Deforming discontinuous subgroups of reduced Heisenberg groups

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Abstract Let $G = \mathbb{H}_{2n+1}^r$ be the $(2n+1)$-dimensional reduced Heisenberg group, and let H be an arbitrary connected Lie subgroup of G. Given any discontinuous subgroup $\Gamma \subset G$ for G/H, we show that resulting deformation space $\mathscr{T}(\Gamma, G, H)$ of the natural action of Γ on G/H is endowed with a smooth manifold structure and is a disjoint union of open smooth manifolds. Unlike the setting of simply connected Heisenberg groups, we show that the stability property holds and that any discrete subgroup of G is stable, following the notion of stability. On the other hand, a local (and hence global) rigidity theorem is obtained. That is, the related parameter space $\mathscr{R}(\Gamma, G, H)$ admits a rigid point if and only if Γ is finite.

1. Introduction

This paper is a continuation of the papers [2], [4], and [5] where the concern was to study the deformation space of a discontinuous group acting on a homogeneous space G/H for a connected subgroup H of the connected and simply connected Heisenberg group $G = \mathbb{H}_{2n+1}$. In the present study, the point is to remove the assumption on G that it is simply connected. The attention here is therefore focused on the reduced Heisenberg group $\mathbb{H}_{2n+1}^r$, for which the universal covering is $\mathbb{H}_{2n+1}$.

The problem of deformation consists in seeking how to deform Γ by means of homomorphisms from Γ to G (thus to consider the set $\mathrm{Hom}(\Gamma, G)$ of all these homomorphisms) in a way such that the deformed discrete subgroup acts properly on G/H. The problem of describing deformations was first advocated by T. Kobayashi in [15] for the general non-Riemannian setting and precisely determines the set of deformation parameters that allow Γ to deform in a way to guarantee the proper discontinuity on G/H. The parameter space

$$\mathscr{R}(\Gamma, G, H) := \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi \text{ is injective, } \varphi(\Gamma) \text{ discrete and acts properly and fixed point freely on } G/H\} \tag{1.1}$$

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(endowed with the pointwise convergence topology), rather than $\text{Hom}(\Gamma, G)$, plays a crucial role in these problems. In order to be precise on parameters, our main goal is to investigate the deformation space $\mathscr{T}(\Gamma, G, H)$ which is the quotient space of the parameter space given above through the equivalence relation arising inner automorphisms.

Unlike the context of simply connected Heisenberg groups (see [2]), we show in this paper that the deformation space $\mathscr{T}(\Gamma, G, H)$ is a Hausdorff space and is even endowed with a smooth manifold structure for any arbitrary connected subgroup H of G and any arbitrary discontinuous subgroup Γ for G/H. Indeed, we will provide a disjoint decomposition of $\mathscr{T}(\Gamma, G, H)$ into open smooth manifolds of a common dimension. On the other hand, we show that the stability property holds for any deformation parameter, which means that in some small neighborhood V_φ of any element φ of the parameter space, the proper action of the discrete subgroup $\psi(\Gamma)$, $\psi \in V_\varphi$, on G/H is preserved.

Concerning the rigidity, we will show that the related parameter space $\mathscr{R}(\Gamma, G, H)$ admits a locally rigid deformation if and only if Γ is finite and that the local rigidity is indeed equivalent to the rigidity. This naturally leads us to ask the following question, which comes out from a question posed in [1].

QUESTION 1.1

Let G be a connected nilpotent Lie group, let H be a connected subgroup of G, and let Γ be a nontrivial discontinuous subgroup for G/H. Then, the local rigidity holds if and only if Γ is a finite group.

Let ε_Γ be the integer given by $\varepsilon_\Gamma = 0$ if Γ is torsion-free, and let $\varepsilon_\Gamma = 1$ otherwise. Let also r_Γ be the rank of Γ, which is the cardinality of a minimal generating set. The nonnegative integer $l_\Gamma := r_\Gamma - \varepsilon_\Gamma$ is called the length of the subgroup Γ.

One of the objectives of this paper is to give an affirmative answer to Question 1.1 in our setting. More precisely, the following main result will be proved.

THEOREM 1.1

Let $G := \mathbb{H}^r_{2n+1}$ be the reduced Heisenberg Lie group, and let Γ be a discontinuous subgroup of length l_Γ for the homogeneous space G/H where H is a connected closed subgroup of G. Then we have the following.

(1) *The stability property holds. That is, any discrete subgroup of G is stable.*

(2) *The parameter space $\mathscr{R}(\Gamma, G, H)$ and the deformation space $\mathscr{T}(\Gamma, G, H)$ are endowed with smooth manifold structures of dimensions $(2n+1)l_\Gamma - \frac{1}{2}l_\Gamma(l_\Gamma - 1)$ and $2nl_\Gamma - \frac{1}{2}l_\Gamma(l_\Gamma - 1)$, respectively.*

(3) *The G-orbits of $\mathscr{R}(\Gamma, G, H)$ have a common dimension equal to l_Γ.*

(4) *The parameter space $\mathscr{R}(\Gamma, G, H)$ admits a locally rigid point if and only if Γ is a finite group.*

The outline of the paper is as follows. Section 2 is devoted to fixing some notation and defining the necessary ingredients. In Section 3, we prove that any closed subgroup of a connected completely solvable Lie group G admits a unique syndetic hull, which contains the maximal compact subgroup of G (cf. Theorem 3.9). Section 4 aims to study the structure of the set $\mathrm{Hom}(\Gamma, G)$ of homomorphisms of Γ in G (see Proposition 4.4) and to prove that the set $\mathrm{Hom}_0^d(\Gamma, G)$ of injective homomorphisms with discrete images is an open set of $\mathrm{Hom}(\Gamma, G)$ (cf. Proposition 4.6 and Corollary 4.8). Section 5 is devoted to characterization of the proper action of a connected subgroup Γ acting on a arbitrary homogeneous space G/H (see Proposition 5.2), to recalling the concept of stable subgroups of nilpotent Lie groups, and to proving our main upshot (cf. Theorem 1.1).

2. Backgrounds

We begin this section with fixing some notation and terminology and recording some basic facts about deformations. The readers could consult the references [3], [11]–[13], [15], [16], [18], and some references therein for broader information about the subject. Concerning the entire subject, we strongly recommend the papers [11] and [16].

2.1. Proper and fixed point actions

Let $\mathscr{M}$ be a locally compact space, and let L be a locally compact topological group. The continuous action of the group L on $\mathscr{M}$ is said to be

(1) *proper* if for each compact subset $S \subset \mathscr{M}$, the set $L_S = \{k \in L : k \cdot S \cap S \neq \emptyset\}$ is compact;

(2) *fixed point free* (or *free*) if for each $m \in \mathscr{M}$, the isotropy group $L_m = \{k \in L : k \cdot m = m\}$ is trivial;

(3) *properly discontinuous* if L is discrete and the action of L on $\mathscr{M}$ is proper and free. In the case where $\mathscr{M} = G/H$ is a homogeneous space and L is a subgroup of G, then the action of L on $\mathscr{M}$ is proper if $SHS^{-1} \cap L$ is compact for any compact set S in G. Likewise the action of L on $\mathscr{M}$ is free if for every $g \in G$, $L \cap gHg^{-1} = \{e\}$. In this context, the subgroup L is said to be a discontinuous group for the homogeneous space $\mathscr{M}$ if L is a discrete subgroup of G and L acts properly and freely on $\mathscr{M}$.

The action of K on $\mathscr{M}$ (or the triple (G, H, K)) is of compact intersection property, denoted by (CI) (introduced in [11]) if for each $m \in \mathscr{M}$, the isotropy group K_m is compact.

As a first example, let M_g be a Riemann surface of genus $g \geq 2$. Let $G = \mathrm{PSL}_2(\mathbb{R})$ and $H = \mathrm{SO}_2$. The fundamental group $\Gamma = \pi_1(M_g)$ of M_g, regarded as a discrete subgroup of G, is a discontinuous group for G/H, and we have $M_g = \Gamma \backslash G/H$.

On the other hand, let M be a smooth manifold with a local structure S (complex structure, affine structure, Lorentz structure, symplectic structure, pseudo-Riemannian structure, etc.). Let $\widetilde{M}$ be the universal covering of M, and let

$$G = \left\{\varphi \in \mathrm{Diff}(\widetilde{M}), \varphi \text{ preserves the structure } S\right\}.$$

If G is a Lie group acting transitively on $\widetilde{M}$ and Γ is the fundamental group of M, then $\widetilde{M} = G/H$, where H is the isotropy group of a point and $M = \Gamma\backslash\widetilde{M}$. Here, Γ is a discontinuous group for G/H.

2.2. Clifford–Klein forms

Let Γ be a discontinuous subgroup for the homogeneous space G/H. The quotient space $\Gamma\backslash G/H$ is said to be a *Clifford–Klein form* for the homogeneous space G/H. The following point was emphasized in [14]. Any Clifford–Klein form is endowed with a smooth manifold structure for which the quotient canonical surjection $\pi : G/H \to \Gamma\backslash G/H$ turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford–Klein form $\Gamma\backslash G/H$ inherits any G-invariant geometric structure (e.g., complex structure, pseudo-Riemanian structure, conformal structure, symplectic structure, etc.) from the homogeneous space G/H through the covering map π.

