

Homological dimensions relative to preresolving subcategories

Zhaoyong Huang

Abstract We introduce relative preresolving subcategories and precoresolving subcategories of an abelian category and define homological dimensions and codimensions relative to these subcategories, respectively. We study the properties of these homological dimensions and codimensions and unify some important properties possessed by some known homological dimensions. Then we apply the obtained properties to special subcategories and in particular to module categories. Finally we propose some open questions and conjectures which are closely related to the Auslander–Reiten conjecture and the strong Nakayama conjecture.

1. Introduction

In classical homological theory, homological dimensions are important and fundamental invariants, and every homological dimension of modules is defined relative to some certain subcategory of modules. For example, projective, flat, and injective dimensions of modules are defined relative to the categories of projective, flat, and injective modules, respectively. When projective, flat, and injective modules are generalized to Gorenstein projective, Gorenstein flat, and Gorenstein injective modules, respectively, in relative homological theory, Gorenstein projective, Gorenstein flat, and Gorenstein injective dimensions emerge; and, in particular, they share many nice properties of projective, flat, and injective dimensions, respectively (e.g., [AB], [C], [CFH], [CI], [DLM], [EJ1], [EJ2], [EJL], [GD], [GT], [H1], [H2], [HuH], [LHX], [MD], [SSW], [Z]). Then a natural question is: if two homological (co)dimensions relative to a category and its subcategory are defined, what is the relation between these two homological (co)dimensions? The purpose of this paper is to study this question. We introduce relative preresolving subcategories and precoresolving subcategories and define homological dimensions and codimensions relative to these subcategories, respectively. Then we study

Kyoto Journal of Mathematics, Vol. 54, No. 4 (2014), 727–757

DOI [10.1215/21562261-2801795](https://doi.org/10.1215/21562261-2801795), © 2014 by Kyoto University

Received March 4, 2013. Revised July 9, 2013. Accepted September 18, 2013.

2010 Mathematics Subject Classification: Primary 18G25; Secondary 18G20, 18G10, 16E10.

Author's work partially supported by the Specialized Research Fund for the Doctoral Program of Higher Education (grant no. 20100091110034), Natural Science Foundation of China (grant no. 11171142), Natural Science Foundation of Jiangsu Province of China (grant no. BK2010007), and a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

their properties and unify some important properties possessed by some known homological dimensions.

This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results; in particular, we give the definition of homological (co)dimension relative to a certain full and additive subcategory of an abelian category.

In Section 3, we first give the definition of (pre)resolving subcategories of an abelian category. Then we give some criteria for computing and comparing homological dimensions relative to different preressolving subcategories. Let \mathcal{E} and \mathcal{T} be additive and full subcategories of an abelian category \mathcal{A} such that \mathcal{T} is \mathcal{E} -preresolving with an \mathcal{E} -proper generator \mathcal{C} . Assume that $0 \rightarrow M \rightarrow T_1 \rightarrow T_0 \rightarrow A \rightarrow 0$ is an exact sequence in \mathcal{A} with both T_0 and T_1 objects in \mathcal{T} . Then there exists an exact sequence $0 \rightarrow M \rightarrow T \rightarrow C \rightarrow A \rightarrow 0$ in \mathcal{A} with T an object in \mathcal{T} and C an object in \mathcal{C} ; and furthermore, if the former exact sequence is $\text{Hom}_{\mathcal{A}}(X, -)$ -exact for some object X in \mathcal{A} , then so is the latter one. As applications of this result, we get that an object in \mathcal{A} is an n - \mathcal{C} -cosyzygy if and only if it is an n - \mathcal{T} -cosyzygy; and we also get that the \mathcal{T} -dimension of an object A in \mathcal{A} is at most n if and only if there exists an exact sequence $0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_0 \rightarrow A \rightarrow 0$ in \mathcal{A} with all C_i 's objects in \mathcal{C} and K_n an object in \mathcal{T} . In addition, we give some sufficient conditions under which the \mathcal{T} -dimension and the \mathcal{C} -dimension of an object in \mathcal{A} are identical.

Section 4 is completely dual to Section 3.

In Section 5, we apply the results in Sections 3 and 4 to special subcategories and in particular to module categories. Some known results are generalized. Finally we propose some questions and conjectures concerning the obtained results, which are closely related to the Auslander–Reiten conjecture and the strong Nakayama conjecture.

Throughout this paper, \mathcal{A} is an abelian category and all subcategories of \mathcal{A} are full and additive.

2. Preliminaries

In this section, we give some terminology and some preliminary results.

DEFINITION 2.1 ([Hu, DEFINITION 5.2, P. 164])

Let \mathcal{C} be a subcategory of \mathcal{A} , and let $n \geq 0$.

(a) If there exists an exact sequence $0 \rightarrow M \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_0 \rightarrow A \rightarrow 0$ in \mathcal{A} with all C_i 's objects in \mathcal{C} , then M is called an n - \mathcal{C} -syzygy object (of A), and A is called an n - \mathcal{C} -cosyzygy object (of M); in this case, we denote $M = \Omega_{\mathcal{C}}^n(A)$ and $A = \Omega_{\mathcal{C}}^{-n}(M)$. We denote by $\Omega_{\mathcal{C}}^n(\mathcal{A})$ (resp., $\Omega_{\mathcal{C}}^{-n}(\mathcal{A})$) the subcategory of \mathcal{A} consisting of n - \mathcal{C} -syzygy (resp., n - \mathcal{C} -cosyzygy) objects.

(b) For an object A in \mathcal{A} , the \mathcal{C} -dimension (resp., \mathcal{C} -codimension), denoted by $\mathcal{C}\text{-dim } A$ (resp., $\mathcal{C}\text{-codim } A$), is defined as $\inf\{n \geq 0 \mid \text{there exists an exact sequence } 0 \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0 \text{ (resp., } 0 \rightarrow A \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow$

$C^n \rightarrow 0$) in \mathcal{A} with all C_i 's (resp., C^i 's) objects in \mathcal{C} . Set $\mathcal{C}\text{-dim } A$ (resp., $\mathcal{C}\text{-codim } A$) = ∞ if no such integer exists.

Let \mathcal{C} be a subcategory of \mathcal{A} . We denote $\mathcal{C}^\perp = \{A \text{ is an object in } \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^i(C, A) = 0 \text{ for any object } C \text{ in } \mathcal{C} \text{ and } i \geq 1\}$ and ${}^\perp\mathcal{C} = \{A \text{ is an object in } \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^i(A, C) = 0 \text{ for any object } C \text{ in } \mathcal{C} \text{ and } i \geq 1\}$.

LEMMA 2.2

Let \mathcal{C} and \mathcal{D} be subcategories of \mathcal{A} , let M be an object in ${}^\perp\mathcal{C}$, and let M' be an object in $\Omega_{\mathcal{C}}^{-n}(\mathcal{A})$ such that some $\Omega_{\mathcal{C}}^n(M')$ is an object in \mathcal{D}^\perp . If $\mathcal{D}\text{-dim } M \leq n (< \infty)$, then $\text{Ext}_{\mathcal{A}}^i(M, M') = 0$ for any $i \geq 1$.

Proof

By assumption, there exists an exact sequence

$$0 \rightarrow M'' \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow M' \rightarrow 0$$

in \mathcal{A} with all C_i 's objects in \mathcal{C} and M'' an object in \mathcal{D}^\perp . Let M be an object in ${}^\perp\mathcal{C}$. Then $\text{Ext}_{\mathcal{A}}^i(M, M') \cong \text{Ext}_{\mathcal{A}}^{n+i}(M, M'')$ for any $i \geq 1$. If $\mathcal{D}\text{-dim } M \leq n (< \infty)$, then there exists an exact sequence

$$0 \rightarrow D_n \rightarrow \cdots \rightarrow D_1 \rightarrow D_0 \rightarrow M \rightarrow 0$$

in \mathcal{A} with all D_i 's objects in \mathcal{D} . So $\text{Ext}_{\mathcal{A}}^{n+i}(M, M'') \cong \text{Ext}_{\mathcal{A}}^i(D_n, M'') = 0$ for any $i \geq 1$ and hence $\text{Ext}_{\mathcal{A}}^i(M, M') = 0$ for any $i \geq 1$. □

Let \mathcal{E} be a subcategory of \mathcal{A} . Recall from [EJ2] that a sequence

$$\mathbb{S} : \cdots \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \cdots$$

in \mathcal{A} is called $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact (resp., $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact) if $\text{Hom}_{\mathcal{A}}(E, \mathbb{S})$ (resp., $\text{Hom}_{\mathcal{A}}(\mathbb{S}, E)$) is exact for any object E in \mathcal{E} . An epimorphism (resp., a monomorphism) in \mathcal{A} is called \mathcal{E} -proper (resp., \mathcal{E} -coproper) if it is $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact (resp., $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact).

PROPOSITION 2.3

Let \mathcal{C} and \mathcal{E} be subcategories of \mathcal{A} , and let \mathcal{C} be closed under kernels of (\mathcal{E} -proper) epimorphisms. If

$$(2.1) \quad 0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$$

is a $(\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact) exact sequence in \mathcal{A} with A_3 an object in \mathcal{C} , then $\mathcal{C}\text{-dim } A_1 \leq \mathcal{C}\text{-dim } A_2$.

Proof

Let $\mathcal{C}\text{-dim } A_2 = n (< \infty)$, and let

$$0 \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow A_2 \rightarrow 0$$

be an exact sequence in \mathcal{A} with all C_i 's objects in \mathcal{C} . By [Hu, Theorem 3.2], there exist exact sequences

$$0 \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C \rightarrow A_1 \rightarrow 0$$

and

$$(2.2) \quad 0 \rightarrow C \rightarrow C_0 \rightarrow A_3 \rightarrow 0.$$

From the proof of [Hu, Theorem 3.2] we see that if (2.1) is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, then so is (2.2). Because \mathcal{C} is closed under kernels of (\mathcal{C} -proper) epimorphisms and A_3 is an object in \mathcal{C} by assumption, C is an object in \mathcal{C} and $\mathcal{C}\text{-dim } A_1 \leq n$. \square

Let \mathcal{C} be a subcategory of \mathcal{A} . We write $\mathcal{C} \perp \mathcal{C}$ if $\text{Ext}_{\mathcal{A}}^i(C_1, C_2) = 0$ for any objects C_1, C_2 in \mathcal{C} and $i \geq 1$, and we denote by $\mathcal{C}\text{-dim}^{<\infty}$ (resp., $\mathcal{C}\text{-codim}^{<\infty}$) the subcategory of \mathcal{A} consisting of objects with finite \mathcal{C} -dimension (resp., \mathcal{C} -codimension).

LEMMA 2.4

Let \mathcal{C} be a subcategory of \mathcal{A} such that $\mathcal{C} \perp \mathcal{C}$ and $\mathcal{C}\text{-dim}^{<\infty}$ is closed under direct summands, and let $0 \rightarrow K \rightarrow C \rightarrow A \rightarrow 0$ be an exact sequence in \mathcal{A} with $\mathcal{C}\text{-dim } A < \infty$ and C an object in \mathcal{C} . If K is an object in \mathcal{C}^\perp , then $\mathcal{C}\text{-dim } K < \infty$.

Proof

Because $\mathcal{C}\text{-dim } A < \infty$, there exists an exact sequence

$$0 \rightarrow M \rightarrow C_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with C_0 an object in \mathcal{C} and $\mathcal{C}\text{-dim } M < \infty$. Consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & M & \xlongequal{\quad} & M & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & C_0 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & A \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 &
 \end{array}$$

Because $\mathcal{C} \perp \mathcal{C}$ and $\mathcal{C}\text{-dim } M < \infty$, it is easy to get that $M \in \mathcal{C}^\perp$ by dimension shifting. So the middle column in the above diagram splits, and hence $\mathcal{C}\text{-dim } N \leq \mathcal{C}\text{-dim } M < \infty$ by [Hu, Lemma 3.1]. Because K is an object in \mathcal{C}^\perp by assumption,

the middle row in the above diagram also splits and K is isomorphic to a direct summand of N . Thus \mathcal{C} -dim $K < \infty$. □

DEFINITION 2.5

Let $\mathcal{C} \subseteq \mathcal{T}$ be subcategories of \mathcal{A} .

(a) \mathcal{C} is called a *generator* (resp., *cogenerator*) for \mathcal{T} if, for any object T in \mathcal{T} , there exists an exact sequence $0 \rightarrow T' \rightarrow C \rightarrow T \rightarrow 0$ (resp., $0 \rightarrow T \rightarrow C \rightarrow T' \rightarrow 0$) in \mathcal{T} with C an object in \mathcal{C} (cf. [SSW]).

