Very good and very bad field generators

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Abstract Let $\mathbf{k}$ be a field. A *field generator* is a polynomial $F \in \mathbf{k}[X,Y]$ satisfying $\mathbf{k}(F,G) = \mathbf{k}(X,Y)$ for some $G \in \mathbf{k}(X,Y)$. If G can be chosen in $\mathbf{k}[X,Y]$, we call F a *good field generator*; otherwise, F is a *bad field generator*. These notions were first studied by Abhyankar, Jan, and Russell in the 1970s. The present paper introduces and studies the notions of "very good" and "very bad" field generators. We give theoretical results as well as new examples of bad and very bad field generators.

1. Introduction

Throughout this paper, $\mathbf{k}$ denotes an arbitrary field unless otherwise specified.

If R is a subring of a ring S, the notation $S = R^{[n]}$ means that S is R-isomorphic to a polynomial algebra in n variables over R. If L/K is a field extension, $L = K^{(n)}$ means that L is a purely transcendental extension of K, of transcendence degree n. We write $\operatorname{Frac} R$ for the field of fractions of a domain R. All curves and surfaces are irreducible and reduced.

DEFINITION 1.1

Let $A = \mathbf{k}^{[2]}$ and $K = \operatorname{Frac} A$. A *field generator of* A is an $F \in A$ satisfying $K = \mathbf{k}(F,G)$ for some $G \in K$. A *good field generator of* A is an $F \in A$ satisfying $K = \mathbf{k}(F,G)$ for some $G \in A$. A field generator which is not good is said to be *bad*.

Field generators are studied in [9], [13], [14], and [4]. The first example of a bad field generator was given in [9], and more examples were given in [14] and [4]. Among other things, [14] showed that 21 and 25 are the smallest integers d such that there exists a bad field generator of degree d.

The notions of good and bad field generators are classical. We shall now introduce the notions of "very good" and "very bad" field generators. Before doing so, let us first adopt a convention that we shall keep throughout this paper. Namely, let us agree that the notation $A \preceq B$ means that all the following

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conditions are satisfied:

$$A = \mathbf{k}^{[2]}, \qquad B = \mathbf{k}^{[2]}, \qquad A \subseteq B, \qquad \text{and} \qquad \operatorname{Frac} A = \operatorname{Frac} B.$$

Observe that if $F \in A \preceq B$, then F is a field generator of A if and only if F is a field generator of B. Moreover, if F is a good field generator of A, then it is a good field generator of B (and consequently, if it is a bad field generator of B, then it is a bad field generator of A). However, it might happen that F will be a bad field generator of A and a good field generator of B. These remarks suggest the following.

DEFINITION 1.2
Let $F \in A = \mathbf{k}^{[2]}$ be a field generator of A.

(1) F is a *very good* field generator of A if it is a good field generator of each A' satisfying $F \in A' \preceq A$.

(2) F is a *very bad* field generator of A if it is a bad field generator of each A' satisfying $A' \succeq A$.

It is interesting to note that the notion of very good field generator suggested itself in a natural way, in our study [6] of "lean factorizations" of morphisms $\mathbb{A}^2 \to \mathbb{A}^1$. The definition of very bad field generator then follows by symmetry.

It is clear that "very good" implies "good" and that "very bad" implies "bad." Examples of very good field generators are easy to find; moreover, it follows from Proposition 5.3 and Remark 5.4 that certain well-studied classes of polynomials are included in that of very good field generators. Example 5.11 gives examples of very bad field generators, of bad field generators which are not very bad, and of good ones which are not very good.

NOTATION 1.3
Let $A = \mathbf{k}^{[2]}$. Given $F \in A \setminus \mathbf{k}$, we let $\Gamma_{\text{alg}}(F, A)$ denote the set of prime ideals $\mathfrak{p}$ of A such that the composite $\mathbf{k}[F] \hookrightarrow A \to A/\mathfrak{p}$ is an isomorphism. We also let $\Gamma(F, A) = \{V(\mathfrak{p}) \mid \mathfrak{p} \in \Gamma_{\text{alg}}(F, A)\}$; that is, $\Gamma(F, A)$ is the set of curves $C \subset \operatorname{Spec} A$ which have the property that the composite $C \hookrightarrow \operatorname{Spec} A \to \operatorname{Spec} \mathbf{k}[F]$ is an isomorphism. Note that $\mathfrak{p} \mapsto V(\mathfrak{p})$ is a bijection $\Gamma_{\text{alg}}(F, A) \to \Gamma(F, A)$.

The set $\Gamma(F, A)$ (or equivalently $\Gamma_{\text{alg}}(F, A)$) plays an important role in our study of field generators. One of our main results is Theorem 4.11, which asserts that if F is a field generator of $A = \mathbf{k}^{[2]}$ satisfying $|\Gamma(F, A)| > 2$, then there exist X, Y such that $A = \mathbf{k}[X, Y]$ and $F = \alpha(Y)X + \beta(Y)$ for some $\alpha(Y), \beta(Y) \in \mathbf{k}[Y]$. (Note that $\alpha(Y)X + \beta(Y)$ is a very good field generator of an especially simple type.) In particular, if F is a bad field generator of A, then $|\Gamma(F, A)| \in \{0, 1, 2\}$, where (by Examples 5.12) the three cases occur and where (by Proposition 5.8) the case $|\Gamma(F, A)| = 0$ is equivalent to F being very bad. This last equivalence is a characterization of very bad field generators which turns out to be easy to use in practice. A characterization of very good field generators is not known, but Proposition 5.3 is a partial result in that direction.

Section 5 also shows how to construct very bad field generators from a given bad field generator. We use that construction method in proofs (for instance, in Proposition 5.10) and also for giving new examples of bad and very bad field generators (see Example 5.11). Our aim, with these examples, is not to give explicit polynomials (which would be in principle easy) but rather to demonstrate the method.

The main results are in Sections 4 and 5, but Theorem 2.5 and Proposition 2.9 are also noteworthy.

REMARK

Section 5 of [5] further develops the theory presented here.

We reiterate that $\mathbf{k}$ denotes an arbitrary field (unless otherwise specified) throughout this paper. We write $\mathbb{A}^n$ or $\mathbb{A}^n_{\mathbf{k}}$ for the affine n-space over $\mathbf{k}$, that is, the scheme $\operatorname{Spec} A$ where $A = \mathbf{k}^{[n]}$.

2. Dicriticals

DEFINITION 2.1

Given a field extension $L \subseteq M$, let $\mathbb{V}(M/L)$ be the set of valuation rings R satisfying $L \subseteq R \subseteq M$, $\operatorname{Frac} R = M$, and $R \neq M$.

Given a pair (F, A) such that $A = \mathbf{k}^{[2]}$ and $F \in A \setminus \mathbf{k}$, define

$$\mathbb{V}^\infty(F, A) = \{R \in \mathbb{V}(K/\mathbf{k}(F)) \mid A \nsubseteq R\} \quad \text{where } K = \operatorname{Frac} A.$$

Then $\mathbb{V}^\infty(F, A)$ is a nonempty finite set which depends only on the pair $(\mathbf{k}(F), A)$. For each $R \in \mathbb{V}^\infty(F, A)$, let $\mathfrak{m}_R$ be the maximal ideal of R. Let $R_1, \ldots, R_t$ be the distinct elements of $\mathbb{V}^\infty(F, A)$, and let $d_i = [R_i/\mathfrak{m}_{R_i} : \mathbf{k}(F)]$ for $i = 1, \ldots, t$. Then we define

$$\Delta(F, A) = [d_1, \ldots, d_t] \qquad \text{and} \qquad \operatorname{dic}(F, A) = |\mathbb{V}^\infty(F, A)| = t,$$

where $[d_1, \ldots, d_t]$ is an unordered t-tuple of positive integers.

Given $A = \mathbf{k}^{[2]}$ and $F \in A \setminus \mathbf{k}$, we call the elements of $\mathbb{V}^\infty(F, A)$ the *dicriticals* of (F, A) or of F in A; given $R \in \mathbb{V}^\infty(F, A)$, we call $[R/\mathfrak{m}_R : \mathbf{k}(F)]$ the *degree of the dicritical* R.

REMARK 2.2

Let $A = \mathbf{k}^{[2]}$ and $F \in A \setminus \mathbf{k}$. Choose a pair $\gamma = (X, Y)$ satisfying $A = \mathbf{k}[X, Y]$, and consider the embedding of $\mathbb{A}^2$ in $\mathbb{P}^2$, $(x, y) \mapsto (1 : x : y)$, determined by γ. That is, identify $\mathbb{A}^2$ with the complement of the line $W = 0$ in $\mathbb{P}^2 = \operatorname{Proj} \mathbf{k}[W, X, Y]$, where $\mathbf{k}[W, X, Y] = \mathbf{k}^{[3]}$ is $\mathbb{N}$-graded by total degree in W, X, Y. Consider the closed subset $V(F)$ of $\mathbb{A}^2$ and its closure $\overline{V(F)}$ in $\mathbb{P}^2$.

For each $R \in \mathbb{V}^\infty(F, A)$, there exists a unique point $Q_R \in \mathbb{P}^2$ such that R is centered at Q_R (i.e., R dominates the local ring of $\mathbb{P}^2$ at Q_R). One can see that

$R \mapsto Q_R$ is a surjective set map $\mathbb{V}^\infty(F,A) \to \overline{V(F)} \setminus V(F)$. It follows that

$$\operatorname{dic}(F,A) \geq \left|\overline{V(F)} \setminus V(F)\right|. \tag{1}$$

Note that (1) is valid for every choice of $\gamma = (X,Y)$. The right-hand side of (1) depends on (F,A,γ), but $\operatorname{dic}(F,A)$ depends only on (F,A).

REMARK 2.3

Except for the notation, our definitions of "dicritical" and of "degree of dicritical" are identical to those given by Abhyankar in [1] (see the last sentence of p. 92). Note, however, that many authors use a definition formulated in terms of horizontal curves at infinity. Let us make the link between those two approaches. For this discussion, we assume that $\mathbf{k}$ is algebraically closed. Consider a pair (F,A) such that $A = \mathbf{k}^{[2]}$ and $F \in A \setminus \mathbf{k}$. Let us use the abbreviation $\mathbb{V} = \mathbb{V}(\operatorname{Frac}(A)/\mathbf{k}(F))$; then $\mathbb{V}^\infty(F,A) = \{R \in \mathbb{V} \mid A \not\subseteq R\}$.

Let $f : \mathbb{A}^2 = \operatorname{Spec} A \to \mathbb{A}^1 = \operatorname{Spec} \mathbf{k}[F]$ be the morphism determined by the inclusion $\mathbf{k}[F] \to A$. Then there exists a (nonunique) commutative diagram

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "\bar{f}"] \\ \mathbb{A}^1 \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd} \tag{2}$$

where X is a nonsingular projective surface, the arrows $\hookrightarrow$ are open immersions, and $\bar{f}$ is a morphism. Let us say that a curve $C \subset X$ is "horizontal" if it satisfies $\bar{f}(C) = \mathbb{P}^1$, let H denote the set of curves $C \subset X$ which are horizontal, and let $H^\infty = \{C \in H \mid C \subseteq X \setminus \mathbb{A}^2\}$. Several authors refer to the elements of H^∞ as the dicriticals of f. For each $C \in H^\infty$, the degree of the morphism $\bar{f}|_C : C \to \mathbb{P}^1$ is then called the "degree of the dicritical" C.

Given $C \in H$, let ξ_C be the generic point of C, and observe that the local ring $\mathcal{O}_{X,\xi_C}$ of X at the point $\xi_C \in X$ is an element of $\mathbb{V}$. In fact, the map $C \mapsto \mathcal{O}_{X,\xi_C}$ from H to $\mathbb{V}$ is bijective, and so is its restriction $H^\infty \to \mathbb{V}^\infty(F,A)$. Thus

$$\operatorname{dic}(F,A) = |H^\infty|,$$

where $\operatorname{dic}(F,A)$ is defined in Definition 2.1. We also note that if $C \in H$ corresponds to $R = \mathcal{O}_{X,\xi_C} \in \mathbb{V}$ by the above bijection $H \to \mathbb{V}$, then the degree of $\bar{f}|_C : C \to \mathbb{P}^1$ is equal to $[\mathbf{k}(C) : \mathbf{k}(\mathbb{P}^1)]$, where the function field $\mathbf{k}(C)$ of C can be identified with the residue field of R; so $\deg(\bar{f}|_C) = [R/\mathfrak{m}_R : \mathbf{k}(F)]$. Consequently, if we write $H^\infty = \{C_1, \dots, C_t\}$, then

$$\Delta(F,A) = \left[\deg(\bar{f}|_{C_1}), \dots, \deg(\bar{f}|_{C_t})\right],$$

where $\Delta(F,A)$ is defined in Definition 2.1. To summarize, our definition (2.1) of dicriticals and of their degrees is equivalent (via the bijection $H^\infty \to \mathbb{V}^\infty(F,A)$, $C \mapsto \mathcal{O}_{X,\xi_C}$) to that given in terms of H^∞.

Let us be precise about the use of language (regarding dicriticals or some equivalent concept) in [3], [4], [13], and [14], since we are going to refer to those papers.

Papers [3] and [4] follow the H^∞-approach but do not use the word "dicritical." The elements of H^∞ are called "horizontal curves" or "horizontal components," and the degree of such a curve C is defined to be the degree of $\bar{f}|_C$.

Papers [13] and [14] simply speak of points at infinity instead of dicriticals. Let us explain this. Let $\mathbf{k}$ be any field, and let $A = \mathbf{k}^{[2]}$, $F \in A \setminus \mathbf{k}$, $L = \operatorname{Frac} A$, and $K = \mathbf{k}(F)$. Then L/K is a function field in one variable. Let C_F be the complete regular curve over K whose function field is L. Let P be a closed point of C_F, and let $(\mathcal{O}_P, \mathfrak{m}_P)$ be the local ring of C_F at P; by definition of the degree of a point on a variety, $\deg P = [\mathcal{O}_P/\mathfrak{m}_P : K]$; if $A \nsubseteq \mathcal{O}_P$, one says (in [13] and [14]) that P is a "point of C_F at infinity" (or a "place of C_F at infinity"). As is well known, the set of closed points of C_F can be identified with $\mathbb{V}(L/K)$ via the bijection $P \mapsto \mathcal{O}_P$; then the set $\mathbb{V}^\infty(F, A)$ of dicriticals is precisely the set of points of C_F at infinity, and the degrees of the dicriticals are the degrees of the points. For instance the sentence "C_F has exactly two places at infinity, one of degree 2 and one of degree 3" in [14, p. 324], means that $\Delta(F, A) = [2, 3]$.

This closes the remark on terminology. Our terminology, throughout, is that of Definition 2.1.

A *function field in one variable* is a finitely generated field extension of transcendence degree 1.

NOTATION 2.4

Given a function field in one variable L/K, we set

$$\eta(L/K) = \gcd\big\{[R/\mathfrak{m}_R : K] \bigm| R \in \mathbb{V}(L/K)\big\},$$

where $\mathbb{V}(L/K)$ is defined in Definition 2.1. Note that if L/K has a K-rational point,† then $\eta(L/K) = 1$. In particular, if $L = K^{(1)}$, then $\eta(L/K) = 1$.

THEOREM 2.5

Let $\mathbf{k}$ be a field, and let $A = \mathbf{k}^{[2]}$, $F \in A \setminus \mathbf{k}$, and $\Delta(F, A) = [d_1, \dots, d_t]$. Then

$$\gcd(d_1, \dots, d_t) = \eta\big(\operatorname{Frac}(A)/\mathbf{k}(F)\big).$$

In particular, if F is a field generator of A, then $\gcd(d_1, \dots, d_t) = 1$.