2.3. Parameter and deformation spaces

The material dealt with in this subsection is taken from the pioneering paper [16] of T. Kobayashi. The reader could also consult the references [12] and [15] for precise definitions. Suppose that Γ is a finitely generated subgroup of G. As in the first introductory section, we designate by $\mathrm{Hom}(\Gamma, G)$ the set of group homomorphisms from Γ to G endowed with the pointwise convergence topology. The same topology is obtained by taking generators $\gamma_1, \dots, \gamma_k$ of Γ, then, using the injective map

$$\mathrm{Hom}(\Gamma, G) \to G \times \cdots \times G, \qquad \varphi \mapsto \big(\varphi(\gamma_1), \dots, \varphi(\gamma_k)\big)$$

to equip $\mathrm{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \cdots \times G$. The parameter space $\mathscr{R}(\Gamma, G, H)$ defined as in (1.1), which is introduced by T. Kobayashi [15] for general settings, stands for an interesting object when the *rigidity* fails. Such a space plays a crucial role as we will see later. For each $\varphi \in \mathscr{R}(\Gamma, G, H)$, the space $\varphi(\Gamma)\backslash G/H$ is a Clifford–Klein form which is a Hausdorff topological space and even equipped with a structure of a smooth manifold for which the quotient canonical map is an open covering. Let now $\varphi \in \mathscr{R}(\Gamma, G, H)$ and $g \in G$; we consider the element φ^g of $\mathrm{Hom}(\Gamma, G)$ defined by $\varphi^g(\gamma) = g\varphi(\gamma)g^{-1}$, $\gamma \in \Gamma$. It is clear that the element $\varphi^g \in \mathscr{R}(\Gamma, G, H)$ and that the map

$$\varphi(\Gamma)\backslash G/H \to \varphi^g(\Gamma)\backslash G/H, \qquad \varphi(\Gamma)xH \mapsto \varphi^g(\Gamma)gxH$$

is a natural diffeomorphism. T. Kobayashi [16] introduced the orbit space

$$\mathscr{T}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, H)/G$$

instead of $\mathscr{R}(\Gamma, G, H)$ to avoid the unessential part of deformations from inner automorphisms. We call the set $\mathscr{T}(\Gamma, G, H)$ the deformation space of the discontinuous group Γ for the homogeneous space G/H.

2.4. The concept of stability in the sense of Kobayashi and Nasrin

Let us come back to the general setting for a while. The homomorphism $\varphi \in \mathscr{R}(\Gamma, G, H)$ is said to be *topologically stable* or merely *stable* in the sense of Kobayashi and Nasrin [17] if there is an open set in $\mathrm{Hom}(\Gamma, G)$ which contains φ and is contained in $\mathscr{R}(\Gamma, G, H)$. When the set $\mathscr{R}(\Gamma, G, H)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$, then, obviously each of its elements is stable, which is the case for any irreducible Riemannian symmetric space with the assumption that Γ is a torsion-free uniform lattice of G (see [17] and [21]). Furthermore, we point out in this setting that the concept of stability may be one fundamental concept to understand the local structure of the deformation space.

2.5. The concept of rigidity

We keep the same notation and assumptions. Generalizing Weil's notion of local rigidity of discontinuous groups for Riemannian symmetric spaces, T. Kobayashi introduced the notion of local rigidity and rigidity of discontinuous groups for non-Riemannian homogeneous spaces (cf. [12]). Notably, he proved in [15] that for the reductive case, the local rigidity may fail even for irreducible symmetric space of high dimensions. We briefly recall here some details. For comprehensible information, we refer the readers to references [10]–[13] and [15]–[17]. For $\varphi \in \mathscr{R}(\Gamma, G, H)$, the discontinuous subgroup $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* (resp., *rigid*; see [12]) as a discontinuous group of G/H if the orbit of φ under the inner conjugation is open in $\mathscr{R}(\Gamma, G, H)$ (resp., in $\mathrm{Hom}(\Gamma, G)$). This means equivalently that any point sufficiently close to φ should be conjugate to φ under an inner automorphism of G. So, the homomorphisms which are locally rigid are those which correspond to isolated points in the deformation space $\mathscr{T}(\Gamma, G, H)$. When every point in $\mathscr{R}(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete and the Clifford–Klein form $\Gamma\backslash G/H$ does not admit *continuous deformations*. If a given $\varphi \in \mathscr{R}(\Gamma, G, H)$ is not locally rigid, it admits continuous deformations and the related Clifford–Klein form is continuously deformable.

3. On connected exponential Lie groups

Let us first recall the notion of the universal covering of a Lie group. Here, we record some results which will be of interest in our study in this paper and prove the existence of a unique syndetic hull for any closed subgroup of a completely solvable Lie group. We first record the following results.

THEOREM 3.1 ([8, THEOREM XII.10])

Let G be a connected Lie group. Then there exists a connected simply connected Lie group $\widetilde{G}$ and a Lie group homomorphism $\pi : \widetilde{G} \to G$ which is a covering. The kernel of π is a discrete, normal subgroup and so is central in $\widetilde{G}$. In addition, up to isomorphism, $\widetilde{G}$ is unique. The set is called the universal covering of G.

THEOREM 3.2 ([9, PROPOSITION C.8])

Let G and H be two Lie groups associated to Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let F be a group homomorphism from G to H. Then, there exists an algebra homomorphism f from $\mathfrak{g}$ to $\mathfrak{h}$ such that $F \circ \exp_G = \exp_H \circ f$ where $\exp_G : \mathfrak{g} \to G$ and $\exp_H : \mathfrak{h} \to H$ are the exponential maps of G and H, respectively.

Throughout this section, $\mathfrak{g}$ will denote a real exponential solvable Lie algebra and $\widetilde{G}$ its simply connected associated Lie group. This means that $\mathfrak{g}$ is solvable and the exponential map $\exp_{\widetilde{G}} : \mathfrak{g} \to \widetilde{G}$ is a global C^∞-diffeomorphism from $\mathfrak{g}$ onto $\widetilde{G}$. Let $\log_{\widetilde{G}}$ designate the inverse map of $\exp_{\widetilde{G}}$. That is, $\widetilde{G}$ is connected and simply connected, and it is the universal covering of a connected Lie group G, for which the exponential mapping may fail to be injective. Besides, G and $\widetilde{G}$ have the same Lie algebra, and G will also be called a connected exponential Lie group. Furthermore, the following is true.

PROPOSITION 3.3

Let G be a connected exponential solvable Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then, the exponential map $\exp_G : \mathfrak{g} \to G$ is surjective.

Proof

Let $\widetilde{G}$ be the universal covering of G, and let $\pi : \widetilde{G} \to G$ be the associated covering as in Theorem 3.1. We have that $\exp_{\widetilde{G}} : \mathfrak{g} \to \widetilde{G}$ is a diffeomorphism. Therefore, according to Theorem 3.2, there exists a Lie algebra endomorphism f of $\mathfrak{g}$ such that $\pi \circ \exp_{\widetilde{G}} = \exp_G \circ f$. As $\pi \circ \exp_{\widetilde{G}}$ is surjective, then so is $\exp_G \circ f$. Thus, the map $\exp_G$ is surjective. □

DEFINITION 3.4

Let $\mathfrak{g}$ be a Lie algebra such that $\dim \mathfrak{g} = n$. When there exists a sequence of ideals

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}, \quad \dim \mathfrak{g}_j = j \ (0 \le j \le n),$$

we say that $\mathfrak{g}$ is completely solvable. Any completely solvable Lie algebra is an exponential solvable Lie algebra.

DEFINITION 3.5

Let G be a Lie group, and let H be a connected subgroup of G. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G and H, respectively. A basis $\{X_1, \dots, X_p\}$, $p = \dim(\mathfrak{g}/\mathfrak{h})$, is said to be coexponential to $\mathfrak{h}$ in $\mathfrak{g}$ if the map

$$\varphi_{\mathfrak{g},\mathfrak{h}} : \mathbb{R}^p \times H \to G,$$
$$\big((t_1, \dots, t_p), h\big) \mapsto \exp t_p X_p \cdots \exp t_1 X_1 \cdot h$$

is a diffeomorphism.

The following theorem will be of interest in the remainder of the paper.

THEOREM 3.6 ([19, PROPOSITION 2])

Let G be a connected, simply connected, and solvable Lie group. Then every connected closed subgroup of G admits a coexponential basis.

DEFINITION 3.7

Let G be a Lie group, let Γ be a closed subgroup of G, and let $Z^c(G)$ be the maximal compact subgroup of $Z(G)$, the center of G. A syndetic hull of Γ is any connected Lie subgroup L of G which contains $\Gamma \cdot Z^c(G)$ cocompactly. Then obviously, L contains Γ cocompactly.

Let G be a connected completely solvable Lie group. When G is simply connected, then $Z^c(G)$ is trivial and the following result is already known.

THEOREM 3.8 ([4], [20])

Let G be a connected, simply connected, completely solvable Lie group, and let Γ be a closed subgroup of G. Then, Γ admits a unique syndetic hull.

The following result generalizes Theorem 3.8 and establishes the existence of the syndetic hull of any closed subgroup of a connected completely solvable Lie group. More precisely, we have the following.

THEOREM 3.9

Any closed subgroup of a connected completely solvable Lie group admits a unique syndetic hull L, where $L = \exp_G \mathfrak{l}$ and $\mathfrak{l} = \mathbb{R}\text{-span}(\log_{\widetilde{G}} \pi^{-1}(\Gamma))$.

Proof

Let Γ be a closed subgroup of a completely solvable Lie group G. We show first that there exists a connected Lie subgroup of G which contains Γ cocompactly. Let $\widetilde{G}$ be the universal covering of G, and let π be the associative covering $\pi : \widetilde{G} \to G$. We denote $\Lambda = \ker \pi$. As π is continuous, $\widetilde{\Gamma} = \pi^{-1}(\Gamma)$ is a closed subgroup of $\widetilde{G}$. Since $\widetilde{G}$ is connected, simply connected, and completely solvable, then according to Theorem 3.8, $\widetilde{\Gamma}$ has unique syndetic hull, $\widetilde{L}$, say. Let us prove that $L = \pi(\widetilde{L})$ is a connected Lie subgroup of G which contains Γ cocompactly. We have $\widetilde{\Gamma} \subset \widetilde{L}$, so, $\Gamma \subset \pi(\pi^{-1}(\Gamma)) \subset L$. In addition, we have $\widetilde{L} = \widetilde{C}\widetilde{\Gamma}$, for some compact set $\widetilde{C}$ of $\widetilde{G}$; then

$$\pi(\widetilde{L}) = \pi(\widetilde{C})\pi(\widetilde{\Gamma}) = \pi(\widetilde{C})\Gamma.$$

We must show that L is closed in G, which means that $\widetilde{L}\Lambda$ is closed in $\widetilde{G}$. Let $\log_{\widetilde{G}}\Lambda = \mathbb{Z}\text{-span}(Z_1, \dots, Z_d)$ for some $d \in \mathbb{N}$, $\mathfrak{a} = \mathbb{R}\text{-span}(Z_1, \dots, Z_d)$, and $\widetilde{A} = \exp_{\widetilde{G}} \mathfrak{a}$. Then $\widetilde{L}\widetilde{A}$ is closed in $\widetilde{G}$ as $\widetilde{L}\widetilde{A}$ is a connected subgroup in a simply connected solvable Lie group $\widetilde{G}$. Then by Theorem 3.6, there exits a coexponential basis of $\widetilde{L}$ in $\widetilde{L}\widetilde{A}$, which means that $\widetilde{L}\widetilde{A}$ is diffeomorphic to $\widetilde{L} \times \mathbb{R}^s$ and conclusively $\widetilde{L}\Lambda$ to $\widetilde{L} \times \mathbb{Z}^s$ for some $s \leqslant d$. Therefore, $\widetilde{L}\Lambda$ is closed in $\widetilde{G}$.