(b) Let \mathcal{E} be a subcategory of \mathcal{A} . Then, \mathcal{C} is called an \mathcal{E} -*proper generator* (resp., \mathcal{E} -*coproper cogenerator*) for \mathcal{T} if, for any object T in \mathcal{T} , there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ (resp., $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$)-exact exact sequence $0 \rightarrow T' \rightarrow C \rightarrow T \rightarrow 0$ (resp., $0 \rightarrow T \rightarrow C \rightarrow T' \rightarrow 0$) in \mathcal{A} such that C is an object in \mathcal{C} and T' is an object in \mathcal{T} .

LEMMA 2.6

Let $\mathcal{C} \subseteq \mathcal{T}$ be subcategories of \mathcal{A} such that \mathcal{C} is a cogenerator for \mathcal{T} , and let $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ be an exact sequence in \mathcal{A} such that both A_2 and A_3 are objects in \mathcal{T}^\perp . If A_1 is an object in \mathcal{C}^\perp , then A_1 is an object in \mathcal{T}^\perp .

Proof

Let $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ be an exact sequence in \mathcal{A} such that both A_2 and A_3 are objects in \mathcal{T}^\perp . Then $\text{Ext}_{\mathcal{A}}^i(T, A_1) = 0$ for any object T in \mathcal{T} and $i \geq 2$. Because \mathcal{C} is a cogenerator for \mathcal{T} by assumption, there exists an exact sequence

$$0 \rightarrow T \rightarrow C \rightarrow T' \rightarrow 0$$

in \mathcal{A} with C an object in \mathcal{C} and T' an object in \mathcal{T} , which yields an exact sequence

$$\text{Ext}_{\mathcal{A}}^i(C, A_1) \rightarrow \text{Ext}_{\mathcal{A}}^i(T, A_1) \rightarrow \text{Ext}_{\mathcal{A}}^{i+1}(T', A_1)$$

for any $i \geq 1$. Note that $\text{Ext}_{\mathcal{A}}^{i+1}(T', A_1) = 0$ for any $i \geq 1$ by the above argument. So, if A_1 is an object in \mathcal{C}^\perp , then $\text{Ext}_{\mathcal{A}}^i(T, A_1) = 0$ for any $i \geq 1$ and A_1 is an object in \mathcal{T}^\perp . □

LEMMA 2.7

Let $\mathcal{C} \subseteq \mathcal{T}$ be subcategories of \mathcal{A} such that \mathcal{C} is a cogenerator for \mathcal{T} and \mathcal{C} is closed under direct summands. Then $\mathcal{T} \cap \mathcal{T}^\perp \subseteq \mathcal{C}$.

Proof

Let T be an object in $\mathcal{T} \cap \mathcal{T}^\perp$. Then there exists a split exact sequence

$$0 \rightarrow T \rightarrow C \rightarrow T' \rightarrow 0$$

in \mathcal{A} with C an object in \mathcal{C} and T' an object in \mathcal{T} . So T is isomorphic to a direct summand of C . Because \mathcal{C} is closed under direct summands by assumption, T is an object in \mathcal{C} . □

Sather-Wagstaff, Sharif, and White introduced the Gorenstein category $\mathcal{G}(\mathcal{C})$ as follows.

DEFINITION 2.8 ([SSW, DEFINITION 4.1])

Let \mathcal{C} be a subcategory of \mathcal{A} . The *Gorenstein subcategory* $\mathcal{G}(\mathcal{C})$ of \mathcal{A} is defined as $\mathcal{G}(\mathcal{C}) = \{A \text{ is an object in } \mathcal{A} \mid \text{there exists an exact sequence } \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \text{ in } \mathcal{A} \text{ with all terms objects in } \mathcal{C}, \text{ which is both } \text{Hom}_{\mathcal{A}}(\mathcal{C}, -)\text{-exact and } \text{Hom}_{\mathcal{A}}(-, \mathcal{C})\text{-exact such that } A \cong \text{Im}(C_0 \rightarrow C^0)\}$.

The Gorenstein category unifies the following notions: modules of Gorenstein dimension zero (see [AB]), Gorenstein projective modules, Gorenstein injective modules (see [EJ1]), V -Gorenstein projective modules, V -Gorenstein injective modules (see [EJL]), \mathcal{W} -Gorenstein modules (see [GD]), and so on (see [Hu] for the details).

3. Computation and comparison of homological dimensions

In this section, we introduce the notion of (pre)resolving subcategories of \mathcal{A} . Then we give some criteria for computing and comparing homological dimensions relative to different prerresolving subcategories.

DEFINITION 3.1

Let \mathcal{E} and \mathcal{T} be subcategories of \mathcal{A} . Then \mathcal{T} is called \mathcal{E} -*preresolving* in \mathcal{A} if the following conditions are satisfied.

- (a) \mathcal{T} admits an \mathcal{E} -proper generator.
- (b) \mathcal{T} is *closed under \mathcal{E} -proper extensions*, that is, for any $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact exact sequence $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ in \mathcal{A} , if both A_1 and A_3 are objects in \mathcal{T} , then A_2 is also an object in \mathcal{T} .

An \mathcal{E} -preresolving subcategory \mathcal{T} of \mathcal{A} is called \mathcal{E} -*resolving* if the following condition is satisfied.

- (c) \mathcal{T} is *closed under kernels of \mathcal{E} -proper epimorphisms*, that is, for any $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact exact sequence $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ in \mathcal{A} , if both A_2 and A_3 are objects in \mathcal{T} , then A_1 is also an object in \mathcal{T} .

The following list shows that the class of \mathcal{E} -(pre)resolving subcategories is rather large.

EXAMPLE 3.2 (a) Let \mathcal{A} admit enough projective objects, and let \mathcal{E} be the subcategory of \mathcal{A} consisting of projective objects. Then a subcategory of \mathcal{A} closed under \mathcal{E} -proper extensions is just a subcategory of \mathcal{A} closed under extensions. Furthermore, if $\mathcal{C} = \mathcal{E}$ in the above definition, then an \mathcal{E} -preresolving subcategory is just a subcategory which contains all projective objects and is closed

under extensions, and an \mathcal{E} -resolving subcategory is just a projectively resolving subcategory in the sense of [H1].

(b) Let \mathcal{C} be a subcategory of \mathcal{A} with $\mathcal{C} \perp \mathcal{C}$. Then by [SSW, Corollary 4.5], the Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of \mathcal{A} is a \mathcal{C} -preresolving subcategory of \mathcal{A} with a \mathcal{C} -proper generator \mathcal{C} ; furthermore, if \mathcal{C} is closed under kernels of epimorphisms, then $\mathcal{G}(\mathcal{C})$ is a \mathcal{C} -resolving subcategory of \mathcal{A} by [SSW, Theorem 4.12(a)].

(c) Let R be a ring, let $\text{Mod } R$ be the category of left R -modules, and let $\mathcal{P}(\text{Mod } R)$ be the subcategory of $\text{Mod } R$ consisting of projective modules. Recall from [EJ2] that a pair of subcategories $(\mathcal{X}, \mathcal{Y})$ of $\text{Mod } R$ is called a *cotorsion pair* if $\mathcal{X} = \{X \in \text{Mod } R \mid \text{Ext}_R^1(X, Y) = 0 \text{ for any } Y \in \mathcal{Y}\}$ and $\mathcal{Y} = \{Y \in \text{Mod } R \mid \text{Ext}_R^1(X, Y) = 0 \text{ for any } X \in \mathcal{X}\}$. If $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in $\text{Mod } R$, then \mathcal{X} is a $\mathcal{P}(\text{Mod } R)$ -preresolving subcategory of $\text{Mod } R$ with a $\mathcal{P}(\text{Mod } R)$ -proper generator $\mathcal{P}(\text{Mod } R)$ (see [EJ2]).

(d) Let R be a ring, and let $\mathcal{F}(\text{Mod } R)$ be the subcategory of $\text{Mod } R$ consisting of flat modules. Then by [Hu, Lemma 3.1 and Theorem 3.2], it is not difficult to see that the subcategory of $\text{Mod } R$ consisting of strongly Gorenstein flat modules (see [DLM] or Section 5 below for the definition) is an $\mathcal{F}(\text{Mod } R)$ -resolving subcategory of $\text{Mod } R$ with an $\mathcal{F}(\text{Mod } R)$ -proper generator $\mathcal{P}(\text{Mod } R)$.

(e) Let R be a ring. Then, the subcategory of $\text{Mod } R$ consisting of the modules A satisfying $\text{Ext}_R^i(A, P) = 0$ for any $P \in \mathcal{P}(\text{Mod } R)$ and $i \geq 1$, is a $\mathcal{P}(\text{Mod } R)$ -resolving subcategory of $\text{Mod } R$ with a $\mathcal{P}(\text{Mod } R)$ -proper generator $\mathcal{P}(\text{Mod } R)$. Let R be a left Noetherian ring, let $\text{mod } R$ be the category of finitely generated left R -modules, and let $\mathcal{P}(\text{mod } R)$ be the subcategory of $\text{mod } R$ consisting of projective modules. Then the subcategory of $\text{mod } R$ consisting of the modules A satisfying $\text{Ext}_R^i(A, R) = 0$ for any $i \geq 1$ is a $\mathcal{P}(\text{mod } R)$ -resolving subcategory of $\text{mod } R$ with a $\mathcal{P}(\text{mod } R)$ -proper generator $\mathcal{P}(\text{mod } R)$.

Unless stated otherwise, in the rest of this section, we fix a subcategory \mathcal{E} of \mathcal{A} and an \mathcal{E} -preresolving subcategory \mathcal{T} of \mathcal{A} admitting an \mathcal{E} -proper generator \mathcal{C} . We will give some criteria for computing the \mathcal{T} -dimension of a given object A in \mathcal{A} , and then compare it with the \mathcal{C} -dimension of A .

The following two propositions play a crucial role in this section.

PROPOSITION 3.3

Let

$$(3.1) \quad 0 \rightarrow M \rightarrow T_1 \xrightarrow{f} T_0 \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with both T_0 and T_1 objects in \mathcal{T} . Then we have the following.

(a) There exists an exact sequence

$$(3.2) \quad 0 \rightarrow M \rightarrow T \rightarrow C \rightarrow A \rightarrow 0$$

in \mathcal{A} with T an object in \mathcal{T} and C an object in \mathcal{C} .

(b) If (3.1) is $\text{Hom}_{\mathcal{A}}(X, -)$ -exact for some object X in \mathcal{A} , then so is (3.2).

Proof

(a) Let

$$0 \rightarrow M \rightarrow T_1 \xrightarrow{f} T_0 \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with both T_0 and T_1 objects in \mathcal{T} . Because there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact exact sequence

$$0 \rightarrow T'_0 \rightarrow C \rightarrow T_0 \rightarrow 0$$

in \mathcal{A} with C an object in \mathcal{C} and T'_0 an object in \mathcal{T} , we have the following pullback diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T'_0 & \xlongequal{\quad} & T'_0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & W & \longrightarrow & C & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{Im } f & \longrightarrow & T_0 & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Then consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T'_0 & \xlongequal{\quad} & T'_0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & T & \longrightarrow & W \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & T_1 & \longrightarrow & \text{Im } f \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Because the middle column in the first diagram is $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact, the first column in the first diagram (that is, the third column in the second diagram) and the middle column in the second diagram are also $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact by [Hu,

Lemma 2.4(1)]. Because both T'_0 and T_1 are objects in \mathcal{T} , T is also an object in \mathcal{T} . Connecting the middle rows in the above two diagrams we get the desired exact sequence.

(b) If (3.1) is $\text{Hom}_{\mathcal{A}}(X, -)$ -exact for some object X in \mathcal{A} , then so are the third rows in the above two diagrams. So the middle rows in the above two diagrams and (3.2) are also $\text{Hom}_{\mathcal{A}}(X, -)$ -exact by [Hu, Lemma 2.4(1)]. \square

As an application of Proposition 3.3, we get the following.

PROPOSITION 3.4

Let $n \geq 1$, and let

$$0 \rightarrow M \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow \cdots \rightarrow T_0 \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with all T_i 's objects in \mathcal{T} . Then there exist an exact sequence

$$0 \rightarrow N \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_0 \rightarrow A \rightarrow 0$$

and a $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact exact sequence

$$0 \rightarrow T \rightarrow N \rightarrow M \rightarrow 0$$

in \mathcal{A} with all C_i 's objects in \mathcal{C} and T an object in \mathcal{T} . In particular, an object in \mathcal{A} is an n - \mathcal{C} -cosyzygy if and only if it is an n - \mathcal{T} -cosyzygy.

