

A few examples of local rings, II

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Abstract This is the second part of our three-part study titled “A few examples of local rings.” In the first part, we collected the basic tools and well-known important examples. But, since almost all *special* formal fibers of the examples there have extremely *good* properties, their derived normal rings are *always* Noetherian and analytically unramified. In this paper, we develop the basic ideas of the first part and get a somewhat improved method, which gives factorial local domains whose completion can be almost *any* complete local rings.

0. Introduction

This paper is the second part of our three-part study titled “A few examples of local rings.” In the first part [8], we collected the basic tools and well-known important examples, that is, the fundamental technique of constructing local rings whose *generic* formal fibers are *bad* enough that they give examples to answer famous long-standing questions in the theory of Noetherian rings due to C. Rotthaus [12] (cf. [1]), T. Ogoma [10], [11], and R. C. Heitmann [3]. But, since almost all *special* formal fibers of the examples above have extremely *good* properties, their derived normal rings are *always* Noetherian and analytically unramified. On the other hand, M. Nagata has already obtained bad *normal* local domains, such as a two-dimensional analytically *ramified* factorial local domain of *positive* characteristic and a three-dimensional local domain of *positive* characteristic whose derived normal ring is *not* Noetherian (see [6, pp. 207–208, Examples 5, 6]; cf. [7, Examples 6.2, 6.3]).

In this intermediate part, we develop the basic ideas of the first part and get a somewhat improved method, which gives factorial local domains whose completion can be almost *any* complete local rings.[†] To be precise, we prove the following result.

THEOREM 0.1

Let K be a purely transcendental extension field of countably infinite degree over a countable field K_0 , let $n, r \in \mathbb{N}$, and let z_0, z_1, \dots, z_n be $n + 1$ indeterminates over K . Let $R := K[z_0, \underline{z}]_{(z_0, \underline{z})}$, where $\underline{z} := (z_1, \dots, z_n)$, and let \widehat{R} denote the

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[†] More details concerning a similar construction are given in [4, Theorem 19].

completion of R ; that is, $\widehat{R} = K[[z_0, \underline{z}]]$. Let $T_0 := K_0[Z_1, \dots, Z_n]$ be the polynomial ring in the n variables Z_1, \dots, Z_n with coefficients in K_0 . For each j with $1 \leq j \leq r$, let $F_j := F_j(\underline{Z})$ be a polynomial in T_0 with zero constant term, where $\underline{Z} := (Z_1, \dots, Z_n)$. Suppose that (0.1.1) and (0.1.2) hold:

$$(0.1.1) \quad \bigcup_{\nu \geq 1} \text{Ass}(T_0/(F_1, \dots, F_r)^\nu) = \text{Ass}(T_0/(F_1, \dots, F_r)).$$

For each $\mathfrak{P} \in \text{Ass}(T_0/(F_1, \dots, F_r))$, there exists a proper subset

$$(0.1.2) \quad \{i_1, \dots, i_t\} \text{ of } \{1, 2, \dots, n\}, \text{ so that } \mathfrak{P} \subset \mathfrak{Q} := (Z_{i_1}, \dots, Z_{i_t})T_0.$$

Then there exist:

(1) elements $\zeta_1, \dots, \zeta_n \in \widehat{R}$ that are analytically independent over K such that, with $\underline{\zeta} := (\zeta_1, \dots, \zeta_n)$, we have $K[[z_0, \underline{\zeta}]] = K[[z_0, \underline{z}]]$, and

(2) a factorial local domain (A, \mathfrak{m}) with \mathfrak{m} -adic completion \widehat{A} and a canonical inclusion map $\iota: R \hookrightarrow A$

such that $\underline{\zeta}$, A , \widehat{A} , and the map ι satisfy the conditions (0.1.3)–(0.1.5) below. Set

$$f_j := F_j(\underline{\zeta}) \in K_0[[z_0, \underline{\zeta}]] \subset K[[z_0, \underline{\zeta}]] = \widehat{R}.$$

The conditions to be satisfied are as follows.

The map $\hat{\iota}: \widehat{R} \rightarrow \widehat{A}$ induced by ι is a surjection with kernel (f_1, \dots, f_r)

$$(0.1.3) \quad \text{and the associated map } \tilde{\iota}: \widehat{R}/(f_1, \dots, f_r) \hookrightarrow \widehat{A} \text{ is an isomorphism.}$$

$$(0.1.4) \quad \hat{\mathfrak{q}} := (\tilde{\iota}(\zeta_{i_1}), \dots, \tilde{\iota}(\zeta_{i_t}))\widehat{A} \text{ is a prime ideal of } \widehat{A} \text{ and } \hat{\mathfrak{q}} \cap A = (0).$$

$$(0.1.5) \quad A/\mathfrak{p} \text{ is essentially of finite type over } K, \text{ for every } \mathfrak{p} \in \text{Spec}(A)$$

that has height greater than one.

This theorem gives, as direct applications, the following *bad* local domains. (However, the detailed discussion of these examples is given in the last part of our study [9].)

EXAMPLE 0.2

We have a two-dimensional analytically *ramified* factorial local domain of *arbitrary* characteristic.

EXAMPLE 0.3

We have a two-dimensional regular local ring (A, \mathfrak{m}) of positive characteristic that is *not* Japanese but, for every height-one prime ideal \mathfrak{p} of A , $A_{\mathfrak{p}}$ is a Nagata (i.e., universally Japanese) ring.

EXAMPLE 0.4

We have a three-dimensional local domain of *arbitrary* characteristic whose derived normal ring is *not* Noetherian.

EXAMPLE 0.5

We have a three-dimensional factorial local domain whose completion has *embedded* associated prime ideal(s).

EXAMPLE 0.6

We have a three-dimensional factorial Nagata local domain that is *not* universally catenary.

Now let us summarize the contents of this paper. First we recall the well-known lemmas on regular sequences and Rees algebras in Section 1. In Sections 2 and 3, after setting the notation, we write down several elementary formulas on the relations that play essential roles later. We recommend skipping the tedious parts, namely, Sections 2.4, 2.5, 2.6, and 3.2, in the first reading and coming back to check the particular formulas when necessary.

The main body of this article consists of Sections 4 and 5. Making use of the results in the preceding sections, we prove some more technical lemmas in Section 4. The proofs of these lemmas make clear how to decompose the *original* prime elements of the ground regular local ring by *suitably chosen* prime elements. Then, we show in Section 5 that the lemmas above enable us to carry out a *factorial* numbering on these *suitably chosen* prime elements, in decomposing the *original* prime elements by these *suitably chosen* prime elements step by step. Once a *good* factorial numbering has been achieved, it is not a hard task to construct the desired factorial local domain, which is done in Section 6.

In closing the Introduction, we recommend reading the beautiful original work of Nagata [6], Rotthaus [12], Ogoma [10], [11], Heitmann [3], and Brodmann and Rotthaus [1], because our study was inspired by their fundamental ideas and because, even though they are not explicitly mentioned each time, they are scattered throughout our work. We hope that our construction will be of help when new examples are wanted in the future.

Throughout this paper, all rings are commutative with 1. A local ring (A, \mathfrak{m}) means a *Noetherian* ring A with a unique maximal ideal \mathfrak{m} . The set of natural numbers and that of nonnegative integers are denoted, respectively, by \mathbf{N} and \mathbf{N}_0 . Concerning other terminology, we refer the reader to Grothendieck [2], Matsumura [5], and Nagata [6].

1. Preliminary lemmas

Two goals of this section are (1) to establish that $(\text{Ker } \varphi) \otimes_A C = \text{Ker}(\varphi \otimes \text{id}_C)$ for rings A, B, C and homomorphisms $\varphi: A \rightarrow B$, $\varphi \otimes \text{id}_C: C \rightarrow B \otimes_A C$ that arise in later sections of the paper, and (2) to obtain prime ideals in rings and homomorphic images of rings that arise later. To this end, we prove 12 technical lemmas in this section. Most of the lemmas are for rings of polynomials, since the examples constructed in this paper involve rings with infinitely many indeterminates adjoined. We begin by describing conventions for dealing with indeterminates that we use in this section and extend in later sections.

CONVENTIONS 1.0

Let $n, r \in \mathbf{N}$. We intend that, whenever $\underline{U} := (U_1, \dots, U_n)$ or $\underline{V} := (V_1, \dots, V_r)$ are adjoined to a ring A , they are indeterminates over A . We also assume the same for other elements that are adjoined, unless we indicate otherwise.

Let Z_0 and $\underline{Z} := (Z_1, \dots, Z_n)$ be variables over a ring A . Let $F_j := F_j(Z_0, \underline{Z}) := F_j(Z_0, Z_1, \dots, Z_n)$ be a polynomial in the $n + 1$ variables Z_0, Z_1, \dots, Z_n with coefficients in A for each $j = 1, \dots, r$. We use the abbreviations $\underline{F} := \underline{F}(Z_0, \underline{Z}) := (F_1(Z_0, \underline{Z}), \dots, F_r(Z_0, \underline{Z}))$.

In the first lemma, we observe that the kernel commutes with the tensor product in the case of a map from a polynomial ring over a ring A to a finitely generated extension of A contained in $A[1/a]$, for some non-zero divisor a of A , and the map is tensored with a flat algebra extension B of A .

LEMMA 1.1 (CF. [3], [8, LEMMA 1.1])

Let S be a Noetherian ring, and let q, s_1, \dots, s_n be an S -sequence. Consider the S -homomorphism

$$\phi: S[U_1, \dots, U_n] \rightarrow S[s_1/q, \dots, s_n/q] \quad \text{with } U_i \mapsto s_i/q.$$

Put $P := \text{Ker } \phi$. Then P is generated by an $S[\underline{U}]$ -sequence. Namely,

$$P = (qU_1 - s_1, \dots, qU_n - s_n) \quad \text{and} \quad P \cap S = (0).$$

Moreover, if $(0) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_k$ is a primary decomposition in S , then we get a canonical primary decomposition $P = \mathfrak{Q}_1 \cap \dots \cap \mathfrak{Q}_k$ in $S[\underline{U}]$. In particular, when S is an integral domain, P is a prime ideal of $S[\underline{U}]$.

LEMMA 1.2

With the notation of Lemma 1.1, assume in addition that S is an integral domain. Then, for every $\varepsilon \in \mathbf{N}$, we see that $P := (s_1 + U_1q^\varepsilon, \dots, s_n + U_nq^\varepsilon)$ is a prime ideal generated by the $S[\underline{U}]$ -sequence $s_1 + U_1q^\varepsilon, \dots, s_n + U_nq^\varepsilon$ and that $P \cap S = (0)$.

LEMMA 1.3 (CF. [8, LEMMA 1.1])

Let R_0 be a local domain, and let $\underline{U}_k := (U_{1k}, \dots, U_{nk})$ be indeterminates over R_0 for $k \in \mathbf{N}$. Put

$$R = \bigcup_{k \geq 1} R_k, \quad \text{where } R_k = R_{k-1}(\underline{U}_k).$$

Take a sequence of non-zero elements $p_1, p_2, \dots, p_k, \dots$ in R such that

$$(1.3.1) \quad p_1 \in R_0 \quad \text{and} \quad p_k \in R_{k-2} \quad \text{for every } k \in \mathbf{N} \text{ with } k \geq 2.$$

Put

$$(1.3.2) \quad q_k := p_1 \cdots p_k.$$

Further, take $r_1, \dots, r_n \in R_0$ such that p_1, r_1, \dots, r_n is an R_0 -sequence, and let

$$r_{ik} := r_i + U_{i1}q_1^{\varepsilon_1} + \dots + U_{ik}q_k^{\varepsilon_k}$$

with $\varepsilon_j \in \mathbf{N}$ such that $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_k$. Then $P_k := (r_{1k}, \dots, r_{nk})R$ is a prime ideal generated by an R -sequence r_{1k}, \dots, r_{nk} and

$$(1.3.3) \quad p_h \notin P_k \quad \text{whenever } h \leq k + 1.$$

LEMMA 1.4

With the notation of Lemma 1.3, we assume in addition that

$$p_1 \in R_0, \quad p_k \in R_{k-1}, \quad \text{and} \quad p_k \notin P_{k-1} \quad \text{for every } k \in \mathbf{N} \text{ with } k \geq 2.$$

Then P_k is a prime ideal generated by an R -sequence r_{1k}, \dots, r_{nk} and

$$p_h \notin P_k \quad \text{whenever } 2 \leq h \leq k + 1.$$

LEMMA 1.5

Let T be a ring, and let A be a flat T -algebra with the structure homomorphism $\varphi: T \rightarrow A$. Assume that $F_1, \dots, F_r \in T$, and assume that Δ is a non-zero divisor for T . Putting $f_j := \varphi(F_j)$ and $q = \varphi(\Delta)$, we define ring homomorphisms, where $\underline{F}/\Delta := (F_1/\Delta, \dots, F_r/\Delta)$ and $\underline{f}/q := (f_1/q, \dots, f_r/q)$:

$$\phi: T[\underline{V}] \rightarrow T[\underline{F}/\Delta] \quad \text{that maps } V_j \mapsto F_j/\Delta,$$

$$\psi: A[\underline{V}] \rightarrow A[\underline{f}/q] \quad \text{that maps } V_j \mapsto f_j/q.$$

Then $\text{Ker } \psi = A \otimes \text{Ker } \phi$.

Proof

We may assume $T[\underline{V}]/\text{Ker } \phi \cong T[\underline{F}/\Delta] \subset T[1/\Delta]$. Then, since A is a flat T -algebra, the induced map $A \otimes (T[\underline{V}]/\text{Ker } \phi) \rightarrow A[\underline{f}/q] \subset A[1/q] = A \otimes T[1/\Delta]$ is injective and this gives the assertion. \square

LEMMA 1.6

Let $T := L[\Delta, \underline{Z}]$ be a polynomial ring in $n + 1$ variables over a field L , and let A be a flat T -algebra with the structure homomorphism $\varphi: T \rightarrow A$ that maps Δ to q and Z_i to z_i . Taking $F_1(\Delta, \underline{Z}), \dots, F_r(\Delta, \underline{Z}) \in T$, we define ring homomorphisms:

$$\phi: T[\underline{V}] \rightarrow T[\underline{F}/\Delta] \quad \text{that maps } V_j \mapsto F_j(\Delta, \underline{Z})/\Delta,$$

$$\psi: A[\underline{V}] \rightarrow A[\underline{F}/q] \quad \text{that maps } V_j \mapsto F_j(q, \underline{z})/q,$$

where $\underline{z} := (z_1, \dots, z_n)$ and $\underline{F}/q := (F_1(q, \underline{z})/q, \dots, F_r(q, \underline{z})/q)$. Assume that $\text{Ker } \phi = (G_1(\Delta, \underline{Z}, \underline{V}), \dots, G_s(\Delta, \underline{Z}, \underline{V}))$. Then

$$\text{Ker } \psi = (G_1(q, \underline{z}, \underline{V}), \dots, G_s(q, \underline{z}, \underline{V})).$$

LEMMA 1.7

With T of Lemma 1.6, assume that $F_1, \dots, F_r \in T_0 = L[\underline{Z}] \subset T$. Let $\phi: T[\underline{V}] \rightarrow T[\underline{F}/\Delta]$ be a T -homomorphism that maps V_j to $F_j(\underline{Z})/\Delta$. Then

$$\text{Ker } \phi = (\Delta V_1 - F_1(\underline{Z}), \dots, \Delta V_r - F_r(\underline{Z}), G_1(\underline{Z}, \underline{V}), \dots, G_s(\underline{Z}, \underline{V})),$$

where $G_h(\underline{Z}, \underline{V}) \in L[\underline{Z}, \underline{V}]$ and they are homogeneous in \underline{V} .

Proof

Take $H(\Delta, \underline{Z}, \underline{V}) \in \text{Ker } \phi$. We may assume that H contains no monomials divided by ΔV_j . That is,

$$H(\Delta, \underline{Z}, \underline{V}) := H_1(\Delta, \underline{Z}) + H_2(\underline{Z}, \underline{V}),$$

where $H_1 \in L[\Delta, \underline{Z}]$ and $H_2 \in L[\underline{Z}, \underline{V}]$. Then, replacing V_j by $F_j(\underline{Z})/\Delta$, we have

$$H(\Delta, \underline{Z}, \underline{F}(\underline{Z})/\Delta) = H_1(\Delta, \underline{Z}) + H_2(\underline{Z}, \underline{F}/\Delta) = 0$$

in $L[\Delta, \underline{Z}, 1/\Delta]$. Regarding $H(\Delta, \underline{Z}, \underline{F}/\Delta)$ as a Laurent polynomial in Δ , we see that each homogeneous part of H with respect to \underline{V} belongs to $\text{Ker } \phi$. \square

LEMMA 1.8

With the notation above, let ${}^a\phi: \text{Spec}(T[\underline{F}/\Delta]) \rightarrow \text{Spec}(T)$ be the canonical map. Then

$$(1.8.1) \quad {}^a\phi(\text{Ass}(T[\underline{F}/\Delta]/\Delta T[\underline{F}/\Delta])) = \bigcup_{\nu \geq 1} \text{Ass}(T/(\Delta, (\underline{F})^\nu)).$$

Proof

By our construction, we have the following canonical isomorphism of T -algebras:

$$T[\underline{F}/\Delta]/\Delta T[\underline{F}/\Delta] \cong \bigoplus_{\nu \geq 0} (\bar{F})^\nu \bar{T}/(\bar{F})^{\nu+1} \bar{T},$$

where $\bar{T} := T/\Delta T$ and $\bar{F} := (\bar{F}_1(\Delta, \underline{Z}), \dots, \bar{F}_r(\Delta, \underline{Z})) = (\bar{F}_1(0, \underline{Z}), \dots, \bar{F}_r(0, \underline{Z}))$. Then the assertion follows from the definition of Ass (i.e., the set of associated prime ideals). \square

LEMMA 1.9

Let (A, \mathfrak{m}) be a local ring, let $\underline{a} := (a_1, \dots, a_r)$ be elements of A , and let q be a non-zero divisor of A . Putting $\alpha_j := a_j/q \in Q(A)$, the total quotient ring of A , let $\underline{\alpha} := (\alpha_1, \dots, \alpha_r)$, and we consider the surjective A -homomorphisms:

$$\phi: A[\underline{V}] \rightarrow A[\underline{\alpha}] \quad \text{that maps } V_j \mapsto \alpha_j,$$

$$\phi_U: A[\underline{V}]_U \rightarrow A[\underline{\alpha}]_{\phi(U)} \quad \text{with } U = A[\underline{V}] \setminus (\mathfrak{m}, \underline{V}).$$

Taking an element $b \in \mathfrak{m}$, we put $\bar{A} = A/bA$ and $\bar{\mathfrak{m}} = \mathfrak{m}/bA$. Let $\bar{q}, \bar{a}_1, \dots, \bar{a}_r$, respectively, be the images of q, a_1, \dots, a_r in \bar{A} . When \bar{q} is a non-zero divisor, putting $\bar{\alpha}_j := \bar{a}_j/\bar{q} \in Q(\bar{A})$, let $\bar{\underline{\alpha}} := (\bar{\alpha}_1, \dots, \bar{\alpha}_r)$, and we can consider the induced surjective \bar{A} -homomorphisms:

$$\bar{\phi}: \bar{A}[\underline{V}] \rightarrow \bar{A}[\bar{\underline{\alpha}}] \quad \text{that maps } V_j \mapsto \bar{\alpha}_j,$$

$$\bar{\phi}_{\bar{U}}: \bar{A}[\underline{V}]_{\bar{U}} \rightarrow \bar{A}[\bar{\underline{\alpha}}]_{\bar{\phi}(\bar{U})} \quad \text{with } \bar{U} = \bar{A}[\underline{V}] \setminus (\bar{\mathfrak{m}}, \underline{V}).$$

Assume that $b \notin \text{Ass}(A[\underline{\alpha}]_{\phi(U)}/qA[\underline{\alpha}]_{\phi(U)})$. Then

$$(1.9.1) \quad \bar{A}[\bar{\underline{\alpha}}]_{\bar{\phi}(\bar{U})} \cong \bar{A} \otimes A[\underline{\alpha}]_{\phi(U)}.$$

That is, $\text{Ker } \bar{\phi}_{\bar{U}} = \bar{A} \otimes \text{Ker } \phi_U$. In particular, if b is a prime element of A , then b is a prime element of $A[\underline{\alpha}]_{\phi(U)}$.

Proof

On the one hand, by the following diagram whose rows are exact

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A[\underline{a}/q]_{\phi(U)} & \xrightarrow{\iota} & A[1/q]_{\phi(U)} & \longrightarrow & A[1/q]_{\phi(U)}/A[\underline{a}/q]_{\phi(U)} & \longrightarrow & 0 \\
 & & \downarrow \times b & & \downarrow \times b & & \downarrow \times b & & \\
 0 & \longrightarrow & A[\underline{a}/q]_{\phi(U)} & \xrightarrow{\iota} & A[1/q]_{\phi(U)} & \longrightarrow & A[1/q]_{\phi(U)}/A[\underline{a}/q]_{\phi(U)} & \longrightarrow & 0
 \end{array}$$

where $\underline{a}/q := (a_1/q, \dots, a_r/q) = (\alpha_1, \dots, \alpha_r) = \underline{\alpha}$, we get an exact sequence

$$\bar{A} \otimes A[\underline{a}/q]_{\phi(U)} \xrightarrow{\bar{\iota}} \bar{A}[1/\bar{q}]_{\bar{\phi}(\bar{U})} \longrightarrow \bar{A}[1/\bar{q}]_{\bar{\phi}(\bar{U})}/\bar{A}[\bar{\underline{a}}/\bar{q}]_{\bar{\phi}(\bar{U})} \longrightarrow 0.$$

On the other hand, we have a commutative diagram with exact diagonal maps

$$\begin{array}{ccc}
 \bar{A} \otimes A[\underline{a}/q]_{\phi(U)} & \xrightarrow{\bar{\iota}} & \bar{A}[1/\bar{q}]_{\bar{\phi}(\bar{U})} \\
 & \searrow & \nearrow \\
 & \bar{A}[\bar{\underline{a}}]_{\bar{\phi}(\bar{U})} & \\
 & \nearrow & \searrow \\
 0 & & 0
 \end{array}$$

where $\bar{\iota}$ is injective by assumption. Thus, $\bar{A} \otimes \text{Ker } \phi_U = \text{Ker } \bar{\phi}_{\bar{U}}$. □

LEMMA 1.10

Let A be a Noetherian ring with $q, a_1, \dots, a_r \in A$, where we assume that q is a non-zero divisor for A . Putting $\alpha_j := a_j/q \in Q(A)$, we set $\tilde{\underline{a}} := (\alpha_1, \dots, \alpha_r)A[\underline{\alpha}]$, where $\underline{\alpha} := (\alpha_1, \dots, \alpha_r)$. Take an ideal $\mathfrak{b} := (b_1, \dots, b_n)$ of A , and denote by \bar{q}, \bar{a}_j , respectively, the images of q, a_j in $\bar{A} := A/\mathfrak{b}$, where we further assume that \bar{q} is a non-zero divisor for \bar{A} . Putting $\bar{\alpha}_j := \bar{a}_j/\bar{q} \in Q(\bar{A})$, we consider an \bar{A} -homomorphism $\bar{\phi}: \bar{A}[\underline{V}] \rightarrow \bar{A}[\bar{\underline{\alpha}}]$ that maps $V_j \mapsto \bar{\alpha}_j$. Suppose that

$$\text{Ker } \bar{\phi} = (\bar{q}V_1 - \bar{a}_1, \dots, \bar{q}V_r - \bar{a}_r, \bar{G}_1(\underline{V}), \dots, \bar{G}_s(\underline{V})).$$

Then $\tilde{\mathfrak{b}} := \mathfrak{b}A[1/q] \cap A[\underline{\alpha}] = (b_1, \dots, b_n, G_1(\underline{\alpha}), \dots, G_s(\underline{\alpha}))$, where $G_h(\underline{V}) \in A[\underline{V}]$ is a preimage of $\bar{G}_h(\underline{V})$. Assume further that $G_h(\underline{V}) \in (\underline{V})A[\underline{V}]$ for $h = 1, \dots, s$. Then

$$\tilde{\mathfrak{a}} \cap \tilde{\mathfrak{b}} = (b_1 \cdot \alpha_1, \dots, b_n \cdot \alpha_r, G_1(\underline{\alpha}), \dots, G_s(\underline{\alpha})).$$

LEMMA 1.11

Let A be a local ring that contains a field L , and let a_1, \dots, a_n be an A -sequence. Take polynomials without constant terms $F_1(\underline{Z}), \dots, F_r(\underline{Z}) \in (\underline{Z})L[\underline{Z}]$, and let $I := (F_1(\underline{a}), \dots, F_r(\underline{a}))$, where $\underline{a} := (a_1, \dots, a_n)$. Assume that the residue field of A is separable over L . Then, for each $\mathfrak{p} \in \text{Ass}(A)$, there exists at least one $P \in \text{Ass}(A/I)$ such that $P \supset \mathfrak{p}$.

Proof

Let $T := L[\underline{Z}]_{(\underline{Z})}$, a regular local ring, and let $\varphi: T \rightarrow A$ be a local L -homomorphism that maps $Z_i \mapsto a_i$. Because a_1, \dots, a_n is an A -sequence, φ makes A a flat T -algebra. We may assume that A is complete. Thus, there exists a canonical projection $\pi: R \rightarrow A \cong R/\text{Ker } \pi$, where R is a regular local ring with $L \subseteq R$. For each i , choose $z_i \in R$ with $\pi(z_i) = a_i$. Then φ lifts to R ; that is, we have a local L -homomorphism $\Phi: T \rightarrow R$ that maps $Z_i \mapsto z_i$ such that $\varphi = \pi \circ \Phi$. Because z_1, \dots, z_n forms an R -sequence by [6, (42.8)], Φ makes R a flat T -algebra too.