Let $\Gamma' = \Gamma Z^c(G)$ which is a closed subgroup of G. Then Γ' admits at least one connected Lie subgroup L of G containing it cocompactly. We now show that L is unique. Indeed, if $L_1 = \exp_G \mathfrak{l}_1$ and $L_2 = \exp_G \mathfrak{l}_2$ are two such Lie groups, we claim that $L_i/(L_1 \cap L_2)$, $i = 1, 2$, are compact. To see that, consider for $i = 1, 2$ the canonical surjection

$$s_i : L_i \to L_i/(L_1 \cap L_2),$$

which factors through the canonical surjection $\rho_i : L_i \to L_i/\Gamma$ to a surjection $\widetilde{s_i} : L_i/\Gamma \to L_i/(L_1 \cap L_2)$ such that $s_i = \widetilde{s_i} \circ \rho_i$. The map $\widetilde{s_i}$ is surjective and continuous, and thus its image $L_i/(L_1 \cap L_2)$ is compact. Moreover, it is obvious that $L_i/(L_1 \cap L_2)$ is homeomorphic to $(L_i/Z^c(G))/((L_1 \cap L_2)/Z^c(G))$, which is homeomorphic to $\mathbb{R}^p$ for some $p \in \mathbb{N}$. Indeed, $G/Z^c(G)$ turns out to be a connected, simply connected, completely solvable Lie group, and the existence of the coexponential basis of $(L_1 \cap L_2)/Z^c(G)$ in $L_i/Z^c(G)$ allows us to conclude. Finally as this quotient is compact, we get conclusively that $p = 0$. Hence $L_1 \cap L_2 = L_1 = L_2$, as was to be shown. □

Let as above $\widetilde{G}$ be the universal covering of a connected Lie group G, and let $\pi : \widetilde{G} \to G$ be the covering map. A pre-abelian subgroup Γ of G is a subgroup such that $\widetilde{\Gamma} = \pi^{-1}(\Gamma)$ is abelian. When more generally G is exponential solvable and connected, the following could also be seen.

PROPOSITION 3.10

Any pre-abelian closed subgroup of a connected exponential solvable Lie group admits a unique syndetic hull.

Proof

Keep the same notation as in the proof of Theorem 3.9. In this situation $\widetilde{\Gamma}$ is abelian and $\widetilde{L}$ exists by [3, Proposition 3.2]. Then $\mathfrak{l} = \log_{\widetilde{G}}(\widetilde{L})$ is an abelian Lie subalgebra of $\mathfrak{g}$. Finally, $\exp_G(\mathfrak{l})$ is a syndetic hull of Γ, and the unicity is immediate. □

4. On the set $\mathrm{Hom}(\Gamma, G)$ for a discrete subgroup of G

4.1. On the structure of reduced Heisenberg Lie groups

Let $\mathfrak{g} := \mathfrak{h}_{2n+1}$ designate the Heisenberg Lie algebra of dimension $2n + 1$, and let $\widetilde{G} := \mathbb{H}_{2n+1}$ be the corresponding Lie group; $\mathfrak{g}$ can be defined as a real vector space endowed with a skew-symmetric bilinear form b of rank $2n$ and a fixed generator Z belonging to the kernel of b. The center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is then the kernel of b, and it is the one-dimensional subspace $[\mathfrak{g}, \mathfrak{g}]$. For any $X, Y \in \mathfrak{g}$, the Lie bracket is given by

$$[X, Y] = b(X, Y)Z.$$

We assume henceforth that $G := \mathbb{H}^r_{2n+1}$ is the reduced Heisenberg Lie group which we can identify to $\mathbb{R}^{2n} \ltimes \mathbb{T}$, where $\mathbb{T}$ is the group of complex numbers of modulus 1. Indeed, G is the quotient of $\widetilde{G}$ by the central discrete subgroup $\exp_{\widetilde{G}}(\mathbb{Z}Z)$. As the exponential mapping $\exp := \exp_G$ is given by

$$\exp(U + \lambda Z) = (U, e^{2i\pi\lambda}), \quad U \in \mathbb{R}^{2n} \text{ and } \lambda \in \mathbb{R},$$

G can be equipped with the following law:

$$(X, Y, e^{2i\pi t}) * (X', Y', e^{2i\pi s}) = (X + X', Y + Y', e^{2i\pi(t+s+\frac{1}{2}(\prec X', Y\succ - \prec X, Y'\succ))}),$$

where $X, Y, X', Y' \in \mathbb{R}^n$, $t, s \in \mathbb{R}$, and $\prec, \succ$ denotes the usual Euclidian scalar product. According to Proposition 3.3, the exponential map $\exp : \mathfrak{g} \to G$ is surjective. The Lie algebra $\mathfrak{g}$ acts on itself by the adjoint representation ad; that is,

$$\mathrm{ad}_T(Y) = [T, Y], \quad T, Y \in \mathfrak{g}.$$

The group G acts on $\mathfrak{g}$ by the adjoint representation Ad, defined by

$$\mathrm{Ad}_g = \exp \circ \mathrm{ad}_T, \quad g = \exp T \in G.$$

Let Γ be a discrete subgroup of G. This subsection aims to describe to a certain extent the set $\mathrm{Hom}(\Gamma, G)$ of homomorphisms from Γ to G. For $\overrightarrow{w} \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$, we adopt this notation:

$$\exp(\overrightarrow{w} + cZ) = \begin{pmatrix} {}^t\overrightarrow{w} \\ e^{2i\pi c} \end{pmatrix}.$$

We first prove the following structural result.

PROPOSITION 4.1

For a discrete subgroup Γ of G, there exist a unique nonnegative integer l_Γ and a linearly independent family of vectors $\{\overrightarrow{w}_1, \ldots, \overrightarrow{w}_{l_\Gamma}\}$ of $\mathbb{R}^{2n}$ such that

(1) *if Γ is torsion-free, then*

$$\Gamma = \left\{ \begin{pmatrix} {}^t\overrightarrow{w}_1 \\ e^{2i\pi c_1} \end{pmatrix}^{n_1} \cdots \begin{pmatrix} {}^t\overrightarrow{w}_{l_\Gamma} \\ e^{2i\pi c_{l_\Gamma}} \end{pmatrix}^{n_{l_\Gamma}} ; n_1, \ldots, n_{l_\Gamma} \in \mathbb{Z} \right\},$$

for some $c_1, \ldots, c_{l_\Gamma} \in \mathbb{R}$;

(2) *otherwise, let $q \in \mathbb{N}^*$ be the order of $\Gamma \cap Z(G)$; then*

$$\Gamma = \left\{ \begin{pmatrix} {}^t\overrightarrow{w}_1 \\ e^{2i\pi c_1} \end{pmatrix}^{n_1} \cdots \begin{pmatrix} {}^t\overrightarrow{w}_{l_\Gamma} \\ e^{2i\pi c_{l_\Gamma}} \end{pmatrix}^{n_{l_\Gamma}} \begin{pmatrix} {}^t\overrightarrow{0} \\ e^{2i\pi \frac{1}{q}} \end{pmatrix}^{s} ; n_1, \ldots, n_{l_\Gamma}, s \in \mathbb{Z} \right\},$$

for some $c_1, \ldots, c_{l_\Gamma} \in \mathbb{R}$.

Proof

We first consider the surjective projection

$$\begin{aligned} \pi_1 : G &\to G/Z(G), \\ \begin{pmatrix} {}^t\overrightarrow{w} \\ e^{2i\pi c} \end{pmatrix} &\longmapsto {}^t\overrightarrow{w}. \end{aligned}$$

Then, $\pi_1(\Gamma)$ is a discrete subgroup of $\mathbb{R}^{2n}$. This gives that there exist a nonnegative integer l_Γ and a family $\{\overrightarrow{w}_1, \dots, \overrightarrow{w}_{l_\Gamma}\}$ of linearly independent vectors of $\mathbb{R}^{2n}$ such that

$$\pi_1(\Gamma) = \{n_1 {}^t\overrightarrow{w}_1 + \cdots + n_{l_\Gamma} {}^t\overrightarrow{w}_{l_\Gamma}; n_1, \dots, n_{l_\Gamma} \in \mathbb{Z}\}$$

is a discrete subgroup of $\mathbb{R}^{2n}$. As for all $j \in \{1, \dots, l_\Gamma\}$, $\overrightarrow{w}_j \in \pi_1(\Gamma)$, there exists $c_j \in \mathbb{R}$ such that $\gamma_j = \left(\begin{smallmatrix} {}^t\overrightarrow{w}_j \\ e^{2i\pi c_j} \end{smallmatrix}\right) \in \Gamma$ and $\pi_1(\gamma_j) = {}^t\overrightarrow{w}_j$. Then

$$\begin{aligned}
\pi_1(\Gamma) &= \{n_1 \pi_{1|_\Gamma}(\gamma_1) + \cdots + n_{l_\Gamma} \pi_{1|_\Gamma}(\gamma_{l_\Gamma}); n_1, \dots, n_{l_\Gamma} \in \mathbb{Z}\} \\
&= \{\pi_{1|_\Gamma}(\gamma_1^{n_1}) + \cdots + \pi_{1|_\Gamma}(\gamma_{l_\Gamma}^{n_{l_\Gamma}}); n_1, \dots, n_{l_\Gamma} \in \mathbb{Z}\} \\
&= \{\pi_{1|_\Gamma}(\gamma_1^{n_1} \cdots \gamma_{l_\Gamma}^{n_{l_\Gamma}}); n_1, \dots, n_{l_\Gamma} \in \mathbb{Z}\} \\
&= \pi_{1|_\Gamma}\big(\{\gamma_1^{n_1} \cdots \gamma_{l_\Gamma}^{n_{l_\Gamma}}; n_1, \dots, n_{l_\Gamma} \in \mathbb{Z}\}\big).
\end{aligned}$$

