# The $T$-equivariant integral cohomology ring of $F_4/T$

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**Abstract** We determine the $T$-equivariant integral cohomology of $F_4/T$ combinatorially by the Goresky, Kottwitz, and MacPherson (GKM) theory, where $T$ is a maximal torus of the exceptional Lie group $F_4$ and acts on $F_4/T$ by the left multiplication.

## 1. Introduction and statement of the result

Let $G$ be a compact connected Lie group, and let $T$ be its maximal torus. The homogeneous space $G/T$ is a flag variety and it plays an important role in topology, algebraic geometry, representation theory, and combinatorics. In particular, the $T$-equivariant integral cohomology ring $H_T^*(G/T) = H^*(ET \times_T G/T)$ is especially important, where $T$ acts on $G/T$ by the left multiplication.

Goresky, Kottwitz, and MacPherson [GKM] gave a powerful method to determine the equivariant cohomology with $\mathbb{Q}$-coefficients of some good spaces. It is called the *GKM theory*. Let us explain how the Goresky, Kottwitz, and MacPherson (GKM) theory works in our situation. Since the fixed point set $(G/T)^T$ is identified with the Weyl group $W(G)$, the inclusion $i\colon (G/T)^T \to G/T$ induces the map

$$i^*\colon H_T^*(G/T) \to H_T^*\big((G/T)^T\big) = \prod_{W(G)} H^*(BT) = \mathrm{Map}\big(W(G), H^*(BT)\big).$$

Upon tensoring with $\mathbb{Q}$, $i^*$ is injective by the localization theorem (see [H, Theorem (III.1)]). The GKM theory gives a way to describe the image of this map $i^*$, which is restated by Guillemin and Zara [GZ] as follows. The image of $i^*$ is completely determined by a graph with additional data obtained from $G$. Precisely they defined the "cohomology" ring of the graph as a subring of $\mathrm{Map}(W(G), H^*(BT))$ and showed that it coincides with the image of $i^*$. This graph is called a *GKM graph*. Harada, Henriques, and Holm [HHH] showed that, with integer coefficients, $i^*$ is injective and its image coincides with the cohomology of the GKM graph.

By concrete computations by the GKM theory, for a simple Lie group $G$ of classical types and of type $G_2$, Fukukawa, Ishida, and Masuda [FIM] and

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Fukukawa [F] determined the cohomology ring of the GKM graph of $G/T$. Hence they determined the equivariant integral cohomology ring $H_T^*(G/T)$ for a Lie group $G$ of types $A$, $B$, $D$, and $G_2$. In this paper we determine the $T$-equivariant integral cohomology ring of $F_4/T$ by the GKM theory.

For $x=(x_1,\dots,x_n)$, let $e_i(x)$ denote the $i$th elementary symmetric polynomial in $x_1,\dots,x_n$. Put $x^k=(x_1^k,\dots,x_n^k)$. For a linear transformation $\alpha$ of $\mathbb{R}x_1\oplus\cdots\oplus\mathbb{R}x_n$, let $\alpha x=(\alpha x_1,\dots,\alpha x_n)$. Then $e_i(x^k)$ and $e_i(\alpha x)$ denote the $i$th elementary symmetric polynomial in $x_1^k,\dots,x_n^k$ and $\alpha x_1,\dots,\alpha x_n$, respectively. The following theorem is the main result of this paper. In this theorem $t=(t_1,t_2,t_3,t_4)$, $\tau=(\tau_1,\tau_2,\tau_3,\tau_4)$, and $\rho$ is the linear transformation of $\mathbb{R}t_1\oplus\cdots\oplus\mathbb{R}t_4$ defined as (3.2).

THEOREM 1.1

*Let $T$ be a maximal torus of $F_4$ which acts on $F_4/T$ by the left multiplication. Then the $T$-equivariant integral cohomology ring of $F_4/T$ is given as*

$$H_T^*(F_4/T)\cong\mathbb{Z}[t_i,\gamma,\tau_i,\gamma_i,\omega\mid 1\le i\le 4]/(r_1',R_i,r_{2i},r_{12}\mid 1\le i\le 4),$$