Proof

Let $\mathcal{A} = S^{-1}A$, where $S = \mathbf{k}[F] \setminus \{0\}$; let $K = \mathbf{k}(F)$ and $L = \operatorname{Frac} A = \operatorname{Frac} \mathcal{A}$. As $\mathcal{A}$ is a domain and a finitely generated K-algebra, its Krull dimension is $\dim \mathcal{A} = \operatorname{trdeg}(\mathcal{A}/K) = 1$; being a 1-dimensional unique factorization domaine (UFD), $\mathcal{A}$ is a principal ideal domaine (PID).

† A K-rational point is an $R \in \mathbb{V}(L/K)$ satisfying $R/\mathfrak{m}_R = K$.

Let $R_1,\dots,R_t$ be the distinct elements of the subset $\mathbb{V}^\infty(F,A)$ of $\mathbb{V}(L/K)$, let $d_i=[R_i/\mathfrak{m}_{R_i}:K]$ for $1\le i\le t$, and let $d=\gcd(d_1,\dots,d_t)$. It is obvious that $\eta(L/K)\mid d$. So it is enough to show that $[R/\mathfrak{m}_R:K]\in d\mathbb{Z}$ for every $R\in\mathbb{V}(L/K)$.

If $R\in\mathbb{V}^\infty(F,A)$, then $[R/\mathfrak{m}_R:K]\in\{d_1,\dots,d_t\}$, so $[R/\mathfrak{m}_R:K]\in d\mathbb{Z}$ is obvious.

Let $R\in\mathbb{V}(L/K)\setminus\mathbb{V}^\infty(F,A)$. Then $A\subseteq R$, so $\mathcal{A}\subseteq R$, and consequently $\mathfrak{m}_R\cap\mathcal{A}$ is a prime ideal of $\mathcal{A}$. If $\mathfrak{m}_R\cap\mathcal{A}=0$, then $\operatorname{Frac}\mathcal{A}\subseteq R$, which contradicts the definition of $\mathbb{V}(L/K)$. So $\mathfrak{m}_R\cap\mathcal{A}$ is a maximal ideal $\mathfrak{m}$ of $\mathcal{A}$; then $R=\mathcal{A}_\mathfrak{m}$. Since $\mathcal{A}$ is a PID, $\mathfrak{m}=(g)$ for some $g\in\mathcal{A}\setminus\{0\}$. Multiplying g by a suitable element of K^*, we may (and we do) arrange that $g\in A\setminus\{0\}$.

As $g\in L^*$, we may consider the principal divisor $\operatorname{div}(g)\in\operatorname{Div}(L/K)$, where $\operatorname{Div}(L/K)$ is the free abelian group on the set $\mathbb{V}(L/K)$, written additively. Note that (i) g is a uniformizing parameter of R; (ii) since $g\in A$, g belongs to each element of $\mathbb{V}(L/K)\setminus\mathbb{V}^\infty(F,A)$; and (iii) if R' is any element of $\mathbb{V}(L/K)\setminus\mathbb{V}^\infty(F,A)$ satisfying $g\in\mathfrak{m}_{R'}$, then $\mathfrak{m}_{R'}\cap\mathcal{A}=\mathfrak{m}$ and consequently $R'=\mathcal{A}_\mathfrak{m}=R$. Thus

$$\operatorname{div}(g)=1R+\sum_{i=1}^{t}v_i(g)R_i,$$

where v_i is the valuation of R_i. Since $\operatorname{div}(g)$ has degree 0,

$$0=[R/\mathfrak{m}_R:K]+\sum_{i=1}^{t}v_i(g)[R_i/\mathfrak{m}_{R_i}:K]=[R/\mathfrak{m}_R:K]+\sum_{i=1}^{t}v_i(g)d_i,$$

so $[R/\mathfrak{m}_R:K]\in d\mathbb{Z}$. It follows that $d\mid\eta(L/K)$ and hence that $d=\eta(L/K)$, as desired.

If F is a field generator of A, then $L=K^{(1)}$, so $\eta(L/K)=1$, and consequently $d=1$. □

Our next objective is to study how $\Delta(F,A)$ behaves under a birational extension of A. Before doing this, we first need to discuss birational morphisms $\mathbb{A}^2\to\mathbb{A}^2$.

DEFINITION 2.6

Let $\Phi:X\to Y$ be a morphism of nonsingular algebraic surfaces over $\mathbf{k}$. Assume that Φ is *birational*, that is, that there exist nonempty Zariski-open subsets $U\subseteq X$ and $V\subseteq Y$ such that Φ restricts to an isomorphism $U\to V$. By a *missing curve* of Φ we mean a curve $C\subset Y$ such that $C\cap\Phi(X)$ is a finite set of closed points. We write $\operatorname{Miss}(\Phi)$ for the set of missing curves of Φ. Note that $\operatorname{Miss}(\Phi)$ is a finite set.

LEMMA 2.7

Let $\Phi:\mathbb{A}^2\to\mathbb{A}^2$ be a birational morphism.

(a) *Let C be a missing curve of Φ. Then C has one place at infinity, and, consequently, the units of the coordinate algebra of C are algebraic over $\mathbf{k}$.*

(b) *Let $C\subset\mathbb{A}^2$ be a curve which is not a missing curve of Φ. Then there exists a unique curve $D\subset\mathbb{A}^2$ such that $\Phi(D)$ is a dense subset of C.*

Proof
First consider the case where $\mathbf{k}$ is algebraically closed. Then (a) is true by [7, 4.3]. For (b), we consider [7, 1.1] (with $f=\Phi$, $X=\mathbb{A}^2$ and $Y=\mathbb{A}^2$) and we observe that, given a curve $C\subset Y$ which is not a missing curve of f, $\tilde{C}\cap X$ (where $\tilde{C}\subset Y_n$ is the strict transform of C) is the only curve in X whose image in Y is a dense subset of C. This proves (b). Our task, then, is to show that the lemma continues to be valid when $\mathbf{k}$ is an arbitrary field.

Let $A=\mathbf{k}^{[2]}$, $\mathbb{A}^2=\operatorname{Spec}A$, and let $\varphi:A\to A$ be the $\mathbf{k}$-homomorphism corresponding to the given $\Phi:\operatorname{Spec}A\to\operatorname{Spec}A$. Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$. Applying $(\bar{\mathbf{k}}\otimes_{\mathbf{k}}\_)$ to φ gives commutative diagrams

$$\begin{array}{ccc} \bar{A} & \xleftarrow{\bar{\varphi}} & \bar{A} \\ \uparrow & & \uparrow \\ A & \xleftarrow[\varphi]{} & A \end{array} \qquad \begin{array}{ccc} \operatorname{Spec}\bar{A} & \xrightarrow{\bar{\Phi}} & \operatorname{Spec}\bar{A} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \operatorname{Spec}A & \xrightarrow[\Phi]{} & \operatorname{Spec}A \end{array} \qquad \alpha\circ\bar{\Phi}=\Phi\circ\alpha,$$

where $\bar{A}=\bar{\mathbf{k}}\otimes_{\mathbf{k}}A=\bar{\mathbf{k}}^{[2]}$ and $\bar{\Phi}$ is a birational morphism.

(a) Since $\bar{A}$ is integral over A, we may choose a curve $\bar{C}\subset\operatorname{Spec}\bar{A}$ such that $\alpha(\bar{C})=C$. Then $\bar{C}$ is a missing curve of $\bar{\Phi}$, so, by [7, 4.3(c)], $\bar{C}$ is a rational curve with one place at infinity. It follows that the coordinate algebra S of $\bar{C}$ satisfies $S^*=\bar{\mathbf{k}}^*$. As the coordinate algebra R of C satisfies $R\subseteq S$, we deduce that C has one place at infinity and that $R^*\subseteq S^*=\bar{\mathbf{k}}^*$, which proves (a).

For the proof of (b), let G be the group of A-automorphisms of $\bar{A}$. It follows from [10, Theorem 5, p. 33] that, for each $\mathfrak{p}\in\operatorname{Spec}A$, the subset $\alpha^{-1}(\mathfrak{p})$ of $\operatorname{Spec}\bar{A}$ is equal to an orbit of the action of G on $\operatorname{Spec}\bar{A}$.

Observe that, for any $\mathfrak{p}\in\operatorname{Spec}A$, the condition "$V(\mathfrak{p})$ is a curve in $\operatorname{Spec}A$ which is not a missing curve of Φ" is equivalent to "$\operatorname{ht}\mathfrak{p}=1$ and $\mathfrak{p}\in\operatorname{im}(\Phi)$." (We are using the fact that $\operatorname{im}(\Phi)$ is a constructible subset of $\operatorname{Spec}A$.) Moreover, if $\mathfrak{p}$ satisfies these equivalent conditions and $\mathfrak{q}\in\Phi^{-1}(\mathfrak{p})$, then $\operatorname{ht}\mathfrak{q}=1$ and Φ maps the curve $V(\mathfrak{q})$ to a dense subset of $V(\mathfrak{p})$. Thus assertion (b) can be stated as follows: *for each* $\mathfrak{p}\in\operatorname{Spec}A$ *such that* $\operatorname{ht}\mathfrak{p}=1$, $|\Phi^{-1}(\mathfrak{p})|\le 1$. Let $\mathfrak{p}\in\operatorname{Spec}A$ be such that $\operatorname{ht}\mathfrak{p}=1$, and let $\mathfrak{q}_1,\mathfrak{q}_2\in\Phi^{-1}(\mathfrak{p})$. Since $\bar{A}$ is integral over A, we may choose $\bar{\mathfrak{q}}_1,\bar{\mathfrak{q}}_2\in\operatorname{Spec}\bar{A}$ such that $\alpha(\bar{\mathfrak{q}}_i)=\mathfrak{q}_i$ for $i=1,2$. Then $\alpha(\bar{\Phi}(\bar{\mathfrak{q}}_i))=\Phi(\alpha(\bar{\mathfrak{q}}_i))=\Phi(\mathfrak{q}_i)=\mathfrak{p}$ $(i=1,2)$; that is, $\bar{\Phi}(\bar{\mathfrak{q}}_1)$ and $\bar{\Phi}(\bar{\mathfrak{q}}_2)$ lie over $\mathfrak{p}$. Then, by [10, Theorem 5, p. 33], there exists $\Theta\in G$ such that $\Theta(\bar{\Phi}(\bar{\mathfrak{q}}_1))=\bar{\Phi}(\bar{\mathfrak{q}}_2)$. As $\Theta\circ\bar{\Phi}=\bar{\Phi}\circ\Theta$, we have $\bar{\Phi}(\Theta(\bar{\mathfrak{q}}_1))=\bar{\Phi}(\bar{\mathfrak{q}}_2)$. If we define $\bar{\mathfrak{p}}=\bar{\Phi}(\bar{\mathfrak{q}}_2)$, then $\{\Theta(\bar{\mathfrak{q}}_1),\bar{\mathfrak{q}}_2\}\subseteq\bar{\Phi}^{-1}(\bar{\mathfrak{p}})$; as $\alpha(\bar{\mathfrak{p}})=\alpha(\bar{\Phi}(\bar{\mathfrak{q}}_2))=\Phi(\alpha(\bar{\mathfrak{q}}_2))=\Phi(\mathfrak{q}_2)=\mathfrak{p}$ and $\operatorname{ht}\mathfrak{p}=1$, we have $\operatorname{ht}\bar{\mathfrak{p}}=1$; by the case $\mathbf{k}=\bar{\mathbf{k}}$ of (b), $|\bar{\Phi}^{-1}(\bar{\mathfrak{p}})|\le 1$, so $\Theta(\bar{\mathfrak{q}}_1)=\bar{\mathfrak{q}}_2$; consequently, $\alpha(\bar{\mathfrak{q}}_1)=\alpha(\bar{\mathfrak{q}}_2)$, that is, $\mathfrak{q}_1=\mathfrak{q}_2$. □

NOTATION 2.8
Consider morphisms $\mathbb{A}^2\xrightarrow{\Phi}\mathbb{A}^2\xrightarrow{f}\mathbb{A}^1$ where Φ is birational and f is dominant. Then we write

$$\operatorname{Miss}_{\mathrm{hor}}(\Phi,f)=\big\{C\in\operatorname{Miss}(\Phi)\bigm| f(C)\text{ is a dense subset of }\mathbb{A}^1\big\}.$$

We refer to the elements of $\mathrm{Miss}_{\mathrm{hor}}(\Phi, f)$ as the "f-horizontal" missing curves of Φ.

Note that, in the following result, $\mathrm{Miss}_{\mathrm{hor}}(\Phi, f)$ may be empty. See the introduction for the notation $A \preceq B$.

PROPOSITION 2.9
Let $A \preceq B$ and $F \in A \setminus \mathbf{k}$, and consider the morphisms

$$\operatorname{Spec} B \xrightarrow{\Phi} \operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} \mathbf{k}[F]$$

determined by the inclusions $\mathbf{k}[F] \hookrightarrow A \hookrightarrow B$. Let $C_1, \dots, C_h$ be the distinct elements of $\mathrm{Miss}_{\mathrm{hor}}(\Phi, f)$ and, for each $i \in \{1, \dots, h\}$, let δ_i be the degree† *of the morphism $f|_{C_i} : C_i \to \operatorname{Spec} \mathbf{k}[F]$.*

(a) *We have $\Delta(F, B) = [\Delta(F, A), \delta_1, \dots, \delta_h]$; that is, $\Delta(F, B)$ is the concatenation of $\Delta(F, A)$ and $[\delta_1, \dots, \delta_h]$. In particular,* $\operatorname{dic}(F, B) = \operatorname{dic}(F, A) + |\mathrm{Miss}_{\mathrm{hor}}(\Phi, f)|$.

(b) *For each $i \in \{1, \dots, h\}$, $\delta_i = 1 \Leftrightarrow C_i \in \Gamma(F, A)$.*

Proof
(a) Given local domains $(\mathcal{O}, \mathfrak{m})$ and $(\mathcal{O}', \mathfrak{m}')$, we write $\mathcal{O} \trianglelefteq \mathcal{O}'$ to indicate that $\mathcal{O}$ is a subring of $\mathcal{O}'$ and that $\mathfrak{m}' \cap \mathcal{O} = \mathfrak{m}$ (i.e., $\mathcal{O}'$ dominates $\mathcal{O}$). Note that if $\mathcal{O} \trianglelefteq \mathcal{O}'$, $\operatorname{Frac} \mathcal{O} = \operatorname{Frac} \mathcal{O}'$, and $\mathcal{O}$ is a valuation ring, then $\mathcal{O} = \mathcal{O}'$. Let $K = \operatorname{Frac} A = \operatorname{Frac} B$, and, for each $i \in \{1, \dots, h\}$, let $\mathfrak{p}_i \in \operatorname{Spec} A$ be the generic point of C_i. We claim that

$$\mathbb{V}^\infty(F, B) = \mathbb{V}^\infty(F, A) \cup \{A_{\mathfrak{p}_1}, \dots, A_{\mathfrak{p}_h}\}. \tag{3}$$

It is obvious that $\mathbb{V}^\infty(F, A) \subseteq \mathbb{V}^\infty(F, B)$. Let $i \in \{1, \dots, h\}$. As C_i is f-horizontal, $\mathfrak{p}_i \cap \mathbf{k}[F] = 0$ and hence $A_{\mathfrak{p}_i} \in \mathbb{V}(K/\mathbf{k}(F))$. Since $C_i \in \mathrm{Miss}(\Phi)$, no $\mathfrak{q} \in \operatorname{Spec} B$ satisfies $\mathfrak{q} \cap A = \mathfrak{p}_i$, so $B \nsubseteq A_{\mathfrak{p}_i}$ and hence $A_{\mathfrak{p}_i} \in \mathbb{V}^\infty(F, B)$. This proves $\supseteq$ in (3).