Hence,

$$\Gamma = \{\gamma_1^{n_1} \cdots \gamma_{l_\Gamma}^{n_{l_\Gamma}}; n_1, \dots, n_{l_\Gamma} \in \mathbb{Z}\} \cdot \big(\Gamma \cap Z(G)\big).$$

We now show that l_Γ is unique. Indeed, if l_Γ and l'_Γ are two distinct such integers with $l_\Gamma < l'_\Gamma$, say, there exist two linearly independent families of vectors $\{\overrightarrow{w}_1, \dots, \overrightarrow{w}_{l_\Gamma}\}$ and $\{\overrightarrow{w}'_1, \dots, \overrightarrow{w}'_{l'_\Gamma}\}$ of $\mathbb{R}^{2n}$ such that

$$\begin{aligned}
\Gamma &= \left\{ \begin{pmatrix} {}^t\overrightarrow{w}_1 \\ e^{2i\pi c_1} \end{pmatrix}^{n_1} \cdots \begin{pmatrix} {}^t\overrightarrow{w}_{l_\Gamma} \\ e^{2i\pi c_{l_\Gamma}} \end{pmatrix}^{n_{l_\Gamma}}; n_1, \dots, n_{l_\Gamma} \in \mathbb{Z} \right\} \cdot \big(\Gamma \cap Z(G)\big) \\
&= \left\{ \begin{pmatrix} {}^t\overrightarrow{w}'_1 \\ e^{2i\pi c'_1} \end{pmatrix}^{m_1} \cdots \begin{pmatrix} {}^t\overrightarrow{w}'_{l'_\Gamma} \\ e^{2i\pi c'_{l'_\Gamma}} \end{pmatrix}^{m_{l'_\Gamma}}; m_1, \dots, m_{l'_\Gamma} \in \mathbb{Z} \right\} \cdot \big(\Gamma \cap Z(G)\big)
\end{aligned}$$

for some $c_1, \dots, c_{l_\Gamma}, c'_1, \dots, c'_{l'_\Gamma} \in \mathbb{R}$. There exist then for all $j \in \{1, \dots, l'_\Gamma\}$ some integers $(n_i^j)_{\substack{1 \leqslant i \leqslant l_\Gamma \\ 1 \leqslant j \leqslant l'_\Gamma}} \in \mathbb{Z}$ such that $\overrightarrow{w}'_j = \sum_{i=1}^{l_\Gamma} n_i^j \overrightarrow{w}_i$. This is impossible given $l_\Gamma < l'_\Gamma$. □

The following is an immediate consequence of Proposition 4.1.

COROLLARY 4.2

Any discrete subgroup of G is finitely generated.

REMARK 4.3

The integer l_Γ is indeed the length of Γ.

4.2. A matrix-like writing of elements of $\mathrm{Hom}(\Gamma, G)$

Let $\mathscr{M}_{r,s}(\mathbb{C})$ be the vector space of matrices of r rows and s columns. When $r = s$, we adopt the notation $\mathscr{M}_r(\mathbb{C})$ instead of $\mathscr{M}_{r,r}(\mathbb{C})$. Let now $\{\gamma_1, \dots, \gamma_k\}$ be a set of generators of Γ. Thanks to the injective map

$$\mathrm{Hom}(\Gamma, G) \to G \times \cdots \times G, \qquad \varphi \mapsto \big(\varphi(\gamma_1), \dots, \varphi(\gamma_k)\big)$$

to equip $\mathrm{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \cdots \times G$ and the identification of $G \times \cdots \times G$ to the space $\mathscr{M}_{2n+1,k}(\mathbb{C})$, it appears clear that the map

$$\Psi : \mathrm{Hom}(\Gamma, G) \to \mathscr{M}_{2n+1,k}(\mathbb{C}), \tag{4.1}$$

which associates to any element $\varphi \in \mathrm{Hom}(\Gamma, G)$ its matrix

$$M_\varphi(A, B, z) = \begin{pmatrix} A \\ B \\ e^{2i\pi z} \end{pmatrix} = \begin{pmatrix} C \\ e^{2i\pi z} \end{pmatrix} \in \mathscr{M}_{2n+1,k}(\mathbb{C}), \tag{4.2}$$

where $C = \left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$, A and $B \in \mathscr{M}_{n,k}(\mathbb{R})$, and $z := (z_1, \ldots, z_k) \in \mathbb{R}^k$, with

$$e^{2i\pi z} := \begin{pmatrix} e^{2i\pi z_1} & \cdots & e^{2i\pi z_k} \end{pmatrix} \in \mathscr{M}_{1,k}(\mathbb{C})$$

is a homeomorphism on its range. Let us write $C = \lfloor C^1, \ldots, C^k \rfloor$, where this symbol merely designs the matrix constituted of the columns $C^1, \ldots, C^k$. This means indeed that

$$\varphi(\gamma_j) := \exp(C^j + z_j Z)$$

for any $1 \leq j \leq k$. Let $\mathscr{E}$ denote the subset of $\mathscr{M}_{2n+1,k}(\mathbb{C})$ consisting of the totality of matrices as in (4.2) which is homeomorphic to the set $\mathscr{M}_{2n,k}(\mathbb{R}) \times \mathbb{T}^k$. Through the coming sections, Γ will serve as a discontinuous subgroup for a homogeneous space G/H, we hence pose the following:

$$\mathrm{Hom}^0(\Gamma, G) = \big\{\varphi \in \mathrm{Hom}(\Gamma, G) : \varphi \text{ is injective}\big\}$$

and

$$\mathrm{Hom}^0_d(\Gamma, G) = \big\{\varphi \in \mathrm{Hom}^0(\Gamma, G) : \varphi(\Gamma) \text{ is discrete}\big\}.$$

The set $\mathrm{Hom}(\Gamma, G)$ is homeomorphically identified to a subset $\mathscr{U}$ of $\mathscr{E}$, and $\mathrm{Hom}^0_d(\Gamma, G)$ is identified to a subset $\mathscr{U}^0_d$ of $\mathscr{U}$. The group G acts on $\mathscr{E}$ through the following law: For $g = \exp X$, with $X \in \mathfrak{g}$ with coordinates ${}^t(\alpha, \beta, \gamma)$, $\alpha, \beta \in \mathscr{M}_{1,n}(\mathbb{R})$, $\gamma \in \mathbb{R}$,

$$g \star \begin{pmatrix} C = \lfloor C^1, \ldots, C^k \rfloor \\ e^{2i\pi z} \end{pmatrix} = \begin{pmatrix} \lfloor g \cdot C^1 \cdot g^{-1}, \ldots, g \cdot C^k \cdot g^{-1} \rfloor \\ e^{2i\pi(z_1 + \alpha C^1_1 - \beta C^1_2)} \ldots e^{2i\pi(z_k + \alpha C^k_1 - \beta C^k_2)} \end{pmatrix},$$

where $C^i = \left(\begin{smallmatrix} C^i_1 \\ C^i_2 \end{smallmatrix}\right)$, $C^i_1, C^i_2 \in \mathscr{M}_{n,1}(\mathbb{R})$, $i \in \{1, \ldots, k\}$. The map $\Psi : \mathrm{Hom}(\Gamma, G) \to \mathscr{E}$ given in equation (4.1) turns out to be G-equivariant.

For

$$M = M(A, B, z) = \begin{pmatrix} A \\ B \\ e^{2i\pi z} \end{pmatrix} \in \mathscr{M}_{2n+1,k}(\mathbb{C}),$$

$$g \star M = \mathrm{Ad}_{\exp X} \cdot M = \begin{pmatrix} A \\ B \\ e^{2i\pi(z - \beta A + \alpha B)} \end{pmatrix}. \tag{4.3}$$

For all $j \in \{1, \dots, l = l_\Gamma\}$, we consider the notation

$$^t\overrightarrow{w}_j = \begin{pmatrix} ^t\overrightarrow{w}_j^1 \\ ^t\overrightarrow{w}_j^2 \end{pmatrix} \in \mathbb{R}^{2n},$$

where $\overrightarrow{w}_j^1, \overrightarrow{w}_j^2 \in \mathbb{R}^n$. Let $p_{ij} = 0$ if Γ is torsion-free, and let

$$p_{ij} = q(\prec \overrightarrow{w}_j^1, \overrightarrow{w}_i^2 \succ - \prec \overrightarrow{w}_i^1, \overrightarrow{w}_j^2 \succ)$$

otherwise. Let also

$$P(\Gamma) = \begin{pmatrix} 0 & p_{12} & \cdots & \cdots & p_{1l} \\ -p_{12} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & p_{l-1\,l} \\ -p_{1l} & \cdots & \cdots & -p_{l-1\,l} & 0 \end{pmatrix} \in \mathscr{M}_l(\mathbb{R}),$$

$\mathscr{A}(l, \mathbb{R})$ the subspace of $\mathscr{M}_l(\mathbb{R})$ of skew-symmetric matrices and $\mathscr{A}(l, \mathbb{Z})$ the subset of $\mathscr{A}(l, \mathbb{R})$ with entries in $\mathbb{Z}$. We now prove the following.