Proof

We proceed by induction on n . The case for $n = 1$ has been proved in the proof of Proposition 3.3. Now suppose that $n \geq 2$, and suppose that we have an exact sequence

$$0 \rightarrow M \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow \cdots \rightarrow T_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with all T_i 's objects in \mathcal{T} . Put $K = \text{Ker}(T_1 \rightarrow T_0)$. By Proposition 3.3, we get an exact sequence

$$0 \rightarrow K \rightarrow T'_1 \rightarrow C_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with T'_1 an object in \mathcal{T} and C_0 an object in \mathcal{C} . Put $A' = \text{Im}(T'_1 \rightarrow C_0)$. Then we get an exact sequence

$$0 \rightarrow M \rightarrow T_{n-1} \rightarrow T_{n-2} \rightarrow \cdots \rightarrow T_2 \rightarrow T'_1 \rightarrow A' \rightarrow 0$$

in \mathcal{A} . Thus the assertion follows from the induction hypothesis. \square

The following corollary is an immediate consequence of Proposition 3.4.

COROLLARY 3.5

Let M be an object in \mathcal{A} , and let $n \geq 0$. If \mathcal{T} -codim $M = n$, then there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact exact sequence $0 \rightarrow T \rightarrow N \rightarrow M \rightarrow 0$ in \mathcal{A} with \mathcal{C} -codim $N \leq n$ and T an object in \mathcal{T} .

Proof

Let M be an object in \mathcal{A} with $\mathcal{T}\text{-codim } M = n$. Applying Proposition 3.4 with $A = 0$ we get a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence $0 \rightarrow T \rightarrow N \rightarrow M \rightarrow 0$ in \mathcal{A} with $\mathcal{C}\text{-codim } N \leq n$ and T an object in \mathcal{T} . \square

We give a criterion for computing the \mathcal{T} -dimension of an object in \mathcal{A} as follows.

THEOREM 3.6

The following statements are equivalent for any object A in \mathcal{A} and $n \geq 0$.

- (a) $\mathcal{T}\text{-dim } A \leq n$.
- (b) *There exists an exact sequence*

$$0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with all C_i 's objects in \mathcal{C} and K_n an object in \mathcal{T} .

Proof

The proof of (b) \Rightarrow (a) is trivial.

For (a) \Rightarrow (b), we proceed by induction on n . The case for $n = 0$ is trivial. If $n = 1$, then there exists an exact sequence

$$0 \rightarrow T_1 \rightarrow T_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with both T_0 and T_1 objects in \mathcal{T} . Applying Proposition 3.3 with $M = 0$, we get an exact sequence

$$0 \rightarrow K_1 \rightarrow C_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with C_0 an object in \mathcal{C} and K_1 an object in \mathcal{T} .

Now suppose that $n \geq 2$. Then there exists an exact sequence

$$0 \rightarrow T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with all T_i 's objects in \mathcal{T} . Put $M = \text{Ker}(T_1 \rightarrow T_0)$. By Proposition 3.3, we get an exact sequence

$$0 \rightarrow M \rightarrow T'_1 \rightarrow C_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with C_0 an object in \mathcal{C} and T'_1 an object in \mathcal{T} . Put $B = \text{Im}(T'_1 \rightarrow C_0)$. Then we get an exact sequence

$$0 \rightarrow T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T'_1 \rightarrow B \rightarrow 0.$$

By the induction hypothesis, we get the exact sequence

$$0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow B \rightarrow 0$$

in \mathcal{A} with all C_i 's objects in \mathcal{C} and K_n an object in \mathcal{T} . Thus we get the desired exact sequence

$$0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0. \quad \square$$

The following result gives a criterion for computing the \mathcal{T} -codimension of an object in \mathcal{A} . To some extent, the proof of this result is dual to that of Theorem 3.6, so we omit it.

THEOREM 3.7

The following statements are equivalent for any object M in \mathcal{A} and $n \geq 0$.

- (a) \mathcal{T} -codim $M \leq n$.
- (b) There exists an exact sequence

$$0 \rightarrow M \rightarrow K^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^{n-1} \rightarrow C^n \rightarrow 0$$

in \mathcal{A} with K^0 an object in \mathcal{T} and all C^i 's objects in \mathcal{C} , that is, there exists an exact sequence

$$0 \rightarrow M \rightarrow T \rightarrow H \rightarrow 0$$

in \mathcal{A} with T an object in \mathcal{T} and \mathcal{C} -codim $H \leq n - 1$.

The following result gives a sufficient condition such that the n - \mathcal{C} -syzygy of an object in \mathcal{A} with \mathcal{T} -dimension at most n is in \mathcal{T} , in which the first assertion generalizes [AB, Lemma 3.12].

THEOREM 3.8

Let \mathcal{T} be closed under kernels of (\mathcal{E} -proper) epimorphisms and $\mathcal{T} \subseteq \mathcal{C}^\perp$, and let $n \geq 0$. Then for any object A in \mathcal{A} with \mathcal{T} -dim $A \leq n$ we have the following.

- (a) For any ($\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact) exact sequence $0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0$ in \mathcal{A} with all C_i 's objects in \mathcal{C} , K_n is an object in \mathcal{T} .
- (b) If $0 \rightarrow K \rightarrow C \rightarrow A \rightarrow 0$ is a ($\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact) exact sequence in \mathcal{A} with C an object in \mathcal{C} , then \mathcal{T} -dim $K \leq n - 1$.

Proof

Let \mathcal{T} -dim $A \leq n$, and let $\mathcal{T} \subseteq \mathcal{C}^\perp$. Then there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence

$$0 \rightarrow T_n \rightarrow C'_{n-1} \rightarrow \dots \rightarrow C'_1 \rightarrow C'_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with all C'_i 's objects in \mathcal{C} and T_n an object in \mathcal{T} by Proposition 3.4.

- (a) Let

$$0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0$$

be a ($\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact) exact sequence in \mathcal{A} with all C_i 's objects in \mathcal{C} . Then by [Hu, Theorem 3.2] we get a ($\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact) exact sequence

$$0 \rightarrow K_n \rightarrow T_n \oplus C_{n-1} \rightarrow C'_{n-1} \oplus C_{n-2} \rightarrow \dots \rightarrow C'_1 \oplus C_0 \rightarrow C'_0 \rightarrow 0.$$

Because \mathcal{T} is closed under kernels of (\mathcal{E} -proper) epimorphisms by assumption, K_n is an object in \mathcal{T} .

(b) Put $T_1 = \text{Im}(C'_1 \rightarrow C'_0)$. Then we have a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence

$$0 \rightarrow T_1 \rightarrow C'_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with $\mathcal{T}\text{-dim} T_1 \leq n-1$. Let $0 \rightarrow K \rightarrow C \rightarrow A \rightarrow 0$ be a $(\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact) exact sequence in \mathcal{A} with C an object in \mathcal{C} . Consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & T_1 & \xlongequal{\quad} & T_1 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & C'_0 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & A \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 &
 \end{array}$$

Because the third column in this diagram is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact, the middle column is also $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact by [Hu, Lemma 2.4(1)]. So the middle column splits and $X \cong T_1 \oplus C$. Then the middle row yields an exact sequence

$$0 \rightarrow K \rightarrow T_1 \oplus C \rightarrow C'_0 \rightarrow 0.$$

By Proposition 2.3, $\mathcal{T}\text{-dim} K \leq \mathcal{T}\text{-dim} T_1 \oplus C \leq n-1$. □

We use $\mathcal{T}\text{-dim}^{\leq n}$ to denote the subcategory of \mathcal{A} consisting of objects with \mathcal{T} -dimension at most n .

COROLLARY 3.9

Let \mathcal{T} be a \mathcal{C} -resolving subcategory of \mathcal{A} with a \mathcal{C} -proper generator \mathcal{C} and $\mathcal{T} \subseteq \mathcal{C}^\perp$. If \mathcal{T} is closed under direct summands, then so is $\mathcal{T}\text{-dim}^{\leq n}$ for any $n \geq 0$.

Proof

The case for $n=0$ follows from assumption. Now, let $n \geq 1$, and let A be an object in \mathcal{A} with $\mathcal{T}\text{-dim} A \leq n$ and $A = A_1 \oplus A_2$. Because $\mathcal{T} \subseteq \mathcal{C}^\perp$ by assumption, there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence

$$0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \rightarrow C_0 \xrightarrow{f_0} A \rightarrow 0$$

in \mathcal{A} with all C_i 's objects in \mathcal{C} and K_n an object in \mathcal{T} by Theorem 3.6. Note that both

$$0 \rightarrow A_2 \xrightarrow{\begin{pmatrix} 0 \\ 1_{A_2} \end{pmatrix}} A \xrightarrow{(1_{A_1}, 0)} A_1 \rightarrow 0$$

and

$$0 \rightarrow A_1 \xrightarrow{\begin{pmatrix} 1_{A_1} \\ 0 \end{pmatrix}} A \xrightarrow{(0, 1_{A_2})} A_2 \rightarrow 0$$

are exact and split. So both

$$C_0 \xrightarrow{(1_{A_1}, 0)f_0} A_1 \rightarrow 0$$

and

$$C_0 \xrightarrow{(0, 1_{A_2})f_0} A_2 \rightarrow 0$$

are $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequences. By [Hu, Theorem 3.6], we get the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequences:

$$C_0 \oplus C_1 \rightarrow C_0 \rightarrow A_1 \rightarrow 0$$

and

$$C_0 \oplus C_1 \rightarrow C_0 \rightarrow A_2 \rightarrow 0.$$

Again by [Hu, Theorem 3.6], we get the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequences:

$$C_0 \oplus C_1 \oplus C_2 \rightarrow C_0 \oplus C_1 \rightarrow C_0 \rightarrow A_1 \rightarrow 0$$

and

$$C_0 \oplus C_1 \oplus C_2 \rightarrow C_0 \oplus C_1 \rightarrow C_0 \rightarrow A_2 \rightarrow 0.$$

Continuing this procedure, we finally get the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequences:

$$0 \rightarrow X_n \rightarrow \bigoplus_{i=0}^{n-1} C_i \rightarrow \bigoplus_{i=0}^{n-2} C_i \rightarrow \cdots \rightarrow C_1 \oplus C_0 \rightarrow C_0 \rightarrow A_1 \rightarrow 0$$

and

$$0 \rightarrow Y_n \rightarrow \bigoplus_{i=0}^{n-1} C_i \rightarrow \bigoplus_{i=0}^{n-2} C_i \rightarrow \cdots \rightarrow C_1 \oplus C_0 \rightarrow C_0 \rightarrow A_2 \rightarrow 0.$$

Put $U_j = \bigoplus_{i=0}^j C_i$ for any $0 \leq j \leq n-1$. Then we get a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence

$$\begin{aligned} 0 \rightarrow X_n \oplus Y_n &\rightarrow U_{n-1} \oplus U_{n-1} \rightarrow U_{n-2} \oplus U_{n-2} \rightarrow \cdots \\ &\rightarrow U_1 \oplus U_1 \rightarrow U_0 \oplus U_0 \rightarrow A \rightarrow 0. \end{aligned}$$

By Theorem 3.8, $X_n \oplus Y_n$ is an object in \mathcal{T} . So both X_n and Y_n are objects in \mathcal{T} and hence $\mathcal{T}\text{-dim } A_1 \leq n$ and $\mathcal{T}\text{-dim } A_2 \leq n$. □

The following result gives some sufficient conditions such that the \mathcal{T} -dimension and the \mathcal{C} -dimension of an object in \mathcal{A} are identical.

THEOREM 3.10

Let $\mathcal{T} \subseteq \mathcal{C}^\perp \cap {}^\perp\mathcal{C}$, and let \mathcal{C} be closed under direct summands. Then for an object A in \mathcal{A} , $\mathcal{T}\text{-dim } A = \mathcal{C}\text{-dim } A$ if one of the following conditions is satisfied.

- (a) $\mathcal{C}\text{-dim } A < \infty$, $\mathcal{E} = \mathcal{C}$, and \mathcal{T} is closed under kernels of \mathcal{C} -proper epimorphisms.
- (b) $\mathcal{C}\text{-dim } A < \infty$, $\mathcal{E} = \mathcal{C}$, and $\mathcal{C}\text{-dim}^{<\infty}$ is closed under direct summands.
- (c) A is an object in \mathcal{T}^\perp and \mathcal{C} is a cogenerator for \mathcal{T} .