Let $\mathfrak{p} \in \text{Ass}(A)$. Compare the heights of $J := (F_1(\underline{Z}), \dots, F_r(\underline{Z}))$, $I_0 := (F_1(\underline{z}), \dots, F_r(\underline{z}))$, where $\underline{z} := (z_1, \dots, z_n)$, and $\bar{I} = (I + \mathfrak{p})/\mathfrak{p} = (I_0 + \mathfrak{P})/\mathfrak{P}$, where \mathfrak{P} is the prime ideal of R that corresponds to \mathfrak{p} . Suppose that $\text{ht } J = h$. Then $\text{ht } I_0 = h$. Hence, $\text{ht } \bar{I} \leq h$ by [6, (42.8)]. Take a prime ideal P of A that contains $I + \mathfrak{p}$ and that satisfies the inequality $\text{ht}(P/\mathfrak{p}) \leq h$. Then $\text{depth } A_P \leq h$ by [5, (15.E)]. Consequently, $\text{depth } T_Q \leq h$ for $Q = P \cap T$. Thus, Q contains J and $\text{ht } Q \leq h$. This means that $Q \in \text{Ass}(T/J)$ and that $\text{ht } Q = \text{depth } T_Q = h$. Therefore, $P \in \text{Ass}(A/I)$. \square

LEMMA 1.12

Let A be a ring. Take ideals I, J of A , and take $a, b \in A$. Assume that a is a non-zero divisor for $A/(I + J)$, and assume that b is a non-zero divisor for $A/(aA + I + J)$. Then

$$(aA + I) \cap (bA + J) = abA + aJ + bI + I \cap J.$$

Proof

It suffices to show that the left-hand side of the displayed equation is contained in the right-hand side. Take $x \in (aA + I) \cap (bA + J)$. Then $x = ay + \alpha = bz + \beta$ with $y, z \in A$, $\alpha \in I$, and $\beta \in J$. Thus, $z \in ((aA + I + J) : bA) = (aA + I + J)$; that is, $z = aw + \gamma + \delta$, where $w \in A$, $\gamma \in I$, and $\delta \in J$. Hence,

$$x = b(aw + \gamma + \delta) + \beta = abw + b\gamma + (b\delta + \beta) \in abA + bI + J.$$

Therefore, $b\delta + \beta \in J \cap (aA + I)$. We claim that $J \cap (aA + I) = aJ + I \cap J$.

Proof of the claim. Take $\xi \in J \cap (aA + I)$, and write $\xi = au + \omega$ with $u \in A$ and $\omega \in I$. Then $u \in ((I + J) : aA) = I + J$; that is, $u = \tau + \sigma$ with $\tau \in I$ and $\sigma \in J$. Thus, $\xi = a(\tau + \sigma) + \omega = a\sigma + (a\tau + \omega) \in aJ + I \cap J$. \square

2. Primary relations

2.1. Notation and terminology

Let K_0 be a countable field. Let K be a purely transcendental extension field of countable degree over K_0 , say,

$$K := \bigcup_{k \in \mathbf{N}} K_0(\{a_{i\ell}, b_{i\ell}, c_{j\ell} \mid i = 1, \dots, n; j = 1, \dots, r; \ell = 1, \dots, k\}),$$

where $n, r \in \mathbf{N}$ and $\{a_{i\ell}, b_{i\ell}, c_{j\ell} \mid i = 1, \dots, n; j = 1, \dots, r; \ell = 1, \dots, k\}$ is a transcendental basis over K_0 . Let $\underline{a}_k := (a_{1k}, \dots, a_{nk})$, let $\underline{b}_k := (b_{1k}, \dots, b_{rk})$, let

$\underline{c}_k := (c_{1k}, \dots, c_{rk})$, let $\underline{a} := (\{a_k\}_{k \in \mathbf{N}})$, let $\underline{b} := (\{b_k\}_{k \in \mathbf{N}})$, and let $\underline{c} := (\{c_k\}_{k \in \mathbf{N}})$. Let $K_k := K_{k-1}[\underline{a}_k, \underline{b}_k, \underline{c}_k]$ for every $k \in \mathbf{N}$. Then

$$K = \bigcup_{k \in \mathbf{N}} K_k = K_0(\underline{a}, \underline{b}, \underline{c}).$$

Let $\underline{z} := (z_1, \dots, z_n)$ and $\underline{w} := (w_1, \dots, w_r)$ be indeterminates over K . Let $S_0 := K_0[z_0, \underline{z}, \underline{w}]$ be a polynomial ring with prime ideal $\mathfrak{N}_0 := (z_0, \underline{z}, \underline{w})S_0$, and let $R_0 := (S_0)_{\mathfrak{N}_0} = K_0[z_0, \underline{z}, \underline{w}]_{(z_0, \underline{z}, \underline{w})}$. Then

$$(2.1.0) \quad R_0 \text{ is a regular local ring with maximal ideal } \mathfrak{n}_0 := (z_0, \underline{z}, \underline{w})R_0.$$

Let $S_k := S_{k-1}[\underline{a}_k, \underline{b}_k, \underline{c}_k]$ be a polynomial ring with prime ideal $\mathfrak{N}_k := (z_0, \underline{z}, \underline{w})S_k$, and let $R_k := (S_k)_{\mathfrak{N}_k} = K_k[z_0, \underline{z}, \underline{w}]_{(z_0, \underline{z}, \underline{w})}$. Then

$$(2.1.1) \quad R_k \text{ is a regular local ring with maximal ideal } \mathfrak{n}_k := (z_0, \underline{z}, \underline{w})R_k.$$

Further, put

$$(2.1.2) \quad S := \bigcup_{k \in \mathbf{N}} S_k = K_0[\underline{a}, \underline{b}, \underline{c}, z_0, \underline{z}, \underline{w}] \text{ with prime ideal } \mathfrak{N} := (z_0, \underline{z}, \underline{w})S,$$

and let $R := S_{\mathfrak{N}} = K[z_0, \underline{z}, \underline{w}]_{(z_0, \underline{z}, \underline{w})}$. Then

$$(2.1.3) \quad R \text{ is a countable regular local ring with maximal ideal } \mathfrak{n} := (z_0, \underline{z}, \underline{w})R$$

such that

$$(2.1.4) \quad R_k = R_{k-1}(\underline{a}_k, \underline{b}_k, \underline{c}_k) \quad \text{and} \quad R = \bigcup_{k \in \mathbf{N}} R_k.$$

2.2. Polynomial relations

With the notation of Section 2.1, we consider r polynomials in n variables over K_0 with zero constant term: $F_1(\underline{Z}), \dots, F_r(\underline{Z}) \in (\underline{Z})K_0[\underline{Z}]$.

Let $\{\Delta_{ig}, \nabla_{ih} \mid 1 \leq i \leq n \text{ and } g, h \in \mathbf{N}_0\}$ be a countably infinite set of indeterminates over $K_0[\underline{Z}]$. Define, for each $\ell, m \in \mathbf{N}$,

$$X_{i\ell} := \sum_{g=0}^{\ell} \Delta_{ig}, \quad Y_{im} := \sum_{h=0}^m \nabla_{ih}, \quad \text{and} \quad Z_{i\ell m} := X_{i\ell} + Y_{im}.$$

Given a monomial in the $n(\ell+1)$ indeterminates Δ_{ig} , where $1 \leq i \leq n$ and $0 \leq g \leq \ell$, we abbreviate the monomial by $\Delta^{E^+(\ell)}$, where $E^+(\ell) := (e_{10}^+, \dots, e_{n\ell}^+) \in \mathbf{N}_0^{n(\ell+1)}$ is the ordered tuple of $n(\ell+1)$ exponents that occur in the monomial. That is,

$$\Delta^{E^+(\ell)} := \Delta_{10}^{e_{10}^+} \dots \Delta_{n\ell}^{e_{n\ell}^+}.$$

Similarly for monomials in the $n(m+1)$ indeterminates ∇_{ih} , where $1 \leq i \leq n$ and $0 \leq h \leq m$, set

$$\nabla^{E^-(m)} := \nabla_{10}^{e_{10}^-} \dots \nabla_{nm}^{e_{nm}^-},$$

where $E^-(m) := (e_{10}^-, \dots, e_{nm}^-) \in \mathbf{N}_0^{n(m+1)}$. Note that in [9] we continue to use $+$ to designate expressions involving or dominated by the indeterminates Δ_{ig} , and

we use $-$ to designate expressions involving or dominated by the indeterminates ∇_{ih} . Let

$$\begin{aligned}\underline{\Delta}_\ell &:= (\Delta_{10}, \Delta_{11}, \dots, \Delta_{1\ell}, \Delta_{20}, \dots, \Delta_{n\ell}), \\ \underline{\nabla}_m &:= (\nabla_{10}, \nabla_{11}, \dots, \nabla_{1m}, \nabla_{20}, \dots, \nabla_{nm}),\end{aligned}$$

and let $\underline{Z}_{\ell m} := (Z_{1\ell m}, \dots, Z_{n\ell m})$. Then, for each j with $1 \leq j \leq r$, we write $F_j(\underline{Z}_{\ell m})$ in terms of monomials in the Δ_{ig} 's and ∇_{ih} 's as follows:

$$\begin{aligned}F_j(\underline{Z}_{\ell m}) &:= F_j\left(\sum_{g=0}^{\ell} \Delta_{1g} + \sum_{h=0}^m \nabla_{1h}, \dots, \sum_{g=0}^{\ell} \Delta_{ng} + \sum_{h=0}^m \nabla_{nh}\right) \\ &= F_j(\Delta_{10} + \dots + \Delta_{1\ell} + \nabla_{10} + \dots + \nabla_{1m}, \dots, \Delta_{n0} + \dots + \nabla_{nm}) \\ &= \sum k_{jE^+(\ell)E^-(m)} \Delta^{E^+(\ell)} \nabla^{E^-(m)} \in K_0[\underline{\Delta}_\ell, \underline{\nabla}_m],\end{aligned}$$

for some elements $k_{jE^+(\ell)E^-(m)} \in K_0$, where the sum is taken over every monomial $\Delta^{E^+(\ell)} \nabla^{E^-(m)}$ that occurs in $F_j(\underline{Z}_{\ell m})$. With the terminology above, we define, for each $E^+(\ell)$ and $E^-(m)$,

$$\begin{aligned}o(E^+(\ell)) &:= \max\{0, g \mid 0 \leq g \leq \ell \text{ and } e_{ig}^+ > 0 \text{ for some } i \text{ with } 1 \leq i \leq n\}, \\ o(E^-(m)) &:= \max\{0, h \mid 0 \leq h \leq m \text{ and } e_{ih}^- > 0 \text{ for some } i \text{ with } 1 \leq i \leq n\}.\end{aligned}$$

That is, $o(E^+(\ell))$ is the largest $g \leq \ell$ such that the exponent of Δ_{ig} is positive in $\Delta^{E^+(\ell)}$ for some i , and similarly for $o(E^-(m))$. Next we write

$$F_j(\underline{Z}_{\ell m}) = F_j^+(\underline{Z}_{\ell m}) + F_j^-(\underline{Z}_{\ell m}),$$

where

$$F_j^+(\underline{Z}_{\ell m}) := \sum_{o(E^+(\ell)) > o(E^-(m))} k_{jE^+(\ell)E^-(m)} \Delta^{E^+(\ell)} \nabla^{E^-(m)}$$

and

$$F_j^-(\underline{Z}_{\ell m}) := \sum_{o(E^+(\ell)) \leq o(E^-(m))} k_{jE^+(\ell)E^-(m)} \Delta^{E^+(\ell)} \nabla^{E^-(m)}.$$

Let B be a ring, and let $\underline{Z} := Z_1, \dots, Z_n$ be variables over B . Take $F(\underline{Z}) \in B[\underline{Z}]$, and take variables $\underline{\Omega} := (\Omega_1, \dots, \Omega_n)$ over $B[\underline{Z}]$. By Taylor's expansion, we get

$$(2.2.1) \quad F(\underline{Z} + \underline{\Omega}) = F(\underline{Z}) + \sum \frac{\partial^D F}{(\partial \underline{Z})^D}(\underline{Z}) \Omega^D \in B[\underline{Z}, \underline{\Omega}],$$

where $\underline{Z} + \underline{\Omega} := (Z_1 + \Omega_1, \dots, Z_n + \Omega_n)$,

$$\frac{\partial^D F}{(\partial \underline{Z})^D}(\underline{Z}) := \frac{1}{d_1! \dots d_n!} \cdot \frac{\partial^{d_1 + \dots + d_n} F}{(\partial Z_1)^{d_1} \dots (\partial Z_n)^{d_n}}(\underline{Z}),$$

and $\Omega^D := \Omega_1^{d_1} \cdots \Omega_n^{d_n}$, and the sum is taken over all $D := (d_1, \dots, d_n) \in \mathbf{N}_0^n \setminus \{(0, \dots, 0)\}$. Note that the formula above works in arbitrary characteristic. Hence, with the notation above, we have

$$\begin{aligned} F_j(\underline{Z}_{(k+1)k}) &:= F_j(Z_{1kk} + \Delta_{1(k+1)}, \dots, Z_{nkk} + \Delta_{n(k+1)}) \\ &= F_j(\underline{Z}_{kk}) + \sum \frac{\partial^D F_j}{(\partial Z)^D}(\underline{Z}_{kk}) \Delta_{k+1}^D. \end{aligned}$$

Thus,

$$\begin{aligned} F_j^+(\underline{Z}_{(k+1)k}) &= F_j^+(\underline{Z}_{kk}) + \sum \frac{\partial^D F_j}{(\partial Z)^D}(\underline{Z}_{kk}) \Delta_{k+1}^D, \\ F_j^-(\underline{Z}_{(k+1)k}) &= F_j^-(\underline{Z}_{kk}). \end{aligned}$$

By the same reasoning, we have

$$\begin{aligned} F_j^+(\underline{Z}_{(k+1)(k+1)}) &= F_j^+(\underline{Z}_{(k+1)k}), \\ F_j^-(\underline{Z}_{(k+1)(k+1)}) &= F_j^-(\underline{Z}_{(k+1)k}) + \sum \frac{\partial^D F_j}{(\partial Z)^D}(\underline{Z}_{(k+1)k}) \nabla_{k+1}^D. \end{aligned}$$

2.3. Numbering, I

With the notation as in Section 2.1, suppose that $\underline{x} := (x_1, \dots, x_n)$ and $\underline{y} := (y_1, \dots, y_n)$ are elements of R_0 that satisfy the following:

$$(2.3.0) \quad z_i = x_i + y_i \quad \text{for } i = 1, \dots, n,$$

$$(2.3.1) \quad z_0, x_1, \dots, x_n, w_1, \dots, w_r \text{ form regular systems of parameters of } R,$$

$$(2.3.2) \quad z_0, y_1, \dots, y_n, w_1, \dots, w_r \text{ form regular systems of parameters of } R.$$

Inductively, we define two sequences of elements of the field of quotients $Q(R)$

$$\pi_1, \pi_2, \dots, \pi_k, \dots \quad \text{and} \quad p_1, p_2, \dots, p_k, \dots$$

in the following manner (for the details, see Section 5):

$$\pi_1 := z_0 \quad \text{and} \quad p_1 \in R_0.$$

Supposing that π_1, \dots, π_k and p_1, \dots, p_k are already chosen, we write

$$q_\ell := p_1 \cdots p_\ell \quad \text{and} \quad \theta_m := \pi_1 \cdots \pi_m \quad \text{for } \ell, m \leq k.$$

By choosing suitable natural numbers $\varepsilon_1, \dots, \varepsilon_k$, we set

$$t_h := q_h^{\varepsilon_h} \theta_h \quad \text{for } h \leq k.$$

For $h, \ell, m \leq k$, we put

$$(2.3.3) \quad x_{i\ell} := x_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{i\ell} q_\ell^{\varepsilon_\ell},$$

$$(2.3.4) \quad \eta_{im} := y_i + b_{i1} \theta_1^{\varepsilon_1} + \cdots + b_{im} \theta_m^{\varepsilon_m},$$

$$(2.3.5) \quad w_{jh} := w_j + c_{j1} t_1^{\varepsilon_1} + \cdots + c_{jh} t_h^{\varepsilon_h},$$

$$(2.3.6) \quad \zeta_{ilm} := x_{i\ell} + \eta_{im}.$$

With the notation above, put

$$(2.3.7) \quad \phi_{j\ell m}^+ := F_j^+(\underline{\zeta}_{\ell m}),$$

$$(2.3.8) \quad \phi_{j\ell m}^- := F_j^-(\underline{\zeta}_{\ell m}).$$

Here we use the abbreviations as in Conventions 1.0, namely, $\underline{x}_\ell := (x_{1\ell}, \dots, x_{n\ell})$, $\underline{\eta}_m := (\eta_{1m}, \dots, \eta_{nm})$, $\underline{w}_h := (w_{1h}, \dots, w_{rh})$, and $\underline{\zeta}_{\ell m} := (\zeta_{1\ell m}, \dots, \zeta_{n\ell m})$. Further, for suitable natural numbers ν_1, \dots, ν_k , let

$$(2.3.9) \quad \alpha_{j\ell m}^+ := \frac{1}{q_\ell^{\nu_\ell}} (\phi_{j\ell m}^+ + w_{j\ell}),$$

$$(2.3.10) \quad \alpha_{j\ell m}^- := \frac{1}{\theta_m^{\nu_m}} (\phi_{j\ell m}^- - w_{jm}).$$

Finally, take π_{k+1}, p_{k+1} that satisfy

$$(2.3.11) \quad p_{k+1} \in R_{k-1},$$

$$(2.3.12) \quad \pi_{k+1} := r_{k+1} + \sum r_{(k+1)H} (q_k \alpha_{kk}^+)^H,$$

where $(q_k \alpha_{kk}^+)^H := (q_k \alpha_{1kk}^+)^{h_1} \cdots (q_k \alpha_{rkk}^+)^{h_r}$ and $r_{k+1}, r_{(k+1)H} \in R_k$, and the sum is taken over all $H := (h_1, \dots, h_r) \in \mathbf{N}_0^r \setminus \{(0, \dots, 0)\}$. Thus, by repeating the process above, we get two sequences

$$\pi_1, \pi_2, \dots, \pi_k, \dots \quad \text{and} \quad p_1, p_2, \dots, p_k, \dots$$

Here we remark that, when the ε_k 's are chosen sufficiently large,

$$q_k := p_1 \cdots p_k \in R_{k-2},$$

$$\theta_k := \pi_1 \cdots \pi_k \in R_{k-1} [q_{k-1} \alpha_{(k-1)(k-1)}^+],$$

where $q_{k-1} \alpha_{(k-1)(k-1)}^+ := (q_{k-1} \alpha_{1(k-1)(k-1)}^+, \dots, q_{k-1} \alpha_{r(k-1)(k-1)}^+)$,

$$(2.3.13) \quad t_k := q_k^{\varepsilon_k} \theta_k \in q_k R_{k-1},$$

$$x_{ik} := x_i + a_{i1} q_1^{\varepsilon_1} + \cdots + a_{ik} q_k^{\varepsilon_k} \in R_{k-1} [a_{ik}],$$

$$\eta_{ik} := y_i + b_{i1} \theta_1^{\varepsilon_1} + \cdots + b_{ik} \theta_k^{\varepsilon_k} \in R_{k-1} [q_{k-1} \alpha_{(k-1)(k-1)}^+] [b_{ik}],$$

$$(2.3.14) \quad w_{jk} := w_j + c_{j1} t_1^{\varepsilon_1} + \cdots + c_{jk} t_k^{\varepsilon_k} \in R_{k-1} [c_{jk}].$$

2.4. Numbering, II

With the notation above, by Taylor's expansion, taking the sums over all $D := (d_1, \dots, d_n) \in \mathbf{N}_0^n \setminus \{(0, \dots, 0)\}$, we get

$$\phi_{j(k+1)k}^+ = \phi_{jkk}^+ + \sum \frac{\partial^D F_j}{(\partial Z)^D} (\underline{\zeta}_{kk}) (a_{k+1} q_{k+1}^{\varepsilon_{k+1}})^D,$$

$$\phi_{j(k+1)k}^- = \phi_{jkk}^-,$$

$$\phi_{j(k+1)(k+1)}^+ = \phi_{j(k+1)k}^+,$$

$$(2.4.1) \quad \phi_{j(k+1)(k+1)}^- = \phi_{j(k+1)k}^- + \sum \frac{\partial^D F_j}{(\partial Z)^D} (\underline{\zeta}_{(k+1)k}) (b_{k+1} \theta_{k+1}^{\varepsilon_{k+1}})^D.$$

Hence,

$$\begin{aligned} \alpha_{j(k+1)k}^+ &= \frac{1}{q_{k+1}^{\nu_{k+1}}} (\phi_{j(k+1)k}^+ + w_{j(k+1)}) \\ &= \frac{1}{q_{k+1}^{\nu_{k+1}}} \left\{ \phi_{jkk}^+ + \sum \frac{\partial^D F_j}{(\partial Z)^D} (\underline{\zeta}_{kk}) (a_{k+1} q_{k+1}^{\varepsilon_{k+1}})^D \right. \\ &\quad \left. + (w_{jk} + c_{j(k+1)} t_{k+1}^{\varepsilon_{k+1}}) \right\}, \\ \alpha_{j(k+1)k}^- &= \frac{1}{\theta_k^{\nu_k}} (\phi_{jkk}^- - w_{jk}), \\ \alpha_{j(k+1)(k+1)}^+ &= \frac{1}{q_{k+1}^{\nu_{k+1}}} (\phi_{j(k+1)k}^+ + w_{j(k+1)}), \\ \alpha_{j(k+1)(k+1)}^- &= \frac{1}{\theta_{k+1}^{\nu_{k+1}}} \left\{ \phi_{j(k+1)k}^- + \sum \frac{\partial^D F_j}{(\partial Z)^D} (\underline{\zeta}_{(k+1)k}) (b_{k+1} \theta_{k+1}^{\varepsilon_{k+1}})^D \right. \\ &\quad \left. - (w_{jk} + c_{j(k+1)} t_{k+1}^{\varepsilon_{k+1}}) \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_{jkk}^+ &= \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \alpha_{j(k+1)k}^+ - c_{j(k+1)} \frac{t_{k+1}^{\varepsilon_{k+1}}}{q_k^{\nu_k}} \\ &\quad - \frac{1}{q_k^{\nu_k}} \sum \frac{\partial^D F_j}{(\partial Z)^D} (\underline{\zeta}_{kk}) (a_{k+1} q_{k+1}^{\varepsilon_{k+1}})^D \\ &= \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \alpha_{j(k+1)k}^+ + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\nu_k}} \kappa_{jkk}^+, \\ \alpha_{jkk}^- &= \alpha_{j(k+1)k}^-, \\ \alpha_{j(k+1)k}^+ &= \alpha_{j(k+1)(k+1)}^+, \\ \alpha_{j(k+1)k}^- &= \frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \alpha_{j(k+1)(k+1)}^- + c_{j(k+1)} \frac{t_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k}} \\ &\quad - \frac{1}{\theta_k^{\nu_k}} \sum \frac{\partial^D F_j}{(\partial Z)^D} (\underline{\zeta}_{(k+1)k}) (b_{k+1} \theta_{k+1}^{\varepsilon_{k+1}})^D \\ &= \frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \alpha_{j(k+1)(k+1)}^- + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k}} \kappa_{j(k+1)k}^-, \end{aligned}$$

where

$$\begin{aligned} \kappa_{jkk}^+ &\in R_0[a_{k+1}, c_{k+1}, q_{k+1}^{\varepsilon_{k+1}}, q_{k+1}^{\varepsilon_{k+1}-1} \theta_{k+1}, \underline{\zeta}_{kk}], \\ \kappa_{j(k+1)k}^- &\in R_0[b_{k+1}, c_{k+1}, q_{k+1}^{\varepsilon_{k+1}}, \theta_{k+1}^{\varepsilon_{k+1}}, \underline{\zeta}_{(k+1)k}]. \end{aligned}$$

Therefore,

$$(2.4.2) \quad \alpha_{jkk}^+ = \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \alpha_{j(k+1)(k+1)}^+ + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\nu_k}} \kappa_{jkk}^+,$$

$$(2.4.3) \quad \alpha_{jkk}^- = \frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \alpha_{j(k+1)(k+1)}^- + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k}} \kappa_{j(k+1)k}^-,$$

where $\kappa_{jkk}^+ \in R_k[\underline{a_{k+1}}, \underline{c_{k+1}}, q_{k-1} \alpha_{(k-1)(k-1)}^+]$ and $\kappa_{j(k+1)k}^- \in R_k[\underline{a_{k+1}}, \underline{b_{k+1}}, \underline{c_{k+1}}, q_k \alpha_{kk}^+]$.