PROPOSITION 4.4

We keep the same notation and hypotheses. Let $M(A, B, z)$ be as in (4.2), where $A, B \in \mathscr{M}_{n,k}(\mathbb{R})$. We have the following.

(1) *If Γ is torsion-free, then $k = l$ and*

$$\mathscr{U} = \left\{ M(A, B, z) \in \mathscr{E} : {}^tAB - {}^tBA \in \mathscr{A}(l, \mathbb{Z}) \right\}.$$

(2) *Otherwise, $k = l + 1$ and*

$$\mathscr{U} = \Big\{ M(A, B, z) \in \mathscr{E} \,\Big|\, A = (A' \quad {}^t\overrightarrow{0}), B = (B' \quad {}^t\overrightarrow{0}), A', B' \in \mathscr{M}_{n,l}(\mathbb{R}),$$
$$z = \left(z_1, \dots, z_l, \frac{p}{q}\right), p \in \{0, \dots, q-1\}, z_1, \dots, z_l \in \mathbb{R}, \text{ and}$$
$$^tA'B' - {}^tB'A' \in \frac{p}{q} P(\Gamma) + \mathscr{A}(l, \mathbb{Z}) \Big\}.$$

Proof

It is sufficient to prove the proposition when Γ is not torsion-free. Indeed, otherwise, $P(\Gamma) = 0$, and the same arguments work. For $\varphi \in \mathrm{Hom}(\Gamma, G)$, $M_\varphi(A, B, z) \in \mathscr{U}$, and $\gamma_{l+1} = \begin{pmatrix} ^t\overrightarrow{0} \\ e^{2i\pi\frac{1}{q}} \end{pmatrix}$, we have $\varphi(\gamma_{l+1}) = \gamma_{l+1}^p$ for some $p \in \{0, \dots, q-1\}$. Now, let $r, j \in \{1, \dots, l\}$. Then

$$\varphi(\gamma_r \gamma_j \gamma_r^{-1} \gamma_j^{-1}) = \varphi(\gamma_r)\varphi(\gamma_j)\varphi(\gamma_r)^{-1}\varphi(\gamma_j)^{-1} = \begin{pmatrix} ^t\overrightarrow{0} \\ e^{2i\pi({}^tA^jB^r - {}^tB^jA^r)} \end{pmatrix}.$$

On the other hand, we have

$$\gamma_r \gamma_j \gamma_r^{-1} \gamma_j^{-1} = \begin{pmatrix} ^t\overrightarrow{0} \\ e^{2i\pi(\prec \overrightarrow{w}_j^1, \overrightarrow{w}_r^2 \succ - \prec \overrightarrow{w}_r^1, \overrightarrow{w}_j^2 \succ)} \end{pmatrix} = \begin{pmatrix} ^t\overrightarrow{0} \\ e^{2i\pi\frac{p_{rj}}{q}} \end{pmatrix} = \gamma_{l+1}^{p_{rj}}.$$

As $\varphi \in \mathrm{Hom}(\Gamma, G)$, then,

$$\varphi(\gamma_r\gamma_j\gamma_r^{-1}\gamma_j^{-1}) = \big(\varphi(\gamma_{l+1})\big)^{p_{rj}} = \gamma_{l+1}^{pp_{rj}} = \begin{pmatrix} {}^t\overrightarrow{0} \\ e^{2i\pi\frac{pp_{rj}}{q}} \end{pmatrix}.$$

This gives ${}^tA^jB^r - {}^tB^jA^r \in \frac{p}{q}p_{rj} + \mathbb{Z}$ for some $p \in \{0,\dots,q-1\}$. Let now $M(A,B,z) = \lfloor g_1,\dots,g_{l+1}\rfloor \in \mathscr{E}$ such that $g_j = {}^t(C^j, e^{2i\pi z_j})$ for $j \in \{1,\dots,l\}$ and $g_{l+1} = {}^t(0, e^{2i\pi\frac{p}{q}})$ for some $p \in \{0,\dots,q-1\}$ with the convention that $g_{l+1} = e$ if Γ is torsion-free, which satisfies the required conditions. Let φ be the map defined by

$$\varphi : \Gamma \to G, \qquad \gamma_1^{n_1}\dots\gamma_l^{n_l}\gamma_{l+1}^{n_{l+1}} \mapsto g_1^{n_1}\dots g_l^{n_l}g_{l+1}^{n_{l+1}}.$$

We need to show that $\varphi \in \mathrm{Hom}(\Gamma, G)$. Let $\gamma = \gamma_1^{n_1}\cdots\gamma_l^{n_l}\gamma_{l+1}^{n_{l+1}}$ and $\gamma' = \gamma_1^{m_1}\cdots\gamma_l^{m_l}\gamma_{l+1}^{m_{l+1}}$ in Γ. Therefore,

$$\begin{aligned}\varphi(\gamma\gamma') &= \varphi(\gamma_1^{n_1}\cdots\gamma_l^{n_l}\gamma_{l+1}^{n_{l+1}}\gamma_1^{m_1}\cdots\gamma_l^{m_l}\gamma_{l+1}^{m_{l+1}})\\ &= \varphi(\gamma_1^{n_1+m_1}\cdots\gamma_l^{n_l+m_l}\gamma_{l+1}^{m}),\end{aligned}$$

where

$$m = n_{l+1} + m_{l+1} - \sum_{1\leqslant j<i\leqslant l} n_i m_j p_{ij}.$$

Then

$$\begin{aligned}\varphi(\gamma\gamma') &= g_1^{n_1+m_1}\cdots g_l^{n_l+m_l}g_{l+1}^{m}\\ &= g_1^{n_1}\cdots g_l^{n_l}g_{l+1}^{n_{l+1}}g_1^{m_1}\cdots g_l^{m_l}g_{l+1}^{m_{l+1}}\\ &= \varphi(\gamma_1^{n_1}\cdots\gamma_l^{n_l}\gamma_{l+1}^{n_{l+1}})\varphi(\gamma_1^{m_1}\cdots\gamma_l^{m_l}\gamma_{l+1}^{m_{l+1}}).\\ &= \varphi(\gamma)\varphi(\gamma').\end{aligned}$$

This shows that $\lfloor g_1,\dots,g_{l+1}\rfloor \in \mathscr{U}$, which is enough to conclude. □

Any information concerning the structures of the spaces $\mathrm{Hom}(\Gamma, G)$ and $R(\Gamma, G, H)$ may help to understand the properties and the structure of the deformation space $\mathscr{T}(\Gamma, G, H)$. The sets $\mathrm{Hom}(\Gamma, G)$ and $R(\Gamma, G, H)$ may have some singularities, and there is no clear reason to say that the parameter space $R(\Gamma, G, H)$ is an analytic or algebraic or smooth manifold. For instance, when the parameter space is a semialgebraic set, it has certainly a finite number of connected components, which means in turn that the deformation space itself enjoys this feature. Corollary 5.7 below will be set toward such a purpose. Up to this step, let $\mathscr{U}$ and $\mathscr{E}$ be as in Section 4.2. We have the following.

COROLLARY 4.5

For a discrete subgroup Γ of G, the set $\mathrm{Hom}(\Gamma, G)$ is homeomorphic to a disjoint union of open (and hence closed) algebraic sets in $\mathscr{U}$. (Disjoint means here with empty pairwise intersection).

Proof
Recall that $\mathrm{Hom}(\Gamma, G)$ is homeomorphically identified to a subset $\mathscr{U}$ of $\mathscr{E}$. It suffices then to show that $\mathscr{U}$ splits to a disjoint union of open algebraic sets in $\mathscr{U}$. We only treat the case where Γ is torsion-free; the other case is handled similarly. For $D \in \mathscr{A}(l,\mathbb{Z})$, let

$$\mathscr{U}_D = \{M(A,B,z) \in \mathscr{E} : {}^tAB - {}^tBA = D\}.$$

We have

$$\mathscr{U} = \coprod_{D \in \mathscr{A}(l,\mathbb{Z})} \mathscr{U}_D.$$

Clearly the sets $\mathscr{U}_D$ are algebraic in $\mathscr{U}$ and $\mathscr{U}_D \cap \mathscr{U}_{D'} \neq \emptyset$ for $D \neq D' \in \mathscr{A}(l,\mathbb{Z})$. We only need to show that $\mathscr{U}_D$ is open in $\mathscr{U}$ for all $D \in \mathscr{A}(l,\mathbb{Z})$. Let $(A_j)_{j\in\mathbb{N}}$, $(B_j)_{j\in\mathbb{N}}$ be some sequences of $\mathscr{M}_{n,k}(\mathbb{R})$, and let $(z_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{R}^k$ such that $(M(A_j,B_j,z_j))_{j\in\mathbb{N}}$ is a sequence in ${}^c\mathscr{U}_D$ which converges to $M(A,B,z)$ in $\mathscr{E}$. This means that there exists a sequence $(D_j)_{j\in\mathbb{N}} \subset \mathscr{A}(l,\mathbb{Z})\backslash\{D\}$ such that ${}^tA_jB_j - {}^tB_jA_j = D_j$ for all $j \in \mathbb{N}$ and $({}^tA_jB_j - {}^tB_jA_j)_{j\in\mathbb{N}}$ converges to ${}^tAB - {}^tBA$. Then $(D_j)_{j\in\mathbb{N}}$ is stationary and $M(A,B,z) \in {}^c\mathscr{U}_D$. □

We next show the following.

PROPOSITION 4.6
Let G be the reduced Heisenberg Lie group, and let Γ be a discrete subgroup of G.