Proof

It is trivial that $\mathcal{C}\text{-dim } A \geq \mathcal{T}\text{-dim } A$. In the following we prove that $\mathcal{C}\text{-dim } A \leq \mathcal{T}\text{-dim } A$. Suppose that $\mathcal{T}\text{-dim } A = n < \infty$.

(a) Let $\mathcal{C}\text{-dim } A = t (< \infty)$. If $n < t$, then consider the following $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence:

$$0 \rightarrow C_t \rightarrow \cdots \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0,$$

in \mathcal{A} with all C_i 's objects in \mathcal{C} . Put $K_n = \text{Im}(C_n \rightarrow C_{n-1})$. So K_n is an object in \mathcal{T} by Theorem 3.8(a) and hence an object in ${}^\perp\mathcal{C}$ by assumption. This yields that the exact sequence

$$0 \rightarrow C_t \rightarrow \cdots \rightarrow C_n \rightarrow K_n \rightarrow 0$$

splits and K_n is isomorphic to a direct summand of C_n . Because \mathcal{C} is closed under direct summands by assumption, K_n is an object in \mathcal{C} and $\mathcal{C}\text{-dim } A \leq n$, which is a contradiction. So $n \geq t$.

In the following, we prove (b) and (c).

Let

$$0 \rightarrow T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with all T_i 's objects in \mathcal{T} . By Proposition 3.4, we get an exact sequence

$$0 \rightarrow K_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0$$

and a $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact exact sequence

$$0 \rightarrow T \rightarrow K_n \rightarrow T_n \rightarrow 0$$

in \mathcal{A} with all C_i 's objects in \mathcal{C} and T an object in \mathcal{T} . So K_n is an object in $\mathcal{T} (\subseteq \mathcal{C}^\perp \cap {}^\perp\mathcal{C})$.

(b) Because $\mathcal{C}\text{-dim } A < \infty$, $\mathcal{C}\text{-dim } K_n < \infty$ by Lemma 2.4. By assumption $\mathcal{T} \subseteq {}^\perp\mathcal{C}$, so it is easy to see that K_n is isomorphic to a direct summand of some object in \mathcal{C} . Since \mathcal{C} is closed under direct summands by assumption, K_n is an object in \mathcal{C} and $\mathcal{C}\text{-dim } A \leq n$.

(c) Let A be an object in \mathcal{T}^\perp , and let $K_i = \text{Im}(C_i \rightarrow C_{i-1})$ for any $1 \leq i \leq n - 1$. Then all K_i 's are objects in \mathcal{C}^\perp . By Lemma 2.6, all K_i 's are objects in \mathcal{T}^\perp , and in particular, K_n is an object in \mathcal{T}^\perp . So K_n is an object in \mathcal{C} by Lemma 2.7, and hence $\mathcal{C}\text{-dim } A \leq n$. □

The following result gives a sufficient condition such that the \mathcal{T} -codimension and the \mathcal{C} -codimension of an object in \mathcal{A} are identical.

THEOREM 3.11

Let \mathcal{D} be a subcategory of \mathcal{A} such that $\mathcal{T} \subseteq {}^\perp\mathcal{C} \cap \mathcal{D}^\perp$, and let $\mathcal{C}\text{-codim}^{\leq n}$ be closed under direct summands for any $n \geq 0$. If M is an object in \mathcal{A} with $\mathcal{D}\text{-dim } M < \infty$, then $\mathcal{T}\text{-codim } M = \mathcal{C}\text{-codim } M$.

Proof

It is clear that $\mathcal{C}\text{-codim } M \geq \mathcal{T}\text{-codim } M$. In the following we prove that $\mathcal{T}\text{-codim } M \geq \mathcal{C}\text{-codim } M$.

Without loss of generality, assume that $\mathcal{T}\text{-codim } M = n < \infty$. If $n = 0$, then M is an object in \mathcal{T} and there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact exact sequence

$$(3.3) \quad 0 \rightarrow M' \rightarrow C \rightarrow M \rightarrow 0$$

in \mathcal{A} with C an object in \mathcal{C} and M' an object in \mathcal{T} . Notice that $\mathcal{T} \subseteq {}^\perp\mathcal{C} \cap \mathcal{D}^\perp$ by assumption, so $\text{Ext}_{\mathcal{A}}^i(M, M') = 0$ for any $i \geq 1$ by Lemma 2.2. It follows that the exact sequence (3.3) splits, which implies that M is isomorphic to a direct summand of C . Because \mathcal{C} is closed under direct summands by assumption, M is an object in \mathcal{C} .

Now suppose that $n \geq 1$. By Theorem 3.7, there exists an exact sequence

$$0 \rightarrow M \rightarrow T \rightarrow H \rightarrow 0$$

in \mathcal{A} with T an object in \mathcal{T} and $\mathcal{C}\text{-codim } H \leq n - 1$. It is easy to see that M is an object in ${}^\perp\mathcal{C}$. Because there exists a $\text{Hom}_{\mathcal{A}}(\mathcal{E}, -)$ -exact exact sequence

$$0 \rightarrow T' \rightarrow C' \rightarrow T \rightarrow 0$$

in \mathcal{A} with C' an object in \mathcal{C} and T' an object in \mathcal{T} , we have the following pullback diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T' & \xlongequal{\quad} & T' & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & N & \longrightarrow & C' & \longrightarrow & H \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & M & \longrightarrow & T & \longrightarrow & H \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By the exactness of the middle row in the above diagram, $\mathcal{C}\text{-codim } N \leq n$. Because $\text{Ext}_{\mathcal{A}}^1(M, T') = 0$ by Lemma 2.2, the first column in the above diagram splits. So M is isomorphic to a direct summand of N . Because $\mathcal{C}\text{-codim}^{\leq n}$ is closed under direct summands by assumption, $\mathcal{C}\text{-codim } M \leq n$. \square

In the following, we fix a subcategory \mathcal{C} of \mathcal{A} .

The following two corollaries give some sufficient conditions such that the $\mathcal{G}(\mathcal{C})$ -dimension and the \mathcal{C} -dimension of an object in \mathcal{A} are identical. The first one is a generalization of [Z, Theorem 2.3].

COROLLARY 3.12

Let $\mathcal{C} \perp \mathcal{C}$, and let \mathcal{C} be closed under direct summands. Then for any object A in $\mathcal{G}(\mathcal{C})^\perp$, $\mathcal{G}(\mathcal{C})\text{-dim } A = \mathcal{C}\text{-dim } A$.

Proof

Let $\mathcal{C} \perp \mathcal{C}$. It is clear that \mathcal{C} is a \mathcal{C} -proper generator and a \mathcal{C} -coproper cogenerator for $\mathcal{G}(\mathcal{C})$. By [SSW, Corollary 4.5], $\mathcal{G}(\mathcal{C})$ is closed under extensions. By [Hu, Lemma 5.7], $\mathcal{G}(\mathcal{C}) \subseteq \mathcal{C}^\perp \cap {}^\perp\mathcal{C}$. Now the assertion follows from Theorem 3.10(c). \square

The following is a generalization of [H2, Theorem 2.2] and [Z, Corollary 2.5].

COROLLARY 3.13

Let $\mathcal{C} \perp \mathcal{C}$, let \mathcal{C} be closed under direct summands, and let \mathcal{D} be a subcategory of $\mathcal{G}(\mathcal{C})^\perp$. Then for any object A in \mathcal{C}^\perp with $\mathcal{D}\text{-codim } A < \infty$, $\mathcal{G}(\mathcal{C})\text{-dim } A = \mathcal{C}\text{-dim } A$.

Proof

Let A be an object in \mathcal{C}^\perp with $\mathcal{D}\text{-codim } A < \infty$. Because \mathcal{D} is a subcategory of $\mathcal{G}(\mathcal{C})^\perp$, it is easy to see that A is an object in $\mathcal{G}(\mathcal{C})^\perp$ by Lemma 2.6. Then the assertion follows from Corollary 3.12. \square

The following result gives a sufficient condition such that the $\mathcal{G}(\mathcal{C})$ -dimension and the ${}^\perp\mathcal{C}$ -dimension of an object in \mathcal{A} are identical.

THEOREM 3.14

Let $\mathcal{C} \perp \mathcal{C}$, and let A an object in \mathcal{A} with $\mathcal{G}(\mathcal{C})\text{-dim } A < \infty$. Then $\mathcal{G}(\mathcal{C})\text{-dim } A = {}^\perp\mathcal{C}\text{-dim } A$.

Proof

By [Hu, Lemma 5.7], $\mathcal{G}(\mathcal{C})\text{-dim } A \geq {}^\perp\mathcal{C}\text{-dim } A$. In the following we prove that $\mathcal{G}(\mathcal{C})\text{-dim } A \leq {}^\perp\mathcal{C}\text{-dim } A$.

Suppose that ${}^\perp\mathcal{C}\text{-dim } A = n < \infty$ and that $\mathcal{G}(\mathcal{C})\text{-dim } A = m < \infty$. If $n = 0$, then A is an object in ${}^\perp\mathcal{C}$. So by [Hu, Theorem 5.8], A is an object in $\mathcal{G}(\mathcal{C})$

and $m = 0$. Let $n \geq 1$. Then $\text{Ext}_{\mathcal{A}}^{n+i}(A, C) = 0$ for any object C in \mathcal{C} and $i \geq 1$. If $m > n$, then consider the following exact sequence:

$$0 \rightarrow G_m \rightarrow \cdots \rightarrow G_n \rightarrow G_{n-1} \rightarrow G_1 \rightarrow G_0 \rightarrow A \rightarrow 0,$$

in \mathcal{A} with all G_i 's objects in $\mathcal{G}(\mathcal{C})$. By putting $K_n = \text{Im}(G_n \rightarrow G_{n-1})$, K_n is an object in ${}^\perp\mathcal{C}$ and $\mathcal{G}(\mathcal{C})\text{-dim } K_n \leq m - n < \infty$. By the above argument, K_n is an object in $\mathcal{G}(\mathcal{C})$. So $\mathcal{G}(\mathcal{C})\text{-dim } A \leq n$, which is a contradiction. Thus $m \leq n$. It follows that $\mathcal{G}(\mathcal{C})\text{-dim } A \leq {}^\perp\mathcal{C}\text{-dim } A$. \square

4. Dual results

In this section, we introduce the notion of (pre)coresolving subcategories of \mathcal{A} . Then we give some criteria for computing and comparing homological codimensions relative to different precoresolving subcategories. The results and their proofs in this section are completely dual to those in Section 3, so we only list the results without proofs.

DEFINITION 4.1

Let \mathcal{E} and \mathcal{T} be subcategories of \mathcal{A} . Then \mathcal{T} is called \mathcal{E} -precoresolving in \mathcal{A} if the following conditions are satisfied.

- (a) \mathcal{T} admits an \mathcal{E} -coproper cogenerator.
- (b) \mathcal{T} is closed under \mathcal{E} -coproper extensions, that is, for any $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact exact sequence $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ in \mathcal{A} , if both A_1 and A_3 are objects in \mathcal{T} , then A_2 is also an object in \mathcal{T} .

An \mathcal{E} -precoresolving subcategory \mathcal{T} of \mathcal{A} is called \mathcal{E} -coresolving if the following condition is satisfied.

- (c) \mathcal{T} is closed under cokernels of \mathcal{E} -coproper monomorphisms, that is, for any $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact exact sequence $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ in \mathcal{A} , if both A_1 and A_2 are objects in \mathcal{T} , then A_3 is also an object in \mathcal{T} .

The following list shows that the class of \mathcal{E} -(pre)coresolving subcategories is rather large.

EXAMPLE 4.2

(a) Let \mathcal{A} admit enough injective objects, and let \mathcal{E} be the subcategory of \mathcal{A} consisting of injective objects. Then a subcategory of \mathcal{A} closed under \mathcal{E} -coproper extensions is just a subcategory of \mathcal{A} closed under extensions. Furthermore, if $\mathcal{C} = \mathcal{E}$ in the above definition, then an \mathcal{E} -precoresolving subcategory is just a subcategory which contains all injective objects and is closed under extensions, and an \mathcal{E} -coresolving subcategory is just an injectively coresolving subcategory in the sense of [H1].