2.5. Numbering, III

Consequently, $\pi_2, \pi_3, \dots, \pi_h, \dots$ are written as

$$\begin{aligned} \pi_h &= r_h + \sum r_{hH} (q_{h-1} \alpha_{(h-1)(h-1)}^+)^H \\ &= r_h + \sum r_{hH} \left(q_{h-1} \left(\frac{q_h^{\nu_h}}{q_{h-1}^{\nu_{h-1}}} \alpha_{hh}^+ + \frac{q_h^{\varepsilon_h}}{q_{h-1}^{\nu_{h-1}}} \kappa_{(h-1)(h-1)}^+ \right) \right)^H \\ &= r_h + q_h^{\varepsilon'_h} u_{hh} + \sum r_{hH} \left(\frac{q_h^{\nu_h-1}}{q_{h-1}^{\nu_{h-1}-1}} q_h \alpha_{hh}^+ \right)^H, \end{aligned}$$

where each $r_h, r_{hH} \in R_{h-1}$, where if we take ε_h sufficiently large, then ε'_h is large enough and $u_{hh} \in R_{h-1}[\underline{a_h}, \underline{c_h}]$ (cf. (2.3.12), (2.4.2)), and where the sums are taken over all $H := (h_1, \dots, h_r) \in \mathbf{N}_0^r \setminus \{(0, \dots, 0)\}$. Hence, if $h \leq m$ and $\varepsilon_h, \dots, \varepsilon_m$ are sufficiently large, then

$$(2.5.1) \quad \begin{aligned} \pi_h &= r_h + q_h^{\varepsilon'_h} u_{hm} + \sum r_{hH} \left(\frac{q_m^{\nu_m-1}}{q_{h-1}^{\nu_{h-1}-1}} q_m \alpha_{mm}^+ \right)^H \\ &\quad \text{with } u_{hm} \in R_{m-1}[\underline{a_m}, \underline{c_m}]. \end{aligned}$$

Putting $s_{hm} := \pi_1(r_2 + q_2^{\varepsilon'_2} u_{2m}) \cdots (r_h + q_h^{\varepsilon'_h} u_{hm}) \in R_{m-1}[\underline{a_m}, \underline{c_m}]$, we have

$$(2.5.2) \quad \begin{aligned} \theta_h &= s_{hm} + \sum t_{hH} \left(\frac{q_m^{\nu_m-1}}{q_{h-1}^{\nu_{h-1}-1}} q_m \alpha_{mm}^+ \right)^H \\ &\quad \text{with } t_{hH} \in \pi_1 R_{m-1}[\underline{a_m}, \underline{c_m}]. \end{aligned}$$

Thus, when $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ are suitably large, we see that

$$\begin{aligned} \eta_{im} &:= y_i + b_{i1} \theta_1^{\varepsilon_1} + b_{i2} \theta_2^{\varepsilon_2} + \cdots + b_{im} \theta_m^{\varepsilon_m} \\ &= y_i + b_{i1} \pi_1^{\varepsilon_1} + \left(b_{i2} s_{2m}^{\varepsilon_2} + \sum v_{i2mH} \left(\frac{q_m^{\nu_m}}{q_1^{\nu_1}} \alpha_{mm}^+ \right)^H \right) + \cdots \\ &\quad + \left(b_{im} s_{mm}^{\varepsilon_m} + \sum v_{immH} \left(\frac{q_m^{\nu_m}}{q_{m-1}^{\nu_{m-1}}} \alpha_{mm}^+ \right)^H \right) \\ &= y_i + b_{i1} \pi_1^{\varepsilon_1} + b_{i2} s_{2m}^{\varepsilon_2} + \cdots + b_{im} s_{mm}^{\varepsilon_m} + \sum \Upsilon_{imH} \left(\frac{q_m^{\nu_m}}{q_{m-1}^{\nu_{m-1}}} \alpha_{mm}^+ \right)^H, \end{aligned}$$

where

$$\Upsilon_{imH} = \sum_{h=2}^m v_{ihmH} (q_{m-1}^{\nu_{m-1}} / q_{h-1}^{\nu_{h-1}})^H \in \pi_1 R_{m-1} [\underline{a}_m, \underline{b}_m, \underline{c}_m].$$

By letting $z_{ilm} := x_{il} + y_{im}$ with $y_{im} := y_i + b_{i1}\pi_1^{\varepsilon_1} + b_{i2}s_{2m}^{\varepsilon_2} + \cdots + b_{im}s_{mm}^{\varepsilon_m}$, we get

$$(2.5.3) \quad \zeta_{ilm} = x_{il} + \eta_{im} = z_{ilm} + \sum \Upsilon_{imH} \left(\frac{q_m^{\nu_m}}{q_{m-1}^{\nu_{m-1}}} \alpha_{mm}^+ \right)^H.$$

When $m \leq \ell$ and $\varepsilon_2, \dots, \varepsilon_\ell$ are sufficiently large, we have

$$(2.5.4) \quad \phi_{j\ell m}^+ = g_{j\ell m}^+ + \sum_{\hat{j}=1}^r d_{j\ell m \hat{j}}^+ q_m^{\nu_m} \alpha_{jmm}^+,$$

where $g_{j\ell m}^+ := F_j^+(z_{\ell m}) \in R_{\ell-1}[\underline{a}_\ell, \underline{c}_\ell]$ with $z_{\ell m} := (z_{1\ell m}, \dots, z_{n\ell m})$, $\underline{a}_\ell := (a_{1\ell}, \dots, a_{n\ell})$, and $\underline{c}_\ell := (c_{1\ell}, \dots, c_{r\ell})$, and where $d_{j\ell m \hat{j}}^+ \in \pi_1 \mathfrak{n}_{\ell-1} R_{\ell-1}[\underline{a}_\ell, \underline{c}_\ell]$ (cf. (2.1.1)). Also, we get

$$(2.5.5) \quad \phi_{j\ell m}^- = g_{j\ell m}^- + \sum \Delta_{j\ell m H}^- (q_m \alpha_{mm}^+)^H,$$

where $g_{j\ell m}^- := F_j^-(z_{\ell m}) \in R_{\ell-1}[\underline{a}_\ell, \underline{b}_m, \underline{c}_\ell]$ with $\underline{b}_m := (b_{1m}, \dots, b_{nm})$ and where $\Delta_{j\ell m H}^- \in \pi_1 \mathfrak{n}_{\ell-1} R_{\ell-1}[\underline{a}_\ell, \underline{b}_m, \underline{c}_\ell]$.

2.6. Numbering, IV

Finally, we remark that

$$(2.6.1) \quad \begin{aligned} \phi_{jkk}^+ + w_{jk} &= \left(g_{jkk}^+ + \sum_{\hat{j}=1}^r d_{jkk\hat{j}}^+ q_k^{\nu_k} \alpha_{jkk}^+ \right) + w_{jk} \\ &= g_{jkk}^+ + w_{jk} + \sum_{\hat{j}=1}^r d_{jkk\hat{j}}^+ (\phi_{jkk}^+ + w_{jk}). \end{aligned}$$

Thus,

$$(2.6.2) \quad \phi_{jkk}^+ + w_{jk} = \sum_{\hat{j}=1}^r r_{j\hat{j}}^{(k)} (g_{jkk}^+ + w_{jk})$$

with an invertible $r \times r$ matrix $(r_{j\hat{j}}^{(k)})$ in $R_{k-1}(\underline{a}_k, \underline{c}_k)$ (cf. (2.5.4)).

If we put $\alpha_{jkk}^\Delta := (1/q_k^{\nu_k})(g_{jkk}^+ + w_{jk})$, then $\alpha_{jkk}^+ = \sum_{\hat{j}=1}^r r_{j\hat{j}}^{(k)} \alpha_{jkk}^\Delta$ and $R_k[\alpha_{kk}^+] = R_k[\alpha_{kk}^\Delta]$ where $\alpha_{kk}^+ := (\alpha_{1kk}^+, \dots, \alpha_{rkk}^+)$ and $\alpha_{kk}^\Delta := (\alpha_{1kk}^\Delta, \dots, \alpha_{rkk}^\Delta)$. Hence,

$$(2.6.3) \quad \begin{aligned} \phi_{jkk}^+ &= g_{jkk}^+ + \sum_{\hat{j}=1}^r d_{jkk\hat{j}}^+ q_k^{\nu_k} \alpha_{jkk}^+ \\ &= g_{jkk}^+ + \sum_{\hat{j}=1}^r d_{jkk\hat{j}}^\Delta q_k^{\nu_k} \alpha_{jkk}^\Delta, \end{aligned}$$

$$(2.6.4) \quad \begin{aligned} \phi_{jkk}^- &= g_{jkk}^- + \sum \Delta_{jkkH}^- (q_k \alpha_{kk}^+)^H \\ &= g_{jkk}^- + \sum \Delta_{jkkH}^\nabla (q_k \alpha_{kk}^\Delta)^H, \end{aligned}$$

where $d_{jkkj}^+, d_{jkkj}^\Delta \in \pi_1 \mathfrak{n}_{k-1} R_{k-1}(\underline{a}_k, \underline{c}_k)$ and $\Delta_{jkkH}^-, \Delta_{jkkH}^\nabla \in \pi_1 \mathfrak{n}_k R_k$.

3. Secondary relations

3.1. Taylor's formulas, I

With notation as in previous sections, let $T := K_0[Z_1, \dots, Z_n, \Delta]$ be a polynomial ring in $n+1$ variables over a field K_0 . Take r polynomials in n variables $\underline{Z} := (Z_1, \dots, Z_n)$ over K_0 with zero constant term:

$$F_1(\underline{Z}), \dots, F_r(\underline{Z}) \in T_0 := K_0[Z_1, \dots, Z_n] \subset T.$$

Let $\underline{V} := (V_1, \dots, V_r)$, and let $\phi: T[\underline{V}] \rightarrow T[1/\Delta]$ be a T -homomorphism that maps V_j to $F_j(\underline{Z})/\Delta$. Then

$$(3.1.0) \quad \text{Ker } \phi = (\Delta V_1 - F_1(\underline{Z}), \dots, \Delta V_r - F_r(\underline{Z}), G_1(\underline{Z}, \underline{V}), \dots, G_s(\underline{Z}, \underline{V})),$$

where $G_h(\underline{Z}, \underline{V}) \in T_0[\underline{V}]$ and homogeneous of degree g_h in \underline{V}

for $h = 1, \dots, s$ (cf. Lemma 1.7). By definition, we have

$$(3.1.1) \quad G_h(\underline{Z}, \underline{F}/\Delta) \equiv 0 \quad \text{in } T[1/\Delta],$$

where $\underline{F}/\Delta := (F_1(\underline{Z})/\Delta, \dots, F_r(\underline{Z})/\Delta)$. Now let $\underline{\Lambda} := (\Lambda_1, \dots, \Lambda_r)$ be new variables over T . Recall that, by Taylor's expansion (see (2.2.1)), taking the sums over all $E := (e_1, \dots, e_r) \in \mathbf{N}_0^r \setminus \{(0, \dots, 0)\}$, we have

$$G_h(\underline{Z}, \underline{V} + \underline{\Lambda}) = G_h(\underline{Z}, \underline{V}) + \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{Z}, \underline{V}) \Lambda^E,$$

where $\underline{V} + \underline{\Lambda} := (V_1 + \Lambda_1, \dots, V_r + \Lambda_r)$,

$$\frac{\partial^E G_h}{(\partial V)^E}(\underline{Z}, \underline{V}) = \frac{1}{e_1! \cdots e_r!} \cdot \frac{\partial^{e_1 + \cdots + e_r} G_h}{(\partial V_1)^{e_1} \cdots (\partial V_r)^{e_r}}(\underline{Z}, \underline{V}),$$

and we abbreviate the monomial $\Lambda_1^{e_1} \cdots \Lambda_r^{e_r}$ by Λ^E . In (3.1.1), we replace Z_i and $F_j(\underline{Z})$, respectively, by

$$Z_{ilm} := (\Delta_{i0} + \Delta_{i1} + \cdots + \Delta_{i\ell}) + (\nabla_{i0} + \nabla_{i1} + \cdots + \nabla_{im}),$$

$$F_j(\underline{Z}_{\ell m}) := (F_j^+(\underline{Z}_{\ell m}) + W_{j\ell}) + (F_j^-(\underline{Z}_{\ell m}) - W_{j\ell}),$$

where $\underline{Z}_{\ell m} := (Z_{1\ell m}, \dots, Z_{n\ell m})$. Further, if we replace Δ by Δ_ℓ , then we get

$$\begin{aligned} &G_h(\underline{Z}_{\ell m}, \underline{F}(\underline{Z}_{\ell m})/\Delta_\ell) \\ &= G_h\left(\underline{Z}_{\ell m}, \frac{(F^+(\underline{Z}_{\ell m}) + W_\ell) + (F^-(\underline{Z}_{\ell m}) - W_\ell)}{\Delta_\ell}\right) \equiv 0 \end{aligned}$$

in $T[\Delta_\ell, \underline{W}_\ell, \underline{\Delta}_\ell, \underline{\nabla}_m, 1/\Delta_\ell]$. Here we denote

$$(3.1.2) \quad \begin{aligned} \underline{F}(\underline{Z}_{\ell m})/\Delta_\ell &:= (F_1(\underline{Z}_{\ell m})/\Delta_\ell, \dots, F_r(\underline{Z}_{\ell m})/\Delta_\ell), \\ \underline{W}_\ell &:= (W_{1\ell}, \dots, W_{r\ell}), \\ \underline{F}^\pm(\underline{Z}_{\ell m}) &:= (F_1^\pm(\underline{Z}_{\ell m}), \dots, F_r^\pm(\underline{Z}_{\ell m})), \\ \underline{\Delta}_\ell &:= (\Delta_{10}, \Delta_{11}, \dots, \Delta_{1\ell}, \Delta_{20}, \dots, \Delta_{n\ell}), \quad \text{and} \\ \underline{\nabla}_m &:= (\nabla_{10}, \nabla_{11}, \dots, \nabla_{1m}, \nabla_{20}, \dots, \nabla_{nm}). \end{aligned}$$

Further, if we replace Δ by ∇_m , then we get

$$\begin{aligned} &G_h(\underline{Z}_{\ell m}, \underline{F}(\underline{Z}_{\ell m})/\nabla_m) \\ &= G_h\left(\underline{Z}_{\ell m}, \frac{(F^-(\underline{Z}_{\ell m}) - W_m) + (F^+(\underline{Z}_{\ell m}) + W_m)}{\nabla_m}\right) \equiv 0 \end{aligned}$$

in $T[\nabla_m, \underline{W}_\ell, \underline{\Delta}_\ell, \underline{\nabla}_m, 1/\nabla_m]$, where

$$\begin{aligned} \underline{F}(\underline{Z}_{\ell m})/\nabla_m &:= (F_1(\underline{Z}_{\ell m})/\nabla_m, \dots, F_r(\underline{Z}_{\ell m})/\nabla_m) \quad \text{and} \\ \underline{W}_m &:= (W_{1m}, \dots, W_{rm}). \end{aligned}$$

Hence,

$$\begin{aligned} &G_h\left(\underline{Z}_{\ell m}, \frac{F^+(\underline{Z}_{\ell m}) + W_\ell}{\Delta_\ell}\right) \\ &= - \sum \frac{\partial^E G_h}{(\partial T)^E}\left(\underline{Z}_{\ell m}, \frac{F^+(\underline{Z}_{\ell m}) + W_\ell}{\Delta_\ell}\right) \left(\frac{F^-(\underline{Z}_{\ell m}) - W_\ell}{\Delta_\ell}\right)^E, \\ &G_h\left(\underline{Z}_{\ell m}, \frac{F^-(\underline{Z}_{\ell m}) - W_m}{\nabla_m}\right) \\ &= - \sum \frac{\partial^E G_h}{(\partial V)^E}\left(\underline{Z}_{\ell m}, \frac{F^-(\underline{Z}_{\ell m}) - W_m}{\nabla_m}\right) \left(\frac{F^+(\underline{Z}_{\ell m}) + W_m}{\nabla_m}\right)^E. \end{aligned}$$

Thus, if we replace Z_{ikk} , $F_j^+(Z_{kk})$, $F_j^-(Z_{kk})$, W_{jk} , Δ_k , and ∇_k by ζ_{ikk} , ϕ_{jkk}^+ , ϕ_{jkk}^- , w_{jk} , $q_k^{\nu_k}$, and $\theta_k^{\nu_k}$, respectively, then we have

$$\begin{aligned} G_h\left(\underline{\zeta}_{kk}, \frac{\phi_{kk}^+ + w_k}{q_k^{\nu_k}}\right) &= - \sum \frac{\partial^E G_h}{(\partial V)^E}\left(\underline{\zeta}_{kk}, \frac{\phi_{kk}^+ + w_k}{q_k^{\nu_k}}\right) \left(\frac{\phi_{kk}^- - w_k}{q_k^{\nu_k}}\right)^E, \\ G_h\left(\underline{\zeta}_{kk}, \frac{\phi_{kk}^- - w_k}{\theta_k^{\nu_k}}\right) &= - \sum \frac{\partial^E G_h}{(\partial V)^E}\left(\underline{\zeta}_{kk}, \frac{\phi_{kk}^- - w_k}{\theta_k^{\nu_k}}\right) \left(\frac{\phi_{kk}^+ + w_k}{\theta_k^{\nu_k}}\right)^E. \end{aligned}$$

That is,

$$\begin{aligned} G_h(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+) &= - \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+) \left(\frac{\theta_k^{\nu_k}}{q_k^{\nu_k}} \alpha_{kk}^-\right)^E, \\ G_h(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^-) &= - \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^-) \left(\frac{q_k^{\nu_k}}{\theta_k^{\nu_k}} \alpha_{kk}^+\right)^E. \end{aligned}$$

In the same manner, we get

$$\begin{aligned}
& G_h(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^+) \\
&= - \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^+) \left(\frac{\theta_k^{\nu_k}}{q_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)k}^- - c_{k+1} \frac{t_{k+1}^{\varepsilon_{k+1}}}{q_{k+1}^{\nu_{k+1}}} \right)^E, \\
& G_h(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^-) \\
&= - \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^-) \left(\frac{q_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \alpha_{(k+1)k}^+ - c_{k+1} \frac{t_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k}} \right)^E, \\
& G_h(\underline{\zeta}_{(k+1)(k+1)}, \underline{\alpha}_{(k+1)(k+1)}^+) \\
&= - \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{(k+1)(k+1)}, \underline{\alpha}_{(k+1)(k+1)}^+) \left(\frac{\theta_{k+1}^{\nu_{k+1}}}{q_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)(k+1)}^- \right)^E, \\
& G_h(\underline{\zeta}_{(k+1)(k+1)}, \underline{\alpha}_{(k+1)(k+1)}^-) \\
&= - \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{(k+1)(k+1)}, \underline{\alpha}_{(k+1)(k+1)}^-) \left(\frac{q_{k+1}^{\nu_{k+1}}}{\theta_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)(k+1)}^+ \right)^E.
\end{aligned}$$

3.2. Taylor's formulas, II

With the notation above, for a suitable natural number μ_k , let

$$\begin{aligned}
(3.2.1) \quad \beta_{hkk}^+ &:= \frac{1}{\theta_k^{\mu_k}} G_h(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+) \\
&= - \frac{1}{\theta_k^{\mu_k}} \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+) \left(\frac{\theta_k^{\nu_k}}{q_k^{\nu_k}} \alpha_{kk}^- \right)^E,
\end{aligned}$$

$$\begin{aligned}
(3.2.2) \quad \beta_{hkk}^- &:= \frac{1}{q_k^{\mu_k}} G_h(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^-) \\
&= - \frac{1}{q_k^{\mu_k}} \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^-) \left(\frac{q_k^{\nu_k}}{\theta_k^{\nu_k}} \alpha_{kk}^+ \right)^E,
\end{aligned}$$

$$\begin{aligned}
\beta_{h(k+1)k}^+ &:= \frac{1}{\theta_k^{\mu_k}} G_h(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^+) \\
&= - \frac{1}{\theta_k^{\mu_k}} \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^+) \\
&\quad \times \left(\frac{\theta_k^{\nu_k}}{q_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)k}^- - c_{k+1} \frac{t_{k+1}^{\varepsilon_{k+1}}}{q_{k+1}^{\nu_{k+1}}} \right)^E,
\end{aligned}$$

$$\begin{aligned}
\beta_{h(k+1)k}^- &:= \frac{1}{q_{k+1}^{\mu_{k+1}}} G_h(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^-) \\
&= - \frac{1}{q_{k+1}^{\mu_{k+1}}} \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^-)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{q_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \alpha_{(k+1)k}^+ - c_{k+1} \frac{t_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k}} \right)^E, \\
\beta_{h(k+1)(k+1)}^+ & := \frac{1}{\theta_{k+1}^{\mu_{k+1}}} G_h(\underline{\zeta_{(k+1)(k+1)}}, \underline{\alpha_{(k+1)(k+1)}^+}) \\
& = -\frac{1}{\theta_{k+1}^{\mu_{k+1}}} \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta_{(k+1)(k+1)}}, \underline{\alpha_{(k+1)(k+1)}^+}) \\
& \quad \times \left(\frac{\theta_{k+1}^{\nu_{k+1}}}{q_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)(k+1)}^- \right)^E, \\
\beta_{h(k+1)(k+1)}^- & := \frac{1}{q_{k+1}^{\mu_{k+1}}} G_h(\underline{\zeta_{(k+1)(k+1)}}, \underline{\alpha_{(k+1)(k+1)}^-}) \\
& = -\frac{1}{q_{k+1}^{\mu_{k+1}}} \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta_{(k+1)(k+1)}}, \underline{\alpha_{(k+1)(k+1)}^-}) \\
& \quad \times \left(\frac{q_{k+1}^{\nu_{k+1}}}{\theta_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)(k+1)}^+ \right)^E.
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta_{hkk}^+ & = -\frac{1}{\theta_k^{\mu_k}} \sum \frac{\partial^E G_h}{(\partial V)^E} \left(\underline{\zeta_{(k+1)k}} - a_{k+1} q_{k+1}^{\varepsilon_{k+1}}, \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \alpha_{(k+1)k}^+ + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\nu_k}} \kappa_{kk}^+ \right) \\
& \quad \times \left\{ \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \left(\frac{\theta_k^{\nu_k}}{q_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)k}^- - c_{k+1} \frac{t_{k+1}^{\varepsilon_{k+1}}}{q_{k+1}^{\nu_{k+1}}} \right) + \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} c_{k+1} \frac{t_{k+1}^{\varepsilon_{k+1}}}{q_{k+1}^{\nu_{k+1}}} \right\}^E \\
& = -\left(\frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \right)^{g_h} \frac{1}{\theta_k^{\mu_k}} \sum \frac{\partial^E G_h}{(\partial V)^E}(\underline{\zeta_{(k+1)k}}, \underline{\alpha_{(k+1)k}^+}) \\
& \quad \times \left(\frac{\theta_k^{\nu_k}}{q_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)k}^- - c_{k+1} \frac{t_{k+1}^{\varepsilon_{k+1}}}{q_{k+1}^{\nu_{k+1}}} \right)^E \\
& \quad + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\nu_k} g_h} \left(\frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\mu_k}} \delta_{hkk}^+ + \frac{\theta_k^{\nu_k}}{\theta_k^{\mu_k}} \sum_{j=1}^r \lambda_{hjk}^+ \alpha_{j(k+1)k}^- \right) \\
& = -\left(\frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\nu_k}} \right)^{g_h} \beta_{h(k+1)k}^+ + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\nu_k} g_h} \left(\frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\mu_k}} \delta_{hkk}^+ + \frac{\theta_k^{\nu_k}}{\theta_k^{\mu_k}} \sum_{j=1}^r \lambda_{hjk}^+ \alpha_{j(k+1)k}^- \right),
\end{aligned}$$

where each $\delta_{hkk}^+ \in R_0[a_{k+1}, c_{k+1}, p_{k+1}, q_k, \theta_{k+1}^{\varepsilon_{k+1}}, \underline{\zeta_{kk}}, q_{k+1}^{\nu_{k+1}} \alpha_{(k+1)(k+1)}^+, \theta_k^{\nu_k} \alpha_{kk}^-]$ and $\lambda_{hjk}^+ \in R_0[a_{k+1}, p_{k+1}, q_k, \underline{\zeta_{kk}}, q_{k+1}^{\nu_{k+1}} \alpha_{(k+1)(k+1)}^+, \theta_k^{\nu_k} \alpha_{kk}^-]$ (cf. (3.1.0)). Then we have

$$\begin{aligned}
\beta_{hkk}^- & = \frac{1}{q_k^{\mu_k}} G_h(\underline{\zeta_{(k+1)k}} - a_{k+1} q_{k+1}^{\varepsilon_{k+1}}, \underline{\alpha_{(k+1)k}^-}) \\
& = \frac{q_{k+1}^{\mu_{k+1}}}{q_k^{\mu_k}} \frac{1}{q_{k+1}^{\mu_{k+1}}} G_h(\underline{\zeta_{(k+1)k}}, \underline{\alpha_{(k+1)k}^-})
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{q_k^{\mu_k}} \sum \frac{\partial^D G_h}{(\partial Z)^D} (\underline{\zeta}_{(k+1)k}, \underline{\alpha}_{(k+1)k}^-) (-a_{k+1} q_{k+1}^{\varepsilon_{k+1}})^D \\
& = \frac{q_{k+1}^{\mu_{k+1}}}{q_k^{\mu_k}} \beta_{h(k+1)k}^- + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\mu_k}} \delta_{hkk}^-,
\end{aligned}$$

where $\delta_{hkk}^- \in R_0[\underline{a}_{k+1}, \underline{q}_{k+1}^{\varepsilon_{k+1}}, \underline{\zeta}_{(k+1)k}][\underline{\alpha}_{kk}^-]$ and homogeneous in $\underline{\alpha}_{kk}^-$. We have

$$\begin{aligned}
\beta_{h(k+1)k}^+ & = \frac{1}{\theta_k^{\mu_k}} G_h(\underline{\zeta}_{(k+1)(k+1)} - \underline{b}_{k+1} \theta_{k+1}^{\varepsilon_{k+1}}, \underline{\alpha}_{(k+1)(k+1)}^+) \\
& = \frac{\theta_{k+1}^{\mu_{k+1}}}{\theta_k^{\mu_k}} \frac{1}{\theta_{k+1}^{\mu_{k+1}}} G_h(\underline{\zeta}_{(k+1)(k+1)}, \underline{\alpha}_{(k+1)(k+1)}^+) \\
& \quad + \frac{1}{\theta_k^{\mu_k}} \sum \frac{\partial^D G_h}{(\partial Z)^D} (\underline{\zeta}_{(k+1)(k+1)}, \underline{\alpha}_{(k+1)(k+1)}^+) (-b_{k+1} \theta_{k+1}^{\varepsilon_{k+1}})^D \\
& = \frac{\theta_{k+1}^{\mu_{k+1}}}{\theta_k^{\mu_k}} \beta_{h(k+1)(k+1)}^+ + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\mu_k}} \delta_{h(k+1)k}^+,
\end{aligned}$$