*where* $|t_i|=|\gamma|=|\tau_i|=2$, $|\gamma_i|=2i$, $|\omega|=8$,

$$r_1'=e_1(t)-2\gamma,\qquad R_i=e_i(\tau)-e_i(t)-2\gamma_i\quad(i=1,2,3),$$

$$r_{12}=\omega\big(\omega-e_4(\rho t)\big)\big(\omega+e_4(\rho^2 t)\big),\qquad R_4=e_4(\tau)-e_4(t)-2\gamma_4-\omega,$$

$$r_2=\sum_{j=1}^{2}(-1)^j\gamma_j\big(\gamma_{2-j}+e_{2-j}(t)\big),\qquad r_4=\sum_{j=1}^{4}(-1)^j\gamma_j\big(\gamma_{4-j}+e_{4-j}(t)\big)-\omega,$$

$$r_6=\sum_{j=2}^{4}(-1)^j\gamma_j\big(\gamma_{6-j}+e_{6-j}(t)\big)+(\gamma_2+\gamma^2)\omega,$$

$$r_8=\gamma_4\big(\gamma_4+e_4(t)\big)+\omega^2+\big(\gamma_4-e_4(\rho t)\big)\omega.$$

The ordinary integral cohomology ring $H^*(F_4/T)$ was determined by Toda and Watanabe [TW]. We can obtain the integral cohomology ring of $F_4/T$ as a corollary of Theorem 1.1 as follows. There is a fibration sequence

$$F_4/T\longrightarrow ET\times_T F_4/T\xrightarrow{\ p\ }BT.$$

Since the projection $p\colon ET\times_T F_4/T\to BT$ restricts to $p\circ i\colon ET\times_T(F_4/T)^T\to BT$, where $i$ is the inclusion $ET\times_T(F_4/T)^T\to ET\times_T F_4/T$, the induced map $(p\circ i)^*\colon H^*(BT)\to H^*(ET\times_T(F_4/T)^T)=\mathrm{Map}(W(F_4),H^*(BT))$ sends elements of $H^*(BT)$ to constant functions. In Theorem 1.1, $t_1$, $t_2$, $t_3$, $t_4$, and $\gamma$ correspond to constant functions (see Section 4). Since the cohomology of $F_4/T$ and $BT$ have vanishing odd parts, the Serre spectral sequence of the fibration $p$ collapses at the $E_2$-term. Hence $H^*(F_4/T)\cong H_T^*(F_4/T)/(t_1,t_2,t_3,t_4,\gamma)$.

COROLLARY 1.1 ([TW, THEOREM A])

*The integral cohomology ring of $F_4/T$ is given as*

$$H^*(F_4/T) \cong \mathbb{Z}[\tau_i, \gamma_1, \gamma_3, \omega \mid 1 \leq i \leq 4]/(\overline{r}_1, \overline{r}_2, \overline{r}_3, \overline{r}_4, \overline{r}_6, \overline{r}_8, \overline{r}_{12}),$$

*where*

$$\overline{r}_1 = 2\gamma_1 - e_1(\tau), \qquad \overline{r}_2 = 2\gamma_1^2 - e_2(\tau),$$
$$\overline{r}_3 = 2\gamma_3 - e_3(\tau), \qquad \overline{r}_4 = e_4(\tau) - 2\gamma_1 e_3(\tau) + 2\gamma_1^4 - 3\omega,$$
$$\overline{r}_6 = -\gamma_1^2 e_4(\tau) + \gamma_3^2, \qquad \overline{r}_8 = 3e_4(\tau)\gamma_1^4 - \gamma_1^8 + 3\omega\big(\omega + e_3(\tau)\gamma_1\big),$$
$$\overline{r}_{12} = \omega^3.$$

Corollary 1.1 will be proved in Section 8. Throughout this paper, all cohomology groups and rings will be taken with integer coefficients.

## 2. GKM graph and its cohomology

Let $G$ be a compact connected Lie group, and let $T$ be its maximal torus. Specializing and abstracting the work of Goresky, Kottwitz, and MacPherson [GKM], Guillemin and Zara [GZ] introduced a certain graph to each of whose edge an element of $H^2(BT)$ is given and showed that the $T$-equivariant cohomology of $G/T$ with complex coefficients is recovered from this graph. Let us introduce this special graph. Recall that there is a natural identification

$$\mathrm{Hom}(T, S^1) \cong H^2(BT),$$

where the left-hand side is the set of weights of $G$. Let $W(G)$ and $\Phi(G)$ denote the Weyl group and the root system of $G$, respectively. Since every root is a weight, we regard $\Phi(G) \subset H^2(BT)$. There is a canonical action of the Weyl group $W(G)$ on $\mathrm{Hom}(T, S^1)$ and it restricts to $\Phi(G)$. We denote this action as $w\alpha$ for $w \in W(G)$ and $\alpha \in H^2(BT)$. Recall that, to each $\alpha \in \Phi(G)$, one can assign a reflection $\sigma_\alpha$ which is an element of the Weyl group $W(G)$.