(b) Let \mathcal{C} be a subcategory of \mathcal{A} with $\mathcal{C} \perp \mathcal{C}$. Then by [SSW, Corollary 4.5], the Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of \mathcal{A} is a \mathcal{C} -precoresolving subcategory of \mathcal{A}

with a \mathcal{C} -coproper cogenerator \mathcal{C} ; furthermore, if \mathcal{C} is closed under cokernels of monomorphisms, then $\mathcal{G}(\mathcal{C})$ is a \mathcal{C} -coresolving subcategory of \mathcal{A} by [SSW, Theorem 4.12(b)].

(c) Let R be a ring, and let $\mathcal{I}(\text{Mod } R)$ be the subcategory of $\text{Mod } R$ consisting of injective modules. If $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair in $\text{Mod } R$, then \mathcal{Y} is an $\mathcal{I}(\text{Mod } R)$ -precoresolving subcategory of $\text{Mod } R$ with an $\mathcal{I}(\text{Mod } R)$ -coproper cogenerator $\mathcal{I}(\text{Mod } R)$ (see [EJ2]).

(d) Let R be a ring. Recall that a module E in $\text{Mod } R$ is called *FP-injective* if $\text{Ext}_R^1(M, E) = 0$ for any finitely presented left R -module M . The FP-injective modules are also known as *absolutely pure modules*. We use $\mathcal{FI}(\text{Mod } R)$ to denote the subcategory of $\text{Mod } R$ consisting of FP-injective modules. Then by [Hu, Lemma 3.1 and Theorem 3.4], it is not difficult to see that the subcategory of $\text{Mod } R$ consisting of Gorenstein FP-injective modules (see [MD] or Section 5 below for the definition) is an $\mathcal{FI}(\text{Mod } R)$ -coresolving subcategory of $\text{Mod } R$ with an $\mathcal{FI}(\text{Mod } R)$ -coproper cogenerator $\mathcal{I}(\text{Mod } R)$.

(e) Let R be a ring. We denote by $\widetilde{\text{cores } \mathcal{P}(\text{Mod } R)} = \{M \in \text{Mod } R \mid \text{there exists a } \text{Hom}_R(-, \mathcal{P}(\text{Mod } R))\text{-exact exact sequence } 0 \rightarrow M \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots \rightarrow P^i \rightarrow \cdots \text{ in } \text{Mod } R \text{ with all } P^i\text{'s projective}\}$. Then by [Hu, Lemma 3.1 and Theorem 3.8], it is easy to see that $\widetilde{\text{cores } \mathcal{P}(\text{Mod } R)}$ is a $\mathcal{P}(\text{Mod } R)$ -coresolving subcategory of $\text{Mod } R$ with a $\mathcal{P}(\text{Mod } R)$ -coproper cogenerator $\mathcal{P}(\text{Mod } R)$. Let R be a left and right Noetherian ring. Then by [AB, Theorem 2.17] and [Hu, Lemma 3.1], the subcategory of $\text{mod } R$ consisting of ∞ -torsion-free modules (see [HuH] or Section 5 below for the definition) is a $\mathcal{P}(\text{mod } R)$ -coresolving subcategory of $\text{mod } R$ with a $\mathcal{P}(\text{mod } R)$ -coproper cogenerator $\mathcal{P}(\text{mod } R)$.

Unless stated otherwise, in the rest of this section, we fix a subcategory \mathcal{E} of \mathcal{A} and an \mathcal{E} -precoresolving subcategory \mathcal{T} of \mathcal{A} admitting an \mathcal{E} -coproper cogenerator \mathcal{C} . We will give some criteria for computing the \mathcal{T} -codimension of a given object A in \mathcal{A} , and then compare it with the \mathcal{C} -codimension of A .

The following two propositions play a crucial role in this section.

PROPOSITION 4.3

Let

$$(4.1) \quad 0 \rightarrow M \rightarrow T^0 \rightarrow T^1 \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with both T^0 and T^1 objects in \mathcal{T} . Then we have the following.

(a) There exists an exact sequence

$$(4.2) \quad 0 \rightarrow M \rightarrow C \rightarrow T \rightarrow A \rightarrow 0$$

in \mathcal{A} with T an object in \mathcal{T} and C an object in \mathcal{C} .

(b) If (4.1) is $\text{Hom}_{\mathcal{A}}(-, X)$ -exact for some object X in \mathcal{A} , then so is (4.2).

As an application of Proposition 4.3, we get the following.

PROPOSITION 4.4

Let $n \geq 1$, and let

$$0 \rightarrow M \rightarrow T^0 \rightarrow T^1 \rightarrow \cdots \rightarrow T^{n-1} \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with all T^i 's objects in \mathcal{T} . Then there exist an exact sequence

$$0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^{n-1} \rightarrow B \rightarrow 0$$

and a $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$$

in \mathcal{A} with all C^i 's objects in \mathcal{C} and T an object in \mathcal{T} . In particular, an object in \mathcal{A} is an n - \mathcal{C} -syzygy if and only if it is an n - \mathcal{T} -syzygy.

The following corollary is an immediate consequence of Proposition 4.4.

COROLLARY 4.5

Let A be an object in \mathcal{A} , and let $n \geq 0$. If $\mathcal{T}\text{-dim } A = n$, then there exists a $\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact exact sequence $0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$ in \mathcal{A} with $\mathcal{C}\text{-dim } B \leq n$ and T an object in \mathcal{T} .

We give a criterion for computing the \mathcal{T} -codimension of an object in \mathcal{A} as follows.

THEOREM 4.6

The following statements are equivalent for any object M in \mathcal{A} and $n \geq 0$.

- (a) $\mathcal{T}\text{-codim } M \leq n$.
- (b) There exists an exact sequence

$$0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^{n-1} \rightarrow K^n \rightarrow 0$$

in \mathcal{A} with all C^i 's objects in \mathcal{C} and K^n an object in \mathcal{T} .

The following result gives a criterion for computing the \mathcal{T} -dimension of an object in \mathcal{A} .

THEOREM 4.7

The following statements are equivalent for any object A in \mathcal{A} and $n \geq 0$.

- (a) $\mathcal{T}\text{-dim } A \leq n$.
- (b) There exists an exact sequence

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow K_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with K_0 an object in \mathcal{T} and all C_i 's objects in \mathcal{C} ; that is, there exists an exact sequence

$$0 \rightarrow H \rightarrow T \rightarrow A \rightarrow 0$$

in \mathcal{A} with T an object in \mathcal{T} and $\mathcal{C}\text{-dim } H \leq n - 1$.

The following result gives a sufficient condition such that the n - \mathcal{C} -cosyzygy of an object in \mathcal{A} with \mathcal{T} -codimension at most n is in \mathcal{T} .

THEOREM 4.8

Let \mathcal{T} be closed under cokernels of (\mathcal{E} -coproper) monomorphisms and $\mathcal{T} \subseteq {}^\perp\mathcal{C}$, and let $n \geq 0$. Then for any object M in \mathcal{A} with \mathcal{T} -codim $M \leq n$ we have the following.

- (a) For any ($\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact) exact sequence $0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^{n-1} \rightarrow K^n \rightarrow 0$ in \mathcal{A} with all C^i 's objects in \mathcal{C} , K^n is an object in \mathcal{T} .
- (b) If $0 \rightarrow M \rightarrow C \rightarrow K \rightarrow 0$ is a ($\text{Hom}_{\mathcal{A}}(-, \mathcal{E})$ -exact) exact sequence in \mathcal{A} with C an object in \mathcal{C} , then \mathcal{T} -codim $K \leq n - 1$.

We use $\mathcal{T}\text{-codim}^{\leq n}$ to denote the subcategory of \mathcal{A} consisting of objects with \mathcal{T} -codimension at most n .

COROLLARY 4.9

Let \mathcal{T} be a \mathcal{C} -coresolving subcategory of \mathcal{A} with a \mathcal{C} -coproper cogenerator \mathcal{C} and $\mathcal{T} \subseteq {}^\perp\mathcal{C}$. If \mathcal{T} is closed under direct summands, then so is $\mathcal{T}\text{-codim}^{\leq n}$ for any $n \geq 0$.

The following result gives some sufficient conditions such that the \mathcal{T} -codimension and the \mathcal{C} -codimension of an object in \mathcal{A} are identical.

THEOREM 4.10

Let $\mathcal{T} \subseteq \mathcal{C}^\perp \cap {}^\perp\mathcal{C}$, and let \mathcal{C} be closed under direct summands. Then for an object M in \mathcal{A} , \mathcal{T} -codim $M = \mathcal{C}$ -codim M if one of the following conditions is satisfied.

- (a) \mathcal{C} -codim $M < \infty$, $\mathcal{E} = \mathcal{C}$, and \mathcal{T} is closed under cokernels of \mathcal{C} -coproper monomorphisms.
- (b) \mathcal{C} -codim $M < \infty$, $\mathcal{E} = \mathcal{C}$, and $\mathcal{C}\text{-dim}^{< \infty}$ is closed under direct summands.
- (c) M is an object in ${}^\perp\mathcal{T}$ and \mathcal{C} is a generator for \mathcal{T} .

The following result gives a sufficient condition such that the \mathcal{T} -dimension and the \mathcal{C} -dimension of an object in \mathcal{A} are identical.

THEOREM 4.11

Let \mathcal{D} be a subcategory of \mathcal{A} such that $\mathcal{T} \subseteq \mathcal{C}^\perp \cap {}^\perp\mathcal{D}$, and let $\mathcal{C}\text{-dim}^{\leq n}$ be closed under direct summands for any $n \geq 0$. If A is an object in \mathcal{A} with \mathcal{D} -codim $A < \infty$, then $\mathcal{T}\text{-dim } A = \mathcal{C}\text{-dim } A$.

In the following, we fix a subcategory \mathcal{C} of \mathcal{A} .

The following two corollaries give some sufficient conditions such that the $\mathcal{G}(\mathcal{C})$ -codimension and the \mathcal{C} -codimension of an object in \mathcal{A} are identical.

COROLLARY 4.12

Let $\mathcal{C} \perp \mathcal{C}$, and let \mathcal{C} be closed under direct summands. Then for any object M in ${}^\perp\mathcal{G}(\mathcal{C})$, $\mathcal{G}(\mathcal{C})$ -codim $M = \mathcal{C}$ -codim M .

COROLLARY 4.13

Let $\mathcal{C} \perp \mathcal{C}$, let \mathcal{C} be closed under direct summands, and let \mathcal{D} be a subcategory of ${}^\perp\mathcal{G}(\mathcal{C})$. Then for any object M in ${}^\perp\mathcal{C}$ with \mathcal{D} -dim $M < \infty$, $\mathcal{G}(\mathcal{C})$ -codim $M = \mathcal{C}$ -codim M .

The following result gives a sufficient condition such that the $\mathcal{G}(\mathcal{C})$ -codimension and the \mathcal{C}^\perp -codimension of an object in \mathcal{A} are identical.

THEOREM 4.14

Let $\mathcal{C} \perp \mathcal{C}$, and let M be an object in \mathcal{A} with $\mathcal{G}(\mathcal{C})$ -codim $M < \infty$. Then $\mathcal{G}(\mathcal{C})$ -codim $M = \mathcal{C}^\perp$ -codim M .

5. Applications and questions

In this section, we will apply the results in Sections 3 and 4 to special subcategories and in particular to module categories. Finally, we propose some open questions and conjectures concerning the obtained results.

5.1. Special subcategories

We define $\text{res } \tilde{\mathcal{C}} = \{A \text{ is an object in } \mathcal{A} \mid \text{there exists a } \text{Hom}_{\mathcal{A}}(\mathcal{C}, -)\text{-exact exact sequence } \cdots \rightarrow C_i \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0 \text{ in } \mathcal{A} \text{ with all } C_i\text{'s objects in } \mathcal{C}\}$. Dually, we define $\text{cores } \tilde{\mathcal{C}} = \{M \text{ is an object in } \mathcal{A} \mid \text{there exists a } \text{Hom}_{\mathcal{A}}(-, \mathcal{C})\text{-exact exact sequence } 0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \rightarrow C^i \rightarrow \cdots \text{ in } \mathcal{A} \text{ with all } C^i\text{'s objects in } \mathcal{C}\}$ (see [SSW]).

We have the following.