where $\delta_{h(k+1)k}^+ \in R_0[\underline{b}_{k+1}, \theta_{k+1}^{\varepsilon_{k+1}}, \underline{\zeta}_{(k+1)(k+1)}][\underline{\alpha}_{(k+1)(k+1)}^+]$ and homogeneous in $\underline{\alpha}_{(k+1)(k+1)}^+$. We have

$$\begin{aligned}
\beta_{h(k+1)k}^- & = -\frac{1}{q_{k+1}^{\mu_{k+1}}} \sum \frac{\partial^E G_h}{(\partial V)^E} \left(\underline{\zeta}_{(k+1)(k+1)} - \underline{b}_{k+1} \theta_{k+1}^{\varepsilon_{k+1}}, \right. \\
& \quad \left. \frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \underline{\alpha}_{(k+1)(k+1)}^- + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k}} \kappa_{(k+1)k}^- \right) \\
& \quad \times \left(\frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \frac{q_{k+1}^{\nu_{k+1}}}{\theta_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)(k+1)}^+ - c_{k+1} \frac{q_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k}} \right)^E \\
& = -\left(\frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \right)^{g_h} \frac{1}{q_{k+1}^{\mu_{k+1}}} \sum \frac{\partial^E G_h}{(\partial V)^E} (\underline{\zeta}_{(k+1)(k+1)}, \\
& \quad \underline{\alpha}_{(k+1)(k+1)}^-) \left(\frac{q_{k+1}^{\nu_{k+1}}}{\theta_{k+1}^{\nu_{k+1}}} \alpha_{(k+1)(k+1)}^+ \right)^E \\
& \quad + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k} g_h} \left(\frac{q_{k+1}^{\varepsilon_{k+1}}}{q_{k+1}^{\mu_{k+1}}} \delta_{h(k+1)k}^- + \frac{q_{k+1}^{\nu_{k+1}}}{q_{k+1}^{\mu_{k+1}}} \sum_{j=1}^r \lambda_{hj}^- \alpha_{j(k+1)(k+1)}^+ \right) \\
& = -\left(\frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}} \right)^{g_h} \beta_{h(k+1)(k+1)}^- \\
& \quad + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k} g_h} \left(\frac{q_{k+1}^{\varepsilon_{k+1}}}{q_{k+1}^{\mu_{k+1}}} \delta_{h(k+1)k}^- + \frac{q_{k+1}^{\nu_{k+1}}}{q_{k+1}^{\mu_{k+1}}} \sum_{j=1}^r \lambda_{hj}^- \alpha_{j(k+1)(k+1)}^+ \right),
\end{aligned}$$

where

$$\begin{aligned}\delta_{h(k+1)k}^- &\in R_0[\underline{b_{k+1}}, \underline{c_{k+1}}, q_{k+1}^{\varepsilon_{k+1}}, \pi_{k+1}, \theta_{k+1}, \underline{\zeta_{(k+1)k}}, \\ &\quad q_{k+1}^{\nu_{k+1}} \underline{\alpha_{(k+1)(k+1)}^+}, \theta_{k+1}^{\nu_{k+1}} \underline{\alpha_{(k+1)(k+1)}^-}], \\ \lambda_{hj(k+1)k}^- &\in R_0[\underline{b_{k+1}}, \pi_{k+1}, \theta_k, \underline{\zeta_{(k+1)k}}, q_{k+1}^{\nu_{k+1}} \underline{\alpha_{(k+1)(k+1)}^+}, \theta_{k+1}^{\nu_{k+1}} \underline{\alpha_{(k+1)(k+1)}^-}].\end{aligned}$$

3.3. Taylor's formulas, III

Summarizing the above, we have

$$(3.3.1) \quad \beta_{hkk}^+ = -\left(\frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\mu_k}}\right)^{g_h} \left(\frac{\theta_{k+1}^{\mu_{k+1}}}{\theta_k^{\mu_k}} \beta_{h(k+1)(k+1)}^+ + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\mu_k}} \delta_{h(k+1)k}^+\right) \\ + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\nu_k g_h}} \left(\frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\mu_k}} \delta_{hkk}^+ + \frac{\theta_k^{\nu_k}}{\theta_k^{\mu_k}} \sum \lambda_{hjk}^+ \alpha_{j(k+1)k}^- \right),$$

$$(3.3.2) \quad \beta_{hkk}^- = -\frac{q_{k+1}^{\mu_{k+1}}}{q_k^{\mu_k}} \left(\frac{\theta_{k+1}^{\nu_{k+1}}}{\theta_k^{\nu_k}}\right)^{g_h} \beta_{h(k+1)(k+1)}^- \\ + \frac{\theta_{k+1}^{\varepsilon_{k+1}}}{\theta_k^{\nu_k g_h}} \left(\frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\mu_k}} \delta_{h(k+1)k}^- + \frac{q_{k+1}^{\nu_{k+1}}}{q_k^{\mu_k}} \sum \lambda_{hj(k+1)k}^- \alpha_{j(k+1)(k+1)}^+\right) \\ + \frac{q_{k+1}^{\varepsilon_{k+1}}}{q_k^{\mu_k}} \delta_{hkk}^-,$$

where

$$\begin{aligned}\delta_{hkk}^+, \lambda_{hjk}^+ &\in R_k[\underline{\alpha_{kk}^+}][\underline{a_{k+1}}, \underline{c_{k+1}}], \\ \delta_{hkk}^- &\in R_k[\underline{\alpha_{kk}^+}][\underline{a_{k+1}}][\underline{\alpha_{kk}^-}] \text{ and homogeneous in } \underline{\alpha_{kk}^-}, \\ \delta_{h(k+1)k}^+ &\in R_k[\underline{\alpha_{kk}^+}][\underline{a_{k+1}}, \underline{b_{k+1}}][\underline{\alpha_{(k+1)(k+1)}^+}] \text{ and} \\ &\text{homogeneous in } \underline{\alpha_{(k+1)(k+1)}^+}, \\ \delta_{h(k+1)k}^-, \lambda_{hj(k+1)k}^- &\in R_{k+1}[\underline{\alpha_{kk}^+}].\end{aligned}$$

4. Prime elements

As mentioned in the Introduction, we consider polynomials in n variables over a countable field K_0 without constant term: $F_1(\underline{Z}), \dots, F_r(\underline{Z}) \in K_0[Z_1, \dots, Z_n] =: T_0$, and we assume the following.

ASSUMPTION 4.1

The F_j 's satisfy the following:

$$(4.1.1) \quad \bigcup_{\nu \geq 1} \text{Ass}(T_0/(F_1, \dots, F_r)^\nu) = \text{Ass}(T_0/(F_1, \dots, F_r)).$$

For each $\mathfrak{P} \in \text{Ass}(T_0/(F_1, \dots, F_r))$, there exists a proper subset

$$(4.1.2) \quad \{i_1, \dots, i_t\} \text{ of } \{1, 2, \dots, n\} \text{ such that } \mathfrak{P} \subset \mathfrak{Q} := (Z_{i_1}, \dots, Z_{i_t})T_0.$$

From now on, we fully use the notation in Sections 2 and 3.

DEFINITION 4.2

Let \mathfrak{q}_k^+ , \mathfrak{q}_k^- and $\mathfrak{q}_k := \mathfrak{q}_k^+ + \mathfrak{q}_k^-$ be ideals of $R_k[\underline{\alpha}_{kk}^+]$ as follows:

$$(4.2.1) \quad \mathfrak{q}_k^+ := (\phi_{1kk}^+ + w_{1k}, \dots, \phi_{rkk}^+ + w_{rk}),$$

$$(4.2.2) \quad \mathfrak{q}_k^- := (\phi_{1kk}^- - w_{1k}, \dots, \phi_{rkk}^- - w_{rk}).$$

We use ϕ_{jkk}^+ from (2.3.7), ϕ_{jkk}^- from (2.3.8), and w_{jk} from (2.3.5).

DEFINITION 4.3

Let \mathfrak{q}_k^Δ , \mathfrak{q}_k^∇ , and \mathfrak{q}_k^\diamond be ideals of R_k as follows:

$$(4.3.1) \quad \mathfrak{q}_k^\Delta := (g_{1kk}^+ + w_{1k}, \dots, g_{rkk}^+ + w_{rk}),$$

$$(4.3.2) \quad \mathfrak{q}_k^\nabla := (g_{1kk}^- - w_{1k}, \dots, g_{rkk}^- - w_{rk}),$$

$$(4.3.3) \quad \mathfrak{q}_k^\diamond := \mathfrak{q}_k^\Delta + \mathfrak{q}_k^\nabla.$$

We remark that $\mathfrak{q}_k^\Delta R_k[\underline{\alpha}_{kk}^+] = \mathfrak{q}_k^+$ by (2.6.2) (cf. (2.6.4)).

Next let $\tilde{R}_k := R_k[\underline{\alpha}_{kk}^+][\mathfrak{N}_k]$ be a local ring with maximal ideal $\tilde{\mathfrak{n}}_k := \mathfrak{N}_k \tilde{R}_k$, where $\mathfrak{N}_k := (z_0, \underline{z}, \underline{w}, \underline{\alpha}_{kk}^+)$ is a maximal ideal of $R_k[\underline{\alpha}_{kk}^+]$ (cf. Lemma 6.2).

DEFINITION 4.4

Let $\tilde{\mathfrak{q}}_k^+$, $\tilde{\mathfrak{q}}_k^-$, and $\tilde{\mathfrak{q}}_k^\nabla$ be ideals of \tilde{R}_k as follows:

$$(4.4.1) \quad \tilde{\mathfrak{q}}_k^+ := \mathfrak{q}_k^+ \tilde{R}_k[1/q_k] \cap \tilde{R}_k = \mathfrak{q}_k^\Delta \tilde{R}_k[1/q_k] \cap \tilde{R}_k =: \tilde{\mathfrak{q}}_k^+,$$

$$(4.4.2) \quad \tilde{\mathfrak{q}}_k^- := \mathfrak{q}_k^- \tilde{R}_k[1/q_k] \cap \tilde{R}_k,$$

$$(4.4.3) \quad \tilde{\mathfrak{q}}_k^\nabla := \mathfrak{q}_k^\nabla \tilde{R}_k[1/q_k] \cap \tilde{R}_k.$$

Further, let

$$(4.4.4) \quad \tilde{\mathfrak{q}}_k := \tilde{\mathfrak{q}}_k^+ + \tilde{\mathfrak{q}}_k^-,$$

$$(4.4.5) \quad \tilde{\mathfrak{q}}_k^\diamond := \tilde{\mathfrak{q}}_k^\Delta + \tilde{\mathfrak{q}}_k^\nabla.$$

LEMMA 4.5

Assume that $p_i \notin \mathfrak{q}_{k-1}^\Delta$ for each $i \leq k$. Then

$$(4.5.1) \quad R_k/\mathfrak{q}_k^\Delta \xrightarrow{\cong} \tilde{R}_k/\tilde{\mathfrak{q}}_k^+ \quad \text{and} \quad \tilde{\mathfrak{q}}_k^+ = (\alpha_{1kk}^+, \dots, \alpha_{rkk}^+).$$

Here $\tilde{\mathfrak{q}}_k^+$ is a prime ideal generated by an \tilde{R}_k -sequence $\alpha_{1kk}^+, \dots, \alpha_{rkk}^+$, and $p_i \notin \tilde{\mathfrak{q}}_k^+$ for each $i \leq k$. Suppose, in addition, that $p_1 \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+, \dots, p_k \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+$ are distinct prime ideals of \tilde{R}_{k-1} . Then $p_1 \tilde{R}_k + \tilde{\mathfrak{q}}_k^+, \dots, p_k \tilde{R}_k + \tilde{\mathfrak{q}}_k^+$ are distinct prime ideals of \tilde{R}_k and, whenever $\varepsilon_k \geq e_i$,

$$(4.5.2) \quad (p_i^{\varepsilon_i} \tilde{R}_k + \tilde{\mathfrak{q}}_k^+) \cap \tilde{R}_{k-1} = p_i^{\varepsilon_i} \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+.$$

Proof

We first remark that \mathfrak{q}_k^Δ is a prime ideal generated by the elements $g_{1kk}^+ + w_{1k}, \dots, g_{rkk}^+ + w_{rk}$ that form a part of a regular system of parameters of the regular local ring R_k (cf. (2.1.1)), and by assumption, it is clear that $p_i \notin \mathfrak{q}_k^\Delta$ for each $i \leq k$. By definition, we have the following commutative diagram with canonical surjective morphisms:

$$\begin{array}{ccc} R_k/\mathfrak{q}_k^\Delta & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^+ \\ & \searrow & \nearrow \\ & \tilde{R}_k/(\underline{\alpha}_{kk}^+) & \end{array}$$

Indeed, the three surjective morphisms above are isomorphisms, because we have the following canonical inclusions:

$$R_k/\mathfrak{q}_k^\Delta \subset \tilde{R}_k/\tilde{\mathfrak{q}}_k^+ \subset (\tilde{R}_k/\mathfrak{q}_k^\Delta \tilde{R}_k)[1/q_k] = ((R_k/\mathfrak{q}_k^\Delta)[1/q_k])_{\mathfrak{q}_k}.$$

In particular, $\tilde{\mathfrak{q}}_k^+ = (\underline{\alpha}_{kk}^+) \tilde{R}_k$. This implies that $\tilde{\mathfrak{q}}_k^+$ is a prime ideal generated by an \tilde{R}_k -sequence $\alpha_{1kk}^+, \dots, \alpha_{rkk}^+$ and that $p_i \notin \tilde{\mathfrak{q}}_k^+$. Consequently, if $\varepsilon_k \geq e_i$, then we have

$$\begin{aligned} \tilde{R}_k/(p_i^{e_i}, \tilde{\mathfrak{q}}_k^+) &\cong R_k/(p_i^{e_i}, \mathfrak{q}_k^\Delta) \\ &= R_k/(p_i^{e_i}, \mathfrak{q}_{k-1}^\Delta) \\ &\cong (\tilde{R}_{k-1}/(p_i^{e_i}, \tilde{\mathfrak{q}}_{k-1}^+))(\underline{a}_k, \underline{b}_k, \underline{c}_k). \end{aligned} \quad \square$$

LEMMA 4.6

With notation and the assumptions as in Definitions 4.3 and 4.4 and Lemma 4.5, assume further that π_1, q_k form an $(R_k/\mathfrak{q}_k^\diamond)$ -sequence. Then

$$(4.6.1) \quad \pi_1 \notin \bigcup \text{Ass}(\tilde{R}_k/\tilde{\mathfrak{q}}_k) = \bigcup \text{Ass}(\tilde{R}_k/\tilde{\mathfrak{q}}_k^\diamond),$$

where $\bigcup \text{Ass}(\tilde{R}_k/\tilde{\mathfrak{q}}_k)$ denotes the union of the prime ideals in $\text{Ass}(\tilde{R}_k/\tilde{\mathfrak{q}}_k)$ for a fixed $k \in \mathbf{N}$. Hence, (4.6.1) implies that π_1 is a non-zero divisor for $\tilde{R}_k/\tilde{\mathfrak{q}}_k$.

Proof

Because $R_k/\mathfrak{q}_k^\Delta \cong \tilde{R}_k/\tilde{\mathfrak{q}}_k^+$ by (4.5.1), we have (cf. (2.3.8))

$$R_k/\mathfrak{q}_k^\diamond \xrightarrow{\cong} \tilde{R}_k/(\tilde{\mathfrak{q}}_k^+ + \mathfrak{q}_k^\nabla \tilde{R}_k) = \tilde{R}_k/(\tilde{\mathfrak{q}}_k^\Delta + \mathfrak{q}_k^- \tilde{R}_k).$$

Thus, we get canonical surjections

$$\tilde{R}_k/(\tilde{\mathfrak{q}}_k^+ + \mathfrak{q}_k^- \tilde{R}_k) \longrightarrow \tilde{R}_k/\tilde{\mathfrak{q}}_k \quad \text{and} \quad \tilde{R}_k/(\tilde{\mathfrak{q}}_k^\Delta + \mathfrak{q}_k^\nabla \tilde{R}_k) \longrightarrow \tilde{R}_k/\tilde{\mathfrak{q}}_k^\diamond.$$

However, the canonical inclusion $R_k/\mathfrak{q}_k^\diamond \subset (R_k/\mathfrak{q}_k^\Delta)[1/q_k]$ and the isomorphism above show that these surjective morphisms are isomorphisms. Namely,

$$(4.6.2) \quad R_k/\mathfrak{q}_k^\diamond \xrightarrow{\cong} \tilde{R}_k/\tilde{\mathfrak{q}}_k = \tilde{R}_k/\tilde{\mathfrak{q}}_k^\diamond.$$

This gives the assertion. □

LEMMA 4.7

With assumptions as in Lemma 4.5, assume further that $\pi_j \notin \tilde{\mathfrak{q}}_{k-1}^-$ for each $j \leq k$. Then $\tilde{\mathfrak{q}}_k^-$ is a prime ideal and $\pi_j \notin \tilde{\mathfrak{q}}_k^-$ for each $j \leq k$.

Proof

By assumption, first we note that $\pi_j \notin \mathfrak{q}_{\ell-1}^- R_{\ell-1}[1/q_{\ell-1}]$ for $j \leq \ell \leq k$ (cf. (2.3.13), (2.4.1)). Next we show by induction that

$$(4.7.1) \quad (\phi_{1\ell\ell}^- - w_{1\ell}, \dots, \phi_{h\ell\ell}^- - w_{h\ell}) R_{\ell}[1/q_{\ell}] \text{ is a prime ideal generated} \\ \text{by an } R_{\ell}[1/q_{\ell}]\text{-sequence for each } h \leq r \text{ and } 0 \leq \ell \leq k.$$

Indeed, the case $\ell = 0$ is true, because $F_1(\underline{z}) - w_1, \dots, F_h(\underline{z}) - w_h$ form a sub-regular system of parameters of R_0 (cf. (2.1.0), (2.3.1), (2.3.2)). Hence, we only show the claim for the case $\ell = k$, assuming that

$$(\phi_{1(k-1)(k-1)}^- - w_{1(k-1)}, \dots, \phi_{h(k-1)(k-1)}^- - w_{h(k-1)}) R_{k-1}[1/q_{k-1}] \text{ is a prime} \\ \text{ideal generated by an } R_{k-1}[1/q_{k-1}]\text{-sequence for each } h \leq r.$$

But, the assumption above implies that $\theta_k, \phi_{1(k-1)(k-1)}^- - w_{1(k-1)}, \dots, \phi_{r(k-1)(k-1)}^- - w_{r(k-1)}$ form an $R_{k-1}[1/q_{k-1}]$ -sequence (cf. [8, Lemma 1.1]). Thus, $\mathfrak{q}_k^- R_k[1/q_k]$ is a prime ideal and $\pi_j \notin \mathfrak{q}_k^- R_k[1/q_k]$. In fact, this results from

$$\phi_{jkk}^- - w_{jk} = (\phi_{j(k-1)(k-1)}^- - w_{j(k-1)}) + \{H_j(\underline{\zeta}_{k(k-1)}, \underline{b}_k, \theta_k) \theta_k^{\varepsilon_k} - c_{jk} (q_k^{\varepsilon_k} \theta_k)^{\varepsilon_k}\} \\ = (\phi_{j(k-1)(k-1)}^- - w_{j(k-1)}) + (H_j - c_{jk} q_k^{\varepsilon_k^2}) \theta_k^{\varepsilon_k},$$

where (cf. (2.4.1), Lemma 1.2)

$$H_j(\underline{\zeta}_{k(k-1)}, \underline{b}_k, \theta_k) = \frac{1}{\theta_k^{\varepsilon_k}} \sum \frac{\partial^D F_j}{(\partial Z)^D}(\underline{\zeta}_{k(k-1)})(b_k \theta_k^{\varepsilon_k})^D \\ \in R_{k-1}[\underline{\alpha}_{(k-1)(k-1)}^+][\underline{a}_k, \underline{b}_k] \subset R_{k-1}[\underline{a}_k, \underline{b}_k][1/q_k].$$

Therefore, $\tilde{\mathfrak{q}}_k^-$ is a prime ideal and $\pi_j \notin \tilde{\mathfrak{q}}_k^-$ for each $j \leq k$. \square

LEMMA 4.8

With assumptions as in Lemmas 4.6 and 4.7, assume further that

$z_k, z_{1kk}, \dots, z_{nkk}$ form an $(R_k/\mathfrak{q}_k^{\Delta})$ -sequence,

$z_k, z_{1kk}, \dots, z_{nkk}$ form an $(R_k/\mathfrak{q}_k^{\nabla})$ -sequence.

Then

$$(4.8.1) \quad \tilde{\mathfrak{q}}_k^- = (\mathfrak{q}_k^-, G_1(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+), \dots, G_s(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+)).$$

Proof

First we show that $\tilde{\mathfrak{q}}_k^- \supset (\mathfrak{q}_k^-, G_1(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+), \dots, G_s(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+))$. Indeed, denoting by $T := K_0[Z_1, \dots, Z_n, \Delta]$ a polynomial ring in $n+1$ variables over a field K_0 , we

consider a K_0 -algebra homomorphism

$$\varphi_0: T \rightarrow R_k[\underline{\alpha}_{kk}^+] \text{ that maps } Z_i \text{ to } \zeta_{ikk} \text{ for } i = 1, \dots, n \text{ and } \Delta \text{ to } q_k^{\nu_k}.$$

Then, putting $C := R_k[\underline{\alpha}_{kk}^+]/\mathfrak{q}_k^-$ and setting \tilde{C} as the image of C in $C[1/q_k]$ with canonical projections $\pi: R_k[\underline{\alpha}_{kk}^+] \rightarrow C$ and $\tilde{\pi}: R_k[\underline{\alpha}_{kk}^+] \rightarrow \tilde{C}$, we have the following commutative diagram, because $(\phi_{jkk}^+ + w_{jk}) + (\phi_{jkk}^- - w_{jk}) = F_j(\underline{\zeta}_{kk})$ and because the image of α_{jkk}^+ in \tilde{C} can be identified with that of $F_j(\underline{\zeta}_{kk})/q_k^{\nu_k}$:

$$\begin{array}{ccccc} T[\underline{V}] & \xrightarrow{\phi} & T[\underline{F}/\Delta] & \hookrightarrow & T[1/\Delta] \\ \downarrow \varphi & & \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} \\ R_k[\underline{\alpha}_{kk}^+][\underline{V}] & \xrightarrow{\tilde{\phi}} & \tilde{C} & \hookrightarrow & C[1/q_k^{\nu_k}] \\ \downarrow \psi & & \parallel & & \\ R_k[\underline{\alpha}_{kk}^+] & \xrightarrow{\tilde{\pi}} & \tilde{C} & & \end{array}$$

where $\varphi := \varphi_0 \otimes K_0[\underline{V}]$, $\tilde{\varphi} := (\pi \circ \varphi_0) \otimes_{K_0[\Delta]} K_0[\Delta, 1/\Delta]$, and ψ is an $R_k[\underline{\alpha}_{kk}^+]$ -algebra homomorphism with $V_j \mapsto \alpha_{jkk}^+$. Further, ϕ is a T -homomorphism with $V_j \mapsto F_j(\underline{Z})/\Delta$, $\tilde{\phi}$ is an $R_k[\underline{\alpha}_{kk}^+]$ -algebra homomorphism that maps V_j to the image of α_{jkk}^+ in \tilde{C} , and $\tilde{\varphi}$ is the K_0 -algebra homomorphism induced by φ_0 that maps $F_j(\underline{Z})/\Delta$ to the image of α_{jkk}^+ in \tilde{C} . Then, by Lemma 1.7,

$$\text{Ker } \phi = (\Delta V_1 - F_1(\underline{Z}), \dots, \Delta V_r - F_r(\underline{Z}), G_1(\underline{Z}, \underline{V}), \dots, G_s(\underline{Z}, \underline{V})).$$

Hence, by definition, $\tilde{\mathfrak{q}}_k^- \supset \text{Ker } \tilde{\pi} \supset (\mathfrak{q}_k^-, G_1(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+), \dots, G_s(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+))$.