Consider $R \in \mathbb{V}^\infty(F, B) \setminus \mathbb{V}^\infty(F, A)$. Then $A \subseteq R$, so $\mathfrak{m}_R \cap A$ is an element $\mathfrak{p}$ of $\operatorname{Spec} A$. As $\mathbf{k}(F) \subset R$, $\mathfrak{p} \cap \mathbf{k}[F] = 0$; so $\operatorname{ht} \mathfrak{p} < 2$. If $\mathfrak{p} = 0$, then $\operatorname{Frac} A \subseteq R$, which contradicts $R \in \mathbb{V}(K/\mathbf{k}(F))$; so $\operatorname{ht} \mathfrak{p} = 1$. Then $A_{\mathfrak{p}} \trianglelefteq R$ where $A_{\mathfrak{p}}$ is a valuation ring, so $A_{\mathfrak{p}} = R$. It also follows that $C = V(\mathfrak{p})$ is a curve in $\operatorname{Spec} A$. If $C \notin \mathrm{Miss}(\Phi)$, then there exists $\mathfrak{q} \in \operatorname{Spec} B$ satisfying $\mathfrak{q} \cap A = \mathfrak{p}$; then $R = A_{\mathfrak{p}} \trianglelefteq B_{\mathfrak{q}}$, so $B_{\mathfrak{q}} = R$ and hence $B \subseteq R$, a contradiction; so $C \in \mathrm{Miss}(\Phi)$. As $\mathfrak{p} \cap \mathbf{k}[F] = 0$, $C \in \mathrm{Miss}_{\mathrm{hor}}(\Phi, f)$, and consequently $\mathfrak{p} = \mathfrak{p}_i$ for some $i \in \{1, \dots, h\}$. Then $R = A_{\mathfrak{p}_i}$, and (3) is proved.

What is meant by $f|_{C_i} : C_i \to \operatorname{Spec} \mathbf{k}[F]$ is really $\operatorname{Spec}(A/\mathfrak{p}_i) \to \operatorname{Spec} \mathbf{k}[F]$, that is, the morphism determined by the injective homomorphism $\mathbf{k}[F] \to A/\mathfrak{p}_i$. Let us abuse notation and write $\mathbf{k}[F] \subseteq A/\mathfrak{p}_i$. Then

$$\delta_i = \deg(f|_{C_i}) = \big[\operatorname{Frac}(A/\mathfrak{p}_i) : \mathbf{k}(F)\big] = \big[A_{\mathfrak{p}_i}/\mathfrak{p}_i A_{\mathfrak{p}_i} : \mathbf{k}(F)\big]$$

† Let $R \subseteq S$ be integral domains and $f : \operatorname{Spec} S \to \operatorname{Spec} R$ the corresponding morphism of schemes. Assume that $\operatorname{Frac} S$ is a finite extension of $\operatorname{Frac} R$. Then we define $\deg f = [\operatorname{Frac} S : \operatorname{Frac} R]$.

for each $i \in \{1,\dots,h\}$. It follows that $\Delta(F,B) = [\Delta(F,A),\delta_1,\dots,\delta_h]$, because (clearly) (3) is a disjoint union and $A_{\mathfrak{p}_1},\dots,A_{\mathfrak{p}_h}$ are distinct. So (a) is proved.

(b) Let $i \in \{1,\dots,h\}$, and consider $\mathbf{k}[F] \subseteq A/\mathfrak{p}_i$. The condition $C_i \in \Gamma(F,A)$ is equivalent to $\mathbf{k}[F] = A/\mathfrak{p}_i$, and $\delta_i = 1$ is equivalent to $\mathbf{k}(F) = \operatorname{Frac}(A/\mathfrak{p}_i)$. So it is clear that $C_i \in \Gamma(F,A)$ implies $\delta_i = 1$. Conversely, assume that $\delta_i = 1$. Then $\mathbf{k}[F] \subseteq A/\mathfrak{p}_i \subset \mathbf{k}(F)$, which implies that $A/\mathfrak{p}_i$ is a localization of $\mathbf{k}[F]$. Since the units of $A/\mathfrak{p}_i$ are algebraic over $\mathbf{k}$ by Lemma 2.7, the localization must be trivial, so $\mathbf{k}[F] = A/\mathfrak{p}_i$, and consequently $C_i \in \Gamma(F,A)$. So (b) is proved. □

3. Rectangular polynomials: Properties of the set $\Gamma(F,A)$

DEFINITION 3.1

Let $A = \mathbf{k}^{[2]}$ and $\mathbb{A}^2 = \mathbb{A}^2_{\mathbf{k}} = \operatorname{Spec} A$.

(i) A *variable* of A is an element $F \in A$ satisfying $A = \mathbf{k}[F,G]$ for some $G \in A$.

(ii) A curve $C \subset \mathbb{A}^2$ is called a *coordinate line* if $C = V(F)$ for some variable F of A.

REMARK 3.2

A curve $C \subset \mathbb{A}^2$ is called a *line* if $C \cong \mathbb{A}^1$. It is clear that every coordinate line is a line, and the Abhyankar–Moh–Suzuki theorem (see [2], [16]) states that the converse is true if $\operatorname{char}\mathbf{k} = 0$. It is known that not all lines are coordinate lines if $\operatorname{char}\mathbf{k} > 0$ (on this subject see, e.g., [8] for a survey).

DEFINITION 3.3

Let $A = \mathbf{k}^{[2]}$.

(1) Given $F \in A$ and a pair $\gamma = (X,Y)$ such that $A = \mathbf{k}[X,Y]$, write $F = \sum_{i,j} a_{ij}X^iY^j$ where $a_{ij} \in \mathbf{k}$ for all i, j; then $\operatorname{supp}_\gamma(F) = \{(i,j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\}$ is called the *support of F with respect to γ*.

(2) Given a subset S of $\mathbb{R}^2$, let $\langle S\rangle$ denote its convex hull.

(3) Given $F \in A$, we write $\operatorname{Rec}(F,A)$ for the set of ordered pairs $\gamma = (X,Y)$ satisfying $A = \mathbf{k}[X,Y]$, and

$$\text{there exist } m,n \geq 1 \text{ such that } (m,n) \in \operatorname{supp}_\gamma(F) \subseteq \big\langle (0,0),(m,0),(0,n),(m,n)\big\rangle.$$

Let $\operatorname{Rec}^+(F,A)$ be the set of $\gamma = (X,Y) \in \operatorname{Rec}(F,A)$ satisfying the additional condition $m \leq n$. Clearly,

$$\operatorname{Rec}^+(F,A) \neq \varnothing \Leftrightarrow \operatorname{Rec}(F,A) \neq \varnothing.$$

(4) By a *rectangular element of A* we mean an $F \in A$ satisfying $\operatorname{Rec}(F,A) \neq \varnothing$.

Some examples: no variable of $A = \mathbf{k}^{[2]}$ is rectangular; the polynomial $X^2 + Y^2$ is a rectangular element of $\mathbb{C}[X,Y]$ but not of $\mathbb{R}[X,Y]$. See Theorem 4.1, below, to understand why the notion of rectangular element is relevant for studying field generators.

LEMMA 3.4

Let F be a rectangular element of $A = \mathbf{k}^{[2]}$.

(a) *If $(X,Y) \in \operatorname{Rec}(F,A)$, then*

$$\operatorname{Rec}(F,A) = \big\{(aX+b, cY+d) \mid a,b,c,d \in \mathbf{k}, ac \neq 0\big\} \\ \cup \big\{(cY+d, aX+b) \mid a,b,c,d \in \mathbf{k}, ac \neq 0\big\}.$$

(b) *Up to order, the pair (m,n) in Definition 3.3(3) depends only on (F,A), that is, is independent of the choice of $\gamma \in \operatorname{Rec}(F,A)$.*

Proof

Consider $\gamma = (X,Y)$, $\gamma' = (U,V) \in \operatorname{Rec}(F,A)$, and $m,n,m',n' \geq 1$ such that

$$(m,n) \in \operatorname{supp}_\gamma(F) \subseteq \big\langle (0,0),(m,0),(0,n),(m,n)\big\rangle, \tag{4}$$

$$(m',n') \in \operatorname{supp}_{\gamma'}(F) \subseteq \big\langle (0,0),(m',0),(0,n'),(m',n')\big\rangle. \tag{5}$$

We have $A = \mathbf{k}[U,V] = \mathbf{k}[X,Y] = R[X]$ where $R = \mathbf{k}[Y]$. Given $G \in A \setminus \{0\}$, write $G = \sum_{i=0}^{d} G_i X^i$ ($G_i \in R$, $G_d \neq 0$), and define $\deg_X(G) = d$ and $\operatorname{lco}(G) = G_d$.

Let $\alpha = \deg_X(U)$, $\beta = \deg_X(V)$. We claim that $\min(\alpha,\beta) = 0$. To see this, assume that $\alpha,\beta > 0$. Since U, V are then variables of $\mathbf{k}[X,Y]$ not belonging to $\mathbf{k}[Y]$, we have $\operatorname{lco}(U), \operatorname{lco}(V) \in \mathbf{k}^*$. For each $(i,j) \in \operatorname{supp}_{\gamma'}(F)$ we have $i \leq m'$ and $j \leq n'$ by (5), so

$$\deg_X(U^i V^j) = \alpha i + \beta j \leq \alpha m' + \beta n' = \deg_X(U^{m'} V^{n'}),$$

where equality holds if and only if $(i,j) = (m',n')$; so $\operatorname{lco}(F) = \lambda \operatorname{lco}(U^{m'}V^{n'})$ for some $\lambda \in \mathbf{k}^*$, and consequently $\operatorname{lco}(F) \in \mathbf{k}^*$. However, (4) implies that $\operatorname{lco}(F)$ is a polynomial in $\mathbf{k}[Y]$ of degree $n \geq 1$, a contradiction. This shows that $\min(\deg_X(U), \deg_X(V)) = 0$, and we obtain $\min(\deg_Y(U), \deg_Y(V)) = 0$ by symmetry. So, in (a), the left-hand side is included in the right-hand side. As the reverse inclusion is trivial, (a) is proved. Assertion (b) follows from (a). $\square$

DEFINITION 3.5

For each rectangular element F of $A = \mathbf{k}^{[2]}$ we define

$$\operatorname{bideg}_A(F) = \big(\deg_X(F), \deg_Y(F)\big) \quad \text{for any } (X,Y) \in \operatorname{Rec}^+(F,A).$$

By Lemma 3.4, $\operatorname{bideg}_A(F)$ is well defined and depends only on (F,A).

Moreover, if $(m,n) = \operatorname{bideg}_A(F)$ and $\gamma \in \operatorname{Rec}^+(F,A)$, then

$$1 \leq m \leq n \qquad \text{and} \qquad (m,n) \in \operatorname{supp}_\gamma(F) \subseteq \big\langle (0,0),(m,0),(0,n),(m,n)\big\rangle.$$

REMARK 3.6

Let F be a rectangular element of $A = \mathbf{k}^{[2]}$, and let $(m,n) = \operatorname{bideg}_A(F)$. It follows from Lemma 3.4(b) that if $m = n$, then $\operatorname{Rec}^+(F,A) = \operatorname{Rec}(F,A)$.

We shall now consider the set $\Gamma_{\text{alg}}(F,A)$ defined in the introduction. We first show that $\Gamma_{\text{alg}}(F,A)$ is easy to describe when F is a rectangular element of A.

LEMMA 3.7

Let F be a rectangular element of $A=\mathbf{k}^{[2]}$, let $\gamma=(X,Y)\in\operatorname{Rec}(F,A)$, and let $(m,n)=(\deg_X(F),\deg_Y(F))$. Recall that

$$(m,n)\in\operatorname{supp}_\gamma(F)\subseteq\big\langle(0,0),(m,0),(0,n),(m,n)\big\rangle.$$

Write $F=\sum_{i,j}a_{ij}X^iY^j$ $(a_{ij}\in\mathbf{k})$, and define

$$F_{\text{ver}}(Y)=\sum_{j=0}^{n}a_{m,j}Y^j \qquad \text{and} \qquad F_{\text{hor}}(X)=\sum_{i=0}^{m}a_{i,n}X^j.$$

(a) *$\Gamma_{\text{alg}}(F,A)$ is equal to*

$$\big\{(X-a)\,\big|\,a\in\mathbf{k}\text{ and }\deg F(a,Y)=1\big\}\cup\big\{(Y-b)\,\big|\,b\in\mathbf{k}\text{ and }\deg F(X,b)=1\big\}.$$

(b) *If $\min(m,n)>1$, then $\Gamma_{\text{alg}}(F,A)$ is included in*

$$\big\{(X-a)\,\big|\,a\in\mathbf{k}\text{ and }F_{\text{hor}}(a)=0\big\}\cup\big\{(Y-b)\,\big|\,b\in\mathbf{k}\text{ and }F_{\text{ver}}(b)=0\big\}.$$

Proof

Let $\mathfrak{p}\in\Gamma_{\text{alg}}(F,A)$. As $A/\mathfrak{p}=\mathbf{k}^{[1]}$, we may consider a surjective $\mathbf{k}$-homomorphism $\pi:A\to\mathbf{k}[t]=\mathbf{k}^{[1]}$ such that $\ker\pi=\mathfrak{p}$. Let $x(t)=\pi(X)$ and $y(t)=\pi(Y)$. Since the composite $\mathbf{k}[F]\hookrightarrow A\xrightarrow{\pi}\mathbf{k}[t]$ is an isomorphism and $\pi(F)=F(x(t),y(t))$, we have $\deg_t F(x(t),y(t))=1$. If $x(t),y(t)\notin\mathbf{k}$, then

$$1=\deg_t F\big(x(t),y(t)\big)=m\deg_t x(t)+n\deg_t y(t)\geq m+n\geq 2,$$

which is absurd, so $x(t)\in\mathbf{k}$ or $y(t)\in\mathbf{k}$, from which we obtain $\mathfrak{p}=(Z-\lambda)$ for some $Z\in\{X,Y\}$ and $\lambda\in\mathbf{k}$. It is easily verified that $(X-\lambda)\in\Gamma_{\text{alg}}(F,A)\Leftrightarrow\deg F(\lambda,Y)=1$ and that $(Y-\lambda)\in\Gamma_{\text{alg}}(F,A)\Leftrightarrow\deg F(X,\lambda)=1$, so (a) is proved.

Assume that $\min(m,n)>1$, and consider $(Z-\lambda)\in\Gamma_{\text{alg}}(F,A)$, where $Z\in\{X,Y\}$ and $\lambda\in\mathbf{k}$. If $Z=X$, then $\deg F(\lambda,Y)=1$; as $n>1$, it follows that $F_{\text{hor}}(\lambda)=0$. Similarly, if $Z=Y$, then $F_{\text{ver}}(\lambda)=0$. This proves (b). □

Note the following obvious consequence of Lemma 3.7

COROLLARY 3.8

Let F be a rectangular element of $A=\mathbf{k}^{[2]}$. Then each element of $\Gamma_{\text{alg}}(F,A)$ is of the form (G) for some variable G of A, and each element of $\Gamma(F,A)$ is a coordinate line in $\mathbb{A}^2=\operatorname{Spec}A$.