FACT 5.1

(a) Note that \mathcal{C} is a \mathcal{C} -proper generator for $\text{res } \tilde{\mathcal{C}}$ and $\text{res } \tilde{\mathcal{C}} \cap {}^\perp\mathcal{C}$. By [Hu, Lemma 3.1(1)], both $\text{res } \tilde{\mathcal{C}}$ and $\text{res } \tilde{\mathcal{C}} \cap {}^\perp\mathcal{C}$ are closed under \mathcal{C} -proper extensions. So both $\text{res } \tilde{\mathcal{C}}$ and $\text{res } \tilde{\mathcal{C}} \cap {}^\perp\mathcal{C}$ are \mathcal{C} -preresolving. We remark that if \mathcal{C} is a \mathcal{C} -proper generator for \mathcal{A} , then $\text{res } \tilde{\mathcal{C}} = \mathcal{A}$ and $\text{res } \tilde{\mathcal{C}} \cap {}^\perp\mathcal{C} = {}^\perp\mathcal{C}$.

(b) If \mathcal{C} is closed under kernels of epimorphisms, then so are both $\text{res } \tilde{\mathcal{C}}$ and $\text{res } \tilde{\mathcal{C}} \cap {}^\perp\mathcal{C}$ (see [Hu, Proposition 4.7(1)]).

Dually, we have the following.

(c) Note that \mathcal{C} is a \mathcal{C} -coproper cogenerator for $\text{cores } \tilde{\mathcal{C}}$ and $\mathcal{C}^\perp \cap \text{cores } \tilde{\mathcal{C}}$. By [Hu, Lemma 3.1(2)], both $\text{cores } \tilde{\mathcal{C}}$ and $\mathcal{C}^\perp \cap \text{cores } \tilde{\mathcal{C}}$ are closed under \mathcal{C} -coproper extensions. So both $\text{cores } \tilde{\mathcal{C}}$ and $\mathcal{C}^\perp \cap \text{cores } \tilde{\mathcal{C}}$ are \mathcal{C} -precoresolving.

We also remark that if \mathcal{C} is a \mathcal{C} -coproper cogenerator for \mathcal{A} , then $\text{cores } \widetilde{\mathcal{C}} = \mathcal{A}$ and $\mathcal{C}^\perp \cap \text{cores } \widetilde{\mathcal{C}} = \mathcal{C}^\perp$.

(d) If \mathcal{C} is closed under cokernels of monomorphisms, then so are both $\text{cores } \widetilde{\mathcal{C}}$ and $\mathcal{C}^\perp \cap \text{cores } \widetilde{\mathcal{C}}$ (see [Hu, Proposition 4.7(2)]).

APPLICATION 5.2

By Fact 5.1, we can apply the results in Section 3 in the cases for $\mathcal{T} = \text{res } \widetilde{\mathcal{C}}$ and $\mathcal{T} = \text{res } \widetilde{\mathcal{C}} \cap {}^\perp \mathcal{C}$, respectively, and apply the results in Section 4 in the cases for $\mathcal{T} = \text{cores } \widetilde{\mathcal{C}}$ and $\mathcal{T} = \mathcal{C}^\perp \cap \text{cores } \widetilde{\mathcal{C}}$, respectively. We will not list these consequences in details.

5.2. Module categories

In this subsection, R is a ring and all subcategories of $\text{Mod } R$ are full and additive. For a module A in $\text{Mod } R$, we denote the projective, injective, and flat dimensions of A by $\text{pd}_R A$, $\text{id}_R A$, and $\text{fd}_R A$, respectively.

We first give the following.

PROPOSITION 5.3

Let \mathcal{T} be a subcategory of $\text{Mod } R$.

(a) If \mathcal{T} is closed under extensions and $\mathcal{P}(\text{Mod } R) \subseteq \mathcal{T} \subseteq {}^\perp \mathcal{P}(\text{Mod } R)$, then $\text{pd}_R A = \mathcal{T}\text{-dim } A$ for any $A \in \text{Mod } R$ with $\text{pd}_R A < \infty$.

(b) If \mathcal{T} is closed under extensions and $\mathcal{I}(\text{Mod } R) \subseteq \mathcal{T} \subseteq \mathcal{I}(\text{Mod } R)^\perp$, then $\text{id}_R A = \mathcal{T}\text{-codim } A$ for any $A \in \text{Mod } R$ with $\text{id}_R A < \infty$.

Proof

(a) Because $\mathcal{T} \subseteq {}^\perp \mathcal{P}(\text{Mod } R) = \mathcal{P}(\text{Mod } R)^\perp \cap {}^\perp \mathcal{P}(\text{Mod } R)$ by assumption, we get the assertion by Theorem 3.10(b).

(b) It is dual to (a). □

Let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair in $\text{Mod } R$. Then $\mathcal{X} \cap \mathcal{Y}$ is called the *heart* of $(\mathcal{X}, \mathcal{Y})$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *hereditary* if $\mathcal{X} = {}^\perp \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^\perp$; in this case, \mathcal{X} is projectively resolving and \mathcal{Y} is injectively coresolving (see [GT, Lemma 2.2.10]). By Proposition 5.3 we immediately have the following.

COROLLARY 5.4

Let $(\mathcal{X}, \mathcal{Y})$ be a hereditary cotorsion pair in $\text{Mod } R$ with the heart \mathcal{C} ($= \mathcal{X} \cap \mathcal{Y}$).

(a) If $\mathcal{C} = \mathcal{P}(\text{Mod } R)$, then for any $A \in \text{Mod } R$ with $\text{pd}_R A < \infty$, $\text{pd}_R A = \mathcal{X}\text{-dim } A$.

(b) If $\mathcal{C} = \mathcal{I}(\text{Mod } R)$, then for any $A \in \text{Mod } R$ with $\text{id}_R A < \infty$, $\text{id}_R A = \mathcal{Y}\text{-codim } A$.

Note that a module in $\mathcal{G}(\mathcal{P}(\text{Mod } R))$ (resp., $\mathcal{G}(\mathcal{I}(\text{Mod } R))$) is just a Gorenstein projective (resp., injective) module in $\text{Mod } R$. So $\mathcal{G}(\mathcal{P}(\text{Mod } R))\text{-dim}_R A$ (resp., $\mathcal{G}(\mathcal{I}(\text{Mod } R))\text{-codim}_R A$) is just the Gorenstein projective (resp., injective) dimension of a module A in $\text{Mod } R$. We denote the Gorenstein projective (resp., injective) dimension of a module A in $\text{Mod } R$ by $\text{Gpd}_R A$ (resp., $\text{Gid}_R A$).

DEFINITION 5.5

Let A be a module in $\text{Mod } R$.

(a) ([DLM]) A is called *strongly Gorenstein flat* if there exists a $\text{Hom}_R(-, \mathcal{F}(\text{Mod } R))$ -exact exact sequence $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$ in $\text{Mod } R$ with all terms projective such that $A \cong \text{Im}(P_0 \rightarrow P^0)$. We use $\mathcal{SGF}(\text{Mod } R)$ to denote the subcategory of $\text{Mod } R$ consisting of strongly Gorenstein flat modules.

The *strongly Gorenstein flat dimension* $\text{SGfd}_R A$ of A is defined to be $\inf\{n \mid \text{there exists an exact sequence } 0 \rightarrow G_n \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow A \rightarrow 0 \text{ in } \text{Mod } R \text{ with all } G_i\text{'s in } \mathcal{SGF}(\text{Mod } R)\}$. Set $\text{SGfd}_R A = \infty$ if no such n exists.

(b) ([MD]) A is called *Gorenstein FP-injective* if there exists a $\text{Hom}_R(\mathcal{FI}(\text{Mod } R), -)$ -exact exact sequence $\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ in $\text{Mod } R$ with all terms injective such that $A \cong \text{Im}(I_0 \rightarrow I^0)$. We use $\mathcal{GFI}(\text{Mod } R)$ to denote the subcategory of $\text{Mod } R$ consisting of Gorenstein FP-injective modules.

The *Gorenstein FP-injective dimension* $\text{GFid}_R A$ of A is defined to be $\inf\{n \mid \text{there exists an exact sequence } 0 \rightarrow A \rightarrow H^0 \rightarrow H^1 \rightarrow \cdots \rightarrow H^n \rightarrow 0 \text{ in } \text{Mod } R \text{ with all } H^i\text{'s in } \mathcal{GFI}(\text{Mod } R)\}$. Set $\text{GFid}_R A = \infty$ if no such n exists.

It is trivial that there exist the following inclusions:

$$\begin{aligned} \mathcal{P}(\text{Mod } R) &\subseteq \mathcal{SGF}(\text{Mod } R) \\ &\subseteq \mathcal{G}(\mathcal{P}(\text{Mod } R)) \begin{cases} \subseteq \text{cores } \widetilde{\mathcal{P}(\text{Mod } R)}, \\ \subseteq {}^\perp(\mathcal{P}(\text{Mod } R)) \supseteq {}^\perp(\mathcal{F}(\text{Mod } R)) \supseteq \mathcal{SGF}(\text{Mod } R), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}(\text{Mod } R) &\subseteq \mathcal{GFI}(\text{Mod } R) \\ &\subseteq \mathcal{G}(\mathcal{I}(\text{Mod } R)) \begin{cases} \subseteq \text{res } \widetilde{\mathcal{I}(\text{Mod } R)}, \\ \subseteq (\mathcal{I}(\text{Mod } R))^\perp \supseteq (\mathcal{FI}(\text{Mod } R))^\perp \supseteq \mathcal{GFI}(\text{Mod } R). \end{cases} \end{aligned}$$

So for any module A in $\text{Mod } R$, we have that

$$\begin{aligned} \text{pd}_R A &\geq \text{SGfd}_R A \\ &\geq \text{Gpd}_R A \begin{cases} \geq \text{cores } \widetilde{\mathcal{P}(\text{Mod } R)}\text{-dim } A, \\ \geq {}^\perp(\mathcal{P}(\text{Mod } R))\text{-dim } A \leq {}^\perp(\mathcal{F}(\text{Mod } R))\text{-dim } A \leq \text{SGfd}_R A, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \text{id}_R A &\geq \text{GFid}_R A \\ &\geq \text{Gid}_R A \begin{cases} \geq \text{res}\mathcal{I}(\widetilde{\text{Mod } R})\text{-codim } A, \\ \geq (\mathcal{I}(\text{Mod } R))^\perp\text{-codim } A \leq (\mathcal{FI}(\text{Mod } R))^\perp\text{-codim } A \leq \text{GFid}_R A. \end{cases} \end{aligned}$$

THEOREM 5.6

Let A be a module in $\text{Mod } R$.

- (a) If $A \in (\mathcal{SGF}(\text{Mod } R))^\perp$, then $\text{pd}_R A = \text{SGfd}_R A$.
- (b) If $A \in (\mathcal{G}(\mathcal{P}(\text{Mod } R)))^\perp$, then $\text{pd}_R A = \text{SGfd}_R A = \text{Gpd}_R A$.
- (c) If $\text{id}_R A < \infty$, then $\text{pd}_R A = \text{SGfd}_R A = \text{Gpd}_R A = \text{cores } \mathcal{P}(\widetilde{\text{Mod } R})\text{-dim } A$.
- (d) If $\text{pd}_R A < \infty$, then $\text{pd}_R A = \text{SGfd}_R A = \text{Gpd}_R A = {}^\perp(\mathcal{P}(\text{Mod } R))\text{-dim } A = {}^\perp(\mathcal{F}(\text{Mod } R))\text{-dim } A$.
- (e) If $\text{SGfd}_R A < \infty$, then $\text{SGfd}_R A = \text{Gpd}_R A = {}^\perp(\mathcal{P}(\text{Mod } R))\text{-dim } A = {}^\perp(\mathcal{F}(\text{Mod } R))\text{-dim } A$.
- (f) If $\text{Gpd}_R A < \infty$, then $\text{Gpd}_R A = {}^\perp(\mathcal{P}(\text{Mod } R))\text{-dim } A$.
- (g) If $\text{fd}_R A < \infty$, then $\text{pd}_R A = \text{SGfd}_R A$.

Proof

(a) (resp., (b)) It is clear that $\mathcal{P}(\text{Mod } R)$ is both a $\mathcal{P}(\text{Mod } R)$ -proper generator and a $\mathcal{P}(\text{Mod } R)$ -coproper cogenerator for $\mathcal{SGF}(\text{Mod } R)$ (resp., $\mathcal{G}(\mathcal{P}(\text{Mod } R))$). Note that $\mathcal{SGF}(\text{Mod } R)$ (resp., $\mathcal{G}(\mathcal{P}(\text{Mod } R))$) is closed under extensions by [Hu, Lemma 3.1] (resp., [H1, Theorem 2.5]). Then by putting $\mathcal{T} = \mathcal{SGF}(\text{Mod } R)$ (resp., $\mathcal{T} = \mathcal{G}(\mathcal{P}(\text{Mod } R))$) and $\mathcal{C} = \mathcal{P}(\text{Mod } R)$ in Theorem 3.10(c), we have $\text{pd}_R A = \text{SGfd}_R A$ (resp., $\text{pd}_R A = \text{Gpd}_R A$) if $A \in (\mathcal{SGF}(\text{Mod } R))^\perp$ (resp., $A \in (\mathcal{G}(\mathcal{P}(\text{Mod } R)))^\perp$).