DEFINITION 2.1

The GKM graph of $G/T$ is the Cayley graph of $W(G)$ with respect to a generating set $\{\sigma_\alpha \in W(G) \mid \alpha \in \Phi(G)\}$ which is equipped with the cohomology classes $\pm w\alpha \in H^2(BT)$ to the edge $ww'$ satisfying $w' = w\sigma_\alpha$. We call $\pm w\alpha$ the *label* of the edge $ww'$.

The ambiguity of the sign of the label $\pm w\alpha$ occurs from the equation $w'\alpha = w\sigma_\alpha\alpha = -w\alpha$. Let us introduce the cohomology of the GKM graph. Consider a function $f : W(G) \to H^*(BT)$ between sets. We say that $f$ *satisfies* the GKM condition or $f$ is a GKM function if, for any $w \in W(G)$ and $\alpha \in \Phi(G)$,

$$f(w) - f(w\sigma_\alpha) \in (w\alpha) \subset H^*(BT),$$

where $(x_1, \ldots, x_n)$ means the ideal generated by $x_1, \ldots, x_n$. It is easy to see that all GKM functions form a subring of $\prod_{W(G)} H^*(BT)$, where we identify the set of all functions $W(G) \to H^*(BT)$ with $\prod_{W(G)} H^*(BT)$. Since the GKM graph of $G/T$ has $W(G)$ as its vertex set, a GKM function assigns an element of $H^*(BT)$ to each vertex of the GKM graph.

DEFINITION 2.2

Let $\mathcal{G}$ be the GKM graph of $G/T$. The cohomology ring $H^*(\mathcal{G})$ is defined as the subring of $\prod_{W(G)} H^*(BT)$ consisting of all GKM functions.

Guillemin and Zara [GZ, Theorem 1.7.3] restated an important theorem of the GKM theory as

$$H_T^*(G/T;\mathbb{C}) \cong H^*(\mathcal{G}) \otimes \mathbb{C}.$$

Harada, Henriques, and Holm refined this result to the integral cohomology. More precisely, we have the following.

THEOREM 2.1 ([HHH, THEOREM 3.1 AND LEMMA 5.2])

*Suppose the Lie group $G$ is simple, and let $\mathcal{G}$ be the GKM graph of $G/T$. If $G$ is not of type $C$, then there is an isomorphism*

$$H_T^*(G/T) \cong H^*(\mathcal{G}).$$

## 3. The GKM graph of $F_4/T$

In this section we describe and analyze the GKM graph of $F_4/T$. First of all let us choose a maximal torus of $F_4$. Let $T^4$ be the standard maximal torus of $\mathrm{SO}(9)$, and let $\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4 \in H^2(BT^4)$ be the canonical basis. For the universal covering $\mu : \mathrm{Spin}(9) \to \mathrm{SO}(9)$ let $T = \mu^{-1}(T^4)$. Then $T$ is a maximal torus of $\mathrm{Spin}(9)$. Since $\mathrm{Spin}(9)$ is a Lie subgroup of $F_4$ (see [A, Chapters 8, 9, and 14]), $T$ is also a maximal torus of $F_4$. We fix a maximal torus of $F_4$ to $T$. Let $t_i$ denote $\mu^*(\bar{t}_i) \in H^2(BT)$. By definition we have that

$$H^*(BT) = \mathbb{Z}[t_1, t_2, t_3, t_4, \gamma]/\bigl(2\gamma - e_1(t)\bigr).$$

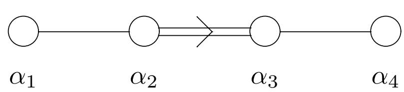
To describe the Weyl group $W(F_4)$ we start with the root system of $F_4$. The root system $\Phi(F_4)$ is given as

$$\Phi(F_4) = \Bigl\{\pm(t_i + t_j), \pm(t_i - t_j), \pm t_k, \frac{1}{2}(\pm t_1 \pm t_2 \pm t_3 \pm t_4) \Bigm| 1 \le i < j \le 4, 1 \le k \le 4\Bigr\}.$$