REMARK 3.9

If G is any element of $A=\mathbf{k}^{[2]}$ such that $A/(G)=\mathbf{k}^{[1]}$ (note that G is not necessarily a variable of A if $\operatorname{char}\mathbf{k}>0$; see Remark 3.2), then there exists $F\in A\setminus\mathbf{k}$ such that $(G)\in\Gamma_{\text{alg}}(F,A)$. (For a proof, write $A/(G)=\mathbf{k}[t]$, and choose

$F \in A$ whose image via $A \to A/(G)$ is t.) So Corollary 3.8 states a nontrivial property of rectangular polynomials.

We shall now ask how $\Gamma(F, A)$ behaves with respect to two operations: extending the base field (Lemma 3.10) and birationally extending the ring A (Lemma 3.11).

LEMMA 3.10

Let $K/\mathbf{k}$ be a field extension; let $A = \mathbf{k}^{[2]}$ and $\bar{A} = K \otimes_{\mathbf{k}} A = K^{[2]}$. Consider $F \in A \subseteq \bar{A}$ such that $F \notin \mathbf{k}$. Then

$$\Gamma_{\text{alg}}(F, A) \to \Gamma_{\text{alg}}(F, \bar{A}), \qquad \mathfrak{p} \mapsto \mathfrak{p}\bar{A} \tag{6}$$

is a well-defined injective map. In particular, $|\Gamma_{\text{alg}}(F, A)| \le |\Gamma_{\text{alg}}(F, \bar{A})|$.

Proof

Consider $\mathfrak{p} \in \Gamma_{\text{alg}}(F, A)$, and write $\mathfrak{p} = GA$ where $G \in A$. Then the composite $\mathbf{k}[F] \to A \to A/GA$ is an isomorphism $\mathbf{k}[F] \to A/GA$. Applying the functor $K \otimes_{\mathbf{k}} \_$ to $\mathbf{k}[F] \to A \to A/GA$ yields $K[F] \to \bar{A} \to \bar{A}/G\bar{A}$ where $K[F] \to \bar{A}/G\bar{A}$ is an isomorphism, so $\mathfrak{p}\bar{A} = G\bar{A}$ is an element of $\Gamma_{\text{alg}}(F, \bar{A})$ and (6) is well defined. Since $A \to \bar{A}$ is a faithfully flat homomorphism, we have $A \cap I\bar{A} = I$ for every ideal I of A (cf. [10, (4.C)(ii), p. 27]), so (6) is injective. □

LEMMA 3.11

Consider $F \in A \preceq A'$, where $F \notin \mathbf{k}$. Let $\Phi : \operatorname{Spec} A' \to \operatorname{Spec} A$ be the morphism determined by $A \hookrightarrow A'$.

(a) *For each $C' \in \Gamma(F, A')$, $\Phi(C')$ is a curve in* $\operatorname{Spec} A$ *and an element of $\Gamma(F, A)$.*

(b) *The set map $\gamma : \Gamma(F, A') \to \Gamma(F, A)$, $C' \mapsto \Phi(C')$, is injective.*

(c) *Let $C \in \Gamma(F, A)$.*

(i) *If $C' \subset \operatorname{Spec} A'$ is a curve such that $\Phi(C') = C$ and such that $\Phi|_{C'} : C' \to C$ is an isomorphism, then $C' \in \Gamma(F, A')$ and $\gamma(C') = C$.*

(ii) *$C \in \operatorname{im} \gamma$ if and only if there exists a curve $C' \subset \operatorname{Spec} A'$ such that $\Phi(C') = C$ and such that $\Phi|_{C'} : C' \to C$ is an isomorphism.*

Proof

Consider an element C' of $\Gamma(F, A')$, and let C be the closure of $\Phi(C')$ in $\operatorname{Spec} A$. As $C' \in \Gamma(F, A')$, we have morphisms $C' \to C \to \operatorname{Spec} \mathbf{k}[F]$ whose composition is an isomorphism (in particular, C is a curve). Considering the coordinate rings $\mathbf{k}[C]$ and $\mathbf{k}[C']$ of C and C', we have injective homomorphisms $\mathbf{k}[F] \to \mathbf{k}[C] \to \mathbf{k}[C']$ such that the composition $\mathbf{k}[F] \to \mathbf{k}[C']$ is an isomorphism; thus $\mathbf{k}[F] = \mathbf{k}[C] = \mathbf{k}[C']$, and hence $\Phi|_{C'} : C' \to C$ and $f|_C : C \to \operatorname{Spec} \mathbf{k}[F]$ are two isomorphisms. This implies that $\Phi(C') = C$ and $C \in \Gamma(F, A)$, which proves (a). Assertion (b) follows from Lemma 2.7(b), and the straightforward verification of (c) is left to the reader. □

4. The cardinality of $\Gamma(F, A)$ for field generators

The aim of this section is to prove Theorem 4.11, which asserts that $|\Gamma(F,A)| \leq 2$ whenever F is a field generator of $A = \mathbf{k}^{[2]}$ which is not of the form $\alpha(Y)X + \beta(Y)$. In the course of proving that result, we obtain Theorem 4.8, which gives new information on "small" field generators.

We shall make essential use of two results of Russell on field generators. The first one (Theorem 4.1) is valid over an arbitrary field $\mathbf{k}$.

THEOREM 4.1 ([13, 3.7 AND 4.5])
If F is a field generator of $A = \mathbf{k}^{[2]}$ which is not a variable of A, then F is a rectangular element of A.

In view of Theorem 4.1 and of the objective of this section, it is interesting to note that there is no upper bound on the cardinality of $\Gamma(F, A)$ for rectangular elements F of $A = \mathbf{k}^{[2]}$.

EXAMPLE 4.2
(a) Let $F = u(Y)X^2 + X \in A = \mathbf{k}[X,Y]$ where $\deg u(Y) > 1$; then Lemma 3.7 implies that $|\Gamma(F,A)|$ equals the number of roots of $u(Y)$.

(b) Assume that $\mathbf{k}$ is infinite, and let $F = \alpha(Y)X + \beta(Y) \in A = \mathbf{k}[X,Y]$ where $\deg \beta(Y) \leq \deg \alpha(Y) > 0$; then Lemma 3.7 implies that $|\Gamma(F,A)| = |\mathbf{k}|$. Note that F is a good field generator of A since $\mathbf{k}(F,Y) = \mathbf{k}(X,Y)$.

The other result of Russell that we need is [14, 1.6]. Because there seems to be an ambiguity in the statement of that result, let us explain the following. (The statement of that result is too long to be reproduced here.) Suppose that $F(X,Y) \in \mathbf{k}[X,Y]$ satisfies all hypotheses of that theorem (in particular $\mu_1 \leq \mu_2$). If $\mu_1 < \mu_2$, then the statement is clear, and all conclusions of [14, 1.6] are true for $F(X,Y)$. If $\mu_1 = \mu_2$, then both $F(X,Y)$ and $F(Y,X)$ satisfy the hypotheses of the theorem, but the proof only shows that the theorem is true *for at least one of these polynomials*. (In the proof, just after (5) on [14, p. 320], one reads, "It follows that $h_1 = 1$ or $l_1 = 1$. Say $h_1 = 1$". The intended meaning of that sentence is: "Replacing $F(X,Y)$ by $F(Y,X)$ if necessary, we may assume that $h_1 = 1$".) There are indeed examples with $\mu_1 = \mu_2$ where the theorem is false for $F(X,Y)$ and true for $F(Y,X)$. We shall use [14, 1.6] in the proofs of Lemma 4.6(d) and Theorem 4.8; the ambiguity arises in the first proof only.

DEFINITION 4.3
Let $F \in A = \mathbf{k}^{[2]}$, and let $\gamma = (X,Y)$ be such that $A = \mathbf{k}[X,Y]$. We say that F is *γ-small* in A if the following conditions are satisfied:

- $F \notin \mathbf{k}[X, u(X)Y]$ for all $u(X) \in \mathbf{k}[X] \setminus \mathbf{k}$,
- $F \notin \mathbf{k}[Xv(Y), Y]$ for all $v(Y) \in \mathbf{k}[Y] \setminus \mathbf{k}$.

REMARK 4.4

It follows from Lemma 3.4 that, for a rectangular element F of $A = \mathbf{k}^{[2]}$, the following are equivalent:

- F is γ-small in A for at least one $\gamma \in \mathrm{Rec}(F, A)$,
- F is γ-small in A for all $\gamma \in \mathrm{Rec}(F, A)$.

DEFINITION 4.5

By a *small* field generator of $A = \mathbf{k}^{[2]}$, we mean a field generator F of A for which there exists $\gamma \in \mathrm{Rec}(F, A)$ such that F is γ-small in A. (Then, by Remark 4.4, F is γ-small in A for every $\gamma \in \mathrm{Rec}(F, A)$.)

LEMMA 4.6

If F is a small field generator of $A = \mathbf{k}^{[2]}$, then the following hold.

(a) *We have $\mathrm{Rec}(F, A) \neq \varnothing$, and, for all $\gamma \in \mathrm{Rec}(F, A)$, F is γ-small in A.*
(b) *F is not a variable of A.*
(c) *The pair $(m, n) = \mathrm{bideg}_A(F)$ (cf. Definition 3.5) satisfies $2 \le m \le n$.*
(d) *If $\mathbf{k}$ is algebraically closed, then $m < n$.*

Proof

(a) The fact that $\mathrm{Rec}(F, A) \neq \varnothing$ is clear, and the other assertion is Remark 4.4. Since no variable of A is rectangular, (b) is true. (c) Pick $\gamma = (X, Y) \in \mathrm{Rec}^+(F, A)$; then F is γ-small and $(m, n) = (\deg_X(F), \deg_Y(F))$. We have $1 \le m \le n$ by definition. If $m = 1$, then $F = a(Y)X + b(Y)$ for some $a(Y), b(Y) \in \mathbf{k}[Y]$, $a(Y) \notin \mathbf{k}$; then $F \in \mathbf{k}[a(Y)X, Y]$, which contradicts the fact that F is γ-small.

(d) Proceeding by contradiction, assume that $m = n$. Pick $\gamma = (X, Y) \in \mathrm{Rec}^+(F, A)$, note that $A = \mathbf{k}[X, Y]$, and view $F \in A$ as a polynomial in X, Y. Then $F(X, Y) \in \mathbf{k}[X, Y]$ satisfies the hypothesis of [14, 1.6] (with $\mu_1 = m$ and $\mu_2 = n$, so $\mu_1 = \mu_2$), and so does $F(Y, X)$. By the discussion just after Example 4.2, there exists $G(X, Y) \in \{F(X, Y), F(Y, X)\}$ such that the conclusions of [14, 1.6] are true for $G(X, Y)$. So, by assertion (6) of that result, there exists $c \in \mathbf{k}$ such that $G(X, Y) \in \mathbf{k}[X, (X - c)Y]$. (Choosing such a $c \in \mathbf{k}$ is equivalent to the choice made in assertion (6).) It follows that $F(X, Y)$ belongs to $\mathbf{k}[X, (X - c)Y]$ or $\mathbf{k}[X(Y - c), Y]$, so F is not γ-small. This contradicts the hypothesis that F is a small field generator of A. □

The next paragraph introduces notation that we need for proving Theorem 4.8.

4.7.

Let $\mathbf{k}$ be an algebraically closed field, let $F = \sum_{i,j} a_{ij} X^i Y^j \in \mathbf{k}[X, Y]$ $(a_{ij} \in \mathbf{k})$ be a polynomial of degree $d > 0$, and let $F^*(W, X, Y) = \sum_{i,j} a_{ij} W^{d-i-j} X^i Y^j \in \mathbf{k}[W, X, Y]$ be the homogenization of F. Consider the pencil $\Lambda = \Lambda(F)$ on $\mathbb{P}^2$

defined by

$$\Lambda = \Lambda(F) = \big\{\mathrm{div}_0(\lambda_0 F^* - \lambda_1 W^d) \,\big|\, (\lambda_0 : \lambda_1) \in \mathbb{P}^1\big\},$$

where div_0 means "divisor of zeros" (of a homogeneous polynomial). Note that $dE_0 \in \Lambda$, where $E_0 = V(W)$ is the line at infinity. Let $\mathcal{B}$ be the set of base points of Λ, including infinitely near ones, and note that $\mathcal{B}$ is a finite set. Consider the minimal resolution of the base points of Λ,

$$S_n \xrightarrow{\pi_n} S_{n-1} \to \cdots \to S_1 \xrightarrow{\pi_1} S_0 = \mathbb{P}^2,$$

where $S_0, \dots, S_n$ are nonsingular projective surfaces and, for each $i \in \{1, \dots, n\}$, $\pi_i : S_i \to S_{i-1}$ is the blowup of S_{i-1} at the point $P_i \in S_{i-1}$ (so $\mathcal{B} = \{P_1, \dots, P_n\}$). Write $E_i = \pi_i^{-1}(P_i) \subset S_i$ for $i \in \{1, \dots, n\}$. Given a point P and a divisor D on some nonsingular surface, the multiplicity of P on D is denoted by $\mu(P, D)$. For each $i \in \{0, \dots, n\}$, let $\Lambda^{(i)}$ be the strict (or proper) transform of Λ on S_i (defined in [13], just before 2.4); note that $\Lambda^{(0)} = \Lambda$.

(a) Let $i \in \{1, \dots, n\}$. Then $\Lambda^{(i-1)}$ is a pencil on S_{i-1} and $P_i \in S_{i-1}$ is a base point of it. The positive integer $\inf\{\mu(P_i, D) \mid D \in \Lambda^{(i-1)}\}$ is called the multiplicity of P_i as a base point of Λ and is denoted by $\mu(P_i, \Lambda)$. We shall abbreviate it $\mu(P_i)$ in the proof of Theorem 4.8.

(b) Let P be a point of S_i, and let D be a divisor on S_j $(i, j \in \{0, \dots, n\})$. Define $\mu(P, D) \in \mathbb{Z}$ by

$$\mu(P, D) = \begin{cases} 0 & \text{if } i < j, \\ \mu(P, \tilde{D}) \text{ where } \tilde{D} \subset S_i \text{ is the strict transform of } D & \text{if } i \geq j. \end{cases}$$

So $\mu(P_i, E_j) \in \{0, 1\}$ for all $i, j \in \{1, \dots, n\}$.

(c) Let $\Lambda_\infty^{(n)}$ be the unique element of $\Lambda^{(n)}$ whose support contains the strict transform of E_0 (so $\Lambda_\infty^{(n)}$ is an effective divisor on S_n). Given $i \in \{0, \dots, n\}$ and an irreducible curve $D \subset S_i$, let $\varepsilon(D) \in \mathbb{N}$ be the coefficient of $\tilde{D}$ in the divisor $\Lambda_\infty^{(n)}$, where $\tilde{D} \subset S_n$ denotes the strict transform of D ($\varepsilon(D)$ is also defined in [13, 3.4]).

We shall now improve Theorem 4.1 in the special case where F is a small field generator and $\mathbf{k}$ is algebraically closed.

