta}\|b\|_{\mathrm{Lip}_\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}$$
$$\leq C2^{-j\alpha+(j-k)(\beta+n\delta_2)}\|b\|_{\mathrm{Lip}_\beta}.$$

So we have

$$G_1 \leq C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{-j\alpha+(j-k)(\beta+n\delta_2)}\Big)^{p_1}$$
$$= C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)(\beta+n\delta_2-\alpha)}\Big)^{p_1}.$$

When $1<p_1<\infty$, take $1/p_1+1/p_1'=1$. Since $\beta+n\delta_2-\alpha>0$, by the Hölder inequality we have

$$G_1 \leq C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big)$$
$$\times\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2}\Big)^{p_1/p_1'}$$
$$\leq C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big) \tag{4.2}$$
$$= C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\Big(\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\Big)$$
$$\leq C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.$$

When $0<p_1\leq 1$, we have

$$G_1 \leq C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\Big)$$
$$= C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\Big(\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\Big) \tag{4.3}$$
$$\leq C\|b\|^{p_1}_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.$$

Next we estimate G_2. By the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b,\mu]$ we have

$$\begin{aligned} G_2 &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\Big(\sum_{j=k-1}^{\infty} |\lambda_j|\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1} \\ &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha}\Big)^{p_1} \\ &\leq C\|b\|_{\mathrm{Lip}_\beta}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{4.4}$$

Thus, by (4.1)–(4.4) we complete the proof of Theorem 4.1. □

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School of Science, Shandong University of Technology, Zibo, Shandong, China;
hbwang_2006@163.com