Now we prove (4.8.1). Note that \mathfrak{q}_k^Δ is generated by an $(R_k/q_k R_k)$ -sequence (cf. (4.3.1)). Further, we have (cf. (2.3.8), (4.3.2))

$$\begin{aligned} \mathfrak{q}_k^\diamond &:= \mathfrak{q}_k^\Delta + \mathfrak{q}_k^\nabla \\ (4.8.2) \quad &= \mathfrak{q}_k^\Delta + (F_1(\underline{z}_{kk}), \dots, F_r(\underline{z}_{kk})) \\ &= (F_1(\underline{z}_{kk}), \dots, F_r(\underline{z}_{kk})) + \mathfrak{q}_k^\nabla \end{aligned}$$

and z_{1kk}, \dots, z_{nkk} form an $(R_k/(q_k, \mathfrak{q}_k^\Delta))$ -sequence. Thus, by Lemma 1.11, we see that

$$\bigcup_{\nu \geq 1} \text{Ass}(R_k/(q_k, (\mathfrak{q}_k^\Delta)^\nu)) = \bigcup \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\Delta)) \subset \bigcup \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\diamond)),$$

where the first union is the union of the prime ideals in $\text{Ass}(R_k/(q_k, (\mathfrak{q}_k^\Delta)^\nu))$ of all the ν for a fixed $k \in \mathbf{N}$ (cf. (0.1.1)), and $\bigcup \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\diamond))$ denotes the union of the prime ideals in $\text{Ass}(R_k/(q_k, \mathfrak{q}_k^\diamond))$ for a fixed $k \in \mathbf{N}$. Hence, by Lemma 4.6

$$\begin{aligned} \pi_1 &\notin \bigcup_{\nu \geq 1} \text{Ass}(R_k/(q_k, (\mathfrak{q}_k^\Delta)^\nu)) \\ &= \bigcup \text{Ass}(R_k[\underline{\alpha}_{kk}^\Delta]/q_k R_k[\underline{\alpha}_{kk}^\Delta]) \cap R_k \end{aligned}$$

$$= \bigcup \text{Ass}(R_k[\underline{\alpha}_{kk}^+]/q_k R_k[\underline{\alpha}_{kk}^+]) \cap R_k.$$

Here $\bigcup \text{Ass}(R_k[\underline{\alpha}_{kk}^\Delta]/q_k R_k[\underline{\alpha}_{kk}^\Delta]) \cap R_k$ means the union of the prime ideals in the set

$$\{\mathfrak{p} = \mathfrak{P} \cap R_k \mid \mathfrak{P} \in \text{Ass}(R_k[\underline{\alpha}_{kk}^\Delta]/q_k R_k[\underline{\alpha}_{kk}^\Delta])\}.$$

Then, letting $\bar{\alpha}_{jkk}^+$, $\bar{\alpha}_{jkk}^\Delta$ be the images of α_{jkk}^+ , α_{jkk}^Δ in the field of quotients $Q(\bar{R}_k)$ of $\bar{R}_k = R_k/\pi_1 R_k$, we get the following isomorphisms (cf. (1.9.1)):

$$\begin{aligned} \bar{R}_k[\underline{\bar{\alpha}}_{kk}^+] &\cong (R_k/\pi_1 R_k) \otimes R_k[\underline{\alpha}_{kk}^+] \\ &= (R_k/\pi_1 R_k) \otimes R_k[\underline{\alpha}_{kk}^\Delta] \cong \bar{R}_k[\underline{\bar{\alpha}}_{kk}^\Delta]. \end{aligned}$$

On the other hand, by assumption (cf. (4.8.2))

$$\bigcup_{\nu \geq 1} \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\nabla, (\mathfrak{q}_k^\Delta)^\nu)) = \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\diamond)).$$

Thus, denoting by $\bar{\alpha}_{jkk}^\Delta$ the image of α_{jkk}^Δ in $Q(R_k/\mathfrak{q}_k^\nabla)$, we have (cf. Lemma 1.8)

$$\begin{aligned} (4.8.3) \quad \pi_1 \notin \bigcup \text{Ass}((R_k/\mathfrak{q}_k^\nabla)[\underline{\bar{\alpha}}_{kk}^\Delta]/q_k(R_k/\mathfrak{q}_k^\nabla)[\underline{\bar{\alpha}}_{kk}^\Delta]) \cap R_k \\ = \bigcup_{\nu \geq 1} \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\nabla, (\mathfrak{q}_k^\Delta)^\nu)). \end{aligned}$$

By mapping V_j to $\bar{\alpha}_{jkk}^\Delta$, consider an R_k -algebra isomorphism

$$(4.8.4) \quad R_k[V_1, \dots, V_r]/J \xrightarrow{\cong} (R_k/\mathfrak{q}_k^\nabla)[\underline{\bar{\alpha}}_{1kk}^\Delta, \dots, \underline{\bar{\alpha}}_{rkk}^\Delta] \subset \bar{R}_k/\bar{\mathfrak{q}}_k^\nabla.$$

Then we have (cf. (4.8.2), Lemma 1.7)

$$\begin{aligned} J = (\mathfrak{q}_k^\nabla, q_k^{\nu_k} V_1 - (g_{1kk}^+ + w_{1k}), \dots \\ \dots, q_k^{\nu_k} V_r - (g_{rkk}^+ + w_{rk}), G_1(\underline{z}_{kk}, \underline{V}), \dots, G_s(\underline{z}_{kk}, \underline{V})). \end{aligned}$$

Hence,

$$\begin{aligned} \bar{R}_k/(\pi_1, \bar{\mathfrak{q}}_k^-, \underline{G}(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+)) &\cong (\bar{R}_k[\underline{\bar{\alpha}}_{kk}^+])_{\bar{\mathfrak{q}}_{\bar{R}_k}}/(\bar{\mathfrak{q}}_k^-, \underline{\bar{G}}(\underline{\bar{\zeta}}_{kk}, \underline{\bar{\alpha}}_{kk}^+)) \\ &\cong (\bar{R}_k[\underline{\bar{\alpha}}_{kk}^\Delta])_{\bar{\mathfrak{q}}_{\bar{R}_k}}/(\bar{\mathfrak{q}}_k^\nabla, \underline{\bar{G}}(\underline{\bar{z}}_{kk}, \underline{\bar{\alpha}}_{kk}^\Delta)) \\ &\cong \bar{R}_k/(\pi_1, \bar{\mathfrak{q}}_k^\nabla), \end{aligned}$$

where $\bar{\mathfrak{q}}_k^-$, $\bar{G}_h(\underline{\bar{\zeta}}_{kk}, \underline{\bar{\alpha}}_{kk}^+)$, $\bar{\mathfrak{q}}_k^\nabla$, and $\bar{G}_h(\underline{\bar{z}}_{kk}, \underline{\bar{\alpha}}_{kk}^\Delta)$ are the images of \mathfrak{q}_k^- , $G_h(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+)$, \mathfrak{q}_k^∇ , and $G_h(\underline{z}_{kk}, \underline{\alpha}_{kk}^\Delta)$ in $Q(\bar{R}_k/\pi_1 \bar{R}_k) = Q((\bar{R}_k[\underline{\bar{\alpha}}_{kk}^+])_{\bar{\mathfrak{q}}_{\bar{R}_k}})$ (cf. (4.8.4), (2.6.3), (2.6.1)). Thus, $(\pi_1, \bar{\mathfrak{q}}_k^-) = (\pi_1, \mathfrak{q}_k^-, \underline{G}(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+))$, because (cf. (4.8.3), (4.4.1))

$$\begin{aligned} \mathfrak{q}_k \notin \bigcup \text{Ass}(R_k[\underline{\alpha}_{kk}^\Delta]/(\pi_1, \mathfrak{q}_k^\nabla, \underline{G}(\underline{z}_{kk}, \underline{\alpha}_{kk}^\Delta))) \\ = \bigcup \text{Ass}(R_k[\underline{\alpha}_{kk}^+]/(\pi_1, \mathfrak{q}_k^-, \underline{G}(\underline{z}_{kk}, \underline{\alpha}_{kk}^+))). \end{aligned}$$

Therefore, $\bar{\mathfrak{q}}_k^- = (\mathfrak{q}_k^-, \underline{G}(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+))$, because $\bar{\mathfrak{q}}_k^- : \pi_1 = \bar{\mathfrak{q}}_k^-$ (cf. (4.7.1)). \square

Finally, note that in the proof above we have also proved

$$(4.8.5) \quad (\pi_1, \tilde{\mathfrak{q}}_k^-) = (\pi_1, \tilde{\mathfrak{q}}_k^\nabla), \text{ and} \\ \tilde{\mathfrak{q}}_k^\nabla = (\mathfrak{q}_k^\nabla, G_1(\underline{z}_{kk}, \underline{\alpha}_{kk}^\Delta), \dots, G_s(\underline{z}_{kk}, \underline{\alpha}_{kk}^\Delta)) \text{ is a prime ideal.}$$

LEMMA 4.9

With the notation and assumptions from Lemma 4.8, we have

$$\tilde{\mathfrak{q}}_k^+ \cap \tilde{\mathfrak{q}}_k^- = (\alpha_{ikk}^+(\phi_{jkk}^- - w_{jk}), G_h(\underline{\zeta}_{kk}, \underline{\alpha}_{kk}^+))$$

for $i = 1, \dots, n$, $j = 1, \dots, r$, and $h = 1, \dots, s$, where $\underline{\zeta}_{kk} := (\zeta_{1kk}, \dots, \zeta_{nkk})$ and $\underline{\alpha}_{kk}^+ := (\alpha_{1kk}^+, \dots, \alpha_{rkk}^+)$.

Proof

Let $I := \tilde{\mathfrak{q}}_k^+ \cap \tilde{\mathfrak{q}}_k^-$, and let $J := \tilde{\mathfrak{q}}_k^\Delta \cap \tilde{\mathfrak{q}}_k^\nabla$. Then by (4.6.1), (4.7.1), and (4.8.4), we get the following commutative diagrams whose rows are exact:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \tilde{R}_k/I & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^+ \oplus \tilde{R}_k/\tilde{\mathfrak{q}}_k^- & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k \longrightarrow 0 \\ & & \downarrow \times \pi_1 & & \downarrow \times \pi_1 & & \downarrow \times \pi_1 \\ 0 & \longrightarrow & \tilde{R}_k/I & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^+ \oplus \tilde{R}_k/\tilde{\mathfrak{q}}_k^- & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \tilde{R}_k/J & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^\Delta \oplus \tilde{R}_k/\tilde{\mathfrak{q}}_k^\nabla & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^\diamond \longrightarrow 0 \\ & & \downarrow \times \pi_1 & & \downarrow \times \pi_1 & & \downarrow \times \pi_1 \\ 0 & \longrightarrow & \tilde{R}_k/J & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^\Delta \oplus \tilde{R}_k/\tilde{\mathfrak{q}}_k^\nabla & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^\diamond \longrightarrow 0 \end{array}$$

Hence, we also get the following commutative diagram whose rows are exact (cf. (4.8.5)):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{R}_k/\bar{I} & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^+ \oplus \tilde{R}_k/\tilde{\mathfrak{q}}_k^- & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k \\ & & & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & \tilde{R}_k/\bar{J} & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^\Delta \oplus \tilde{R}_k/\tilde{\mathfrak{q}}_k^\nabla & \longrightarrow & \tilde{R}_k/\tilde{\mathfrak{q}}_k^\diamond \end{array}$$

where

$$\begin{aligned}\bar{I} &:= (\pi_1, I), & \bar{\mathfrak{q}}_k^+ &:= (\pi_1, \tilde{\mathfrak{q}}_k^+), & \bar{\mathfrak{q}}_k^- &:= (\pi_1, \tilde{\mathfrak{q}}_k^-), & \bar{\mathfrak{q}}_k &:= \bar{\mathfrak{q}}_k^+ + \bar{\mathfrak{q}}_k^- = (\pi_1, \tilde{\mathfrak{q}}_k), \\ \bar{J} &:= (\pi_1, J), & \bar{\mathfrak{q}}_k^\Delta &:= (\pi_1, \tilde{\mathfrak{q}}_k^\Delta), & \bar{\mathfrak{q}}_k^\nabla &:= (\pi_1, \tilde{\mathfrak{q}}_k^\nabla), & \bar{\mathfrak{q}}_k^\diamond &:= \bar{\mathfrak{q}}_k^\Delta + \bar{\mathfrak{q}}_k^\nabla = (\pi_1, \tilde{\mathfrak{q}}_k^\diamond).\end{aligned}$$

Thus, it suffices to show that $J = (\alpha_{ikk}^\Delta (g_{jkk}^- - w_{jk}), G_h(z_{kk}, \alpha_{kk}^\Delta))$ for $i = 1, \dots, n$, $j = 1, \dots, r$, and $h = 1, \dots, s$. This follows from Lemma 1.10. \square

LEMMA 4.10

With assumptions as in Lemma 4.8, we have

$$(4.10.1) \quad \begin{aligned}\text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)/q_k(\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)) \cap R_k \\ = \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k)) \cap R_k,\end{aligned}$$

$$(4.10.2) \quad \begin{aligned}\text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)/q_k(\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)) \cap R_k[q_k\alpha_{kk}^+] \\ = \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k)) \cap R_k[q_k\alpha_{kk}^+].\end{aligned}$$

Here $\text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k)) \cap R_k := \{\mathfrak{p} = \mathfrak{P} \cap R_k \mid \mathfrak{P} \in \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k))\}$.

Proof

First we show that there exists an R_k -algebra isomorphism

$$(4.10.3) \quad \tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^-) \xrightarrow{\cong} \tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^\nabla).$$

Indeed, we have an R_k -algebra isomorphism $R_k[\underline{V}^+]/J^+ \xrightarrow{\cong} R_k[\underline{\alpha}_{kk}^+]$, mapping V_j^+ to α_{jkk}^+ as in (4.8.4), and $J^+ = (q_k^{\nu_k} V_j^+ - (\phi_{jkk}^+ + w_{jk}))$. Further, $\tilde{\mathfrak{q}}_k^- = (\mathfrak{q}_k^-, G_h(\zeta_{kk}, \alpha_{kk}^+))$, where (cf. (4.8.1), (2.6.4), (2.5.3))

$$\mathfrak{q}_k^- = (\phi_{jkk}^- - w_{jk}) = \left(g_{jkk}^- + \sum \Delta_{jkkH}^- (q_k \alpha_{kk}^+)^H - w_{jk} \right),$$

$$G_h(\zeta_{kk}, \alpha_{kk}^+) = G_h\left(z_{kk} + \sum \Upsilon_{kH} (q_k^{\nu_k}/q_{k-1}^{\nu_{k-1}}) \alpha_{kk}^+ \right)^H, \alpha_{kk}^+.$$

Moreover, we have an R_k -algebra isomorphism $R_k[\underline{V}^\Delta]/J^\Delta \xrightarrow{\cong} R_k[\underline{\alpha}_{kk}^\Delta]$ with $V_j^\Delta \mapsto \alpha_{jkk}^\Delta$ and $J^\Delta = (q_k^{\nu_k} V_j^\Delta - (g_{jkk}^+ + w_{jk}))$ for $j = 1, \dots, r$. Also, we have $\tilde{\mathfrak{q}}_k^\nabla = (\mathfrak{q}_k^\nabla, G_h(z_{kk}, \alpha_{kk}^\Delta))$, where $\mathfrak{q}_k^\nabla := (g_{jkk}^- - w_{jk})$ (cf. (4.8.5)). Thus, we have R_k -algebra isomorphisms

$$(R_k[\underline{V}^+]/(q_k, \phi_{jkk}^+ + w_{jk}, g_{jkk}^- - w_{jk}, G_h(z_{kk}, \underline{V}^+)))_{\mathfrak{N}_{V^+}} \xrightarrow{\cong} \tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^-),$$

$$(R_k[\underline{V}^\Delta]/(q_k, g_{jkk}^+ + w_{jk}, g_{jkk}^- - w_{jk}, G_h(z_{kk}, \underline{V}^\Delta)))_{\mathfrak{N}_{V^\Delta}} \xrightarrow{\cong} \tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^\nabla),$$

where $\mathfrak{N}_{V^+} := (z_0, \underline{z}, \underline{w}, \underline{V}^+)$ is a maximal ideal of $R_k[\underline{V}^+]$ and $\mathfrak{N}_{V^\Delta} := (z_0, \underline{z}, \underline{w}, \underline{V}^\Delta)$ is a maximal ideal of $R_k[\underline{V}^\Delta]$. Therefore, we get the assertion (4.10.3) (cf. (2.6.2)). Now (4.10.1) results from

$$\begin{aligned}\text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)/q_k(\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)) \cap R_k \\ = \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^-)) \cap R_k \\ = \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^\nabla)) \cap R_k\end{aligned}$$

$$\begin{aligned}
&= \text{Ass}(((R_k/\mathfrak{q}_k^\nabla)[\underline{\tilde{\alpha}}_{kk}^\Delta]/q_k(R_k/\mathfrak{q}_k^\nabla)[\underline{\tilde{\alpha}}_{kk}^\Delta])_{\mathfrak{R}_k}) \cap R_k \\
&= \bigcup_{\nu \geq 1} \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\nabla, (\mathfrak{q}_k^\Delta)^\nu)) \\
&= \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\nabla, \mathfrak{q}_k^\Delta)) \\
&= \text{Ass}(R_k/(q_k, \mathfrak{q}_k^\diamond)) \\
&= \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k)) \cap R_k,
\end{aligned}$$

where $\tilde{\alpha}_{jkk}^\Delta$ is the image of α_{jkk}^Δ in $Q(R_k/\mathfrak{q}_k^\nabla)$ (cf. (4.8.4), (4.1.1), (4.6.2)). Finally, (4.10.2) follows from (4.10.1), because $q_k \tilde{R}_k \cap R_k[q_k \underline{\alpha}_{kk}^+]$ contains all the $q_k \alpha_{jkk}^+$'s and because $R_k[q_k \underline{\alpha}_{kk}^+]/(q_k \underline{\alpha}_{kk}^+) R_k[q_k \underline{\alpha}_{kk}^+] \cong R_k/\mathfrak{q}_k^\Delta$ (cf. (4.5.1)). \square

LEMMA 4.11

With assumptions as in Lemma 4.8, suppose further that one of the following equivalent conditions is satisfied (cf. (2.5.2), (2.3.6), Lemma 4.5):

Either $s_{kk}, z_{1kk}, \dots, z_{nkk}$ form an $(R_k/\mathfrak{q}_k^\Delta)$ -sequence,
or $\theta_k, \zeta_{1kk}, \dots, \zeta_{nkk}$ form an $(\tilde{R}_k/\tilde{\mathfrak{q}}_k^+)$ -sequence.

Then

$$(4.11.1) \quad \pi_j \notin \bigcup \text{Ass}(\tilde{R}_k/\tilde{\mathfrak{q}}_k) \quad \text{for every } j \leq k \text{ or } s_{kk} \notin \bigcup \text{Ass}(R_k/\mathfrak{q}_k^\diamond),$$

$$(4.11.2) \quad \bigcup_{\nu \geq 1} \text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^+)/(\theta_k, (\mathfrak{q}_k^-)^\nu)) \cap R_k = \text{Ass}(\tilde{R}_k/(\theta_k, \tilde{\mathfrak{q}}_k)) \cap R_k.$$

Proof

First, we remark that (4.11.1) follows from the assumption and the facts that (cf. [8, Lemma 1.1])

$$\mathfrak{q}_k^\nabla = (g_{jkk}^- - w_{jk}) \quad \text{and} \quad g_{jkk}^- - w_{jk} \equiv F_j(z_{kk}) \pmod{\mathfrak{q}_k^\Delta}.$$

Then

$$\begin{aligned}
&\bigcup_{\nu \geq 1} \text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^+)/(\theta_k, (\mathfrak{q}_k^-)^\nu)) \cap R_k \\
&= \text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^+)[\underline{\tilde{\alpha}}_{kk}^-]/\theta_k(\tilde{R}_k/\tilde{\mathfrak{q}}_k^+)[\underline{\tilde{\alpha}}_{kk}^-]) \cap R_k \\
&= \text{Ass}((R_k/\mathfrak{q}_k^\Delta)[\underline{\tilde{\alpha}}_{kk}^-]/s_{kk}(R_k/\mathfrak{q}_k^\Delta)[\underline{\tilde{\alpha}}_{kk}^-]) \cap R_k \\
&= \bigcup_{\nu \geq 1} \text{Ass}(R_k/(s_{kk}, \mathfrak{q}_k^\Delta, (\mathfrak{q}_k^\nabla)^\nu)) \\
&= \text{Ass}(R_k/(s_{kk}, \mathfrak{q}_k^\Delta, \mathfrak{q}_k^\nabla)) \\
&= \text{Ass}(R_k/(s_{kk}, \mathfrak{q}_k^\diamond)) \\
&= \text{Ass}(\tilde{R}_k/(\theta_k, \tilde{\mathfrak{q}}_k)) \cap R_k,
\end{aligned}$$

where $\tilde{\alpha}_{jk}^-$ is the image of α_{jk}^- in $Q(\tilde{R}_k/\tilde{\mathfrak{q}}_k^+) = Q(R_k/\mathfrak{q}_k^\Delta)$ (cf. (1.8.1), (4.5.1), (4.1.1), (4.11.1), (4.6.2)). \square

LEMMA 4.12

With notation and assumptions as in Lemma 4.11, we have

$$\begin{aligned} & \bigcup \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^-)) \cap R_k \supset \bigcup \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^+)) \cap R_k, \\ & \bigcup \left\{ \bigcup_{\nu \geq 1} \text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^+)/(\theta_k, (\mathfrak{q}_k^-)^\nu)) \right\} \cap R_k \supset \bigcup \text{Ass}(\tilde{R}_k/(\theta_k, \tilde{\mathfrak{q}}_k^+)) \cap R_k. \end{aligned}$$

Proof

We remark that, by (2.3.6) and (4.5.1), the first of the assumptions in Lemma 4.8 is equivalent to $q_k, \zeta_{1kk}, \dots, \zeta_{nkk}$ forming an $(\tilde{R}_k/\tilde{\mathfrak{q}}_k^+)$ -sequence. The assertions above follow from Lemmas 4.11 and 1.11. Indeed, we have (cf. (4.10.1), (4.11.2))

$$\begin{aligned} & \bigcup \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k)) \cap R_k \supset \bigcup \text{Ass}(\tilde{R}_k/(q_k, \tilde{\mathfrak{q}}_k^+)) \cap R_k, \\ & \bigcup \text{Ass}(\tilde{R}_k/(\theta_k, \tilde{\mathfrak{q}}_k)) \cap R_k \supset \bigcup \text{Ass}(\tilde{R}_k/(\theta_k, \tilde{\mathfrak{q}}_k^+)) \cap R_k. \quad \square \end{aligned}$$

LEMMA 4.13

With notation and assumptions as in Lemma 4.11, take $a \in R_{k-2}$ or $a \in R_{k-1}$. If ε_k is sufficiently large, then the following statements are equivalent to each other for every $i \leq k$:

$$(4.13.1) \quad a \in p_i^{\varepsilon_i} R_{k-1} + \mathfrak{q}_{k-1}^\Delta \text{ and } a \notin p_i^{\varepsilon_i+1} R_{k-1} + \mathfrak{q}_{k-1}^\Delta;$$

$$(4.13.2) \quad a \in p_i^{\varepsilon_i} \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+ \text{ and } a \notin p_i^{\varepsilon_i+1} \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+;$$

$$(4.13.3) \quad a \in p_i^{\varepsilon_i} \tilde{R}_k + \tilde{\mathfrak{q}}_k \text{ and } a \text{ is not contained in any primary component of } p_i^{\varepsilon_i+1} \tilde{R}_k + \tilde{\mathfrak{q}}_k;$$

$$(4.13.4) \quad a \in p_i^{\varepsilon_i} \tilde{R}_k + \tilde{\mathfrak{q}}_k \text{ and if we have } a \equiv p_i^{\varepsilon_i} \beta \pmod{\tilde{\mathfrak{q}}_k} \text{ with } \beta \in \tilde{R}_k, \text{ then } \beta \notin \bigcup \text{Ass}(\tilde{R}_k/(p_i, \tilde{\mathfrak{q}}_k));$$

$$(4.13.5) \quad a \in p_i^{\varepsilon_i} \tilde{R}_k + \tilde{\mathfrak{q}}_k^+ \text{ and } a \notin p_i^{\varepsilon_i+1} \tilde{R}_k + \tilde{\mathfrak{q}}_k^+;$$

$$(4.13.6) \quad a \in p_i^{\varepsilon_i} R_k + \mathfrak{q}_k^\Delta \text{ and } a \notin p_i^{\varepsilon_i+1} R_k + \mathfrak{q}_k^\Delta.$$

Proof

The equivalence of (4.13.1), (4.13.2), (4.13.5), and (4.13.6) follows from Lemma 4.5. That (4.13.3) implies (4.13.1) is a consequence of the following:

$$(p_i^{\varepsilon_i}, \tilde{\mathfrak{q}}_k) \cap R_{k-1} = (p_i^{\varepsilon_i}, \mathfrak{q}_{k-1}^\Delta) \quad \text{and} \quad \Omega \cap R_{k-1} = (p_i^{\varepsilon_i+1}, \mathfrak{q}_{k-1}^\Delta)$$

for at least one primary component Ω of $(p_i^{\varepsilon_i+1}, \tilde{\mathfrak{q}}_k)$, because θ_k is a non-zero divisor for $\tilde{R}_k/(p_i^{\varepsilon_i+1}, \tilde{\mathfrak{q}}_k)$ and $(p_i^{\varepsilon_i+1}, \mathfrak{q}_{k-1}^\Delta)$ is an irreducible ideal of R_{k-1} (cf. Lemma 1.2). Because (4.13.4) clearly implies (4.13.3), to complete the proof, it

suffices to show that (4.13.2) implies (4.13.4). Let

$$a = p_i^{e_i} b + \sum_{j=1}^r c_j \alpha_{j(k-1)(k-1)}^+$$

with $b, c_j \in \tilde{R}_{k-1}$ such that $b \notin (p_i, \tilde{\mathfrak{q}}_{k-1}^+)$. Then $a = p_i^{e_i} (b + q_k \kappa) + \sum_{j=1}^r d_j \alpha_{jkk}^+$ with $d_j, \kappa \in R_k[\underline{\alpha_{kk}^+}]$, because

$$\alpha_{j(k-1)(k-1)}^+ = \frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \alpha_{jkk}^+ + \frac{q_k^{\varepsilon_k}}{q_{k-1}^{\nu_{k-1}}} \kappa_{j(k-1)(k-1)}^+$$

with $\kappa_{j(k-1)(k-1)}^+ \in R_k[\underline{\alpha_{(k-1)(k-1)}^+}]$.