(c) It is clear that $\mathcal{P}(\text{Mod } R)$ is a $\mathcal{P}(\text{Mod } R)$ -coproper cogenerator for $\text{cores } \mathcal{P}(\widetilde{\text{Mod } R})$. Note that $\text{cores } \mathcal{P}(\widetilde{\text{Mod } R})$ is closed under $\mathcal{P}(\text{Mod } R)$ -coproper extensions by [Hu, Lemma 3.1]. Then by putting $\mathcal{T} = \text{cores } \mathcal{P}(\widetilde{\text{Mod } R})$, $\mathcal{C} = \mathcal{P}(\text{Mod } R)$, and $\mathcal{D} = \mathcal{I}(\text{Mod } R)$ in Theorem 4.11, we have $\text{pd}_R A = \text{cores } \mathcal{P}(\widetilde{\text{Mod } R})\text{-dim } A$ if $\text{id}_R A < \infty$.

(d) By Proposition 5.3(a), we have $\text{pd}_R A = {}^\perp(\mathcal{P}(\text{Mod } R))\text{-dim } A$ if $\text{pd}_R A < \infty$.

(e) Note that $\mathcal{SGF}(\text{Mod } R)$ is closed under extensions (see the proof of (a)). Let $\text{SGfd}_R A = n < \infty$. Then by Theorem 3.6, there exists an exact sequence

$$0 \rightarrow G_n \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

in $\text{Mod } R$ with all P_i 's projective and G_n strongly Gorenstein flat. Put $K_i = \text{Im}(P_i \rightarrow P_{i-1})$ for any $1 \leq i \leq n - 1$. Suppose that ${}^\perp(\mathcal{P}(\text{Mod } R))\text{-dim } A = m < \infty$. It suffices to show that $m \geq n$. If $m < n$, then $K_m \in {}^\perp(\mathcal{P}(\text{Mod } R))$ and $K_{n-1} \in$

${}^\perp(\mathcal{P}(\text{Mod } R))$ by Theorem 3.8(a). Because there exists a $\text{Hom}_R(-, \mathcal{F}(\text{Mod } R))$ -exact exact sequence $0 \rightarrow G_n \rightarrow P \rightarrow G \rightarrow 0$ in $\text{Mod } R$ with P projective and G strongly Gorenstein flat, we have the following pushout diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & G_n & \longrightarrow & P_{n-1} & \longrightarrow & K_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & P & \longrightarrow & G' & \longrightarrow & K_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & G & \xlongequal{\quad} & G & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By using an argument similar to that in the proof of [H1, Theorem 2.5], we get that $\mathcal{SGF}(\text{Mod } R)$ is closed under direct summands. Because both the middle column and the middle row in the above diagram split, $G' \cong P_{n-1} \oplus G$ is strongly Gorenstein flat and K_{n-1} is isomorphic to a direct summand of G' , which implies that K_{n-1} is strongly Gorenstein flat and $\text{SGfd}_R A \leq n - 1$. It is a contradiction.

(f) We get the assertion by putting $\mathcal{C} = \mathcal{P}(\text{Mod } R)$ in Theorem 3.14 or $\mathcal{C} = \mathcal{P}(\text{Mod } R)$ and $\mathcal{T} = \mathcal{G}(\mathcal{P}(\text{Mod } R))$ in Theorem 3.10.

(g) By the definition of strongly Gorenstein flat modules, it is easy to see that $A \in (\mathcal{SGF}(\text{Mod } R))^\perp$ if $\text{fd}_R A < \infty$. Then the assertion follows from (a). \square

REMARK 5.7

Theorem 5.6(b) is [Z, Theorem 2.3]. Theorem 5.6(c) generalizes [H2, Theorem 2.2], which states that for a module A in $\text{Mod } R$, if $\text{id}_R A < \infty$, then $\text{pd}_R A = \text{Gpd}_R A$. Notice that a module in $\text{Mod } R$ with finite injective dimension is in $(\mathcal{G}(\mathcal{P}(\text{Mod } R)))^\perp$, so we may also get [H2, Theorem 2.2] by Theorem 5.6(b) (see [Z, Corollary 2.5]). Theorem 5.6(f) is well known (see [H1, Theorem 2.20]).

Let A be a module in $\text{Mod } R$. Recall that A is called *Gorenstein flat* if there exists an exact sequence

$$\dots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$$

in $\text{Mod } R$ with all terms flat such that $A \cong \text{Im}(F_0 \rightarrow F^0)$ and the sequence remains still exact after applying the functor $I \otimes_R -$ for any injective right R -module I . The *Gorenstein flat dimension* of A , denoted by $\text{Gfd}_R A$, is defined to be $\inf\{n \mid \text{there exists an exact sequence } 0 \rightarrow H_n \rightarrow \dots \rightarrow H_1 \rightarrow H_0 \rightarrow A \rightarrow 0 \text{ with all } H_i\text{'s Gorenstein flat}\}$. Set $\text{Gfd}_R A = \infty$ if no such n exists (see [EJT], [H1]).

The following is an open question: is every Gorenstein projective module over any ring Gorenstein flat? Holm proved in [H1, Proposition 2.4] that if R is a right coherent ring with finite left finitistic projective dimension, then every Gorenstein projective module in $\text{Mod } R$ is Gorenstein flat. As an immediate consequence of Theorem 5.6, we have the following.

COROLLARY 5.8

Let R be a right coherent ring, and let A be a module in $\text{Mod } R$. Then $\text{Gpd}_R A \geq \text{Gfd}_R A$ if either of the following conditions is satisfied:

- (a) $A \in (\mathcal{G}(\mathcal{P}(\text{Mod } R)))^\perp$ (in particular, if $\text{pd}_R A < \infty$ or $\text{id}_R A < \infty$), or
- (b) $\text{SGfd}_R A < \infty$.

Proof

Let R be a right coherent ring, and let A be a module in $\text{Mod } R$. Then $\text{SGfd}_R A \geq \text{Gfd}_R A$ by [DLM, Proposition 2.3]. So the assertions follow from Theorems 5.6(b) and 5.6(e), respectively. \square

Recall from [B1] that R is called *left GF-closed* if the subcategory of $\text{Mod } R$ consisting of Gorenstein flat modules is closed under extensions.

COROLLARY 5.9

Let A be a module in $\text{Mod } R$, and let n be a nonnegative integer.

- (a) ([CFH, Lemma 2.17]) If $\text{Gpd}_R A = n$, then there exists an exact sequence $0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$ in $\text{Mod } R$ with $\text{pd}_B A = n$ and T Gorenstein projective.
- (b) If $\text{SGfd}_R A = n$, then there exists an exact sequence $0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$ in $\text{Mod } R$ with $\text{pd}_B A = n$ and T strongly Gorenstein flat.
- (c) If R is left GF-closed and $\text{Gfd}_R A = n$, then there exists an exact sequence $0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$ in $\text{Mod } R$ with $\text{fd}_R B = n$ and T Gorenstein flat.

Proof

- (a) (resp., (b)) Let $\text{Gpd}_R A = n$ (resp., $\text{SGfd}_R A = n$). By Corollary 4.5, there exists an exact sequence $0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$ in $\text{Mod } R$ with $\text{pd}_R B \leq n$ and T Gorenstein projective (resp., strongly Gorenstein flat). Then by [H1, Theorem 2.5] (resp., Example 3.2(d)) and Proposition 2.3, we have $\text{Gpd}_R B \geq \text{Gpd}_R A = n$ (resp., $\text{SGfd}_R B \geq \text{SGfd}_R A = n$). So $\text{pd}_R B = n$ by Theorem 5.6(d).
- (c) Let R be left GF-closed, and let $\text{Gfd}_R A = n$. By Corollary 4.5, there exists an exact sequence $0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$ in $\text{Mod } R$ with $\text{fd}_R B \leq n$ and T Gorenstein flat. Then by [B1, Theorem 2.3] and Proposition 2.3, we have that $\text{Gfd}_R B \geq \text{Gfd}_R A = n$. So $\text{fd}_R B = n$ by [B2, Theorem 2.2]. \square

Recall that the *FP-injective dimension* $\text{FP-id}_R A$ of A in $\text{Mod } R$ is defined to be $\inf\{n \mid \text{there exists an exact sequence } 0 \rightarrow A \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^n \rightarrow 0 \text{ in } \text{Mod } R \text{ with all } E^i \text{ in } \mathcal{FI}(\text{Mod } R)\}$. Set $\text{FP-id}_R A = \infty$ if no such n exists.

The following result is the dual of Theorem 5.6.

THEOREM 5.10

Let A be a module in $\text{Mod } R$.

- (a) If $A \in {}^\perp(\mathcal{GF}\mathcal{I}(\text{Mod } R))$, then $\text{id}_R A = \text{GFid}_R A$.
- (b) If $A \in {}^\perp(\mathcal{G}(\mathcal{I}(\text{Mod } R)))$, then $\text{id}_R A = \text{GFid}_R A = \text{Gid}_R A$.
- (c) If $\text{pd}_R A < \infty$, then $\text{id}_R A = \text{GFid}_R A = \text{Gid}_R A = \text{res } \widetilde{\mathcal{I}(\text{Mod } R)}\text{-codim } A$.
- (d) If $\text{id}_R A < \infty$, then $\text{id}_R A = \text{GFid}_R A = \text{Gid}_R A = (\mathcal{I}(\text{Mod } R))^\perp\text{-codim } A = (\mathcal{F}\mathcal{I}(\text{Mod } R))^\perp\text{-codim } A$.
- (e) If $\text{GFid}_R A < \infty$, then $\text{GFid}_R A = \text{Gid}_R A = (\mathcal{I}(\text{Mod } R))^\perp\text{-codim } A = (\mathcal{F}\mathcal{I}(\text{Mod } R))^\perp\text{-codim } A$.
- (f) If $\text{Gid}_R A < \infty$, then $\text{Gid}_R A = (\mathcal{I}(\text{Mod } R))^\perp\text{-codim } A$.
- (g) If $\text{FP-id}_R A < \infty$, then $\text{id}_R A = \text{GFid}_R A$.

REMARK 5.11

Theorem 5.10(c) generalizes [H2, Theorem 2.1], which states that for a module A in $\text{Mod } R$, if $\text{pd}_R A < \infty$, then $\text{id}_R A = \text{Gid}_R A$. Notice that a module in $\text{Mod } R$ with finite projective dimension is in ${}^\perp(\mathcal{G}(\mathcal{I}(\text{Mod } R)))$, so we may also get [H2, Theorem 2.1] by Theorem 5.10(b). Theorem 5.10(f) is well known (see [H1, Theorem 2.22]).

5.3. Questions

In view of Theorems 5.6(c) and 5.6(d), it is natural to ask the following.

QUESTION 5.12

If A is a module in $\text{Mod } R$ with $\text{id}_R A < \infty$, does $\text{pd}_R A = {}^\perp(\mathcal{P}(\text{Mod } R))\text{-dim } A$ hold?

QUESTION 5.13

If A is a module in $\text{Mod } R$ with $\text{pd}_R A < \infty$, does $\text{pd}_R A = \text{cores } \widetilde{\mathcal{P}(\text{Mod } R)}\text{-dim } A$ hold?

From now on, R is a left and right Noetherian ring (unless stated otherwise). We write ${}^\perp_R R = \{M \in \text{mod } R \mid \text{Ext}_R^i({}_R M, {}_R R) = 0 \text{ for any } i \geq 1\}$ (resp., ${}^\perp R_R = \{N \in \text{mod } R^{\text{op}} \mid \text{Ext}_{R^{\text{op}}}^i(N_R, R_R) = 0 \text{ for any } i \geq 1\}$).