(1) *If Γ is torsion-free, then*

$$\mathscr{U}_d^0 = \{M(A,B,z) \in \mathscr{U} : \mathrm{rk}(C) = l\}.$$

(2) *Otherwise, if the symbol $\wedge$ means the greatest common divisor, the set $\mathscr{U}_d^0$ reads*

$$\Big\{M(A,B,z) \in \mathscr{U} \,\Big|\, A = (A' \quad {}^t\overrightarrow{0}), B = (B' \quad {}^t\overrightarrow{0}), A', B' \in \mathscr{M}_{n,l}(\mathbb{R}),$$

$$z = \Big(z_1, \ldots, z_l, \frac{p}{q}\Big), p \in \{1, \ldots, q-1\}, p \wedge q = 1, z_1, \ldots, z_l \in \mathbb{R}, \mathrm{rk}\begin{pmatrix} A' \\ B' \end{pmatrix} = l\Big\}.$$

Proof
As in Proposition 4.4, it is sufficient to consider the case where Γ is not torsion-free. Let us first recall the following well-known result.

LEMMA 4.7 ([7, COROLLARY TG VII.3])
Let $(\overrightarrow{a}_i)_{1\leqslant i\leqslant p}$ be a linearly independent family of p vectors of $\mathbb{R}^n$, and let $\overrightarrow{b} = \sum_{i=1}^p t_i \overrightarrow{a}_i$ be a linear combination of real coefficients t_i. Then, the subgroup of $\mathbb{R}^n$ generated by $\{\overrightarrow{a}_1, \ldots, \overrightarrow{a}_p, \overrightarrow{b}\}$ is discrete if and only if t_i are rational.

Let $\varphi \in \mathrm{Hom}_d^0(\Gamma, G)$ and $M_\varphi(A,B,z) \in \mathscr{U}_d^0$; then $\varphi(\Gamma \cap Z(G)) = \Gamma \cap Z(G)$. Therefore

$$A = (A' \quad {}^t\overrightarrow{0}), \qquad B = (B' \quad {}^t\overrightarrow{0}), \quad A', B' \in \mathscr{M}_{n,l}(\mathbb{R}),$$

$z = (z_1, \ldots, z_l, \frac{p}{q})$, $z_1, \ldots, z_l \in \mathbb{R}$, $p \in \{1, \ldots, q-1\}$, and $p \wedge q = 1$. We now show that $\mathrm{rk}\big(C' = \binom{A'}{B'}\big) = l$. As Γ is not torsion-free, then according to Proposition 4.1,

$$\Gamma = \left\{ \begin{pmatrix} {}^t\overrightarrow{w}_1 \\ e^{2i\pi c_1} \end{pmatrix}^{n_1} \cdots \begin{pmatrix} {}^t\overrightarrow{w}_l \\ e^{2i\pi c_l} \end{pmatrix}^{n_l} \begin{pmatrix} {}^t\overrightarrow{0} \\ e^{2i\pi \frac{1}{q}} \end{pmatrix}^{n} ; n_1, \ldots, n_l, n \in \mathbb{Z} \right\},$$

where $\{\overrightarrow{w}_1, \ldots, \overrightarrow{w}_l\}$ is a linearly independent family of $\mathbb{R}^{2n}$ and $c_1, \ldots, c_l \in \mathbb{R}$. The columns of the matrix C' generate a discrete subgroup of $\mathbb{R}^{2n}$. According to Lemma 4.7, if $\mathrm{rk}(C') = l' < l$, then the columns of C' are $\mathbb{Q}$-linearly dependent. We can and do assume that $\mathrm{rk}\lfloor C'^1, \ldots, C'^{l'} \rfloor = l'$. We denote $I = \{1, \ldots, l'\}$. Let $j_0 \in \{1, \ldots, l\} \backslash I$ be such that $C'^{j_0} = \sum_{j \in I} \lambda_j C'^j$, where $\lambda_j \in \mathbb{Q}$ for $j \in I$. We denote $\lambda_j = \frac{p_j}{q_j}$, $Q = \prod_{j \in I} q_j$, $Q_j = \frac{Q}{q_j}$, and $\gamma = \gamma_1^{Q_1 p_1} \cdots \gamma_{l'}^{Q_{l'} p_{l'}} \gamma_{j_0}^{-Q}$. As $\mathrm{rk}\lfloor {}^t\overrightarrow{w}_1, \ldots, {}^t\overrightarrow{w}_{l'}, {}^t\overrightarrow{w}_{j_0} \rfloor = l' + 1$, we have $\gamma \neq e$. Moreover, it is not hard to see that

$$\exp\Big(Q\Big(-\sum_{j \in I} \lambda_j z_j + z_{j_0}\Big) Z\Big) \in \varphi(\Gamma) \cap Z(G).$$

This gives $-\sum_{j \in I} \lambda_j z_j + z_{j_0} \in \mathbb{Q}$, which contradicts the fact that φ is injective. Conversely, let $\varphi \in \mathrm{Hom}(\Gamma, G)$ and $M_\varphi(A,B,z) \in \mathscr{U}$ be such that

$$A = (A' \quad {}^t\overrightarrow{0}), \qquad B = (B' \quad {}^t\overrightarrow{0}), \quad A', B' \in \mathscr{M}_{n,l}(\mathbb{R}), \qquad \mathrm{rk}(C') = l,$$

$z = (z_1, \ldots, z_l, \frac{p}{q})$, $p \in \{1, \ldots, q-1\}$, $p \wedge q = 1$, and $z_1, \ldots, z_l \in \mathbb{R}$. Let us show that φ is injective. Let $\gamma \in \ker\varphi$; then $\gamma \in \Gamma \cap Z(G)$. Therefore $\gamma = \exp(\frac{p'}{q} Z)$ for some $p' \in \mathbb{Z}$. Hence,

$$\ker\varphi = \Big\{\exp\Big(\frac{p'}{q} Z\Big) \in \Gamma : \exp\Big(\frac{pp'}{q} Z\Big) = e\Big\} = \{e\},$$

which entails that φ is injective. We now show that $\varphi(\Gamma)$ is discrete. For $A' = (a_{ij})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l}}$ and $B' = (b_{ij})_{\substack{1 \leqslant i \leqslant n, \\ 1 \leqslant j \leqslant l}}$, let $(m_j^1)_{j \in \mathbb{N}}, \ldots, (m_j^l)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ be some integer sequences such that the sequence $(u_j)_{j \in \mathbb{N}}$ of $\varphi(\Gamma)$ defined by

$$u_j = \exp\Big(m_j^1 \Big(\sum_{i=1}^n (a_{i1} X_i + b_{i1} Y_i) + z_1 Z\Big)\Big) \cdots \exp\Big(m_j^l \Big(\sum_{i=1}^n (a_{il} X_i + b_{il} Y_i) + z_l Z\Big)\Big)$$
$$\times \exp\Big(m_j \frac{p}{q} Z\Big)$$

converges. Hence, the sequence

$$\exp\Big(\sum_{i=1}^n (m_j^1 a_{i1} + \cdots + m_j^l a_{il}) X_i + (m_j^1 b_{i1} + \cdots + m_j^l b_{il}) Y_i\Big)$$

converges, which implies that $m_j^1 a_{i1} + \cdots + m_j^l a_{il}$ and $m_j^1 b_{i1} + \cdots + m_j^l b_{il}$ converge for all $i \in \{1, \ldots, n\}$. As $\mathrm{rk}(C') = l$, these sequences converge. Therefore $(u_j)_{j\in\mathbb{N}}$ is stationary. □

Proposition 4.6 shows that $\mathscr{U}_d^0$ is open in $\mathscr{U}$. So the following becomes clear.

COROLLARY 4.8

Let G be the reduced Heisenberg Lie group, and let Γ be a discrete subgroup of G. Then the set $\mathrm{Hom}_d^0(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$.

DEFINITION 4.9 ([6])

A subset V of $\mathbb{R}^n$ is called semialgebraic if it admits some representation of the form

$$V = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : P_{i,j}(x)\ s_{i,j}\ 0\},$$

where for each $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$, $P_{i,j}$ are some polynomials on $\mathbb{R}^n$ and $s_{ij} \in \{>, =, <\}$.

We now show the following result.

THEOREM 4.10

Let G be the reduced Heisenberg Lie group, and let Γ be a discrete subgroup of G of length l. Then we have the following.

(1) *$\mathrm{Hom}_d^0(\Gamma, G)$ is homeomorphic to a disjoint union of semialgebraic and open smooth manifolds in $\mathscr{U}_d^0$ of a common dimension equal to $(2n+1)l - \frac{1}{2}l(l-1)$.*

(2) *$\mathrm{Hom}_d^0(\Gamma, G)$ and $\mathrm{Hom}_d^0(\Gamma, G)/G$ are endowed with smooth manifold structures of dimensions $(2n+1)l - \frac{1}{2}l(l-1)$ and $2nl - \frac{1}{2}l(l-1)$, respectively.*

The following elementary result is proved in [2, Lemma 4.2].

LEMMA 4.11

Let $M \in \mathscr{M}_{n,l}(\mathbb{R})$ $(l \leqslant n)$ of maximal rank. Then the map

$$\varphi_M : \mathscr{M}_{n,l}(\mathbb{R}) \to \mathscr{A}(l, \mathbb{R}),$$
$$H \mapsto {}^tMH - {}^tHM$$

is surjective.