Thus, $\beta \equiv b + q_k \kappa \pmod{\tilde{\mathfrak{q}}_k}$, because $a \equiv p_i^{e_i} (b + q_k \kappa) \pmod{\tilde{\mathfrak{q}}_k}$. Therefore, the assertion results from the following: if $b \in \tilde{R}_{k-1}$ and $b \notin p_i \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+$, then $b + q_k \kappa'$ is a non-zero divisor for $\tilde{R}_k / (p_i, \tilde{\mathfrak{q}}_k)$ for every $\kappa' \in \tilde{R}_k$ (cf. Lemma 1.2, (4.1.1)). \square

LEMMA 4.14

With assumptions as in Lemma 4.11, suppose that

$$\pi_1 \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^-, \dots, \pi_k \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^- \text{ are distinct prime ideals.}$$

Assume further that, for every $j \leq k$,

$$(4.14.1) \quad \pi_j \notin \bigcup \text{Ass}((\tilde{R}_{k-1}/\tilde{\mathfrak{q}}_{k-1}^-)/q_k(\tilde{R}_{k-1}/\tilde{\mathfrak{q}}_{k-1}^-)),$$

$$(4.14.2) \quad \pi_j \notin \bigcup \text{Ass}((\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)/q_k(\tilde{R}_k/\tilde{\mathfrak{q}}_k^-)).$$

Then $\pi_1 \tilde{R}_k + \tilde{\mathfrak{q}}_k^-, \dots, \pi_k \tilde{R}_k + \tilde{\mathfrak{q}}_k^-$ are distinct prime ideals, and

$$(4.14.3) \quad (\pi_j^{f_j} \tilde{R}_k + \tilde{\mathfrak{q}}_k^-) \cap \tilde{R}_{k-1} = \pi_j^{f_j} \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^- \quad \text{whenever } \varepsilon_k \geq f_j.$$

Proof

By assumptions (4.14.1) and (4.14.2), we have

$$q_k \notin \bigcup \text{Ass}(\tilde{R}_{k-1}/(\pi_j, \tilde{\mathfrak{q}}_{k-1}^-)) \cup \bigcup \text{Ass}(\tilde{R}_k/(\pi_j, \tilde{\mathfrak{q}}_k^-)) \quad \text{for each } j \leq k.$$

Then $(\pi_j^{f_j}, \tilde{\mathfrak{q}}_k^-) \tilde{R}_\ell[1/q_\ell] \cap \tilde{R}_\ell = \pi_j^{f_j} \tilde{R}_\ell + \tilde{\mathfrak{q}}_k^-$ for $\ell = k-1$ and k . Moreover,

$$\begin{aligned} (\pi_j^{f_j}, \tilde{\mathfrak{q}}_k^-) \tilde{R}_k[1/q_k] &= (\pi_j^{f_j}, \mathfrak{q}_k^-) \tilde{R}_k[1/q_k] \\ &= (\pi_j^{f_j}, \mathfrak{q}_{k-1}^-) \tilde{R}_k[1/q_k] \\ &= (\pi_j^{f_j}, \tilde{\mathfrak{q}}_{k-1}^-) \tilde{R}_k[1/q_k] \end{aligned}$$

whenever $\varepsilon_k \geq f_j$ (cf. (2.3.14), (2.4.1), (4.2.2), (4.4.3)). This completes the proof. \square

LEMMA 4.15

With assumptions as in Lemma 4.14, we have

$$\begin{aligned} \bigcap_{i=1}^k (p_i^{e_i}, \tilde{\mathfrak{q}}_k^+) \cap \bigcap_{j=1}^k (\pi_j^{f_j}, \tilde{\mathfrak{q}}_k^-) &= p_1^{e_1} \cdots p_k^{e_k} \pi_1^{f_1} \cdots \pi_k^{f_k} \tilde{R}_k \\ &\quad + p_1^{e_1} \cdots p_k^{e_k} \tilde{\mathfrak{q}}_k^- + \pi_1^{f_1} \cdots \pi_k^{f_k} \tilde{\mathfrak{q}}_k^+ + \tilde{\mathfrak{q}}_k^+ \cap \tilde{\mathfrak{q}}_k^-. \end{aligned}$$

Proof

The assertion follows from Lemma 1.12. Indeed, by Lemmas 4.5 and 4.14, we get

$$\bigcap_{i=1}^k (p_i^{e_i}, \tilde{\mathfrak{q}}_k^+) = p_1^{e_1} \cdots p_k^{e_k} \tilde{R}_k + \tilde{\mathfrak{q}}_k^+ \quad \text{and} \quad \bigcap_{j=1}^k (\pi_j^{f_j}, \tilde{\mathfrak{q}}_k^-) = \pi_1^{f_1} \cdots \pi_k^{f_k} \tilde{R}_k + \tilde{\mathfrak{q}}_k^-.$$

Moreover, $p_1^{e_1} \cdots p_k^{e_k}$ is a non-zero divisor on $\tilde{R}_k/\tilde{\mathfrak{q}}_k$ (cf. (4.6.2)) and $\pi_1^{f_1} \cdots \pi_k^{f_k}$ is a non-zero divisor for $\tilde{R}_k/(p_1^{e_1} \cdots p_k^{e_k}, \tilde{\mathfrak{q}}_k)$ (cf. (4.14.2), (4.10.2)). \square

LEMMA 4.16

With assumptions as in Lemma 4.14, take $a \in R_{k-1}$. Suppose that

$$(4.16.1) \quad a \in \bigcap_{i=1}^k (p_i^{e_i}, \mathfrak{q}_{k-1}^+) \cap \bigcap_{j=1}^k (\pi_j^{f_j}, \tilde{\mathfrak{q}}_{k-1}^-).$$

If ε_k, ν_k are sufficiently large, then there exist $\pi \in R_k[q_k \alpha_{kk}^+]$ and a unit $u \in \tilde{R}_{k-1}$ such that

$$(4.16.2) \quad a \equiv p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot \pi \cdot u \pmod{\tilde{\mathfrak{q}}_k^-}.$$

Further, assume that

$$(4.16.3) \quad a \notin \bigcup_{i=1}^k (p_i^{e_i+1}, \mathfrak{q}_{k-1}^\Delta) \cup \bigcup_{j=1}^k (\pi_j^{f_j+1}, \tilde{\mathfrak{q}}_{k-1}^-).$$

Then

$$(4.16.4) \quad \pi \notin \bigcup \text{Ass}(\tilde{R}_k/(p_i, \tilde{\mathfrak{q}}_k^+)) \cup \bigcup \text{Ass}(\tilde{R}_k/(\pi_j, \tilde{\mathfrak{q}}_k^-))$$

for every $i, j \leq k$. Moreover, when a is a prime element of R_{k-1} , the image of π is either a prime element or a unit in $\tilde{R}_k/\tilde{\mathfrak{q}}_k^-$.

Proof

By the first assumption and by Lemma 4.15, we can express

$$a = p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot b + p_1^{e_1} \cdots p_k^{e_k} \cdot c + \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot d + e$$

with $b \in \tilde{R}_{k-1}$, $c \in \tilde{\mathfrak{q}}_{k-1}^-$, $d \in \tilde{\mathfrak{q}}_{k-1}^+$, and $e \in \tilde{\mathfrak{q}}_{k-1}^+ \cap \tilde{\mathfrak{q}}_{k-1}^-$. That is,

$$c = \sum c_j (\phi_{j(k-1)(k-1)}^- - w_{j(k-1)}),$$

$$d = \sum d_j \alpha_{j(k-1)(k-1)}^+,$$

$$e = \sum e_{ij} \alpha_{i(k-1)(k-1)}^+ (\phi_{j(k-1)(k-1)}^- - w_{j(k-1)}) \\ + \sum e'_h G_h(\zeta_{(k-1)(k-1)}, \alpha_{(k-1)(k-1)}^+).$$

Then

$$c = \sum c_j (\phi_{jkk}^- - w_{jk}) + \theta_k^{\varepsilon_k} \kappa_{k-1}^-, \\ d = \sum d_j \left(\frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \alpha_{jkk}^+ + \frac{q_k^{\varepsilon_k}}{q_{k-1}^{\nu_{k-1}}} \kappa_{j(k-1)(k-1)}^+ \right) \\ = \frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \sum d_j \alpha_{jkk}^+ + \frac{q_k^{\varepsilon_k}}{q_{k-1}^{\nu_{k-1}}} \kappa_{k-1}^+,$$

and

$$e = \sum e_{ij} \left(\frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \alpha_{ikk}^+ + \frac{q_k^{\varepsilon_k}}{q_{k-1}^{\nu_{k-1}}} \kappa_{i(k-1)(k-1)}^+ \right) (\phi_{jkk}^- - w_{jk} + \theta_k^{\varepsilon_k} \kappa_{j(k-1)(k-1)}^-) \\ + \sum e'_h \left\{ \left(\frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \right)^{gh} [G_h(\zeta_{kk}, \alpha_{kk}^+) + \theta_k^{\varepsilon_k} \delta_{hk(k-1)}^+] \right. \\ \left. + \frac{q_k^{\varepsilon_k}}{q_{k-1}^{\nu_{k-1}gh}} [\theta_k^{\varepsilon_k} \delta_{h(k-1)(k-1)}^+ \right. \\ \left. + \sum \lambda_{hj(k-1)(k-1)}^+ (\phi_{jkk}^- - w_{jk} + \theta_k^{\varepsilon_k} \kappa_{j(k-1)(k-1)}^-)] \right\} \\ = \frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \sum e_j (\phi_{jkk}^- - w_{jk}) \\ + \left(\frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \right)^{gh} \sum e'_h G_h(\zeta_{kk}, \alpha_{kk}^+) + \frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \theta_k^{\varepsilon_k} \lambda_{k-1}^+,$$

where $\kappa_{k-1}^- \in \tilde{R}_{k-1}$, $\kappa_{k-1}^+ \in \tilde{R}_{k-1}[a_k, c_k]$, $e_j \in \tilde{R}_{k-1}[a_k, c_k, \alpha_{kk}^+]$, and $\lambda_{k-1}^+ \in \tilde{R}_{k-1}[a_k, b_k, \alpha_{kk}^+]$ (cf. (2.4.2), (3.3.1), (3.3.2)).

Thus,

$$a = p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \\ \times \left[b + \frac{\theta_k^{\varepsilon_k}}{\pi_1^{f_1} \cdots \pi_k^{f_k}} \kappa_{k-1}^- + \frac{q_k^{\nu_k}}{p_1^{e_1} \cdots p_k^{e_k} \cdot q_{k-1}^{\nu_{k-1}}} \sum d_j \alpha_{jkk}^+ \right. \\ \left. + \frac{q_k^{\varepsilon_k}}{p_1^{e_1} \cdots p_k^{e_k} \cdot q_{k-1}^{\nu_{k-1}}} \kappa_{k-1}^+ + \frac{q_k^{\nu_k} \theta_k^{\varepsilon_k}}{p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot q_{k-1}^{\nu_{k-1}}} \lambda_{k-1}^+ \right] \\ + p_1^{e_1} \cdots p_k^{e_k} \sum c_j (\phi_{jkk}^- - w_{jk}) \\ + \frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \sum e_j (\phi_{jkk}^- - w_{jk}) + \left(\frac{q_k^{\nu_k}}{q_{k-1}^{\nu_{k-1}}} \right)^{gh} \sum e'_h G_h(\zeta_{kk}, \alpha_{kk}^+).$$

Now put

$$(4.16.5) \quad \begin{aligned} \pi' &= b + \frac{\theta_k^{\varepsilon_k}}{\pi_1^{f_1} \cdots \pi_k^{f_k}} \kappa_{k-1}^- \\ &+ \frac{q_k^{\nu_k}}{p_1^{e_1} \cdots p_k^{e_k} \cdot q_{k-1}^{\nu_{k-1}}} \sum d_j \alpha_{jkk}^+ + \frac{q_k^{\varepsilon_k}}{p_1^{e_1} \cdots p_k^{e_k} \cdot q_{k-1}^{\nu_{k-1}}} \kappa_{k-1}^+ \\ &+ \frac{q_k^{\nu_k} \theta_k^{\varepsilon_k}}{p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot q_{k-1}^{\nu_{k-1}}} \lambda_{k-1}^+. \end{aligned}$$

Then there exists $w \in R_{k-1}[\alpha_{(k-1)(k-1)}^+] \setminus \mathfrak{N}_{k-1}$ such that $\pi = w\pi' \in R_k[q_k \alpha_{kk}^+]$. Hence, by letting $u = 1/w$, we get (4.16.2).

Assume further that a satisfies (4.16.3). Then by Lemma 4.13, we see that

$$\pi \notin \bigcup \text{Ass}(\tilde{R}_k/(p_i, \tilde{q}_k)) \quad \text{for each } i \leq k.$$

Thus, π satisfies (4.16.4) by Lemmas 4.10 and 4.14. Finally, when a is a prime element of R_{k-1} , the image of π in $\tilde{R}_k/\tilde{q}_k^-$ is either a prime element or a unit, because

$$\begin{aligned} (\tilde{R}_k/(\pi, \tilde{q}_k^-)) \left[\frac{1}{q_k \theta_k} \right] &\cong (\tilde{R}_k/(a, \tilde{q}_k^-)) \left[\frac{1}{q_k \theta_k} \right] \\ &= \left((R_{k-1}/aR_{k-1}) \left[\frac{1}{q_k \theta_k} \right] [\underline{a}_k, \underline{b}_k, \underline{c}_k] / \mathfrak{q}_k^- \right)_{U_k} \\ &\cong \left((R_{k-1}/aR_{k-1}) [\underline{a}_k, \underline{b}_k] \right)_{U'_k}, \end{aligned}$$

where U_k and U'_k are multiplicative sets. \square

LEMMA 4.17

With notation and assumptions as in Lemma 4.16, take $a \in R_{k-2}$. Assume that

$$a \equiv p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot \epsilon \pmod{\tilde{q}_{k-1}^-}$$

with $\epsilon \in \tilde{R}_{k-1}$ such that the image of ϵ in $\tilde{R}_{k-1}/\tilde{q}_{k-1}^-$ is a unit. Then, for sufficiently large ε_k , we have

$$(4.17.1) \quad a \equiv p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot \epsilon' \pmod{\tilde{q}_k^-}$$

where $\epsilon' \in \tilde{R}_k$ and the image of ϵ' in $\tilde{R}_k/\tilde{q}_k^-$ is a unit.

Proof

By assumption, we can write

$$a = p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot b + p_1^{e_1} \cdots p_k^{e_k} \cdot c + \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot d + e$$

with $b \in \tilde{R}_{k-1}$, $c \in \tilde{q}_{k-1}^-$, $d \in \tilde{q}_{k-1}^+$, $e \in \tilde{q}_{k-1}^+ \cap \tilde{q}_{k-1}^-$ (cf. Lemmas 4.13, 4.16). Hence, using the same notation as in the proof of Lemma 4.16, we get

$$a \equiv p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot \pi' \pmod{\tilde{q}_k^-}.$$

Incidentally, $b \equiv \epsilon \pmod{\tilde{\mathfrak{q}}_{k-1}}$. Indeed, $q_k \theta_k$ is a non-zero divisor for $\tilde{R}_{k-1}/\tilde{\mathfrak{q}}_{k-1}$ and

$$a \equiv p_1^{e_1} \cdots p_k^{e_k} \cdot \pi_1^{f_1} \cdots \pi_k^{f_k} \cdot b \pmod{\tilde{\mathfrak{q}}_{k-1}}.$$

Thus, the image of b in $\tilde{R}_{k-1}/\tilde{\mathfrak{q}}_{k-1}$ is a unit; that is, $b \notin \tilde{\mathfrak{n}}_{k-1}$. Consequently, $\pi' \notin \tilde{\mathfrak{n}}_k$. Therefore, the image of ϵ' in $\tilde{R}_k/\tilde{\mathfrak{q}}_k^-$ is a unit, because $\epsilon' \equiv \pi' \pmod{\tilde{\mathfrak{q}}_k^-}$. \square

5. Factorial numbering

5.1. Simple numbering on \mathcal{P}

With notation as in Section 2, because $R_k = R_{k-1}(\underline{a}_k, \underline{b}_k, \underline{c}_k)$ and $R = \bigcup_k R_k$ (cf. (2.1.1)), each height-one prime ideal \mathfrak{p} of R is generated by a unique prime element (up to a unit multiple) $\rho \in S_k$. Thus, choosing such uniquely determined $\rho \in S_k$ for each height-one prime ideal \mathfrak{p} of R , we get a set of prime elements that represents the height-one prime ideals of R , and we denote the set by \mathcal{P} (cf. (2.1.2)). That is,

$$(5.1.1) \quad \mathcal{P} \subset \mathfrak{N} \setminus \{0\},$$

which contains, for each height-one prime ideal $\mathfrak{p} \in \text{Spec}(R)$, exactly one $\rho \in S_k$ such that $\rho R = \mathfrak{p}$. Then, \mathcal{P} is a countable set and we may assume that it contains z_0 and an infinite number of elements of S_0 .

On the other hand, with notation as in Section 4, let

$$(5.1.2) \quad \tilde{R}_{h-1}(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \tilde{R}_{h-1}(a_{ik}, b_{ik}, c_{jk})$$

where $i = 1, \dots, n$ and $j = 1, \dots, r$ with $k = h, h+1, \dots$, and let

$$(5.1.3) \quad \tilde{R}_{h-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq \ell} := \tilde{R}_{h-1}(a_{i\tau}, b_{i\tau}, c_{j\tau})$$

where $i = 1, \dots, n$ and $j = 1, \dots, r$ with $\tau = h, \dots, \ell$ (i.e., $\tau \leq \ell$).

Now, before giving *factorial* numbering, we fix a *simple* numbering $\rho_1, \rho_2, \dots, \rho_h, \dots$ to the members of \mathcal{P} as follows:

$$(5.1.4) \quad \rho_1 = z_0 \quad \text{and} \quad \rho_h \in R_{h-2} \quad \text{for } h \geq 2.$$

PROPOSITION 5.2 (FACTORIAL NUMBERING)

With notation as in Section 2 and with simple numbering (5.1.4), we can choose two sequences of elements of $Q(R)$, namely, $\pi_1, \pi_2, \dots, \pi_j, \dots$ and $p_1, p_2, \dots, p_i, \dots$, such that

$$\pi_1 := \rho_1 (= z_0) \quad \text{and} \quad \pi_j \in R_{j-1}[q_{j-1} \alpha_{(j-1)(j-1)}^+] \quad \text{for } j \geq 2,$$

$$p_1 \in \mathcal{P} \cap R_0 \quad \text{and} \quad p_i \in \mathcal{P} \cap R_{i-2} \quad \text{for } i \geq 2,$$

and for each natural number k , they satisfy the following:

$$(5.2.1) \quad q_k, \theta_k \text{ form an } (\tilde{R}_{k-1}/\tilde{\mathfrak{q}}_{k-1})\text{-sequence;}$$

$$(5.2.2) \quad q_k, \theta_k \text{ form an } (\tilde{R}_k/\tilde{\mathfrak{q}}_k)\text{-sequence;}$$

$$(5.2.3) \quad p_1 \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+, \dots, p_k \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^+ \text{ and } \pi_1 \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^-, \dots,$$

$\pi_k \tilde{R}_{k-1} + \tilde{\mathfrak{q}}_{k-1}^-$ are mutually distinct prime ideals of \tilde{R}_{k-1} ;

$$(5.2.4) \quad q_k, z_{1kk}, \dots, z_{nkk} \text{ form an } (R_k/\mathfrak{q}_k^\Delta)\text{-sequence};$$

$$(5.2.5) \quad s_{kk} z_{1kk}, \dots, z_{nkk} \text{ form an } (R_k/\mathfrak{q}_k^\Delta)\text{-sequence};$$

$$(5.2.6) \quad q_k, z_{1kk}, \dots, z_{nkk} \text{ form an } (R_k/\mathfrak{q}_k^\nabla)\text{-sequence}.$$

We can take these elements so that each $\rho_h \in \mathcal{P}$ is decomposed as follows:

$$(5.2.7) \quad \rho_h \equiv p_1^{e_{h1}} \cdots p_{2h-3}^{e_{h(2h-3)}} \cdot \pi_1^{f_{h1}} \cdots \pi_{2h-3}^{f_{h(2h-3)}} \cdot \pi_{2h-2}^{f_{h(2h-2)}} \cdot u_{hk} \pmod{\tilde{\mathfrak{q}}_k^-},$$

where $u_{hk} \in \tilde{R}_k \setminus \tilde{\mathfrak{n}}_k$ for $k \geq 2h-3$ (cf. (4.16.2), (4.17.1)).

On the other hand, we can find natural numbers μ_k, ν_k , and ε_k such that

$$(5.2.8) \quad \begin{array}{ccccccc} \mu_1 & < & \mu_2 & < & \cdots & < & \mu_k & < & \mu_{k+1} & < & \cdots \\ \wedge & & \wedge & & & & \wedge & & \wedge & & \\ \nu_1 & < & \nu_2 & < & \cdots & < & \nu_k & < & \nu_{k+1} & < & \cdots \\ \wedge & & \wedge & & & & \wedge & & \wedge & & \\ \varepsilon_1 & < & \varepsilon_2 & < & \cdots & < & \varepsilon_k & < & \varepsilon_{k+1} & < & \cdots \end{array}$$

and they satisfy

$$q_k^{\varepsilon_k} \theta_k \in R_{k-1} \quad \text{and} \quad \nu_{2k} > N_{2k} := \max\{e_{hi} + \nu_{2k-1}, f_{hj}, f_{h(2h-2)}\}$$

for $1 \leq i, j \leq 2h-3$; $1 \leq h \leq k+1$ (cf. (4.5.2), (4.14.3), (4.16.2)). Finally, we can give enumerations on

$$\mathcal{X}_i^+ := \text{the set of height-one prime ideals of } \tilde{R}_{i-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_i, \tilde{\mathfrak{q}}_{i-1}^+),$$

$$\mathcal{X}_j^- := \text{the set of height-one prime ideals of } \tilde{R}_{j-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_j, \tilde{\mathfrak{q}}_{j-1}^-).$$

Namely, we have (cf. (5.1.2), (5.1.3))

$$\mathcal{X}_i^+ := \{\tilde{\mathfrak{p}}_{ih}^+ := \tilde{\mathfrak{P}}_{ih}^+/(p_i, \tilde{\mathfrak{q}}_{i-1}^+)\} \text{ such that } \tilde{\mathfrak{P}}_{ih}^+ \subset \tilde{R}_{i-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2},$$

$$\mathcal{X}_j^- := \{\tilde{\mathfrak{p}}_{jh}^- := \tilde{\mathfrak{P}}_{jh}^-/(\pi_j, \tilde{\mathfrak{q}}_{j-1}^-)\} \text{ such that } \tilde{\mathfrak{P}}_{jh}^- \subset \tilde{R}_{j-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}.$$

That is, because \mathcal{X}_i^+ and \mathcal{X}_j^- are defined to be the set of height-one prime ideals of $\tilde{R}_{i-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_i, \tilde{\mathfrak{q}}_{i-1}^+)$ and of $\tilde{R}_{j-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_j, \tilde{\mathfrak{q}}_{j-1}^-)$, $\tilde{\mathfrak{P}}_{ih}^+$ should be an ideal of $\tilde{R}_{i-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ generated by the elements of $\tilde{R}_{i-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}$ and $\tilde{\mathfrak{P}}_{jh}^-$ should be an ideal of $\tilde{R}_{j-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ generated by the elements of $\tilde{R}_{j-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}$.

Arranging the members of $\mathcal{X}_1^+, \mathcal{X}_2^+, \dots, \mathcal{X}_i^+, \dots$ and those of $\mathcal{X}_1^-, \mathcal{X}_2^-, \dots, \mathcal{X}_j^-, \dots$ in the following manner:

$$\begin{array}{ccccccc} \tilde{\mathfrak{p}}_{11}^\pm & \tilde{\mathfrak{p}}_{12}^\pm & \tilde{\mathfrak{p}}_{13}^\pm & \cdots & \tilde{\mathfrak{p}}_{1h}^\pm & \cdots & \\ \tilde{\mathfrak{p}}_{21}^\pm & \tilde{\mathfrak{p}}_{22}^\pm & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \tilde{\mathfrak{p}}_{\ell 1}^\pm & \tilde{\mathfrak{p}}_{\ell 2}^\pm & \cdots & \cdots & \tilde{\mathfrak{p}}_{\ell h}^\pm & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{array}$$

we get numberings on $\mathcal{X}^+ := \bigcup \mathcal{X}_i^+$ and on $\mathcal{X}^- := \bigcup \mathcal{X}_j^-$, specifically,

$$\tilde{\mathfrak{p}}_{11}^+, \tilde{\mathfrak{p}}_{12}^+, \tilde{\mathfrak{p}}_{21}^+, \tilde{\mathfrak{p}}_{13}^+, \tilde{\mathfrak{p}}_{22}^+, \tilde{\mathfrak{p}}_{31}^+, \dots \quad \text{and} \quad \tilde{\mathfrak{p}}_{11}^-, \tilde{\mathfrak{p}}_{12}^-, \tilde{\mathfrak{p}}_{21}^-, \tilde{\mathfrak{p}}_{13}^-, \tilde{\mathfrak{p}}_{22}^-, \tilde{\mathfrak{p}}_{31}^-, \dots$$

Further, if $\tilde{\mathfrak{p}}_{\ell h}^-$ (or $\tilde{\mathfrak{p}}_{\ell h}^+$) is the k th member in \mathcal{X}^- (or in \mathcal{X}^+), then we can take π_j, p_i such that

$$(5.2.9) \quad \pi_{2k+1} \in \tilde{\mathfrak{P}}_{\ell h}^+ \quad \text{and} \quad p_{2k+1} \in \tilde{\mathfrak{P}}_{\ell h}^-.$$

In the rest of this section, we carry out the proof of Proposition 5.2 by induction on k .