For any module A in $\text{mod } R$, there exists a projective presentation

$$P_1 \xrightarrow{f} P_0 \rightarrow A \rightarrow 0$$

of A in $\text{mod } R$. (Note that if R is an Artinian algebra, then this projective presentation of A is chosen to be the minimal one.) Then we get an exact sequence

$$0 \rightarrow A^* \rightarrow P_0^* \xrightarrow{f^*} P_1^* \rightarrow \text{Tr } A \rightarrow 0$$

in $\text{mod } R^{\text{op}}$, where $(-)^* = \text{Hom}(-, R)$ and $\text{Tr } A = \text{Coker } f^*$ is the *transpose* of A . Auslander and Bridger generalized the notions of finitely generated projective modules and the projective dimension of finitely generated modules as follows. A module A in $\text{mod } R$ is said to *have Gorenstein dimension zero* if $A \in {}^\perp_R R$ and $\text{Tr } A \in {}^\perp_R R$ (see [AB]). It is well known that, over a left and right Noetherian ring, a finitely generated module is Gorenstein projective if and only if it has Gorenstein dimension zero (see [EJ2, Proposition 10.2.6]).

Let A be a module in $\text{mod } R$. Recall from [HuH] that A is called ∞ -*torsion-free* if $\text{Tr } A \in {}^\perp_R R$. We use $\mathcal{T}(\text{mod } R)$ to denote the subcategory of $\text{mod } R$ consisting of ∞ -torsion-free modules. The *torsion-free dimension* of A , denoted by $\mathcal{T}\text{-dim}_R A$, is defined to be $\inf\{n \mid \text{there exists an exact sequence } 0 \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0 \text{ in } \text{mod } R \text{ with all } X_i\text{'s in } \mathcal{T}(\text{mod } R)\}$. Set $\widetilde{\mathcal{T}\text{-dim}_R A} = \infty$ if no such n exists. By [AB, Theorem 2.17], a module is in $\text{cores } \mathcal{P}(\text{mod } R)$ if and only if it is in $\mathcal{T}(\text{mod } R)$. So $\text{cores } \mathcal{P}(\text{mod } R)\text{-dim } A = \mathcal{T}\text{-dim}_R A$ for any module A in $\text{mod } R$.

By Example 4.2(e) and Corollary 4.5, we immediately have the following.

COROLLARY 5.14 ([HuH, COROLLARY 3.5])

Let A be a module in $\text{mod } R$ with $\mathcal{T}\text{-dim}_R A = n$. Then there exists an exact sequence $0 \rightarrow A \rightarrow B \rightarrow T \rightarrow 0$ in $\text{mod } R$ with $\text{pd}_R B \leq n$ and T ∞ -torsion-free.

The following result is analogous to Theorems 5.6(c) and 5.6(d).

THEOREM 5.15

Let R be a left and right Noetherian ring, and let A be a module in $\text{mod } R$.

- (a) *If $\text{id}_R A < \infty$, then $\text{pd}_R A = \text{Gpd}_R A = \mathcal{T}\text{-dim}_R A$.*
- (b) *If $\text{pd}_R A < \infty$, then $\text{pd}_R A = \text{Gpd}_R A = {}^\perp_R R\text{-dim } A$.*

In view of the assertions in Theorem 5.15, it is natural to ask the following questions, which are finitely generated versions of Questions 5.12 and 5.13, respectively.

QUESTION 5.16

If A is a module in $\text{mod } R$ with $\text{id}_R A < \infty$, does $\text{pd}_R A = {}^\perp_R R\text{-dim } A$ hold? (Equivalently, does $\text{Gpd}_R A = {}^\perp_R R\text{-dim } A$ hold?)

QUESTION 5.17

If A is a module in $\text{mod } R$ with $\text{pd}_R A < \infty$, does $\text{pd}_R A = \mathcal{T}\text{-dim}_R A$ hold? (Equivalently, does $\text{Gpd}_R A = \mathcal{T}\text{-dim}_R A$ hold?)

Let R be an Artinian algebra, let $C(R)$ be the center of R , let J be the Jacobson radical of $C(R)$, and let $I(C(R)/J)$ be the injective envelope of $C(R)/J$. Then the Matlis duality $\mathbb{D}(-) = \text{Hom}_{C(R)}(-, I(C(R)/J))$ between $\text{mod } R$ and

$\text{mod } R^{\text{op}}$ induces a duality between projective (resp., injective) modules in $\text{mod } R$ and injective (resp., projective) modules in $\text{mod } R^{\text{op}}$. As a special case of Question 5.16, we propose the following.

CONJECTURE 5.18

Let R be an Artinian algebra.

- (a) A module A in $\text{mod } R$ is projective if A is injective and $A \in {}^\perp_R R$.
- (b) R is self-injective if $\mathbb{D}(R_R) \in {}^\perp_R R$.

The Auslander–Reiten conjecture (ARC for short) states that, over any Artinian algebra R , a module A in $\text{mod } R$ is projective if $\text{Ext}_R^i(A \oplus R, A \oplus R) = 0$ for any $i \geq 1$ (see [AR]). The strong Nakayama conjecture (SNC for short) states that, over any Artinian algebra R , for any $0 \neq A$ in $\text{mod } R$ there exists an $i \geq 0$ such that $\text{Ext}_R^i(A, R) \neq 0$ (see [CoF]). These two conjectures still remain open. Observe that an equivalent version of ARC states that, over any Artinian algebra R , for any simple module S in $\text{mod } R$ there exists $i \geq 0$ such that $\text{Ext}_R^i(S, R) \neq 0$ (see [AR]). So $\text{SNC} \Rightarrow \text{ARC}$. It is easy to see that $\text{ARC} \Rightarrow \text{Conjecture 5.18(a)} \Rightarrow \text{Conjecture 5.18(b)}$.

The following result shows that Question 5.17 is closely related to SNC.

PROPOSITION 5.19

Let R be an Artinian algebra. Then the following statements are equivalent.

- (a) SNC holds for R^{op} .
- (b) A module in $\text{mod } R$ is projective if $A \in \mathcal{T}(\text{mod } R)$ and $\text{pd}_R A \leq 1$.

Proof

(a) \Rightarrow (b) Let $A \in \mathcal{T}(\text{mod } R)$, and let $\text{pd}_R A \leq 1$. Then $\text{Tr } A \in {}^\perp_R R$ and there exists a minimal projective presentation

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

in $\text{mod } R$, which induces an exact sequence

$$0 \rightarrow A^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow \text{Tr } A \rightarrow 0$$

in $\text{mod } R^{\text{op}}$. So we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
 0 & \longrightarrow & (\text{Tr } A)^* & \longrightarrow & P_1^{**} & \longrightarrow & P_0^{**} & &
 \end{array}$$

Thus $(\text{Tr } A)^* = 0$ and $\text{Ext}_R^i(\text{Tr } A, R) = 0$ for any $i \geq 0$. Then $\text{Tr } A = 0$ by (a), which implies that A is projective by [ARS, Chapter IV, Proposition 1.7(b)].

(b) \Rightarrow (a) Let B be a module in $\text{mod } R^{\text{op}}$ such that $\text{Ext}_{R^{\text{op}}}^i(B, R) = 0$ for any $i \geq 0$. Then B has no nonzero projective summands. So $B \cong \text{Tr } \text{Tr } B$ by

[ARS, Chapter IV, Proposition 1.7(c)] and hence $\text{Ext}_{R^{\text{op}}}^i(\text{Tr Tr } B, R) = 0$ for any $i \geq 0$. So $\text{Tr } B \in \mathcal{T}(\text{mod } R)$. From a minimal projective presentation $Q_1 \rightarrow Q_0 \rightarrow B \rightarrow 0$ of B in $\text{mod } R^{\text{op}}$, we get an exact sequence

$$0 \rightarrow B^* \rightarrow Q_0^* \rightarrow Q_1^* \rightarrow \text{Tr } B \rightarrow 0$$

in $\text{mod } R$ with Q_0^*, Q_1^* projective. Because $B^* = 0$, $\text{pd}_R \text{Tr } B \leq 1$. Then $\text{Tr } B$ is projective by (b), which implies that B is projective. Again because $B^* = 0$, $B = 0$. Therefore SNC holds for R^{op} . \square

Acknowledgment. The author thanks the referee for the useful suggestions.

References

- [AB] M. Auslander and M. Bridger, *Stable Module Theory*, Mem. Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, 1969. [MR 0269685](#).
- [AR] M. Auslander and I. Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69–74. [MR 0389977](#).
- [ARS] M. Auslander, I. Reiten, and S. O. Smolø, *Representation Theory of Artin Algebras*, corrected reprint, Cambridge Stud. Adv. Math. **36**, Cambridge Univ. Press, Cambridge, 1997. [MR 1476671](#).
- [B1] D. Bennis, *Rings over which the class of Gorenstein flat modules is closed under extensions*, Comm. Algebra **37** (2009), 855–868. [MR 2503181](#).
[DOI 10.1080/00927870802271862](#).
- [B2] ———, *A note on Gorenstein flat dimension*, Algebra Colloq. **18** (2011), 155–161. [MR 2753665](#). [DOI 10.1142/S1005386711000095](#).
- [C] L. W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Math. **1747**, Springer, Berlin, 2000. [MR 1799866](#). [DOI 10.1007/BFb0103980](#).
- [CFH] L. W. Christensen, A. Frankild, and H. Holm, *On Gorenstein projective, injective and flat dimensions—a functorial description with applications*, J. Algebra **302** (2006), 231–279. [MR 2236602](#).
[DOI 10.1016/j.jalgebra.2005.12.007](#).
- [CI] L. W. Christensen and S. Iyengar, *Gorenstein dimension of modules over homomorphisms*, J. Pure Appl. Algebra **208** (2007), 177–188. [MR 2269838](#).
[DOI 10.1016/j.jpaa.2005.12.005](#).
- [CoF] R. R. Colby and K. R. Fuller, *A note on the Nakayama conjectures*, Tsukuba J. Math. **14** (1990), 343–352. [MR 1085203](#).
- [DLM] N. Ding, Y. Li, and L. Mao, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. **86** (2009), 323–338. [MR 2529328](#). [DOI 10.1017/S1446788708000761](#).
- [EJ1] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), 611–633. [MR 1363858](#).
[DOI 10.1007/BF02572634](#).
- [EJ2] ———, *Relative Homological Algebra*, de Gruyter Exp. Math. **30**, de Gruyter, Berlin, 2000. [MR 1753146](#). [DOI 10.1515/9783110803662](#).

- [EJL] E. E. Enochs, O. M. G. Jenda, and J. A. López-Ramos, *Covers and envelopes by V -Gorenstein modules*, *Comm. Algebra* **33** (2005), 4705–4717. MR 2188336. DOI 10.1080/00927870500328766.
- [EJT] E. E. Enochs, O. M. G. Jenda, and B. Torrecillas, *Gorenstein flat modules*, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **10** (1993), 1–9. MR 1248299.
- [GD] Y. Geng and N. Ding, *W -Gorenstein modules*, *J. Algebra* **325** (2011), 132–146. MR 2745532. DOI 10.1016/j.jalgebra.2010.09.040.
- [GT] R. Göbel and J. Triifaj, *Approximations and Endomorphism Algebras of Modules*, de Gruyter Exp. Math. **41**, de Gruyter, Berlin, 2006. MR 2251271.
- [H1] H. Holm, *Gorenstein homological dimensions*, *J. Pure Appl. Algebra* **189** (2004), 167–193. MR 2038564. DOI 10.1016/j.jpaa.2003.11.007.
- [H2] ———, *Rings with finite Gorenstein injective dimension*, *Proc. Amer. Math. Soc.* **132** (2004), 1279–1283. MR 2053331. DOI 10.1090/S0002-9939-03-07466-5.
- [Hu] Z. Huang, *Proper resolutions and Gorenstein categories*, *J. Algebra* **393** (2013), 142–169. MR 3090064. DOI 10.1016/j.jalgebra.2013.07.008.
- [HuH] C. Huang and Z. Huang, *Torsionfree dimension of modules and self-injective dimension of rings*, *Osaka J. Math.* **49** (2012), 21–35. MR 2903252.
- [LHX] Z. Liu, Z. Huang, and A. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, *Comm. Algebra* **41** (2013), 1–18. MR 3010518. DOI 10.1080/00927872.2011.602782.
- [MD] L. Mao and N. Ding, *Gorenstein FP -injective and Gorenstein flat modules*, *J. Algebra Appl.* **7** (2008), 491–506. MR 2442073. DOI 10.1142/S0219498808002953.
- [SSW] S. Sather-Wagstaff, T. Sharif, and D. White, *Stability of Gorenstein categories*, *J. Lond. Math. Soc. (2)* **77** (2008), 481–502. MR 2400403. DOI 10.1112/jlms/jdm124.
- [Z] P. Zhang, *Gorensteinness and Buchweitz theorem*, preprint, 2012.

Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, China; huangzy@nju.edu.cn