THEOREM 4.8

Assume that $\mathbf{k}$ is algebraically closed, and let F be a small field generator of $A = \mathbf{k}^{[2]}$. Then there exist $\gamma = (X, Y) \in \mathrm{Rec}^+(F)$ and $(m, n) \in \mathbb{N}^2$ satisfying

$$(*) \quad \begin{aligned} &(m, n) \in \mathrm{supp}_\gamma(F) \subseteq \big\langle (0,0), (m,0), (0, n-m), (m,n) \big\rangle \quad \text{and} \\ &(m, 0) \notin \mathrm{supp}_\gamma(F). \end{aligned}$$

Moreover, for any such $\gamma = (X, Y)$ and (m, n), the following hold.

(a) *We have $(m, n) = \mathrm{bideg}_A(F)$ and $2 \leq m < n$.*

(b) *If we write $F = \sum_{i,j} a_{ij} X^i Y^j$ $(a_{ij} \in \mathbf{k})$, then the polynomial $H(Y) = \sum_{j=0}^{n} a_{mj} Y^j$ has either one or two roots.*

(c) *If* $H(Y)$ *has one root, then* $\operatorname{supp}_\gamma(F) \subseteq \langle(0,0),(m-1,0),(0,n-m),(m,n)\rangle$ *and* $\Gamma_{\text{alg}}(F,A) \subseteq \{(X),(Y)\}$.

(d) *If* $H(Y)$ *has two roots, then the following hold.*

(i) *We have* $m \mid n$.

(ii) *Consider the homogenization* $F^*(W,X,Y) = \sum_{i,j} a_{ij} W^{m+n-i-j} X^i Y^j \in \mathbf{k}[W,X,Y]$ *of* F, *and note that* $(0:0:1) \in \mathbb{P}^2$ *is a common point of the curve* $V(F^*) \subset \mathbb{P}^2$ *and of the line at infinity* $V(W)$. *Then exactly one element* R *of* $\mathbb{V}^\infty(F,A)$ *is centered at* $(0:0:1)$, *and* $[R/\mathfrak{m}_R : \mathbf{k}(F)] = m$ *(cf. 2.2). That is,* F *has exactly one dicritical over the point* $(0:0:1)$ *and that dicritical has degree* m.

(iii) *Let* $k = \deg F(0,Y)$ *(where* $k=0$ *if* $F(0,Y)=0$*). Then* $m \mid k$, $k \le n-m$, *and* $\operatorname{supp}_\gamma(F) \subseteq \langle(0,0),(m,0),(0,k),(m,n)\rangle$.

(iv) *We have* $\Gamma_{\text{alg}}(F,A) \subseteq \{(Y-r_1),(Y-r_2)\}$, *where* r_1, r_2 *are the roots of* $H(Y)$.

The above theorem immediately implies the following (to be improved in Theorem 4.11).

COROLLARY 4.9

Assume that $\mathbf{k}$ *is algebraically closed, and let* F *be a small field generator of* $A = \mathbf{k}^{[2]}$. *Then* $|\Gamma(F,A)| \le 2$.

Proof of Theorem 4.8

Since F is a small field generator of A, $\operatorname{Rec}^+(F,A) \neq \varnothing$. So $(m,n) = \operatorname{bideg}_A(F)$ is defined, and $2 \le m < n$ by Lemma 4.6. For any choice of $\gamma_1 = (X_1,Y_1) \in \operatorname{Rec}^+(F,A)$, we have $(m,n) \in \operatorname{supp}_{\gamma_1}(F) \subseteq \langle(0,0),(m,0),(0,n),(m,n)\rangle$. The pair γ_1 being given, there exist $a,b \in \mathbf{k}$ such that the element $\gamma = (X,Y) = (X_1 - a, Y_1 - b)$ of $\operatorname{Rec}^+(F,A)$ satisfies $(m,0),(0,n) \notin \operatorname{supp}_\gamma(F)$. (We use the fact that $\mathbf{k}$ is algebraically closed here.) So there exists $\gamma = (X,Y) \in \operatorname{Rec}^+(F,A)$ satisfying

$$(m,n) \in \operatorname{supp}_\gamma(F) \subseteq \big\langle(0,0),(m,0),(0,n),(m,n)\big\rangle \quad \text{and} \quad (m,0),(0,n) \notin \operatorname{supp}_\gamma(F), \tag{7}$$

and we fix such a γ from now on. We shall prove that $(*)$ and (a)–(d) hold. In fact we have already noted that (a) is true.

Consider the pencil $\Lambda = \Lambda(F)$ on $\mathbb{P}^2$ (cf. Section 4.7). Then Λ has two base points on $\mathbb{P}^2$, namely, $p_0 = (0:0:1)$ and $q_0 = (0:1:0)$. Let $\mathcal{B}$ be the (finite) set of all base points of Λ, including infinitely near ones. Define a partial order $\le$ on the set $\mathcal{B}$ by declaring that $q < q' \Leftrightarrow q'$ is infinitely near q; note that p_0 and q_0 are exactly the minimal elements of the poset $(\mathcal{B},\le)$. For each $q \in \mathcal{B}$, let $\mu(q,\Lambda)$ denote the multiplicity of q as a base point of Λ (cf. Section 4.7), and let us use the abbreviation $\mu(q) = \mu(q,\Lambda)$. Given $q \in \mathcal{B}$, let E_q denote the exceptional curve created by blowing up q (in the notation of Section 4.7, if $q = P_i$, then $E_q = E_i$).

Given $p, q \in \mathcal{B}$, $\mu(p, E_q) \in \{0,1\}$ is defined in Section 4.7(b). We also define

$$\mathcal{B}_q = \{p \in \mathcal{B} \mid \mu(p, E_q) > 0\} \quad \text{for each } q \in \mathcal{B}. \tag{8}$$

Write $F = \sum_{i,j} a_{ij} X^i Y^j$ ($a_{ij} \in \mathbf{k}$) and $F^*(W, X, Y) = \sum_{i,j} a_{ij} W^{m+n-i-j} \times X^i Y^j$ as in the statement. Define $F_{p_0}(W, X) = F^*(W, X, 1)$ and $F_{q_0}(W, Y) = F^*(W, 1, Y)$; then

$$\begin{aligned} F_{p_0}(W, X) &= \sum_{i,j} a_{ij} W^{m+n-i-j} X^i \\ &= \sum_{i=0}^{m} a_{i,n} W^{m-i} X^i + \text{higher-order terms}, \end{aligned} \tag{9}$$

$$\begin{aligned} F_{q_0}(W, Y) &= \sum_{i,j} a_{ij} W^{m+n-i-j} Y^j \\ &= \sum_{j=0}^{n} a_{m,j} W^{n-j} Y^j + \text{higher-order terms}, \end{aligned} \tag{10}$$

where $a_{m,n} \neq 0$, so $\mu(p_0) = m < n = \mu(q_0)$. Let us define $\mu_1 = m$ and $\mu_2 = n$; then our notation is compatible with [14, 1.6], and the hypothesis of that result is satisfied. Since $\mu_1 < \mu_2$, the ambiguity noted just after Examples 4.2 does not arise here, so assertions (1)–(6) of [14, 1.6] are true for F. By part (2) of that result, there is a unique integer $s \geq 1$ such that (i) there are $s+1$ base points $p_0 < \cdots < p_s$ infinitely near to p_0 with $\mu(p_i) = \mu_1$ for $i = 0, \dots, s$, and (ii) $|\mathcal{B}_{p_s}| \neq 1$ (see (8)). This defines $s \geq 1$ and $p_0, p_1, \dots, p_s$. By [14, 1.6(3)], if we define $\nu = \mu_2 - s\mu_1$ then (since F is small by assumption) $0 \leq \nu < \mu_1$. That is,

$$\mu_2 = s\mu_1 + \nu \qquad \text{and} \qquad 0 \leq \nu < \mu_1 < \mu_2. \tag{11}$$

Since $s \geq 1$, there holds $\mu(p_0) = \mu(p_1) = m$; so, if we write $F_{p_0}(W, X) = \sum_{u,v} b_{u,v} W^u X^v$ ($b_{u,v} \in \mathbf{k}$), taking into account that $(0, n) \notin \operatorname{supp}_\gamma(F)$ we obtain $u + 2v \geq 2m$ for all (u, v) satisfying $b_{u,v} \neq 0$; it follows that $j - i \leq n - m$ for all $(i, j) \in \operatorname{supp}_\gamma(F)$. In view of (7), we conclude that condition $(*)$ is satisfied. There remains to prove (b)–(d).

Let us write $\mathcal{B}_{p_s} = \{p_{11}, \dots, p_{1h}\}$ and $\mathcal{B}_{q_0} = \{q_{11}, \dots, q_{1\ell}\}$ (see (8)); this defines h, ℓ, p_{1j}, q_{1j} in a way that is completely compatible with [14, 1.6]. Since F is small,

$$\ell = s + 1 \tag{12}$$

by [14, 1.6(5)], so in particular $\ell > 0$; however, h may be zero. Let us also set $\mathcal{B}' = \mathcal{B} \setminus (\{p_0, \dots, p_s\} \cup \{p_{11}, \dots, p_{1h}\} \cup \{q_0\} \cup \{q_{11}, \dots, q_{1\ell}\})$. Then we have the disjoint union

$$\mathcal{B} = \{p_0, \dots, p_s\} \cup \{p_{11}, \dots, p_{1h}\} \cup \{q_0\} \cup \{q_{11}, \dots, q_{1\ell}\} \cup \mathcal{B}'. \tag{13}$$

We claim that

(14) $H(Y)$ has either one or two roots, and if it has two, then $h = 0 = \nu$.

To see this, consider the number $\varepsilon(E_{q_0})$ defined in Section 4.7(c). By [13, 3.5.2, 3.5.4], $\varepsilon(E_{q_0}) = \varepsilon(E_0) - \mu(q_0) = d - \mu_2$, where E_0 is the line $\mathbb{P}^2 \setminus \mathbb{A}^2$ and $d = \deg(F) = \mu_1 + \mu_2$, so

$$\varepsilon(E_{q_0}) = \mu_1 > 0. \tag{15}$$

Let $\Lambda^{(q_0)}$ denote the strict transform of Λ with respect to the blowup of $\mathbb{P}^2$ at q_0. Since $\varepsilon(E_{q_0}) > 0$, [13, 3.5.6] implies that

(16) for a general member D of $\Lambda^{(q_0)}$, all points of $D \cap E_{q_0}$ belong to $\mathcal{B}$.

Let ρ be the number of distinct roots of $H(Y)$. Then the initial form $\sum_{j=0}^{n} a_{m,j} W^{n-j} Y^j$ of $F_{q_0}(W,Y)$ is a product $\prod_{i=1}^{\rho} L_i^{e_i}$ where $L_1, \dots, L_\rho \in \mathbf{k}[W,Y]$ are pairwise relatively prime linear forms and $e_i \ge 1$ for all $i = 1, \dots, \rho$. So, for a general member D of $\Lambda^{(q_0)}$, $D \cap E_{q_0}$ consists of exactly ρ points. In view of (16), these ρ points belong to $\mathcal{B}$; in fact they must be the minimal elements of $\{q_{11}, \dots, q_{1\ell}\}$ with respect to the order relation of the poset $(\mathcal{B}, \le)$. So $\{q_{11}, \dots, q_{1\ell}\}$ has exactly ρ minimal elements. Consequently, to prove (14) it suffices to show that

(17) $\{q_{11}, \dots, q_{1\ell}\}$ has 1 or 2 minimal elements, and if it has 2, then $h = 0 = \nu$.

Let us state the facts that we need for proving (17). We use the abbreviation $\mu(\mathcal{S}) = \sum_{p \in \mathcal{S}} \mu(p)$ for any subset $\mathcal{S}$ of $\mathcal{B}$:

(i) $\mu(\mathcal{B}_q) = \mu(q)$ for all $q \in \mathcal{B}$ satisfying $\varepsilon(E_q) > 0$;
(ii) $\mu(q) < \mu_1$ for all $q \in \mathcal{B} \setminus \{p_0, \dots, p_s, q_0\}$;
(iii) $\varepsilon(E_q) > 0$ for all $q \in \{q_{11}, \dots, q_{1\ell}\}$;
(iv) $\mu(p_i) = \mu_1$ for $0 \le i \le s$;
(v) $\sum_{i=1}^{h} \mu(p_{1i}) = \delta\mu_1$ where we define $\delta = \begin{cases} 0 & \text{if } h = 0, \\ 1 & \text{if } h \neq 0; \end{cases}$
(vi) $\mu(q_0) = \mu_2$ and $\sum_{i=1}^{\ell} \mu(q_{1i}) = \mu_2$;
(vii) $\mu(\mathcal{B}) = 3d - 2$, where $d = \deg(F) = \mu_1 + \mu_2$.

Proof of (i)–(vii). Assertion (i) is a well-known consequence of the intersection formula. Since F is small, [14, 1.6(5)] implies that $\mu(q) < \mu_1$ for all $q \in \{q_{11}, \dots, q_{1\ell}\}$; (ii) easily follows. Given $q \in \{q_{11}, \dots, q_{1\ell}\}$ we have $\mu(q, E_{q_0}) > 0$, so [13, 3.5.4] gives $\varepsilon(E_q) \ge \varepsilon(E_{q_0}) - \mu(q)$; then $\varepsilon(E_q) \ge \mu_1 - \mu(q) > 0$ by (15) and (ii), proving (iii); (iv) follows from the definition of $p_0, \dots, p_s$; (v) is obvious if $h = 0$, and it follows from either one of (i) or [14, 1.6(4)] if $h \neq 0$. We already noted that $\mu(q_0) = n = \mu_2$; the second part of (vi) then follows from either one of (i) or [14, 1.6(5)]; (vii) follows from [13, 3.3] with $g = 0$. This proves (i)–(vii).

Since (13) is a disjoint union,

$$\begin{aligned} \mu(\mathcal{B}) &= \sum_{i=0}^{s} \mu(p_i) + \sum_{i=1}^{h} \mu(p_{1i}) + \mu(q_0) + \sum_{i=1}^{\ell} \mu(q_{1i}) + \mu(\mathcal{B}') \\ &= (s+1)\mu_1 + \delta\mu_1 + 2\mu_2 + \mu(\mathcal{B}'). \end{aligned}$$

On the other hand, $\mu(\mathcal{B}) = 3d-2$, $d = \mu_1+\mu_2$ and $\mu_2 = s\mu_1+\nu$; so

$$\mu(\mathcal{B}') = (2-\delta)\mu_1 + \nu - 2. \tag{18}$$

Let $\mathcal{M}$ be the set of maximal elements of $\{q_{11},\dots,q_{1\ell}\}$, and let $\mathcal{N} = \{q_{11},\dots,q_{1\ell}\} \setminus \mathcal{M}$. For each $q \in \mathcal{M}$, we have $\varepsilon(E_q) > 0$ by (iii), so $\mu(q) = \mu(\mathcal{B}_q)$ by (i). Moreover, $\mathcal{B}_q \subseteq \mathcal{B}'$ for all $q \in \mathcal{M}$ and $\mathcal{B}_q \cap \mathcal{B}_{q'} = \varnothing$ for all choices of distinct $q, q' \in \mathcal{M}$. Thus

$$\mu(\mathcal{B}') \geq \mu\Big(\bigcup_{q\in\mathcal{M}} \mathcal{B}_q\Big) = \sum_{q\in\mathcal{M}} \mu(\mathcal{B}_q) = \sum_{q\in\mathcal{M}} \mu(q) = \mu(\mathcal{M}),$$

and combining this with (18) gives

$$\mu(\mathcal{M}) \leq (2-\delta)\mu_1 + \nu - 2. \tag{19}$$

We have $\mu(q) \leq \mu_1$ for all $q \in \mathcal{N}$ by (ii) and $|\mathcal{N}| = \ell - |\mathcal{M}| = s+1-|\mathcal{M}|$ by (12), so

$$\mu(\mathcal{N}) \leq \big(s+1-|\mathcal{M}|\big)\mu_1. \tag{20}$$

By (19) and (20),

$$\begin{aligned} s\mu_1 + \nu = \mu_2 = \mu\big(\{q_{11},\dots,q_{1\ell}\}\big) &= \mu(\mathcal{M}) + \mu(\mathcal{N}) \\ &\leq (2-\delta)\mu_1 + \nu - 2 + \big(s+1-|\mathcal{M}|\big)\mu_1, \end{aligned}$$

so $(|\mathcal{M}|+\delta-3)\mu_1 \leq -2$, and consequently

$$|\mathcal{M}| + \delta \leq 2. \tag{21}$$

Let ρ be the number of minimal elements of $\mathcal{B}_{q_0} = \{q_{11},\dots,q_{1\ell}\}$. Then it is clear that $\rho \leq |\mathcal{M}|$ (actually, $\rho = |\mathcal{M}|$ but we do not need to know this), so $\rho+\delta \leq 2$ by (21). So ρ is either 1 or 2, and if it is 2, then $\delta = 0$, so $h = 0$, so $\nu = 0$ (for the fact that $h = 0$ implies $\nu = 0$, see the line just after (8) in [14, p. 321]). This proves (17) and hence (14). In particular, assertion (b) is proved.