5.3. Choice of π_1 and p_1 and the case $k = 1$

Let $\pi_1 = z_0$. Then $\pi_1 \notin \bigcup \text{Ass}(\tilde{R}_0/\tilde{\mathfrak{q}}_0)$ and $\pi_1 \tilde{R}_0 + \tilde{\mathfrak{q}}_0^-$ is a prime ideal of \tilde{R}_0 . Next take $p_1 \in \mathcal{P} \cap R_0$ such that $p_1 \notin \bigcup \text{Ass}(\tilde{R}_0/(\pi_1, \tilde{\mathfrak{q}}_0))$ and such that $p_1, x_1, \dots, x_n, w_1, \dots, w_r, y_1, \dots, y_n, w_1, \dots, w_r$, and $p_1, z_1, \dots, z_n, w_1, \dots, w_r$, respectively, form part of the regular systems of parameters of R . Then

- $\pi_1 \notin \bigcup \text{Ass}(\tilde{R}_0/\tilde{\mathfrak{q}}_0)$ and $\pi_1 \tilde{R}_0 + \tilde{\mathfrak{q}}_0^-$ is a prime ideal of \tilde{R}_0 ,
- q_1, θ_1 form an $(\tilde{R}_0/\tilde{\mathfrak{q}}_0)$ -sequence,
- q_1, θ_1 form an $(\tilde{R}_1/\tilde{\mathfrak{q}}_1)$ -sequence,
- $p_1 \tilde{R}_0 + \tilde{\mathfrak{q}}_0^+$ and $\pi_1 \tilde{R}_0 + \tilde{\mathfrak{q}}_0^-$ are mutually distinct prime ideals of \tilde{R}_0 ,
- $q_1, z_{111}, \dots, z_{n11}$ form an $(R_1/\mathfrak{q}_1^\Delta)$ -sequence,
- $s_{11}, z_{111}, \dots, z_{n11}$ form an $(R_1/\mathfrak{q}_1^\Delta)$ -sequence,
- $q_1, z_{111}, \dots, z_{n11}$ form an $(R_1/\mathfrak{q}_1^\nabla)$ -sequence.

Finally, we can give enumerations on the sets of height-one prime ideals

$$\mathcal{X}_1^+ \text{ of } \tilde{R}_0(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_1, \tilde{\mathfrak{q}}_0^+) \quad \text{and} \quad \mathcal{X}_1^- \text{ of } \tilde{R}_0(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_1, \tilde{\mathfrak{q}}_0^-).$$

Namely,

$$\mathcal{X}_1^+ := \{\tilde{\mathfrak{p}}_{1h}^+ := \tilde{\mathfrak{P}}_{1h}^+/(p_1, \tilde{\mathfrak{q}}_0^+)\} \quad \text{and} \quad \mathcal{X}_1^- := \{\tilde{\mathfrak{p}}_{1h}^- := \tilde{\mathfrak{P}}_{1h}^-/(\pi_1, \tilde{\mathfrak{q}}_0^-)\}$$

such that $\tilde{\mathfrak{P}}_{1h}^+ \subset \tilde{R}_0(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}$ and $\tilde{\mathfrak{P}}_{1h}^- \subset \tilde{R}_0(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}$.

5.4. Inductive assumptions up to $k = 2\kappa - 1$

For a natural number κ , we may assume by induction that $\pi_1, \dots, \pi_{2\kappa-1}$ and $p_1, \dots, p_{2\kappa-1}$ have been chosen as follows:

$$(5.4.1) \quad \pi_j \in R_{j-1}[q_{j-1} \alpha_{(j-1)(j-1)}^+] \quad \text{and} \quad p_i \in \mathcal{P} \cap R_{i-2} \quad \text{for } i, j \leq 2\kappa - 1,$$

and

- q_m, θ_m form an $(\tilde{R}_{m-1}/\tilde{\mathfrak{q}}_{m-1})$ -sequence,
- q_m, θ_m form an $(\tilde{R}_m/\tilde{\mathfrak{q}}_m)$ -sequence,
- $p_1 \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^+, \dots, p_m \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^+$ and $\pi_1 \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^-, \dots, \pi_m \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^-$ are mutually distinct prime ideals of \tilde{R}_{m-1} ,
- $q_m, z_{1mm}, \dots, z_{nmm}$ form an $(R_m/\mathfrak{q}_m^\Delta)$ -sequence,
- $s_{mm}, z_{1mm}, \dots, z_{nmm}$ form an $(R_m/\mathfrak{q}_m^\Delta)$ -sequence,
- $q_m, z_{1mm}, \dots, z_{nmm}$ form an $(R_m/\mathfrak{q}_m^\nabla)$ -sequence

for $m \leq 2\kappa - 1$. Further, we may assume that, for $2h - 2 \leq 2\kappa - 1$, each $\rho_h \in \mathcal{P}$ is decomposed by the elements above as follows:

$$\rho_h \equiv p_1^{e_{h1}} \cdots p_{2h-3}^{e_h(2h-3)} \cdot \pi_1^{f_{h1}} \cdots \pi_{2h-3}^{f_h(2h-3)} \cdot \pi_{2h-2}^{f_h(2h-2)} \cdot u_{hm} \pmod{\tilde{q}_m^-},$$

where $u_{hm} \in \tilde{R}_m \setminus \tilde{\mathfrak{n}}_m$ with $2h - 3 \leq m \leq 2\kappa - 1$ (cf. (4.16.2), (4.17.1)). On the other hand, we may assume that we find natural numbers μ_k , ν_k , and ε_k such that

$$\begin{array}{ccccccc} \mu_1 & < & \mu_2 & < & \cdots & < & \mu_m & < & \cdots & < & \mu_{2\kappa-1} \\ \wedge & & \wedge & & & & \wedge & & & & \wedge & & \\ \nu_1 & < & \nu_2 & < & \cdots & < & \nu_m & < & \cdots & < & \nu_{2\kappa-1} \\ \wedge & & \wedge & & & & \wedge & & & & \wedge & & \\ \varepsilon_1 & < & \varepsilon_2 & < & \cdots & < & \varepsilon_m & < & \cdots & < & \varepsilon_{2\kappa-1} \end{array}$$

that satisfy

$$(5.4.2) \quad q_m^{\varepsilon_m} \theta_m \in R_{m-1} \text{ and } \nu_{2\ell} > N_{2\ell} := \max\{e_{hi} + \nu_{2\ell-1}, f_{hj}, f_{h(2h-2)}\}$$

for $m \leq 2\kappa - 1$, $i, j \leq 2h - 3$, and $h \leq \ell + 1 \leq \kappa$ (cf. (4.5.2), (4.14.3), (4.16.2)). Finally, we may assume that, for $i, j \leq 2\kappa - 1$, we can give enumerations

$$\mathcal{X}_i^+ := \text{the set of height-one prime ideals of } \tilde{R}_{i-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_i, \tilde{q}_{i-1}^+),$$

$$\mathcal{X}_j^- := \text{the set of height-one prime ideals of } \tilde{R}_{j-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_j, \tilde{q}_{j-1}^-).$$

Namely (cf. (5.1.2), (5.1.3)),

$$\mathcal{X}_i^+ := \{\tilde{\mathfrak{p}}_{ih}^+ := \tilde{\mathfrak{P}}_{ih}^+/(p_i, \tilde{q}_{i-1}^+)\} \text{ such that } \tilde{\mathfrak{P}}_{ih}^+ \subset \tilde{R}_{i-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2},$$

$$\mathcal{X}_j^- := \{\tilde{\mathfrak{p}}_{jh}^- := \tilde{\mathfrak{P}}_{jh}^-/(\pi_j, \tilde{q}_{j-1}^-)\} \text{ such that } \tilde{\mathfrak{P}}_{jh}^- \subset \tilde{R}_{j-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}.$$

Arranging the members of $\mathcal{X}_1^+, \mathcal{X}_2^+, \dots, \mathcal{X}_i^+, \dots$ and those of $\mathcal{X}_1^-, \mathcal{X}_2^-, \dots, \mathcal{X}_j^-, \dots$ in the following manner:

$$\begin{array}{ccccccc} \tilde{\mathfrak{p}}_{11}^\pm & & \tilde{\mathfrak{p}}_{12}^\pm & & \tilde{\mathfrak{p}}_{13}^\pm & \cdots & \tilde{\mathfrak{p}}_{1h}^\pm & \cdots \\ \tilde{\mathfrak{p}}_{21}^\pm & & \tilde{\mathfrak{p}}_{22}^\pm & & \cdots & \cdots & \cdots & \cdots \\ \cdots & & \cdots & & \cdots & \cdots & \cdots & \cdots \\ \cdots & & \cdots & & \cdots & \cdots & \cdots & \cdots \\ \tilde{\mathfrak{p}}_{\ell 1}^\pm & & \tilde{\mathfrak{p}}_{\ell 2}^\pm & & \cdots & \cdots & \tilde{\mathfrak{p}}_{\ell h}^\pm & \cdots \\ \cdots & & \cdots & & \cdots & \cdots & \cdots & \cdots \\ \tilde{\mathfrak{p}}_{(2\kappa-1)1}^\pm & & \tilde{\mathfrak{p}}_{(2\kappa-1)2}^\pm & & \cdots & \cdots & \tilde{\mathfrak{p}}_{(2\kappa-1)h}^\pm & \cdots \end{array}$$

we get numberings on $\mathcal{X}_{(2\kappa-1)}^+ := \bigcup_{i=1}^{2\kappa-1} \mathcal{X}_i^+$ and $\mathcal{X}_{(2\kappa-1)}^- := \bigcup_{j=1}^{2\kappa-1} \mathcal{X}_j^-$:

$$\tilde{\mathfrak{p}}_{11}^+, \tilde{\mathfrak{p}}_{12}^+, \tilde{\mathfrak{p}}_{21}^+, \tilde{\mathfrak{p}}_{13}^+, \tilde{\mathfrak{p}}_{22}^+, \tilde{\mathfrak{p}}_{31}^+, \dots \quad \text{and} \quad \tilde{\mathfrak{p}}_{11}^-, \tilde{\mathfrak{p}}_{12}^-, \tilde{\mathfrak{p}}_{21}^-, \tilde{\mathfrak{p}}_{13}^-, \tilde{\mathfrak{p}}_{22}^-, \tilde{\mathfrak{p}}_{31}^-, \dots$$

Then, if $\tilde{\mathfrak{p}}_{\ell h}^-$ (or $\tilde{\mathfrak{p}}_{\ell h}^+$) is the m th member in $\mathcal{X}_{(2\kappa-1)}^-$ (or in $\mathcal{X}_{(2\kappa-1)}^+$), then we may also assume that π_j, p_i satisfy

$$\pi_{2m+1} \in \tilde{\mathfrak{P}}_{\ell h}^+ \quad \text{and} \quad p_{2m+1} \in \tilde{\mathfrak{P}}_{\ell h}^- \quad \text{for } m+1 \leq \kappa.$$

5.5. Decomposition of $\rho_{\kappa+1}$ and choice of $\pi_{2\kappa}, p_{2\kappa}$

Let $e_{(\kappa+1)1}, e_{(\kappa+1)2}, \dots, e_{(\kappa+1)(2\kappa-1)}, f_{(\kappa+1)1}, f_{(\kappa+1)2}, \dots, f_{(\kappa+1)(2\kappa-1)}$ be natural numbers that satisfy the following

$$\rho_{\kappa+1} \in \bigcap_{i=1}^{2\kappa-1} (p_i^{e_{(\kappa+1)i}}, \tilde{q}_{2\kappa-2}^+) \cap \bigcap_{j=1}^{2\kappa-1} (\pi_j^{f_{(\kappa+1)j}}, \tilde{q}_{2\kappa-2}^-)$$

and

$$\rho_{\kappa+1} \notin \bigcup_{i=1}^{2\kappa-1} (p_i^{e_{(\kappa+1)i}+1}, \tilde{q}_{2\kappa-2}^+) \cup \bigcup_{j=1}^{2\kappa-1} (\pi_j^{f_{(\kappa+1)j}+1}, \tilde{q}_{2\kappa-2}^-).$$

Then

$$\rho_{\kappa+1} \equiv p_1^{e_{(\kappa+1)1}} \cdots p_{2\kappa-1}^{e_{(\kappa+1)(2\kappa-1)}} \cdot \pi_1^{f_{(\kappa+1)1}} \cdots \pi_{2\kappa-1}^{f_{(\kappa+1)(2\kappa-1)}} \cdot \beta_{2\kappa} \cdot u_{2\kappa} \pmod{\tilde{q}_{2\kappa-1}^-}$$

with $\beta_{2\kappa} \in R_{2\kappa-1}[\underline{q_{2\kappa-1}\alpha_{(2\kappa-1)(2\kappa-1)}^+}]$.

Here $\beta_{2\kappa}$ is either a prime element or a unit in $\tilde{R}_{2\kappa-1}/\tilde{q}_{2\kappa-1}^-$ and $u_{2\kappa}$ is a unit in $\tilde{R}_{2\kappa-2}$ (cf. Lemma 4.16). If $\beta_{2\kappa}$ is a prime element, then put $\pi_{2\kappa} := \beta_{2\kappa}$. Then

$$\pi_{2\kappa} \notin \bigcup \text{Ass}(\tilde{R}_{2\kappa-1}/(q_{2\kappa-1}, \tilde{q}_{2\kappa-1}^-))$$

and $\pi_{2\kappa}\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^-$ is a prime ideal different from $p_i\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^+$ and $\pi_j\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^-$ for $i, j \leq 2\kappa - 1$. Otherwise, take $\pi_{2\kappa} \in \mathcal{P} \cap R_0$ such that the image of $\pi_{2\kappa}$ is a prime element in $\tilde{R}_{2\kappa-1}/\tilde{q}_{2\kappa-1}^-$,

$$\pi_{2\kappa} \notin \bigcup \text{Ass}(\tilde{R}_{2\kappa-1}/(q_{2\kappa-1}, \tilde{q}_{2\kappa-1}^-)),$$

and $\pi_{2\kappa}\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^-$ is a prime ideal different from $p_i\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^+$ and $\pi_j\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^-$ for $i, j \leq 2\kappa - 1$.

In both cases, let $p_{2\kappa} \in \mathcal{P} \cap R_{2\kappa-2}$ (or $\mathcal{P} \cap R_0$) such that the image of $p_{2\kappa}$ is a prime element in $\tilde{R}_{2\kappa-1}/\tilde{q}_{2\kappa-1}^+$,

$$p_{2\kappa} \notin \bigcup \text{Ass}(\tilde{R}_{2\kappa-1}/(\theta_{2\kappa}, \tilde{q}_{2\kappa-1}^-)),$$

and $p_{2\kappa}\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^+$ is a prime ideal different from $p_i\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^+$ and $\pi_j\tilde{R}_{2\kappa-1} + \tilde{q}_{2\kappa-1}^-$ for $i+1, j \leq 2\kappa$. Thus,

$$(5.5.1) \quad \pi_j \in R_{j-1}[\underline{q_{j-1}\alpha_{(j-1)(j-1)}^+}] \quad \text{and} \quad p_i \in \mathcal{P} \cap R_{i-2} \quad \text{for } i, j \leq 2\kappa,$$

and, for $m \leq 2\kappa$,

- q_m, θ_m form an $(\tilde{R}_{m-1}/\tilde{q}_{m-1}^-)$ -sequence,
- q_m, θ_m form an $(\tilde{R}_m/\tilde{q}_m)$ -sequence,
- $p_1\tilde{R}_{m-1} + \tilde{q}_{m-1}^+, \dots, p_m\tilde{R}_{m-1} + \tilde{q}_{m-1}^+$ and $\pi_1\tilde{R}_{m-1} + \tilde{q}_{m-1}^-, \dots, \pi_m\tilde{R}_{m-1} + \tilde{q}_{m-1}^-$ are mutually distinct prime ideals of \tilde{R}_{m-1} .

Now we show that if the ε_m 's are large enough, for $m \leq 2\kappa$,

$$(5.5.2) \quad q_m, z_{1mm}, \dots, z_{nmm} \text{ form an } (R_m/\mathfrak{q}_m^\Delta)\text{-sequence,}$$

$$(5.5.3) \quad s_{mm}, z_{1mm}, \dots, z_{nmm} \text{ form an } (R_m/\mathfrak{q}_m^\Delta)\text{-sequence.}$$

Proof of (5.5.2) and (5.5.3)

By definition, it is enough to show that, for $m \leq 2\kappa$,

$$q_m, z_{1mm}, \dots, z_{nmm}, g_{1mm}^+ + w_{1m}, \dots, g_{rmm}^+ + w_{rm} \text{ form an } R_m\text{-sequence,}$$

$$s_{mm}, z_{1mm}, \dots, z_{nmm}, g_{1mm}^+ + w_{1m}, \dots, g_{rmm}^+ + w_{rm} \text{ form an } R_m\text{-sequence.}$$

By induction, we may assume for $\ell \leq 2\kappa - 1$ that

$$q_\ell, z_{1\ell\ell}, \dots, z_{n\ell\ell}, g_{1\ell\ell}^+ + w_{1\ell}, \dots, g_{r\ell\ell}^+ + w_{r\ell} \text{ form an } R_\ell\text{-sequence,}$$

$$s_{\ell\ell}, z_{1\ell\ell}, \dots, z_{n\ell\ell}, g_{1\ell\ell}^+ + w_{1\ell}, \dots, g_{r\ell\ell}^+ + w_{r\ell} \text{ form an } R_\ell\text{-sequence.}$$

Because $z_{1mm}, \dots, z_{nmm}, g_{1mm}^+ + w_{1m}, \dots, g_{rmm}^+ + w_{rm}$ is a subregular system of parameters of R_m , $\mathfrak{P}_m := (z_{1mm}, \dots, z_{nmm}, g_{1mm}^+ + w_{1m}, \dots, g_{rmm}^+ + w_{rm})$ is a prime ideal generated by an R_m -sequence. By construction, we have

$$s_{mm} = s_{(m-1)(m-1)} r_m + q_m^{\varepsilon'_m} \tau_m,$$

$$z_{imm} = x_{im} + \left(y_i + \sum_{h=1}^m b_{ih} s_{hm}^{\varepsilon_h} \right)$$

$$= z_{i(m-1)(m-1)} + q_m^{\varepsilon'_m} \lambda_{im} + s_{(m-1)(m-1)}^{\varepsilon_m} r_{im}^*,$$

$$z_{i(m-1)(m-1)} = z_{i(m-1)(m-2)} + b_{i(m-1)} s_{(m-1)(m-1)}^{\varepsilon_{m-1}},$$

$$g_{jmm}^+ + w_{jm} = g_{j(m-1)(m-1)}^+ + w_{j(m-1)} + q_m^{\varepsilon'_m} \kappa_{jm},$$

where (cf. (2.5.1), (2.3.6), (2.5.4))

$$r_m \in R_{m-1}, r_{im}^* \in R_{m-1}[\underline{b}_m], w_{jm} \in R_{m-1}[\underline{c}_m], z_{im\ell} \in R_{m-1}[\underline{a}_m, \underline{b}_\ell],$$

$$s_{mm}, \tau_m, g_{jmm}^+, \kappa_{jm} \in R_{m-1}[\underline{a}_m, \underline{c}_m], \text{ and } \lambda_{im} \in R_{m-1}[\underline{a}_m, \underline{b}_m, \underline{c}_m].$$

Thus, to get the assertion, it suffices to show that $q_m, r_m \notin \mathfrak{P}_m$ for sufficiently large ε_m . Namely,

$$q_m, r_m \notin \mathfrak{P}_{m-1}.$$

In fact, this results from

$$\begin{aligned} (z_{(m-1)(m-1)}, \tilde{\mathfrak{q}}_{m-1}) \cap R_{m-2} &= (z_{(m-1)(m-1)}, \tilde{\mathfrak{q}}_{m-1}^\Delta) \cap R_{m-2} \\ &= (z_{(m-1)(m-1)}, \mathfrak{q}_{m-1}^\Delta) \cap R_{m-2} \\ &= \mathfrak{P}_{m-1} \cap R_{m-2} \\ &= (0) \end{aligned}$$

and from either $q_m, \pi_m \in R_{m-2}$ or

$$\rho \equiv p_1^{e_1} \cdots \pi_{m-1}^{f_{m-1}} \cdot \pi_m \cdot u_m \pmod{\tilde{\mathfrak{q}}_{m-1}^-},$$

where $\rho \in R_{m-2}$ with $\pi_m \in R_{m-1}[q_{m-1}\alpha_{(m-1)(m-1)}^+]$ (cf. (5.4.1), (5.5.1)). This completes the proof of (5.5.2) and (5.5.3). \square

Moreover, with the notation and assumptions above, we show that

$$(5.5.4) \quad q_m, z_{1mm}, \dots, z_{nmm} \text{ form an } (R_m/\mathfrak{q}_m^\nabla)\text{-sequence for } m \leq 2\kappa.$$

Proof of (5.5.4)

First of all, by inductive hypothesis, we may assume that

$$q_\ell, z_{1\ell\ell}, \dots, z_{n\ell\ell} \text{ form an } (R_\ell/\mathfrak{q}_\ell^\nabla)\text{-sequence for } \ell \leq 2\kappa - 1.$$

By construction, $R_m/(\pi_1, \mathfrak{q}_m^\nabla) \cong (R_{m-1}/(\pi_1, \mathfrak{q}_{m-1}^\nabla))(a_m, b_m, c_m)$ (cf. (2.3.8)). Then, because $g_{1mm} - w_{1m}, \dots, g_{rmm} - w_{rm}, z_{1(m-1)(m-1)}, \dots, z_{n(m-1)(m-1)}$ form a subregular system of parameters of $R_m/\pi_1 R_m$, $\tilde{\mathfrak{P}}_{m-1} := (\tilde{z}_{1(m-1)(m-1)}, \dots, \tilde{z}_{n(m-1)(m-1)})$ is a prime ideal generated by an $(R_{m-1}/(\pi_1, \mathfrak{q}_{m-1}^\nabla))$ -sequence. If we denote by \bar{q}_m and $\tilde{z}_{i(m-1)(m-1)}$ the images of q_m and $z_{i(m-1)(m-1)}$ in $R_{m-1}/(\pi_1, \mathfrak{q}_{m-1}^\nabla)$, $\tilde{z}_{imm} = \tilde{z}_{i(m-1)(m-1)} + a_{im}\bar{q}_m^{\varepsilon_m}$ in $(R_{m-1}/(\pi_1, \mathfrak{q}_{m-1}^\nabla))(a_m, b_m, c_m)$. On the other hand, $p_j \in R_{j-2}$ for $j \geq 2$. Hence, we may further assume that

$$\tilde{\mathfrak{P}}_\ell \cap (R_{\ell-1}/(\pi_1, \mathfrak{q}_{\ell-1}^\nabla)) = (0) \text{ for } \ell \leq 2\kappa - 1.$$

In particular, $\bar{q}_m \notin \tilde{\mathfrak{P}}_{m-1}$. Thus by Lemma 1.1, $\pi_1, q_m, z_{1mm}, \dots, z_{nmm}$ form an $(R_m/\mathfrak{q}_m^\nabla)$ -sequence, and we get the assertion. \square

Next choose sufficiently large $\mu_{2\kappa}, \nu_{2\kappa}, \varepsilon_{2\kappa}$ such that

$$\begin{array}{cccccc} \mu_1 & < & \mu_2 & < & \cdots & < & \mu_m & < & \cdots & < & \mu_{2\kappa-1} & < & \mu_{2\kappa} \\ \wedge & & \wedge & & & & \wedge & & & & & \wedge & & \wedge \\ \nu_1 & < & \nu_2 & < & \cdots & < & \nu_m & < & \cdots & < & \nu_{2\kappa-1} & < & \nu_{2\kappa} \\ \wedge & & \wedge & & & & \wedge & & & & & \wedge & & \wedge \\ \varepsilon_1 & < & \varepsilon_2 & < & \cdots & < & \varepsilon_m & < & \cdots & < & \varepsilon_{2\kappa-1} & < & \varepsilon_{2\kappa} \end{array}$$

and

$$(5.5.5) \quad q_{2\kappa}^{\varepsilon_{2\kappa}} \theta_{2\kappa} \in R_{2\kappa-1} \quad \text{and} \quad \nu_{2\kappa} > N_{2\kappa} := \max\{e_{hi} + \nu_{2\kappa-1}, f_{hj}, f_{h(2h-2)}\}$$

for $i, j \leq 2h - 3$ and $h \leq \kappa + 1$ (cf. (4.5.2), (4.14.3), (4.16.2), (5.4.2), (5.5.2), (5.5.3)). Finally, we give enumerations on

$$\mathcal{X}_{2\kappa}^+ := \text{the set of height-one prime ideals of } \tilde{R}_{2\kappa-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_{2\kappa}, \tilde{\mathfrak{q}}_{2\kappa-1}^+),$$

$$\mathcal{X}_{2\kappa}^- := \text{the set of height-one prime ideals of } \tilde{R}_{2\kappa-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_{2\kappa}, \tilde{\mathfrak{q}}_{2\kappa-1}^-).$$

Namely,

$$\mathcal{X}_{2\kappa}^+ := \{\tilde{\mathfrak{p}}_{(2\kappa)h}^+ := \tilde{\mathfrak{P}}_{(2\kappa)h}^+/(p_{2\kappa}, \tilde{\mathfrak{q}}_{2\kappa-1}^+)\} \text{ such that } \tilde{\mathfrak{P}}_{(2\kappa)h}^+ \subset \tilde{R}_{2\kappa-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2},$$

$$\mathcal{X}_{2\kappa}^- := \{\tilde{\mathfrak{p}}_{(2\kappa)h}^- := \tilde{\mathfrak{P}}_{(2\kappa)h}^-/(\pi_{2\kappa}, \tilde{\mathfrak{q}}_{2\kappa-1}^-)\} \text{ such that } \tilde{\mathfrak{P}}_{(2\kappa)h}^- \subset \tilde{R}_{2\kappa-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}.$$

5.6. Choice of $\pi_{2\kappa+1}$ and $p_{2\kappa+1}$

When $\tilde{\mathfrak{p}}_{\ell h}^+ := \tilde{\mathfrak{P}}_{\ell h}^+ / (p_\ell, \tilde{\mathfrak{q}}_{\ell-1}^+) \subset \tilde{R}_{\ell-1}(\mathbf{a}, \mathbf{b}, \mathbf{c}) / (p_\ell, \tilde{\mathfrak{q}}_{\ell-1}^+)$ is the κ th member in $\mathcal{X}_{(2\kappa)}^+ = \bigcup_{i=1}^{2\kappa} \mathcal{X}_i^+$, take $\pi_{2\kappa+1}$ as a prime element of $\mathcal{P} \cap R_{2\kappa-1}$ such that

- $\pi_{2\kappa+1} \in \tilde{\mathfrak{P}}_{\ell h}^+$ whose image is a prime element in $\tilde{R}_{2\kappa} / \tilde{\mathfrak{q}}_{2\kappa}^-$,
- $\pi_{2\kappa+1} \notin \bigcup \text{Ass}(\tilde{R}_{2\kappa} / (q_{2\kappa}, \tilde{\mathfrak{q}}_{2\kappa}))$,
- $\pi_{2\kappa+1} \tilde{R}_{2\kappa} + \tilde{\mathfrak{q}}_{2\kappa}^-$ is a prime ideal different from $p_i \tilde{R}_{2\kappa} + \tilde{\mathfrak{q}}_{2\kappa}^+$, and $\pi_j \tilde{R}_{2\kappa} + \tilde{\mathfrak{q}}_{2\kappa}^-$ for $i, j \leq 2\kappa$.