Proof

We only treat the case where Γ is torsion-free; the other case is handled similarly. For $D \in \mathscr{A}(l, \mathbb{Z})$, let

$$\mathscr{U}_{d,D}^0 = \mathscr{U}_d^0 \cap \{M(A, B, z) \in \mathscr{E} : {}^tAB - {}^tBA = D\}.$$

We have

$$\mathscr{U}_d^0 = \coprod_{D\in\mathscr{A}(l,\mathbb{Z})} \mathscr{U}_{d,D}^0. \tag{4.4}$$

Clearly, the sets $\mathscr{U}_{d,D}^0$ are semialgebraic and open in $\mathscr{U}_d^0$, and we only need to show that $\mathscr{U}_{d,D}^0$ is endowed with a smooth manifold structure for all $D\in\mathscr{A}(l,\mathbb{Z})$. Let

$$\nu = \left\{ M(A,B,z)\in\mathscr{E} : \mathrm{rk}\begin{pmatrix} A \\ B\end{pmatrix} = l \right\},$$

and let ψ_L be the smooth map

$$\begin{aligned} \psi_D : \nu &\to \mathscr{A}(l,\mathbb{R}), \\ M &\mapsto {}^tAB - {}^tBA - D. \end{aligned}$$

Clearly, $\mathscr{U}_{d,D}^0 = \psi_D^{-1}(\{0\})$. The goal now is to show that zero is a regular value of the map ψ_D. The derivative of ψ_D at a point $M = M(A,B,z)\in\mathscr{U}_{d,D}^0$, is given by

$$\begin{aligned} d(\psi_D)_M : \mathscr{E} &\to \mathscr{A}(l,\mathbb{R}), \\ X = M(H,K,h) &\mapsto {}^tHB - {}^tBH + {}^tAK - {}^tKA. \end{aligned}$$

So, clearly we have

$$d(\psi_D)_M(X) = {}^t\begin{pmatrix} H \\ -K\end{pmatrix}\begin{pmatrix} B \\ A\end{pmatrix} - {}^t\begin{pmatrix} B \\ A\end{pmatrix}\begin{pmatrix} H \\ -K\end{pmatrix},$$

which is enough to conclude thanks to Lemma 4.11. This shows the first point.

For the second point, the set $\mathscr{U}_d^0$ splits to a disjoint union of open smooth manifolds of dimension $(2n+1)l - \frac{1}{2}l(l-1)$ and therefore is endowed with a smooth manifold structure. To conclude that $\mathrm{Hom}_d^0(\Gamma, G)$ is endowed with a smooth manifold structure, it is sufficient to make use of the following elementary result for which the proof is immediate: Let $\mathscr{X}$ and $\mathscr{Y}$ be two Hausdorff topological spaces, and let $h:\mathscr{X}\to\mathscr{Y}$ be a homeomorphism. If one of these spaces is endowed with a smooth manifold structure, then so is the second.

Now, we focus attention to the space $\mathrm{Hom}_d^0(\Gamma, G)/G$. For any $X = {}^t(\alpha,\beta,\gamma)\in\mathfrak{g}$ and $M(A,B,z)\in\mathscr{U}_d^0$, we have as in equation (4.3),

$$\mathrm{Ad}_{\exp X}\cdot M(A,B,z) = M(A,B,z-\beta A+\alpha B).$$

Here $\gamma\in\mathbb{R}$, α, and β are in $\mathbb{R}^n$. For $M(A,B,z)\in\mathscr{U}_0^d$, we can easily see that the matrix through the canonical basis of $\mathbb{R}^{2n}$ and $\mathbb{R}^l$ of the map $\Phi_{A,B}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^l$, $(\alpha,\beta)\mapsto\beta A-\alpha B$ is $M(\Phi_{A,B}) = \left(-{}^tB \ \ {}^tA\right)$, which means that $\mathrm{rk}(M(\Phi_{A,B})) = l$ and that $\Phi_{A,B}$ is surjective. Let $\widetilde{\mathscr{U}_d^0} = \{M(A,B,0)\in\mathscr{U} : \mathrm{rk}(C) = l\}$, which, as above, is endowed with a smooth manifold structure; then the mapping

$$\widetilde{\pi} : \mathscr{U}_d^0/G \to \widetilde{\mathscr{U}_d^0}; \qquad \big[M(A,B,z)\big]\mapsto M(A,B,0)$$

is a continuous bijection. In addition, its inverse coincides with the restriction of the canonical quotient surjection to $\widetilde{\mathscr{U}_d^0}$ regarded as a subset of $\mathscr{U}_d^0$. This shows that $\mathrm{Hom}_d^0(\Gamma, G)/G$ is endowed with a smooth manifold structure. □

5. Proof of the main result

5.1. Proper action of connected subgroups on homogeneous spaces

The following result is proved in [5, Proposition 3.1] and provides a direct way to construct a basis of $\mathfrak{g}$ starting from a given subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and referred to be adapted to $\mathfrak{l}$.

PROPOSITION 5.1

Let $\mathfrak{g}$ be the Heisenberg Lie algebra, and let $\mathfrak{l}$ be a Lie subalgebra of $\mathfrak{g}$. Then there exists a basis $\mathscr{B} = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ of $\mathfrak{g}$ with the Lie commutation relations

$$[X_i, Y_j] = \delta_{i,j} Z, \quad i, j = 1, \ldots, n,$$

and satisfying the following.

(1) *If $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{l}$, then there exist two integers $p, q \geqslant 0$ such that the family*

$$\{X_1, \ldots, X_{p+q}, Y_1, \ldots, Y_p, Z\}$$

constitutes a basis of $\mathfrak{l}$.

(2) *If $\mathfrak{z}(\mathfrak{g}) \not\subseteq \mathfrak{l}$, then $\dim \mathfrak{l} \leqslant n$ and $\mathfrak{l}$ is generated by $X_1, \ldots, X_s$, where $s = \dim \mathfrak{l}$. The symbol $\delta_{i,j}$ here designates the Kronecker symbol. The basis $\mathscr{B}$ is said to be a symplectic basis of $\mathfrak{g}$ adapted to $\mathfrak{l}$.*

Let $H = \exp \mathfrak{h}$ be a closed connected Lie subgroup of the reduced Heisenberg group G, let Γ be a discrete subgroup, and let L be the syndetic hull of Γ. We need to characterize the proper action of the closed connected subgroup L on the homogeneous space G/H. As L contains the center of G, our goal is to prove the following.

PROPOSITION 5.2

Let $H = \exp \mathfrak{h}$ and $L = \exp \mathfrak{l}$ be closed connected subgroups of G such that L contains the center of G. We have the following.

(1) *The action of L on G/H is free if and only if $\mathfrak{l} \cap \mathfrak{h} = \{0\}$.*

(2) *If $\mathfrak{z}(\mathfrak{g}) \subseteqq \mathfrak{h}$, then L acts properly on G/H if and only if $\mathfrak{l} \cap \mathfrak{h} = \mathfrak{z}(\mathfrak{g})$.*

(3) *If $\mathfrak{z}(\mathfrak{g}) \not\subseteq \mathfrak{h}$, then the action of L on G/H is proper if and only if the action of L on G/H is free.*

Proof

(1) We first prove that $\exp(\mathfrak{l} \cap \mathfrak{h}) = L \cap H$. Let $t \in L \cap H$; then there exist $T_1 \in \mathfrak{l}$ and $T_2 \in \mathfrak{h}$ such that $t = \exp T_1 = \exp T_2$. This implies that $T_1 = T_2 \mod (\mathfrak{z}(\mathfrak{g}))$. As $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{l}$, then $T_2 \in \mathfrak{h} \cap \mathfrak{l}$, which implies that $t \in \exp(\mathfrak{l} \cap \mathfrak{h})$. Now suppose that L

acts on G/H freely. As $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{l}$, then $\mathfrak{z}(\mathfrak{g}) \nsubseteq \mathfrak{h}$, and therefore H is nilpotent and simply connected. As $\exp(\mathfrak{l} \cap \mathfrak{h}) = L \cap H = \{e\}$, we get $\mathfrak{l} \cap \mathfrak{h} = \{0\}$. The converse implication is trivial as $\exp(\mathfrak{h} \cap \mathfrak{l}) = H \cap L$ and $Z(G) \subset L$.

(2) Suppose that L acts properly on G/H, which implies that the triplet (L, G, H) is (CI). Then, $\mathfrak{h} \cap \mathfrak{l} \subsetneq \mathfrak{z}(\mathfrak{g})$, and therefore $\mathfrak{z}(\mathfrak{g}) = \mathfrak{h} \cap \mathfrak{l}$. Conversely, let us assume that L and H are not compact; otherwise our assertion is clear. We consider the norm $\|g\| = \inf\{\|X\|, \exp X = g\}$, for $g \in G$. Suppose that the action of L on G/H is not proper; then, there exists a compact set $S \subset G$ such that $SHS^{-1} \cap L$ is not relatively compact. Hence, one can find sequences $V_j \in \mathfrak{h}$, $W_j \in \mathfrak{l}$, A_j, and $B_j \in \mathfrak{g}$ such that

(a) $\exp A_j \in S$ and $\exp B_j \in S$,
(b) $\lim_{j\to+\infty} \|\exp V_j\| = \lim_{j\to+\infty} \|\exp W_j\| = +\infty$,
(c) $\exp W_j = \exp A_j \exp V_j \exp(-B_j)$.

Moreover, G is a two-step nilpotent Lie group; then the last equation gives

$$W_j = V_j + (A_j - B_j) \mod \big(\mathfrak{z}(\mathfrak{g})\big). \tag{5.1}$$

Let $W_j = W_j' \mod (\mathfrak{z}(\mathfrak{g}))$, $V_j = V_j' \mod (\mathfrak{z}(\mathfrak{g}))$, $A_j = A_j' \mod (\mathfrak{z}(\mathfrak{g}))$, and $B_j = B_j' \mod (\mathfrak{z}(\mathfrak{g}))$.

Obviously assertion (b) gives

$$\lim_{j\to+\infty} \|V_j'\| = \lim_{j\to+\infty} \|W_j'\| = +\infty.$$

Then we can assume that

$$\lim_{j\to+\infty} \frac{V_j'}{\|V_j'\|} = V', \qquad \lim_{j\to+\infty} \frac{W_j'}{\|W_j'\|} = W',$$

where $V' \in \mathfrak{h}, W' \in \mathfrak{l}, \|V'\| = \|W'\| = 1$. Let $\alpha_j' = \frac{\|V_j'\|}{\|W_j'\|}$; then equation (5.1) gives

$$\frac{W_j'}{\|W_j'\|} = \alpha_j \frac{V_j'}{\|V_j'\|} + \frac{A_j' - B_j'}{\|W_j'\|}.$$

Thus, $(\alpha_j')_j$ converges to $\alpha' \in \mathbb{R}^*$. Then, $W' \in \mathfrak{h} \cap \mathfrak{l} = \mathfrak{z}(\mathfrak{g})$, which is impossible as $W' \notin \mathfrak{z}(\mathfrak{g})$.