If $H(Y)$ has one root, then, since we arranged that $(m,0) \notin \operatorname{supp}_\gamma(F)$, $H(Y) = a_{mn}Y^n$. This proves the assertion about $\operatorname{supp}_\gamma(F)$, in (c). The assertion about $\Gamma_{\text{alg}}(F,A)$ then follows from Lemma 3.7(b). So (c) is proved.

To prove (d), consider the diagram

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, hook] \arrow[d, "f"'] & \mathbb{P}^2 \arrow[d, dashed, "\varphi_\Lambda"'] & X \arrow[l, "\pi"'] \arrow[dl, "\bar{f}"] \\ \mathbb{A}^1 \arrow[r, hook] & \mathbb{P}^1 & \end{tikzcd} \tag{22}$$

where f is the morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{k}[F]$, $\varphi_\Lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is the rational map determined by f (with domain $\mathbb{P}^2 \setminus \{p_0, q_0\}$), π is the blowup of $\mathbb{P}^2$ along $\mathcal{B}$, and $\bar{f}$ is a morphism; this gives rise to a diagram (2), which (as explained in Remark 2.3) allows us to identify the set $\mathbb{V}^\infty(F,A)$ of dicriticals with the set H^∞ of horizontal curves at infinity.

Assume that $H(Y)$ has two roots, $r_1, r_2 \in \mathbf{k}$. Then $h = \nu = 0$ by (14). Since $\nu = 0$, (11) implies that $m \mid n$, so (d-i) is true. The fact that $h = 0$ implies that $\{q \in \mathcal{B} \mid q \geq p_0\} = \{p_0, p_1, \ldots, p_s\}$; since $\mu(p_i) = \mu_1$ for all $i = 0, \ldots, s$, we see that $\tilde{E}_{p_s}$ (the strict transform of E_{p_s} on X) is the only element $C \in H^\infty$ satisfying $\pi(C) = \{p_0\}$; it follows that (d-ii) is true.

To prove (d-iii), we consider the elements $(m, 0)$ and $(0, m+n-k)$ of the support of $F_{p_0}(W, X)$ and the line segment $L \subset \mathbb{R}^2$ joining those two points. It follows from (d-ii) that L is an edge of the support of $F_{p_0}(W, X)$ and that m divides $m + n - k$. This implies that $m \mid k$ and that $\operatorname{supp}_\gamma(F) \subseteq \langle (0,0), (m,0), (0,k), (m,n) \rangle$, so (d-iii) is true.

By Lemma 3.7(b), we have $\Gamma_{\mathrm{alg}}(F, A) \subseteq \{(X), (Y - r_1), (Y - r_2)\}$. We have $(X) \notin \Gamma_{\mathrm{alg}}(F, A)$ because $\deg F(0, Y)$ is equal to the integer k of (d-iii), and $k \neq 1$. So (d-iv) is true. $\square$

LEMMA 4.10

Let $\mathbf{k}$ be an algebraically closed field, and let F be a rectangular element of $A = \mathbf{k}^{[2]}$ satisfying $|\Gamma_{\mathrm{alg}}(F, A)| \geq 2$. Suppose that $(X, Y) \in \operatorname{Rec}(F, A)$ and $v(Y) \in \mathbf{k}[Y] \setminus \mathbf{k}$ are such that F is a variable of $A' = \mathbf{k}[Xv(Y), Y]$. Then $F = cXv(Y) + w(Y)$ for some $c \in \mathbf{k}^*$ and $w(Y) \in \mathbf{k}[Y]$ such that $\deg w(Y) \leq \deg v(Y)$.

Proof

We may write $F(X, Y) = G(Xv(Y), Y)$ where $G(S, T)$ is a variable of $\mathbf{k}[S, T] = \mathbf{k}^{[2]}$. Since F is rectangular, we have $F \notin \mathbf{k}[Y]$, so $G(S, T) \notin \mathbf{k}[T]$. It follows that $\deg_S(G) \geq 1$, and it suffices to show that $\deg_S(G) = 1$. Proceeding by contradiction, we assume that

$$\deg_S(G) > 1.$$

Since $|\Gamma_{\mathrm{alg}}(F, A)| \geq 2$ and $(X, Y) \in \operatorname{Rec}(F, A)$, Lemma 3.7(a) implies that at least one of the following conditions holds:

(i) There exists $b \in \mathbf{k}$ such that $(Y - b) \in \Gamma_{\mathrm{alg}}(F, A)$;
(ii) there exist distinct $a_1, a_2 \in \mathbf{k}$ such that $(X - a_1), (X - a_2) \in \Gamma_{\mathrm{alg}}(F, A)$.

Suppose that (i) holds. Write $G(S, T) = \sum_{i=0}^m G_i(T) S^i$ where $G_m(T) \neq 0$. Then $m = \deg_S(G) \geq 2$; by a well-known property of variables, $G_m(T) \in \mathbf{k}^*$. Let $b \in \mathbf{k}$ be such that $(Y - b) \in \Gamma_{\mathrm{alg}}(F, A)$; then $\deg F(X, b) = 1$ and $F(X, b) = G(Xv(b), b) = \sum_{i=0}^m G_i(b)(v(b)X)^i$; since $m > 1$ we have $G_m(b) v(b)^m = 0$, so $v(b) = 0$, and consequently $F(X, b) = G_0(b) \in \mathbf{k}$, a contradiction.

Suppose that (ii) holds. Let $d = \deg v(Y) \geq 1$, and define an $\mathbb{N}$-grading $\mathbf{k}[S, T] = \bigoplus_{i \in \mathbb{N}} R_i$ by stipulating that S is homogeneous of degree d and T is homogeneous of degree 1. Write $G(S, T) = \sum_i H_i(S, T)$ where $H_i(S, T) \in R_i$ for all i. By (ii), we may choose $a \in \mathbf{k}^*$ such that $\deg F(a, Y) = 1$. Then

$$F(a, Y) = G\big(av(Y), Y\big) = \sum_i H_i\big(av(Y), Y\big), \quad \text{where } \deg H_i\big(av(Y), Y\big) \leq i.$$

Let $m = \deg_S(G)$ and $N = \deg_T(G)$, and note that $m \geq 2$ and $N \geq 1$. Since G is a variable of $\mathbf{k}[S,T]$,

$$(m,0),(0,N) \in \operatorname{supp}_{(S,T)}(G) \subseteq \langle (0,0),(m,0),(0,N) \rangle. \tag{23}$$

If $N \neq md$ then, by (23), the highest $H_i(S,T)$ is either[†] $\ominus S^m$ (if $md > N$) or $\ominus T^N$ (if $md < N$), and in both cases we have $\deg F(a,Y) \geq md > 1$, a contradiction. So $N = md$, and the highest $H_i(S,T)$ is $H_{md} = (\lambda T^d + \mu S)^m$ for some $\lambda, \mu \in \mathbf{k}^*$. Then for each $j \in \{1,2\}$ we have $\deg F(a_j,Y) = 1 < md$, so the right-hand side of

$$\big(\lambda Y^d + \mu a_j v(Y)\big)^m = F(a_j,Y) - \sum_{i<md} H_i\big(a_j v(Y),Y\big)$$

has degree less than md; then $\deg(\lambda Y^d + \mu a_j v(Y)) < d$ for all $j \in \{1,2\}$, but clearly there can be only one a_j for which this holds. This contradiction completes the proof. □

In the following, $\mathbf{k}$ is an arbitrary field.

THEOREM 4.11

Let F be a field generator of $A = \mathbf{k}^{[2]}$ satisfying $|\Gamma(F,A)| > 2$. Then there exists (X,Y) such that $A = \mathbf{k}[X,Y]$ and $F = \alpha(Y)X + \beta(Y)$ for some $\alpha(Y)$, $\beta(Y) \in \mathbf{k}[Y]$.

Proof

We first prove the case where $\mathbf{k}$ is algebraically closed. Note the following consequence of Lemma 3.7(a), which we will use several times:

$$\text{(24)}\quad \begin{array}{l} \textit{Suppose that } F \textit{ is a rectangular element of } R = \mathbf{k}^{[2]} \textit{ and that} \\ D_1,\ D_2,\ D_3 \textit{ are distinct elements of } \Gamma(F,R). \textit{ Then } D_i \cap D_j = \varnothing \\ \textit{for some } i,j \in \{1,2,3\}. \end{array}$$

We may assume that F is not a variable of A; otherwise the conclusion is obvious. Then F is rectangular by Theorem 4.1. Choose $\gamma = (X,Y) \in \operatorname{Rec}(F,A)$, let $(m,n) = (\deg_X(F), \deg_Y(F))$, and recall that $\min(m,n) \geq 1$. By Corollary 4.9, F is not a small field generator of A, so F is not γ-small; interchanging X and Y if necessary, it follows that $F \in \mathbf{k}[Xv(Y),Y]$ for some $v(Y) \in \mathbf{k}[Y] \setminus \mathbf{k}$. Any such $v(Y)$ satisfies $\deg v(Y) \leq n/m$, so we may choose $v(Y) \in \mathbf{k}[Y] \setminus \mathbf{k}$ satisfying $F \in \mathbf{k}[Xv(Y),Y]$ and

$$\deg w(Y) \leq \deg v(Y) \quad \text{for every } w(Y) \in \mathbf{k}[Y] \text{ satisfying } F \in \mathbf{k}\big[Xw(Y),Y\big]. \tag{25}$$

Let $A_1 = \mathbf{k}[Xv(Y),Y]$, and let $\Phi : \operatorname{Spec} A \to \operatorname{Spec} A_1$ be the birational morphism determined by the inclusion homomorphism $A_1 \hookrightarrow A$. By Lemma 3.11,

[†] We use Abhyankar's symbol $\ominus$ to represent an arbitrary element of $\mathbf{k}^*$. Note that different occurrences of $\ominus$ may represent different elements of $\mathbf{k}^*$.

there is an injective set map $\Gamma(F,A) \to \Gamma(F,A_1)$ given by $C \mapsto \Phi(C)$. Pick distinct elements $C_1, C_2, C_3 \in \Gamma(F,A)$; then $\Phi(C_i)$ $(1 \le i \le 3)$ are distinct elements of $\Gamma(F,A_1)$. In particular,

$$\big|\Gamma(F,A_1)\big| \ge 3. \tag{26}$$

We may assume that F is not a variable of A_1, otherwise the desired conclusion follows from Lemma 4.10. As F is a field generator of A_1 which is not a variable, F is a rectangular element of A_1 by Theorem 4.1. Relabeling C_1, C_2, C_3 if necessary, we get

$$\Phi(C_1) \cap \Phi(C_2) = \varnothing \tag{27}$$

by (24). In particular, $C_1 \cap C_2 = \varnothing$. So, by Lemma 3.7(a), there exist $Z \in \{X, Y\}$ and $\lambda_1, \lambda_2 \in \mathbf{k}$ such that $C_1 = V(Z - \lambda_1)$ and $C_2 = V(Z - \lambda_2)$. Let $b \in \mathbf{k}$ be a root of $v(Y)$; then Φ contracts the line $V(Y-b) \subset \operatorname{Spec} A$ to a point. By (27), $V(Y-b)$ cannot meet both C_1, C_2, so

$$C_1 = V(Y-\lambda_1) \qquad \text{and} \qquad C_2 = V(Y-\lambda_2) \quad \text{for some distinct } \lambda_1, \lambda_2 \in \mathbf{k}.$$

For later use, let us also record that one of conditions (28) and (29) holds (again by Lemma 3.7(a)),

$$C_3 = V(Y-\lambda_3) \quad \text{for some } \lambda_3 \in \mathbf{k}; \tag{28}$$

$$C_3 = V(X-\lambda_3) \quad \text{for some } \lambda_3 \in \mathbf{k}. \tag{29}$$

Pick $(X_1, Y_1) \in \operatorname{Rec}(F, A_1)$. Then Lemma 3.7(a) implies that, for each $i = 1, 2, 3$, there exist $Z_i \in \{X_1, Y_1\}$ and $\mu_i \in \mathbf{k}$ such that $\Phi(C_i) = V(Z_i - \mu_i)$. It is easy to see that $\Phi(C_1) = V(Y - \lambda_1)$ in $\operatorname{Spec} A_1$, so $Z_1 - \mu_1 = \ominus(Y - \lambda_1)$. So we can choose $(X_1, Y_1) \in \operatorname{Rec}(F, A_1)$ in such a way that $Y_1 = Y$. Then $\mathbf{k}[Xv(Y), Y] = \mathbf{k}[X_1, Y]$, and consequently $X_1 = \ominus Xv(Y) + P(Y)$ for some $P(Y) \in \mathbf{k}[Y]$. Multiplying X_1 and $P(Y)$ by a unit if necessary, we find that $(Xv(Y) + P(Y), Y) \in \operatorname{Rec}(F, A_1)$ for some $P(Y) \in \mathbf{k}[Y]$. We set

$$\gamma_1 = (X_1, Y_1) = \big(Xv(Y) + P(Y), Y\big) \in \operatorname{Rec}(F, A_1).$$

By (26) and Corollary 4.9, F is not a small-field generator of A_1. So F is not γ_1-small, and consequently one of (30) or (31) holds:

$$F \in \mathbf{k}\big[X_1 u(Y_1), Y_1\big] \quad \text{for some } u(Y_1) \in \mathbf{k}[Y_1] \setminus \mathbf{k}, \tag{30}$$

$$F \in \mathbf{k}\big[X_1, u(X_1) Y_1\big] \quad \text{for some } u(X_1) \in \mathbf{k}[X_1] \setminus \mathbf{k}. \tag{31}$$

If (30) holds, then $F \in \mathbf{k}[X_1 u(Y_1), Y_1] = \mathbf{k}[(Xv(Y) + P(Y))u(Y), Y] = \mathbf{k}[Xv(Y)u(Y), Y]$, which contradicts (25). So (31) must hold. Pick $c \in \mathbf{k}$ such that $u(c) = 0$; then $F \in \mathbf{k}[X_1, u(X_1)Y_1] \subseteq \mathbf{k}[X_1, (X_1 - c)Y_1]$. Let $A_2 = \mathbf{k}[X_1, (X_1 - c)Y_1]$, and consider the birational morphism $\Psi : \operatorname{Spec} A_1 \to \operatorname{Spec} A_2$ determined by $A_2 \hookrightarrow A_1$. By Lemma 3.11, $\Psi(\Phi(C_1))$, $\Psi(\Phi(C_2))$, and $\Psi(\Phi(C_3))$ are distinct elements of $\Gamma(F, A_2)$. We claim that

$$\Psi\big(\Phi(C_i)\big) \cap \Psi\big(\Phi(C_j)\big) \neq \varnothing \quad \text{for all choices of } i, j \in \{1, 2, 3\}. \tag{32}$$

We prove this in each of the cases (28) and (29). Note that Ψ contracts the line $L = V(X_1 - c) \subset \operatorname{Spec} A_1$ to a point. If (28) holds, then $\Phi(C_i) = V(Y_1 - \lambda_i) \subset \operatorname{Spec} A_1$ $(i = 1, 2, 3)$, so L meets $\Phi(C_i)$ for $i = 1, 2, 3$; then $\Psi(\Phi(C_1)) \cap \Psi(\Phi(C_2)) \cap \Psi(\Phi(C_3)) \neq \varnothing$, so (32) holds in this case. If (29) holds, then $\Phi(C_i) = V(Y_1 - \lambda_i)$ for $i = 1, 2$; as L meets these two lines, we get $\Psi(\Phi(C_1)) \cap \Psi(\Phi(C_2)) \neq \varnothing$. Moreover, $C_i \cap C_3 \neq \varnothing$ for $i = 1, 2$, so $\Psi(\Phi(C_i)) \cap \Psi(\Phi(C_3)) \neq \varnothing$ for $i = 1, 2$, and again (32) is true. So (32) is true in all cases.