Next when $\tilde{\mathfrak{p}}_{\ell h}^- := \tilde{\mathfrak{P}}_{\ell h}^- / (\pi_\ell, \tilde{\mathfrak{q}}_{\ell-1}^-) \subset \tilde{R}_{\ell-1}(\mathbf{a}, \mathbf{b}, \mathbf{c}) / (\pi_\ell, \tilde{\mathfrak{q}}_{\ell-1}^-)$ is the κ th member in $\mathcal{X}_{(2\kappa)}^- = \bigcup_{j=1}^{2\kappa} \mathcal{X}_j^-$, choose $p_{2\kappa+1} \in \mathcal{P} \cap R_{2\kappa-1}$ that satisfies the following:

- $p_{2\kappa+1} \in \tilde{\mathfrak{P}}_{\ell h}^-$ whose image is a prime element in $\tilde{R}_{2\kappa} / \tilde{\mathfrak{q}}_{2\kappa}^+$,
- $p_{2\kappa+1} \notin \bigcup \text{Ass}(\tilde{R}_{2\kappa} / (\theta_{2\kappa+1}, \tilde{\mathfrak{q}}_{2\kappa}))$,
- $p_{2\kappa+1} \tilde{R}_{2\kappa} + \tilde{\mathfrak{q}}_{2\kappa}^+$ is a prime ideal different from $p_i \tilde{R}_{2\kappa} + \tilde{\mathfrak{q}}_{2\kappa}^+$, and $\pi_j \tilde{R}_{2\kappa} + \tilde{\mathfrak{q}}_{2\kappa}^-$ for $i+1, j \leq 2\kappa+1$.

Hence,

$$(5.6.1) \quad \pi_j \in R_{j-1} [q_{j-1} \alpha_{(j-1)(j-1)}^+] \quad \text{and} \quad p_i \in \mathcal{P} \cap R_{i-2} \quad \text{for } i, j \leq 2\kappa+1.$$

Further, for $m \leq 2\kappa+1$

- q_m, θ_m form an $(\tilde{R}_{m-1} / \tilde{\mathfrak{q}}_{m-1})$ -sequence,
- q_m, θ_m form an $(\tilde{R}_m / \tilde{\mathfrak{q}}_m)$ -sequence,
- $p_1 \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^+, \dots, p_m \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^+$ and $\pi_1 \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^-, \dots, \pi_m \tilde{R}_{m-1} + \tilde{\mathfrak{q}}_{m-1}^-$ are mutually distinct prime ideals of \tilde{R}_{m-1} .

Now, by the same arguments as in (5.5.2), (5.5.3), and (5.5.4), we get for $m \leq 2\kappa+1$

$$(5.6.2) \quad q_m, z_{1mm}, \dots, z_{nmm} \text{ form an } (R_m / \mathfrak{q}_m^\Delta)\text{-sequence,}$$

and if the ε_m 's are large enough, then

$$(5.6.3) \quad s_{mm} z_{1mm}, \dots, z_{nmm} \text{ form an } (R_m / \mathfrak{q}_m^\Delta)\text{-sequence,}$$

$$(5.6.4) \quad q_m, z_{1mm}, \dots, z_{nmm} \text{ form an } (R_m / \mathfrak{q}_m^\nabla)\text{-sequence.}$$

Moreover, choose sufficiently large $\mu_{2\kappa+1}, \nu_{2\kappa+1}, \varepsilon_{2\kappa+1}$ such that

$$\begin{array}{cccccc} \mu_1 < \mu_2 < \cdots < \mu_m < \cdots < \mu_{2\kappa} < \mu_{2\kappa+1} \\ \wedge & \wedge & & \wedge & & \wedge \\ \nu_1 < \nu_2 < \cdots < \nu_m < \cdots < \nu_{2\kappa} < \nu_{2\kappa+1} \\ \wedge & \wedge & & \wedge & & \wedge \\ \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m < \cdots < \varepsilon_{2\kappa} < \varepsilon_{2\kappa+1} \end{array}$$

and $q_{2\kappa+1}^{\varepsilon_{2\kappa+1}} \theta_{2\kappa+1} \in R_{2\kappa}$ (cf. (4.5.2), (4.14.3), (4.16.2), (5.4.2), (5.5.5), (5.6.2), (5.6.3)).

Finally, we give enumerations on

$\mathcal{X}_{2\kappa+1}^+ :=$ the set of height-one prime ideals of $\tilde{R}_{2\kappa}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_{2\kappa+1}, \tilde{\mathfrak{q}}_{2\kappa}^+)$,

$\mathcal{X}_{2\kappa+1}^- :=$ the set of height-one prime ideals of $\tilde{R}_{2\kappa}(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_{2\kappa+1}, \tilde{\mathfrak{q}}_{2\kappa}^-)$.

Namely,

$\mathcal{X}_{2\kappa+1}^+ := \{\tilde{\mathfrak{p}}_{(2\kappa+1)h}^+ := \tilde{\mathfrak{P}}_{(2\kappa+1)h}^+/(p_{2\kappa+1}, \tilde{\mathfrak{q}}_{2\kappa}^+)\}$ with $\tilde{\mathfrak{P}}_{(2\kappa+1)h}^+ \subset \tilde{R}_{2\kappa}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}$,

$\mathcal{X}_{2\kappa+1}^- := \{\tilde{\mathfrak{p}}_{(2\kappa+1)h}^- := \tilde{\mathfrak{P}}_{(2\kappa+1)h}^-/(\pi_{2\kappa+1}, \tilde{\mathfrak{q}}_{2\kappa}^-)\}$ with $\tilde{\mathfrak{P}}_{(2\kappa+1)h}^- \subset \tilde{R}_{2\kappa}(\mathbf{a}, \mathbf{b}, \mathbf{c})_{\leq h-2}$.

This completes the proof of Proposition 5.2. \square

6. Construction

6.1. Notation and terminology

With the notation, assumptions, and numberings as in the previous sections, let

$$B_k := \tilde{R}_k[\underline{\alpha}_{kk}^-, \underline{\beta}_{kk}^+, \underline{\beta}_{kk}^-], \quad \mathfrak{M}_k := (z_0, \underline{z}, \underline{w}, \underline{\alpha}_{kk}^+, \underline{\alpha}_{kk}^-, \underline{\beta}_{kk}^+, \underline{\beta}_{kk}^-)B_k,$$

$$B := \bigcup B_k, \quad \text{and} \quad M := \bigcup \mathfrak{M}_k = (z_0, \underline{z}, \underline{w})B,$$

where $\underline{w} := (w_1, \dots, w_r)$, $\underline{z} := (z_1, \dots, z_n)$, $\underline{\alpha}_{kk}^+ := (\alpha_{1kk}^+, \dots, \alpha_{rkk}^+)$, $\underline{\alpha}_{kk}^- := (\alpha_{1kk}^-, \dots, \alpha_{rkk}^-)$, $\underline{\beta}_{kk}^+ := (\beta_{1kk}^+, \dots, \beta_{skk}^+)$, $\underline{\beta}_{kk}^- := (\beta_{1kk}^-, \dots, \beta_{skk}^-)$, and $k = 1, 2, \dots$. Then we have the following result.

LEMMA 6.2

With the notation above, M is a maximal ideal of B . And we have the canonical isomorphism $R/\mathfrak{n} \xrightarrow{\cong} B/M$.

Proof

By definitions and by relations among α_{jkk}^+ , α_{jkk}^- , β_{hkk}^+ , and β_{hkk}^- , it is clear that all these elements are in M (cf. (2.4.2), (3.2.1), (3.2.2)). To get the assertion, it remains to verify that $M \neq B$.

Assume the contrary. That is, $M = B$. Then there exist $\gamma_0, \gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_r \in B$ that satisfy

$$\gamma_0 \cdot z_0 + \gamma_1 \cdot z_1 + \dots + \gamma_n \cdot z_n + \delta_1 \cdot w_1 + \dots + \delta_r \cdot w_r = 1.$$

We may assume that $\gamma_i, \delta_j \in B_k$ for sufficiently large k . Thus there exist $c_i, d_j \in \tilde{R}_k$ and $\nu \in \mathbf{N}$ such that

$$q_k^\nu \cdot \theta_k^\nu(\gamma_i - c_i), q_k^\nu \cdot \theta_k^\nu(\delta_j - d_j) \in \tilde{\mathfrak{q}}_k \quad (\text{cf. (2.3.10), (3.2.1), (3.2.2)}).$$

Hence,

$$q_k^\nu \cdot \theta_k^\nu(c_0 \cdot z_0 + c_1 \cdot z_1 + \dots + c_n \cdot z_n + d_1 \cdot w_1 + \dots + d_r \cdot w_r - 1) \in \tilde{\mathfrak{q}}_k$$

and therefore $q_k^\nu \cdot \theta_k^\nu \in \tilde{\mathfrak{q}}_k$. This contradicts the choice of p_ℓ and π_m (cf. [8, Lemma 1.4], (5.2.2)). \square

LEMMA 6.3

With the notation above, p_i and π_j are prime elements of B that satisfy the following:

$$(6.3.1) \quad \bigcup_k (\tilde{R}_i(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_i, \tilde{\mathfrak{q}}_i^+))[\underline{\alpha_{kk}^-}] \xrightarrow{\cong} B/p_i B,$$

$$(6.3.2) \quad \bigcup_k ((\tilde{R}_j(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_j, \tilde{\mathfrak{q}}_j^-))[\underline{\alpha_{kk}^+}])_{U_{jk}} \xrightarrow{\cong} B/\pi_j B,$$

where $\underline{\alpha_{kk}^-}$ is the image of α_{kk}^- in the field of quotients of $\tilde{R}_i(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_i, \tilde{\mathfrak{q}}_i^+)$ and $\underline{\alpha_{kk}^+}$ is the image of α_{kk}^+ in the field of quotients of $\tilde{R}_j(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_j, \tilde{\mathfrak{q}}_j^-)$ with $U_{jk} := 1 + (\tilde{\mathfrak{n}}_j, \underline{\alpha_{kk}^+})$, respectively.

Proof

First of all, putting $\tilde{R} := \bigcup_k \tilde{R}_k$, we note that $\tilde{R}_k(\mathbf{a}, \mathbf{b}, \mathbf{c})/(p_i, \tilde{\mathfrak{q}}_k^+) \xrightarrow{\cong} \tilde{R}/p_i \tilde{R}$ for every $k \geq i$. In fact, this results from the following commutative diagram:

$$\begin{array}{ccccc} \tilde{R}_k/(p_i, \tilde{\mathfrak{q}}_k^+) & \longrightarrow & \tilde{R}_{k+1}/p_i \tilde{R}_{k+1} & & \\ & \searrow & \downarrow & \searrow & \\ & & \tilde{R}_{k+1}/(p_i, \tilde{\mathfrak{q}}_{k+1}^+) & \longrightarrow & \tilde{R}_{k+2}/p_i \tilde{R}_{k+2} \end{array}$$

where the left diagonal map is injective (cf. (4.5.2)). This also shows that $p_i \tilde{R}_{k+1} \cap \tilde{R}_k = (p_i, \tilde{\mathfrak{q}}_k^+) \tilde{R}_k$ for every $k \geq i$. \square

Proof of (6.3.1)

By the remark just above, we get the following commutative diagram, because $\tilde{R}_k[1/\theta_{k+1}] = B_k[1/\theta_{k+1}]$ (cf. (3.2.1), (3.2.2), (5.2.8)):

$$\begin{array}{ccc} (\tilde{R}_k/(p_i, \tilde{\mathfrak{q}}_k^+))[\underline{\alpha_{kk}^-}] & \longrightarrow & B_k/(p_i B_{k+1} \cap B_k) \\ \downarrow & & \downarrow \\ (\tilde{R}_{k+1}/(p_i, \tilde{\mathfrak{q}}_{k+1}^+))[\underline{\alpha_{(k+1)(k+1)}^-}] & \longrightarrow & (\tilde{R}_{k+1}/(p_i, \tilde{\mathfrak{q}}_{k+1}^+))\left[\frac{1}{\theta_{k+2}}\right] \end{array}$$

where the upper horizontal map is surjective (cf. (2.4.2), (3.3.1), (3.3.2)) and the left vertical map and the lower horizontal map are injective (cf. (5.2.3), (5.2.1)). \square

Proof of (6.3.2)

By construction, we get the following commutative diagram, because $\tilde{R}_k[\underline{\alpha_{kk}^-}][1/q_k] = B_k[1/q_k]$ (cf. (3.2.1), (3.2.2), (5.2.8)):

$$\begin{array}{ccc}
\tilde{R}_k/(\pi_j, \tilde{q}_k^-) & \longrightarrow & B_k/(\pi_j B_{k+1} \cap B_k) \\
\downarrow & & \downarrow \\
\tilde{R}_{k+1}/(\pi_j, \tilde{q}_{k+1}^-) & \longrightarrow & (\tilde{R}_{k+1}/(\pi_j, \tilde{q}_{k+1}^-)) \left[\frac{1}{q_{k+1}} \right]
\end{array}$$

where the upper horizontal map is surjective (cf. (2.4.2), (3.3.1), (3.3.2)) and the left vertical map and the lower horizontal map are injective (cf. (4.14.3), (5.2.3), (5.2.1)). \square

LEMMA 6.4

With the notation above, let $A = B_M$, and let $\mathfrak{m} = MA$. Then (A, \mathfrak{m}) is a Noetherian unique factorization domain.

Proof

Let \mathfrak{p} be a non-zero prime ideal of A . Because A and R have the common field of quotients, $\mathfrak{p} \cap R$ contains a prime element of R , say, ρ . We may assume that $\rho = \rho_{k+1} \in \mathcal{P}$. Then

$$\begin{aligned}
\rho &= p_1^{e_1} \cdots p_{2k-1}^{e_{2k-1}} \cdot \pi_1^{f_1} \cdots \pi_{2k}^{f_{2k}} \\
&\times \left[b + \frac{\theta_{2k}^{\varepsilon_{2k}}}{\pi_1^{f_1} \cdots \pi_{2k}^{f_{2k}}} \kappa_{2k-1}^- + \frac{q_{2k}^{\nu_{2k}}}{p_1^{e_1} \cdots p_{2k-1}^{e_{2k-1}} \cdot q_{2k-1}^{\nu_{2k-1}}} \sum d_j \alpha_{j2k}^+ \right. \\
&\quad \left. + \frac{q_{2k}^{\varepsilon_{2k}}}{p_1^{e_1} \cdots p_{2k-1}^{e_{2k-1}} \cdot q_{2k-1}^{\nu_{2k-1}}} \kappa_{2k-1}^+ + \frac{q_{2k}^{\nu_{2k}} \theta_{2k}^{\varepsilon_{2k}}}{p_1^{e_1} \cdots p_{2k-1}^{e_{2k-1}} \pi_1^{f_1} \cdots \pi_{2k}^{f_{2k}} \cdot q_{2k-1}^{\nu_{2k-1}}} \lambda_{2k-1}^+ \right] \\
&+ p_1^{e_1} \cdots p_{2k-1}^{e_{2k-1}} \sum c_j (\phi_{j2k}^- - w_{j2k}) \\
&+ \frac{q_{2k}^{\nu_{2k}}}{q_{2k-1}^{\nu_{2k-1}}} \sum e_j (\phi_{j2k}^- - w_{j2k}) + \left(\frac{q_{2k}^{\nu_{2k}}}{q_{2k-1}^{\nu_{2k-1}}} \right)^{g_h} \sum e'_h G_h(\zeta_{2k2k}, \alpha_{2k2k}^+),
\end{aligned}$$

where b is a unit element in \tilde{R}_{2k-1} (cf. (4.16.5), Lemma 4.17, (5.2.7)). Hence,

$$\rho = p_1^{e_1} \cdots p_{2k-1}^{e_{2k-1}} \pi_1^{f_1} \cdots \pi_{2k}^{f_{2k}} \left(b + \theta_{2k} \varepsilon' + q_{2k} \delta' + \sum c'_j \alpha_{j2k}^- + \sum d'_h \beta_{h2k}^+ \right),$$

where ε' , δ' , c'_j , and $d'_h \in \tilde{R}_{2k-1}(\underline{a_{2k}}, \underline{b_{2k}}, \underline{c_{2k}})$. Thus,

$$\rho A = p_1^{e_1} \cdots p_{2k-1}^{e_{2k-1}} \pi_1^{f_1} \cdots \pi_{2k}^{f_{2k}} A.$$

That is, \mathfrak{p} contains either p_i for some $i \leq 2k-1$ or π_j for some $j \leq 2k$. Because both $A/p_i A$ and $A/\pi_j A$ are Noetherian domains (cf. (6.3.1), (6.3.2), (5.2.9)), \mathfrak{p} is finitely generated. Therefore, A is Noetherian. Moreover, the argument above shows that if, in addition, \mathfrak{p} is a height-one prime ideal, then \mathfrak{p} is principal (cf. [8, Proposition 5.6]). \square

6.5. Notation

With the same notation as in previous sections, let $\widehat{R} := K[[z_0, z_1, \dots, z_n, w_1, \dots, w_r]]$. Take elements of \widehat{R} :

$$\begin{aligned}\xi_i &:= x_i + a_{i1}q_1^{\varepsilon_1} + \cdots + a_{ik}q_k^{\varepsilon_k} + \cdots = x_i + \sum_{k=1}^{\infty} a_{ik}q_k^{\varepsilon_k}, \\ \omega_j &:= w_j + c_{j1}t_1^{\varepsilon_1} + \cdots + c_{jk}t_k^{\varepsilon_k} + \cdots = w_j + \sum_{k=1}^{\infty} c_{jk}t_k^{\varepsilon_k}.\end{aligned}$$

Let $\varphi_j^+ := \lim_{k \rightarrow \infty} \phi_{jkk}^+$, $\psi_j^+ := \varphi_j^+ + \omega_j$, and let

$$\tilde{\alpha}_{jkk}^+ := \alpha_{jkk}^+ - \frac{\psi_j^+}{q_k^{\nu_k}} = \frac{1}{q_k^{\nu_k}} \left(\phi_{jkk}^+ - \sum_{h>k}^{\infty} c_{jh}t_h^{\varepsilon_h} - \varphi_j^+ \right).$$

By letting $\tilde{\theta}_k := \tilde{\pi}_1 \cdots \tilde{\pi}_k$ with

$$\tilde{\pi}_k := r_k + \sum r_{kH} (q_{k-1} \tilde{\alpha}_{(k-1)(k-1)}^+)^H \in \widehat{R} \quad (\text{cf. (2.3.12)}),$$

we put

$$\begin{aligned}\tilde{\eta}_{im} &:= y_i + b_{i1}\tilde{\theta}_1^{\varepsilon_1} + \cdots + b_{im}\tilde{\theta}_m^{\varepsilon_m}, \\ \tilde{\eta}_i &:= y_i + \sum_{k=1}^{\infty} b_{ik}\tilde{\theta}_k^{\varepsilon_k}, \\ \tilde{\zeta}_{i\ell m} &:= x_{i\ell} + \tilde{\eta}_{im},\end{aligned}$$

and

$$\tilde{\zeta}_i := \xi_i + \tilde{\eta}_i.$$

Further, let

$$\begin{aligned}\tilde{\varphi}_{jkk} &:= F_j(\tilde{\zeta}_{kk}) \\ &= F_j^+(\tilde{\zeta}_{kk}) + F_j^-(\tilde{\zeta}_{kk}) =: \tilde{f}_{jkk}^+ + \tilde{f}_{jkk}^-, \\ \tilde{\varphi}_j &:= F_j(\tilde{\zeta}) \\ &= \lim_{k \rightarrow \infty} \tilde{f}_{jkk}^+ + \lim_{k \rightarrow \infty} \tilde{f}_{jkk}^- =: \tilde{\varphi}_j^+ + \tilde{\varphi}_j^-\end{aligned}$$

and

$$\tilde{\psi}_j^- := \tilde{\varphi}_j^- - \omega_j.$$

LEMMA 6.6

With the notation above, we get

$$K[[z_0, \tilde{\zeta}_1, \dots, \tilde{\zeta}_n]]/(\tilde{\varphi}_1, \dots, \tilde{\varphi}_r) \cong \widehat{R}/(\psi_1^+, \dots, \psi_r^+, \tilde{\psi}_1^-, \dots, \tilde{\psi}_r^-) \xrightarrow{\cong} \widehat{A}.$$

Proof

By Lemma 6.2, we have a canonical surjection $\hat{\iota}: \widehat{R} \rightarrow \widehat{A}$ that gives the following commutative diagram

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\hat{\iota}} & \widehat{A} \\ \rho_R \uparrow & & \uparrow \rho_A \\ R & \xrightarrow{\iota} & A \end{array}$$

with canonical injections ι , ρ_R , and ρ_A . Thus, we are only to determine $\text{Ker } \hat{\iota}$. By definition, $\psi_j^+ - (\phi_{jkk}^+ + w_{jk}) \in q_{k+1}^{\nu_{k+1}} \widehat{R}$ and $\phi_{jkk}^+ + w_{jk} = q_k^{\nu_k} \alpha_{jkk}^+ \in q_k^{\nu_k} A$. Hence, $\hat{\iota}(\psi_j^+) \in q_k^{\nu_k} \widehat{A}$ for every $k \in \mathbf{N}$. Namely, $\psi_j^+ \in \text{Ker } \hat{\iota}$.

On the other hand, $\alpha_{jkk}^+ = \tilde{\alpha}_{jkk}^+$ in \widehat{A} , because $q_k^{\nu_k} (\alpha_{jkk}^+ - \tilde{\alpha}_{jkk}^+) = \psi_j^+ \in \widehat{R}$ and because q_k is a non-zero divisor for \widehat{A} . Thus, $\zeta_i = \tilde{\zeta}_i$, $\phi_{jkk}^- = \tilde{f}_{jkk}^-$, $\tilde{\varphi}_j^- = \lim_{k \rightarrow \infty} \phi_{jkk}^-$, and $\tilde{\varphi}_j^- = F_j(\zeta)$ in \widehat{A} . Moreover, $\tilde{\psi}_j^- - (\tilde{f}_{jkk}^- - w_{jk}) \in (q_{k+1}^{\nu_{k+1}}, \tilde{\pi}_{k+1}^{\nu_{k+1}}) \widehat{R}$ and $\hat{\iota}(\tilde{f}_{jkk}^- - w_{jk}) = \phi_{jkk}^- - w_{jk} = \theta_k^{\nu_k} \alpha_{jkk}^- \in \theta_k^{\nu_k} \widehat{A}$. Hence, $\hat{\iota}(\tilde{\psi}_j^-) \in (q_{k+1}^{\nu_{k+1}}, \pi_k^{\nu_k}) \widehat{A}$ for every $k \in \mathbf{N}$. That is, $\tilde{\psi}_j^- \in \text{Ker } \hat{\iota}$.

Finally, $\widehat{A}/\pi_1 \widehat{A} \cong \widehat{R}/(\pi_1, \mathfrak{q}_0^-, \psi^+)$ and $(\pi_1, \mathfrak{q}_0^-, \psi^+) \widehat{R} = (\pi_1, \tilde{\psi}^-, \psi^+) \widehat{R}$, because

$$A/\pi_1 A \cong \bigcup_k ((\tilde{R}_1(\mathbf{a}, \mathbf{b}, \mathbf{c})/(\pi_1, \tilde{\mathfrak{q}}_1^-))[\underline{\tilde{\alpha}}_{kk}^+])_{U_{1k}} \cong \left(\bigcup_k (R/(\pi_1, \mathfrak{q}_0^-))[\underline{\tilde{\alpha}}_{kk}^+] \right)_U$$

with $U = 1 + (\mathbf{n}, \underline{\tilde{\alpha}}_{kk}^+)$ (cf. (6.3.2), (4.8.1)). Therefore, $\text{Ker } \hat{\iota} = (\psi_1^+, \dots, \psi_r^+, \tilde{\psi}_1^-, \dots, \tilde{\psi}_r^-)$. This completes the proof of Theorem 0.1. \square

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