(3) As $\mathfrak{h}$ does not contain the center of $\mathfrak{g}$, then H is simply connected. If the action of L on G/H is proper, we have that for all $g \in G$, $gLg^{-1} \cap H$ is a compact subgroup of H and therefore $gLg^{-1} \cap H = \{e\}$. Then, the action of L on G/H is free. Conversely, we can and do assume that L and H are not compact; otherwise our assertion is clear. Suppose that the action of L on G/H is not proper; then, there exists a compact $S \subset G$ such that $SHS^{-1} \cap L$ is not relatively compact. Hence, one can find sequences $V_j \in \mathfrak{h}$, $W_j \in \mathfrak{l}$, A_j, and $B_j \in \mathfrak{g}$ meeting conditions (a)–(c) of assertion (2). Moreover, if G is two-step nilpotent Lie group, then equation (c) gives the equation (5.1). We have $\mathfrak{z}(\mathfrak{g}) \nsubseteq \mathfrak{h}$; then according to Proposition 5.1, there exists a basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ of $\mathfrak{g}$ satisfying $[X_i, Y_j] = \delta_{ij} Z$ and $\mathfrak{h} = \mathbb{R}\text{-span}(X_1, \dots, X_s)$, where $s = \dim \mathfrak{h}$. Let $W_j =$

$W'_j \mod (\mathfrak{z}(\mathfrak{g}))$ where $W'_j \in \mathfrak{l}$, $A_j = A'_j \mod (\mathfrak{z}(\mathfrak{g}))$, and $B_j = B'_j \mod (\mathfrak{z}(\mathfrak{g}))$ where $A'_j, B'_j \in \mathfrak{g}$. Hence, the same procedure as in the proof of assertion (2) gives a contradiction. □

5.2. On the parameter space

Let G be the reduced Heisenberg Lie group, let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G, and let Γ be a discontinuous subgroup for the homogeneous space G/H. This section aims to study the parameter space defined as in (1.1) and to study the stability property. For $\varphi \in \mathrm{Hom}_d^0(\Gamma, G)$, let $L_\varphi = \exp_G \mathfrak{l}_\varphi$ be the syndetic hull of $\varphi(\Gamma)$. We first prove the following.

LEMMA 5.3

We keep the notation of Section 4. For any $M_\varphi(A, B, z) \in \mathscr{U}_d^0$, we have

$$\mathfrak{l}_\varphi = \mathbb{R}\text{-span}(Z, C^1, \dots, C^l).$$

Proof

We still adopt the notation of the proof of Theorem 3.9. The closed subgroup $\widetilde{\varphi(\Gamma)}$ coincides with the closed subgroup $\exp_{\widetilde{G}}(\mathbb{Z}(C^1 + z_1 Z)) \cdots \exp_{\widetilde{G}}(\mathbb{Z}(C^l + z_l Z)) \exp_{\widetilde{G}}(\mathbb{Z}Z)$. It is then clear that the Lie algebra of the syndetic hull $\widetilde{L_\varphi}$ of $\widetilde{\varphi(\Gamma)}$ is the Lie subalgebra $\mathfrak{l}_\varphi = \mathbb{R}\text{-span}\{C^1, \dots, C^l, Z\}$. As $L_\varphi = \exp_G \mathfrak{l}_\varphi$, we are done. □

As a direct consequence of Proposition 5.2, we get the following description of the parameter space $\mathscr{R}(\Gamma, G, H)$.

PROPOSITION 5.4

Let G be the reduced Heisenberg Lie group, let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G, and let Γ be a discontinuous subgroup for the homogeneous space G/H. Then

$$\mathscr{R}(\Gamma, G, H) = \{\varphi \in \mathrm{Hom}_d^0(\Gamma, G) : \mathfrak{h} \cap \mathfrak{l}_\varphi \subseteq \mathfrak{z}(\mathfrak{g})\}.$$

More precisely:

(1) *if $\mathfrak{z}(\mathfrak{g}) \nsubseteq \mathfrak{h}$, then $\mathscr{R}(\Gamma, G, H) = \{\varphi \in \mathrm{Hom}_d^0(\Gamma, G) : \mathfrak{h} \cap \mathfrak{l}_\varphi = \{0\}\}$;*
(2) *otherwise, $\mathscr{R}(\Gamma, G, H) = \{\varphi \in \mathrm{Hom}_d^0(\Gamma, G) : \mathfrak{h} \cap \mathfrak{l}_\varphi = \mathfrak{z}(\mathfrak{g})\}$.*

Proof

Let $\varphi \in \mathscr{R}(\Gamma, G, H)$. We first show that the proper action of $\varphi(\Gamma)$ on G/H implies its free action. It is clear that the proper action implies that the triplet $(G, H, \varphi(\Gamma))$ is (CI), which gives that for all $g \in G$, the subgroup $K := \varphi(\Gamma) \cap gHg^{-1}$ is central and then finite as $\varphi(\Gamma)$ is discrete. As the map $\varphi : \Gamma \to \varphi(\Gamma)$ is a group isomorphism and K is finite and cyclic, we get that $\varphi^{-1}(K) = K$. Therefore, $K \subset \Gamma \cap H = \{e\}$. Thus the action of $\varphi(\Gamma)$ on G/H is free. As L_φ

contains $\varphi(\Gamma)$ cocompactly,

$$\mathscr{R}(\Gamma, G, H) = \{\varphi \in \mathrm{Hom}_d^0(\Gamma, G) : L_\varphi \text{ acts properly on } G/H\}.$$

Now, Proposition 5.2 allows us to conclude. □

5.3. Stability of discrete subgroups

Let G be a locally compact group, and let Γ be a discrete subgroup of G. In [13, (5.2.1)], T. Kobayashi defines the set $\pitchfork(\Gamma : G)$ consisting of subsets H for which $SHS^{-1} \cap \Gamma$ is compact for any compact set S in G. Let $\pitchfork_{\mathrm{gp}}(\Gamma : G)$ be the set of all closed connected subgroups belonging to $\pitchfork(\Gamma : G)$. So $H \in \pitchfork_{\mathrm{gp}}(\Gamma : G)$ if and only if Γ acts properly discontinuously on G/H. The following notion of stability of discrete subgroups was defined in [1].

DEFINITION 5.5

(1) Let Γ be a discrete subgroup of G. We set $\mathrm{Stab}(\Gamma : G)$ as the set of all subgroups $H \in \pitchfork_{\mathrm{gp}}(\Gamma : G)$ for which the parameter space $\mathscr{R}(\Gamma, G, H)$ is open in $\mathrm{Hom}(\Gamma, G)$.

(2) A discrete subgroup Γ of G is said to be stable, if $\mathrm{Stab}(\Gamma : G) = \pitchfork_{\mathrm{gp}}(\Gamma : G)$. This means that the space $\mathscr{R}(\Gamma, G, H)$ is open for any $H \in \pitchfork_{\mathrm{gp}}(\Gamma : G)$.

REMARK 5.6

The notion of stability is defined for discrete subgroups. For a discrete subgroup $\Gamma \subset G$, Γ becomes a discontinuous subgroup for G/H for any $H \in \pitchfork_{\mathrm{gp}}(\Gamma : G)$.

The question whether it is possible to characterize all stable discrete subgroups of connected nilpotent Lie groups is also posed in [1]. Now, we are ready to answer this question in our context and to prove our main theorem.

5.4. Proof of Theorem 1.1

(1) Let $\{T_1, \dots, T_r\}$ be a basis of $\mathfrak{h}$, and let $\mathfrak{l}_\varphi = \mathbb{R}\text{-span}\{C^1, \dots, C^l, Z\}$ be as in Lemma 5.3. Thanks to Proposition 5.4, it is clear that $\mathscr{R}(\Gamma, G, H)$ is homeomorphic to the set

$$\{M_\varphi(A, B, z) \in \mathscr{U}_d^0 : \mathrm{rk}\lfloor T_1, \dots, T_r, C^1, \dots, C^l, Z\rfloor = r + l + 1 - \dim(\mathfrak{h} \cap \mathfrak{l}_\varphi)\},$$

which is a Zariski-open set in $\mathscr{U}_d^0$. This completes the proof as $\mathscr{U}_d^0$ is open in $\mathscr{U}$ as in Corollary 4.8. This also entails that the parameter space is open in $\mathrm{Hom}(\Gamma, G)$ and that any discrete subgroup of G is stable.

(2) The parameter and the deformation spaces are open in $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}_d^0(\Gamma, G)/G$, respectively; they are therefore endowed with a smooth manifold structure with the mentioned dimensions thanks to Theorem 4.10.

(3) We immediately see that $\dim G \star M(A, B, z) = l$ for any $M(A, B, z) \in \mathscr{R}(\Gamma, G, H)$.

(4) Assume first that Γ is a finite group; then it is a central and cyclic group. As such, we have the following:

$$\mathscr{T}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, H) = \mathrm{Hom}^0(\Gamma, G) = \mathrm{Aut}(\Gamma),$$

where the last means the automorphism group of Γ, which is a finite group. So the global rigidity property holds on the parameter space.

Let now Γ be infinite. As $\mathscr{R}(\Gamma, G, H)$ is endowed with a smooth manifold structure and $\dim G \star M(A, B, z) \lneq \dim \mathscr{R}(\Gamma, G, H)$ for any $M(A, B, z) \in \mathscr{R}(\Gamma, G, H)$, then the local rigidity fails to hold.

As a direct consequence of the decomposition (4.4) and the proof of Theorem 1.1, we set the following.

COROLLARY 5.7

The parameter and the deformation spaces split into semialgebraic smooth manifolds.

COROLLARY 5.8

Let G be the reduced Heisenberg Lie group, let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G, and let Γ be a discontinuous subgroup for the homogeneous space G/H of length l_Γ. Then the deformation space $\mathscr{T}(\Gamma, G, H)$ splits into open smooth manifolds of common dimension equal to $2nl_\Gamma - \frac{1}{2}l_\Gamma(l_\Gamma - 1)$.

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