From (32) and (24), we deduce that F is not a rectangular element of A_2. As F is a field generator of A_2, Theorem 4.1 implies that

$$F \text{ is a variable of } A_2 = \mathbf{k}\big[X_1, (X_1 - c)Y_1\big].$$

Then Lemma 4.10 gives $F = a_1(X_1 - c)Y_1 + a_2X_1 + a_3$ for some $a_1 \in \mathbf{k}^*$, $a_2, a_3 \in \mathbf{k}$. Then

$$F = \big(Xv(Y) + P(Y)\big)(a_1Y + a_2) - a_1cY + a_3,$$

from which the desired conclusion follows. This proves the theorem in the case where $\mathbf{k}$ is algebraically closed.

To prove the general case, consider a field $\mathbf{k}$ and a field generator F of $A = \mathbf{k}^{[2]}$ satisfying $|\Gamma(F, A)| > 2$. We may assume that F is not a variable of A, otherwise the conclusion is obvious. Then F is a rectangular element of A by Theorem 4.1, and we may choose $(X, Y) \in \operatorname{Rec}^+(F, A)$.

Let $\bar{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$, and let $\bar{A} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} A = \bar{\mathbf{k}}^{[2]}$. Then F is a field generator of $\bar{A}$ and $(X, Y) \in \operatorname{Rec}^+(F, \bar{A})$. In particular, F is not a variable of $\bar{A}$ (since it is a rectangular element of $\bar{A}$). Note that $\operatorname{bideg}_{\bar{A}}(F) = (\deg_X(F), \deg_Y(F)) = \operatorname{bideg}_A(F)$.

We have $|\Gamma(F, \bar{A})| \geq |\Gamma(F, A)| > 2$ by Lemma 3.10, so the case "$\mathbf{k}$ algebraically closed" of the theorem implies that there exists (X_1, Y_1) such that $\bar{A} = \bar{\mathbf{k}}[X_1, Y_1]$ and $F = \alpha_1(Y_1)X_1 + \beta_1(Y_1)$ for some $\alpha_1(Y_1), \beta_1(Y_1) \in \bar{\mathbf{k}}[Y_1]$. Observe that $\deg \alpha_1(Y_1) > 0$, for otherwise F would be a variable of $\bar{A}$. Write $\beta_1(Y_1) = q(Y_1)\alpha_1(Y_1) + \rho(Y_1)$ with $q(Y_1), \rho(Y_1) \in \bar{\mathbf{k}}[Y_1]$ and $\deg \rho(Y_1) < \deg \alpha_1(Y_1)$. Then $F = \alpha_1(Y_1)(X_1 + q(Y_1)) + \rho(Y_1) = \alpha_1(Y_2)X_2 + \rho(Y_2)$ where we define $X_2 = X_1 + q(Y_1)$ and $Y_2 = Y_1$. Since $\deg \rho(Y_2) < \deg \alpha_1(Y_2)$, we have $(X_2, Y_2) \in \operatorname{Rec}(F, \bar{A})$. This shows that $\operatorname{bideg}_{\bar{A}}(F) = (1, n)$ for some $n \geq 1$. Then $\operatorname{bideg}_A(F) = (1, n)$, so $F = \alpha(Y)X + \beta(Y)$ for some $\alpha(Y), \beta(Y) \in \mathbf{k}[Y]$, as desired. $\square$

COROLLARY 4.12

If F is a bad field generator of $A = \mathbf{k}^{[2]}$ then the following hold:

(a) $|\Gamma(F, A)| \leq 2$;
(b) $\operatorname{Rec}(F, A) \neq \varnothing$, *and the pair* $(m, n) = \operatorname{bideg}_A(F)$ *satisfies* $2 \leq m \leq n$;
(c) *there exists* $(X, Y) \in \operatorname{Rec}(F, A)$ *such that*

$$\Gamma_{\text{alg}}(F, A) \subseteq \big\{(X), (Y)\big\} \qquad \textit{or} \qquad \Gamma_{\text{alg}}(F, A) \subseteq \big\{(X), (X - 1)\big\}.$$

Proof

If $|\Gamma(F, A)| > 2$ then Theorem 4.11 implies that there exists (X, Y) such that $A =$

$\mathbf{k}[X,Y]$ and $F = \alpha(Y)X + \beta(Y)$ for some $\alpha(Y), \beta(Y) \in \mathbf{k}[Y]$. This contradicts the hypothesis that F is bad. Indeed, if $\alpha(Y) = 0$, then $F \in \mathbf{k}[Y]$, so F is not a bad field generator of A; and if $\alpha(Y) \neq 0$, then $\mathbf{k}(F,Y) = \mathbf{k}(X,Y)$, so again F is good. This proves (a).

Since F is a field generator of A which is not a variable, $\mathrm{Rec}(F,A) \neq \varnothing$ by Theorem 4.1. The pair $(m,n) = \mathrm{bideg}_A(F)$ is defined and satisfies $1 \leq m \leq n$. Since F is bad, we must have $m > 1$ (if $m = 1$ then pick $(X,Y) \in \mathrm{Rec}^+(F,A)$; then $\deg_X(F) = m = 1$, so $\mathbf{k}(F,Y) = \mathbf{k}(X,Y)$, so F is good), which proves (b).

The proof of (c) is an easy consequence of $|\Gamma(F,A)| \leq 2$ and Lemma 3.7(a). □

EXAMPLE 4.13

Let $F = X(X-1)Y^2 + Y \in A = \mathbf{k}[X,Y]$. Then $\mathbf{k}(F, XY) = \mathbf{k}(X,Y)$, so F is a good field generator of A. Moreover,

$$\Gamma_{\mathrm{alg}}(F,A) = \big\{(X),(X-1)\big\}. \tag{33}$$

At present, we do not know an example of a bad field generator satisfying (33) (cf. Corollary 4.12(c)).

REMARK 4.14

Let $\gamma = |\Gamma_{\mathrm{alg}}(F,A)|$ where F is a bad field generator of $A = \mathbf{k}^{[2]}$. Then $\gamma \in \{0,1,2\}$ by Corollary 4.12. We shall see in Examples 5.12 that the three cases arise.

5. Very good and very bad field generators

We begin by studying good and very good field generators. We shall need the following fact, valid over an arbitrary field $\mathbf{k}$.

OBSERVATION 5.1 ([13, REMARK AFTER 1.3])

Let F be a field generator of $A = \mathbf{k}^{[2]}$. Then F is a good field generator of A if and only if 1 occurs in the list $\Delta(F,A)$.

Also note the following consequence of Theorem 4.1.

COROLLARY 5.2

Let F be a field generator of $A = \mathbf{k}^{[2]}$, and let $\Delta(F,A) = [d_1, \dots, d_t]$. Then F is a variable of A if and only if $t = 1$.

Proof

If F is a variable, then $\Delta(F,A) = [1]$, so $t = 1$. If F is not a variable, then, by Theorem 4.1, there exists $(X,Y) \in \mathrm{Rec}(F,A)$. The embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ determined by the pair (X,Y) has the property that the closure of $V(F)$ in $\mathbb{P}^2$ meets the line at infinity in exactly two points. Then $t \geq 2$ by Remark 2.2. □

Let us agree that the gcd of the empty set is $+\infty$. Then we have the following.

PROPOSITION 5.3

Let F be a field generator of $A = \mathbf{k}^{[2]}$, and let $\Delta(F, A) = [d_1, \dots, d_t]$.

(a) *If $\gcd(\{d_1, \dots, d_t\} \setminus \{1\}) > 1$, then F is a very good field generator of A. In particular, if at most one $i \in \{1, \dots, t\}$ satisfies $d_i > 1$, then F is a very good field generator of A.*

(b) *If at least three $i \in \{1, \dots, t\}$ satisfy $d_i = 1$, then F is a very good field generator of A.*

(c) *If F is a good but not very good field generator of A, then*

$$\Delta(F, A) = [1, \dots, 1, e_1, \dots, e_s],$$

where 1 occurs either 1 or 2 times, $s \geq 2$, $\min(e_1, \dots, e_s) > 1$, and $\gcd(e_1, \dots, e_s) = 1$.

Proof

Assertion (c) follows from (a) and (b). To prove (a) and (b), we consider A' such that $F \in A' \preceq A$; in each of cases (a) and (b), it has to be shown that F is a good field generator of A'. Clearly, F is a field generator of A'.

(a) It follows from Proposition 2.9 that $\Delta(F, A') = [d'_1, \dots, d'_s]$ is a sublist of $\Delta(F, A)$, so $\gcd(\{d'_1, \dots, d'_s\} \setminus \{1\}) > 1$. As $\gcd(d'_1, \dots, d'_s) = 1$ by Theorem 2.5, 1 occurs in $\Delta(F, A')$, so Observation 5.1 implies that F is a good field generator of A'.

(b) Consider the morphisms $\operatorname{Spec} A \xrightarrow{\Phi} \operatorname{Spec} A' \xrightarrow{f} \operatorname{Spec} \mathbf{k}[F]$ determined by the inclusions $\mathbf{k}[F] \hookrightarrow A' \hookrightarrow A$. Let $C_1, \dots, C_h$ be the distinct elements of $\operatorname{Miss}_{\mathrm{hor}}(\Phi, f)$, and, for each $i \in \{1, \dots, h\}$, let δ_i be the degree of $f|_{C_i} : C_i \to \operatorname{Spec} \mathbf{k}[F]$. Then Proposition 2.9 gives $\Delta(F, A) = [\Delta(F, A'), \delta_1, \dots, \delta_h]$ and

$$\text{for all } i \in \{1, \dots, h\}, \quad \delta_i = 1 \Leftrightarrow C_i \in \Gamma(F, A'). \tag{34}$$

Arguing by contradiction, assume that F is a bad field generator of A'; then (by Corollary 4.12) $|\Gamma(F, A')| \leq 2$, so (by (34)) 1 occurs at most twice in $[\delta_1, \dots, \delta_h]$; as (by Observation 5.1) 1 does not occur in $\Delta(F, A')$, it follows that 1 occurs at most twice in $\Delta(F, A)$, which contradicts the assumption of (b). So (b) is proved. $\square$

REMARK 5.4

By Proposition 5.3(a), the polynomials classified in [11], [12], and [15] are special cases of very good field generators. This gives many complicated examples of very good field generators.

EXAMPLE 5.5

Let F be a bad field generator of $A = \mathbb{C}[X, Y]$ such that $\Gamma_{\mathrm{alg}}(F, A) = \{(X), (Y)\}$ and $\Delta(F, A) = [3, 2]$ (such an F exists by Examples 5.12(c)). Let $B = \mathbb{C}[X/Y, Y^2/X]$, and note that $A \preceq B$. Consider the morphisms $\operatorname{Spec} B \xrightarrow{\Phi} \operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} \mathbb{C}[F]$ determined by the inclusions $\mathbb{C}[F] \hookrightarrow A \hookrightarrow B$. Then the missing curves of Φ are $C_1 = V(X)$ and $C_2 = V(Y)$, and these are f-horizontal, so

$\mathrm{Miss}_{\mathrm{hor}}(\Phi, f) = \{C_1, C_2\}$. In the notation of Proposition 2.9 we have $\delta_1 = \delta_2 = 1$ (because $C_1, C_2 \in \Gamma(F, A)$), so that result implies that $\Delta(F, B) = [3, 2, 1, 1]$. Note that F *is not a very good field generator of* B (because it is bad in A). This shows that, in Proposition 5.3(b), one cannot replace "at least three" by "at least two"; and in the second part of Proposition 5.3(a), one cannot replace "at most one" by "at most two."

EXAMPLE 5.6

Let $\mathbf{k}$ be any field, and let $F = XY \in A = \mathbf{k}[X, Y] = \mathbf{k}^{[2]}$. It is easy to see that F is a good field generator of A with $\Delta(F) = [1, 1]$; so by Proposition 5.3(a), F is a very good field generator of A. Define $B = \mathbf{k}[X + Y, \frac{Y}{(X+Y)(X+Y+1)}]$, and note that $F \in A \preceq B$. Consider the morphisms $\operatorname{Spec} B \xrightarrow{\Phi} \operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} \mathbf{k}[F]$ determined by the inclusions $\mathbf{k}[F] \hookrightarrow A \hookrightarrow B$. Then the missing curves of Φ are $C_1 = V(X + Y)$ and $C_2 = V(X + Y + 1)$ and these are f-horizontal, so $\mathrm{Miss}_{\mathrm{hor}}(\Phi, f) = \{C_1, C_2\}$. In the notation of Proposition 2.9 we have $\delta_1 = \delta_2 = 2$ so that result implies that $\Delta(F, B) = [1, 1, 2, 2]$. By Proposition 5.3(a), F is a very good field generator of B. So there exist very good field generators with more than one dicritical of degree greater than 1. The polynomial f given on [3, p. 298] is another example, this one with $\Delta(f, \mathbb{C}[X, Y]) = [1, 1, 2, 4]$.

The question of characterizing very good field generators is not settled by Proposition 5.3 and is still open. Moreover, it is known that the degree list $\Delta(F, A)$ does not characterize very good field generators among good field generators. Indeed, [5, 4.4] states that *there exist good field generators* F, G *of* $A = \mathbf{k}^{[2]}$ *such that* F *is very good,* G *is not very good, and* $\Delta(F, A) = [3, 4, 1] = \Delta(G, A)$.

However, very bad field generators can be characterized: result Proposition 5.8(a) gives such a characterization and, in fact, makes it easy to decide whether a given bad field generator is very bad. We shall derive that characterization from the following result.

PROPOSITION 5.7

Consider $F \in A \preceq A'$, *where* F *is a field generator of* A. *Let* $\Phi : \operatorname{Spec} A' \to \operatorname{Spec} A$ *be the morphism determined by* $A \hookrightarrow A'$. *The following are equivalent.*

(a) F *is a bad field generator of* A'.

(b) F *is a bad field generator of* A, *and no element of* $\Gamma(F, A)$ *is a missing curve of* Φ.

Proof

Consider the morphisms $\operatorname{Spec} A' \xrightarrow{\Phi} \operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} \mathbf{k}[F]$ determined by $\mathbf{k}[F] \hookrightarrow A \hookrightarrow A'$, let $C_1, \dots, C_h$ be the distinct elements of $\mathrm{Miss}_{\mathrm{hor}}(\Phi, f)$, and, for each $i \in \{1, \dots, h\}$, let δ_i be the degree of the dominant morphism $f|_{C_i} : C_i \to \mathbb{A}^1$. We have $\Delta(F, A') = [\Delta(F, A), \delta_1, \dots, \delta_h]$ by Proposition 2.9, and the same result also implies that

$$\text{for all } i \in \{1, \ldots, h\}, \quad \delta_i = 1 \iff C_i \in \Gamma(F, A). \tag{35}$$

Now (a) is true if and only if (by Observation 5.1) 1 does not occur in $\Delta(F, A') = [\Delta(F, A), \delta_1, \ldots, \delta_h]$, if and only if (by (35)) 1 does not occur in $\Delta(F, A)$ and $C_i \notin \Gamma(F, A)$ for all i, if and only if (b) is true. The last equivalence uses Observation 5.1 and the fact that, for any $C \in \Gamma(F, A)$, $C \in \operatorname{Miss}(\Phi) \Leftrightarrow C \in \operatorname{Miss}_{\text{hor}}(\Phi, f)$. $\square$

PROPOSITION 5.8

Let F be a bad field generator of $A = \mathbf{k}^{[2]}$.

(a) *F is a very bad field generator of A if and only if $\Gamma_{\text{alg}}(F, A) = \varnothing$.*

(b) *Suppose that $\Gamma_{\text{alg}}(F, A) \neq \varnothing$. Then there exists (X, Y) such that $A = \mathbf{k}[X, Y]$ and $(X) \in \Gamma_{\text{alg}}(F, A)$. For any such pair (X, Y), F is a good field generator of $\mathbf{k}[X, Y/X]$.*

Proof

We first prove (b). Suppose that $\Gamma_{\text{alg}}(F, A) \neq \varnothing$, and pick $\mathfrak{p} \in \Gamma_{\text{alg}}(F, A)$. As F is not a variable of A, it is a rectangular element of A (by Theorem 4.1), so Corollary 3.8 implies that $\mathfrak{p} = (X)$ for some variable X of A; this shows that there exists (X, Y) such that $A = \mathbf{k}[X, Y]$ and $(X) \in \Gamma_{\text{alg}}(F, A)$. Given such a pair (X, Y), let $B = \mathbf{k}[X, Y/X]$, and consider the birational morphism $\Phi : \operatorname{Spec} B \to \operatorname{Spec} A$ determined by the inclusion $A \to B$. Then $C = V(X) \subset \operatorname{Spec} A$ is the unique missing curve of Φ and the fact that $(X) \in \Gamma_{\text{alg}}(F, A)$ implies that $C \in \Gamma(F, A)$. By Proposition 5.7, it follows that F is a good field generator of B; so (b) is proved.

To prove (a), it's enough to show that

$$F \text{ is not a very bad field generator of } A \implies \Gamma_{\text{alg}}(F, A) \neq \varnothing \tag{36}$$

is true, because we already know, by (b), that the converse of (36) is true. Suppose that F is not very bad. Then there exists a ring B such that $A \preceq B$ and F is a good field generator of B. Consider the birational morphism $\Phi : \operatorname{Spec} B \to \operatorname{Spec} A$ determined by the inclusion $A \to B$. By Proposition 5.7, some element of $\Gamma(F, A)$ is a missing curve of Φ; so $\Gamma(F, A) \neq \varnothing$ (and hence $\Gamma_{\text{alg}}(F, A) \neq \varnothing$). This proves (36), from which (a) follows. $\square$

LEMMA 5.9

Consider $F \in A \preceq A'$, where F is a field generator of A, and let $\Phi : \operatorname{Spec} A' \to \operatorname{Spec} A$ denote the morphism determined by $A \hookrightarrow A'$. Consider the following conditions on the triple (F, A, A').

(i) *F is a bad field generator of A, and, for all $C \in \Gamma(F, A)$, C is not a missing curve of Φ and $C \nsubseteq \operatorname{im}(\Phi)$.*

(ii) *F is a bad field generator of A, and, for all $C \in \Gamma(F, A)$, C is not a missing curve of Φ and no curve $D \subset \operatorname{Spec} A'$ is such that $\Phi|_D$ is an isomorphism from D to C.*

(iii) *F is a very bad field generator of A'.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof

Clearly, (i) $\Rightarrow$ (ii). In view of Proposition 5.7 and Lemma 3.11(c), condition (ii) is equivalent to the following:

> F is a bad field generator of A', and no element of $\Gamma(F, A)$ is in the image of the map $\gamma : \Gamma(F, A') \to \Gamma(F, A)$ defined in Lemma 3.11,

which is equivalent to

> F is a bad field generator of A', and $\Gamma(F, A') = \varnothing$.

So, by Proposition 5.8, (ii) $\Leftrightarrow$ (iii). □

PROPOSITION 5.10

For a field generator F of $A = \mathbf{k}^{[2]}$, the following are equivalent.

(a) *F is a bad field generator of A.*

(b) *There exists $A' \succeq A$ such that F is a very bad field generator of A'.*

(c) *There exists (X, Y) such that $A = \mathbf{k}[X, Y]$ and F is a very bad field generator of $\mathbf{k}[X, Y/X]$.*

Proof

Implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are obvious. Assume that (a) holds, and let us prove (c). By Corollary 4.12, there exists a pair (X, Y) satisfying $A = \mathbf{k}[X, Y]$ and

$$\text{(i)} \quad \Gamma_{\mathrm{alg}}(F, A) \subseteq \big\{(X), (Y)\big\} \qquad \text{or} \qquad \text{(ii)} \quad \Gamma_{\mathrm{alg}}(F, A) \subseteq \big\{(X), (X-1)\big\}.$$

Define

$$(X_1, Y_1) = \begin{cases} (X - Y, Y - 1) & \text{in case (i)}, \\ (X - Y^2 + Y, Y) & \text{in case (ii)}. \end{cases}$$

Note that $A = \mathbf{k}[X_1, Y_1]$, and define $A' = \mathbf{k}[X_1, Y_1/X_1]$; then $F \in A \preceq A'$. We claim that F is a very bad field generator of A'. To see this, consider the birational morphism $\Phi : \operatorname{Spec} A' \to \operatorname{Spec} A$ determined by the inclusion $A \hookrightarrow A'$. Consider the curve $D \subset \operatorname{Spec} A$ and the point $P \in \operatorname{Spec} A$ defined by $D = V(X_1)$ and $\{P\} = V(X_1, Y_1)$; then D is the unique missing curve of Φ and $\operatorname{im} \Phi = (\operatorname{Spec}(A) \setminus D) \cup \{P\}$.

In case (i), we have $V(X) \cap D \nsubseteq \{P\}$ and $V(Y) \cap D \nsubseteq \{P\}$; in case (ii), $V(X) \cap D \nsubseteq \{P\}$ and $V(X-1) \cap D \nsubseteq \{P\}$. So, in both cases, we have $C \cap D \nsubseteq \{P\}$ for all $C \in \Gamma(F, A)$; consequently,

$$C \nsubseteq \operatorname{im} \Phi \quad \text{for all } C \in \Gamma(F, A).$$

Moreover, $D \notin \Gamma(F,A)$, so no element of $\Gamma(F,A)$ is a missing curve of Φ. So (F,A,A') satisfies condition (i) of Lemma 5.9; by that result, F is a very bad field generator of A'. So condition (c) holds. $\square$

To our knowledge, [9], [14], and [4] are the only publications giving examples of bad field generators. Example 5.11, below, shows how the results of this paper can be used to easily produce new examples from old ones. It also gives examples of very bad field generators, of bad field generators which are not very bad, and of good ones which are not very good. (Very good field generators are easy to find; see Remark 5.4.)

EXAMPLE 5.11

Let $F \in A = \mathbf{k}[X,Y] = \mathbf{k}^{[2]}$ (where $\mathbf{k}$ is any field) be the following polynomial of degree 21:

$$\big(Y^2(XY+1)^4 + Y(2XY+1)(XY+1)+1\big)\big(Y(XY+1)^5 + 2XY(XY+1)^2 + X\big).$$

It is shown in [14] (modulo a typo corrected in [4]) that F is a bad field generator of A with $\Delta(F,A) = [2,3]$. Note that $(X,Y) \in \operatorname{Rec}^+(F,A)$ and that $\operatorname{bideg}_A(F) = (9,12)$. It is not difficult to deduce from Lemma 3.7 that $\Gamma_{\text{alg}}(F,A) = \{(Y)\}$.

If $\mathbf{k} = \mathbb{C}$, then the fact that $\Delta(F,A) = [2,3]$ can also be deduced from the proof of [4, 2.3.10], essentially by noting that

$$F(-t^3 + t^4u, 1/t^3) = 2u^3 - 3u^6 + u^9 + th_1(t,u) = \varphi_1(u^3) + th_1(t,u),$$
$$F(1/t^6, -t^4 + t^5/3 + t^6u) = 36u^2 + 54u + 20 + th_2(t,u) = \varphi_2(u) + th_2(t,u),$$

where $\varphi_1, \varphi_2, h_1, h_2$ are polynomials, $\deg \varphi_1 = 3$, and $\deg \varphi_2 = 2$ (refer to the proof of [4, 2.3.10]).

In the following three paragraphs we regard Russell's polynomial F as an element of certain overrings of A. By doing so, we obtain new examples of field generators (which could be called "the good, the very bad, and the ugly").

(a) Since $\Gamma_{\text{alg}}(F,A) = \{(Y)\}$, Proposition 5.8 implies that F is not a very bad field generator of A. By the same result, F is a good (but not very good) field generator of $\mathbf{k}[X/Y,Y]$. In particular, *there exist bad field generators that are not very bad and good field generators that are not very good.*

(b) Let $A' = \mathbf{k}[X,(Y-1)/X]$; then (F,A,A') satisfies condition (i) of Lemma 5.9; by that result, F is a very bad field generator of A'. So, *very bad field generators do exist.* One can see that $\deg_{(X',Y')}(F) = 33$, where $X' = X$ and $Y' = (Y-1)/X$. Using Proposition 2.9, one sees that $\Delta(F,A') = [2,3,3]$.

(c) Choose distinct elements $\lambda_1,\dots,\lambda_N$ of $\mathbf{k}$ $(N \geq 1)$, and let $P(T) = \prod_{i=1}^N (T-\lambda_i)^{e_i} \in \mathbf{k}[T]$ (where T is an indeterminate and $e_i \geq 1$ for all i). Pick $d \geq 1$, and define

$$A'' = \mathbf{k}\big[X+Y^d, Y/P(X+Y^d)\big].$$

Then $A = \mathbf{k}[X,Y] = \mathbf{k}[X+Y^d,Y] \preceq A''$. Consider the morphisms $\operatorname{Spec} A'' \xrightarrow{\Phi} \operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} \mathbf{k}[F]$ determined by the inclusions $\mathbf{k}[F] \hookrightarrow A \hookrightarrow A''$. Then the missing curves of Φ are

$$C_i : X + Y^d = \lambda_i, \quad i = 1, \dots, N.$$

Each C_i is f-horizontal, and the degree of $C_i \hookrightarrow \operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} \mathbf{k}[F]$ is $\delta_i = \deg_t F(\lambda_i - t^d, t) = 9d + 12$. So, by Proposition 2.9,

(37) $\quad \Delta(F, A'') = [2, 3, 9d + 12, \dots, 9d + 12], \quad$ where $9d + 12$ occurs N times.

By Observation 5.1, F is a bad field generator of A''. If we write $A'' = \mathbf{k}[u, v]$ with $u = X + Y^d$ and $v = Y/P(X + Y^d)$, then one can see that $\Gamma_{\text{alg}}(F, A'') = \{(v)\}$, so (by Proposition 5.8) F is not a very bad field generator of A''.

Let $A''' = \mathbf{k}[u, (v - 1)/(u - \lambda_1)]$, and let $\operatorname{Spec} A''' \xrightarrow{\Psi} \operatorname{Spec} A'' \xrightarrow{f''} \operatorname{Spec} \mathbf{k}[F]$ be the morphisms determined by $\mathbf{k}[F] \hookrightarrow A'' \hookrightarrow A'''$. Define $C \subset \operatorname{Spec} A''$ and $Q \in \operatorname{Spec} A''$ by $C = V(u - \lambda_1)$ and $\{Q\} = V(u - \lambda_1, v - 1)$. Then C is the unique missing curve of Ψ and $\operatorname{im} \Psi = (\operatorname{Spec}(A'') \setminus C) \cup \{Q\}$. It follows that (F, A'', A''') satisfies condition (i) of Lemma 5.9; by that result, F is a very bad field generator of A'''. As $f''(C)$ is a point, $\operatorname{Miss}_{\text{hor}}(\Psi, f'') = \varnothing$; so, by Proposition 2.9, $\Delta(F, A''') = \Delta(F, A'')$ (refer to (37)). In particular, *there exist very bad field generators with arbitrarily large numbers of dicriticals of arbitrarily large degrees.*

EXAMPLES 5.12

Let $\gamma = |\Gamma_{\text{alg}}(F, A)|$ where F is a bad field generator of $A = \mathbf{k}^{[2]}$. Then $\gamma \in \{0, 1, 2\}$ by Corollary 4.12. The three cases arise.

(a) By Proposition 5.8, the condition $\gamma = 0$ is equivalent to F being very bad, and very bad field generators do exist by Example 5.11. So $\gamma = 0$ arises.

(b) The polynomial $F \in A = \mathbf{k}[X, Y]$ given at the beginning of Example 5.11 (Russell's polynomial) has $\Gamma_{\text{alg}}(F, A) = \{(Y)\}$, so $\gamma = 1$.

(c) It is stated in [4, 2.3.11] and proved in [4, 2.3.10] that the polynomial

$$F = X(X^5Y^3 + 1)^3 + Y(X^2Y + 1)^8 - X^{16}Y^9 + 4XY + 6X^2Y + 19X^3Y^2$$
$$+ 8X^4Y^2 + 36X^5Y^3 + 34X^7Y^4 + 16X^9Y^5$$

is a bad field generator of $A = \mathbb{C}[X, Y]$. It follows from the proof of [4, 2.3.10] that $\Delta(F, A) = [3, 2]$; indeed, in the setting of that proof, this follows by noting that

$$F(1/t^3, -t^5 + t^6u) = 27u^3 + 72u^2 + 66u + 20 + tk_1(t, u) = \varphi_1(u) + tk_1(t, u),$$
$$F(it^4 + t^5u, 1/t^8) = 256u^8 + 161u^4 - 1 + tk_2(t, u) = \varphi_2(u^4) + tk_2(t, u),$$

where φ_1, φ_2, k_1, k_2 are polynomials, $\deg \varphi_1 = 3$, and $\deg \varphi_2 = 2$. It is obvious that $(X, Y) \in \operatorname{Rec}(F, A)$ and that $\operatorname{bideg}_A(F) = (\deg_Y F, \deg_X F) = (9, 16)$. It follows from 3.7 that $\Gamma_{\text{alg}}(F, A) = \{(X), (Y)\}$, so $\gamma = 2$.

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