The roots $\pm(t_i + t_j)$ and $\pm(t_i - t_j)$ are called *long roots*, and $\pm t_k$ and $\frac{1}{2}(\pm t_1 \pm t_2 \pm t_3 \pm t_4)$ are called *short roots*. Put

$$\alpha_1 = t_2 - t_3, \qquad \alpha_2 = t_3 - t_4,$$
$$\alpha_3 = t_4, \qquad \alpha_4 = \frac{1}{2}(t_1 - t_2 - t_3 - t_4).$$

Then the Dynkin diagram of $F_4$ is as follows:



Then $W(F_4)$ is generated by the reflections $\sigma_{\alpha_i}$ for $i=1,2,3,4$. Since Spin(8) is a Lie subgroup of $F_4$, the root system of Spin(8) is contained in $\Phi(F_4)$, which is given as

$$\Phi(\mathrm{Spin}(8)) = \{\pm(t_i+t_j), \pm(t_i-t_j) \mid 1 \le i < j \le 4\}.$$

It consists of all the long roots of the root system $\Phi(F_4)$. Then the Weyl group $W(\mathrm{Spin}(8))$ is generated by the reflections associated with the long roots, and $W(\mathrm{Spin}(8))$ is a subgroup of $W(F_4)$.

Put $W = W(\mathrm{Spin}(8))$. The vertex set $W(F_4)$ of the GKM graph of $F_4/T$ is decomposed into six cosets by the next theorem.

THEOREM 3.1 ([A, THEOREM 14.2])

*The Weyl group $W$ of* Spin(8) *is a normal subgroup of $W(F_4)$ and there is an isomorphism $W(F_4)/W \cong \mathfrak{S}_3$, where $\mathfrak{S}_n$ is the symmetric group on $n$-letters. Moreover, $W(F_4)/W$ permutes the three root pairs*

$$\pm\frac{1}{2}(t_1+t_2+t_3-t_4), \qquad \pm\frac{1}{2}(t_1+t_2+t_3+t_4), \qquad \pm t_4. \tag{3.1}$$

Let us describe the representatives of $W(F_4)/W$. First we define an element $\rho$ of $W(F_4)$ as

$$\rho = \sigma_{\alpha_3}\sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_0}\sigma_{\alpha_3}\sigma_{\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_3}\sigma_{\alpha_2}\sigma_{\alpha_4}, \tag{3.2}$$

where $\alpha_0$ denotes the root $t_1 - t_2$ of Spin(8). By a straightforward calculation, we have that

$$\begin{aligned} \rho t_i &= \begin{cases} -\gamma + t_i, & i=1,2,3, \\ \gamma - t_4, & i=4, \end{cases} \\ \rho^2 t_i &= \begin{cases} -\gamma + t_4 + t_i, & i=1,2,3, \\ -\gamma, & i=4, \end{cases} \end{aligned} \tag{3.3}$$

and

$$\rho^3 = \mathrm{id}.$$

By the above equations the root system $\Phi(F_4)$ can be rewritten as

$$\Phi(F_4) = \{\pm(t_i+t_j), \pm(t_i-t_j), \pm\rho^\varepsilon t_k \mid 1 \le i < j \le 4, 1 \le k \le 4, 0 \le \varepsilon \le 2\}.$$

Note that $\rho$ permutes the three root pairs (3.1) cyclically and that $\kappa = \sigma_{t_4}$ interchanges $\pm\frac{1}{2}(t_1+t_2+t_3-t_4) = \pm\rho t_4$ and $\pm\frac{1}{2}(t_1+t_2+t_3+t_4) = \pm\rho^2 t_4$. Hence $W(F_4)/W \cong \mathfrak{S}_3$ is generated by $\rho$ and $\kappa$. Since the equation

$$\kappa\rho = \rho^2\kappa \tag{3.4}$$

holds, we have a coset decomposition

$$W(F_4) = \coprod_{\substack{\varepsilon=0,1,2 \\ \delta=0,1}} \rho^\varepsilon \kappa^\delta W.$$

We will describe the GKM graph $\mathcal{F}_4$ of $F_4/T$. There are 24 $(= \#\Phi(F_4)/2)$ edges out of each vertex of $\mathcal{F}_4$. Half of these edges correspond to the long roots $\pm(t_i \pm t_j)$ and the other half correspond to the short roots $\pm\rho^\varepsilon t_i$.

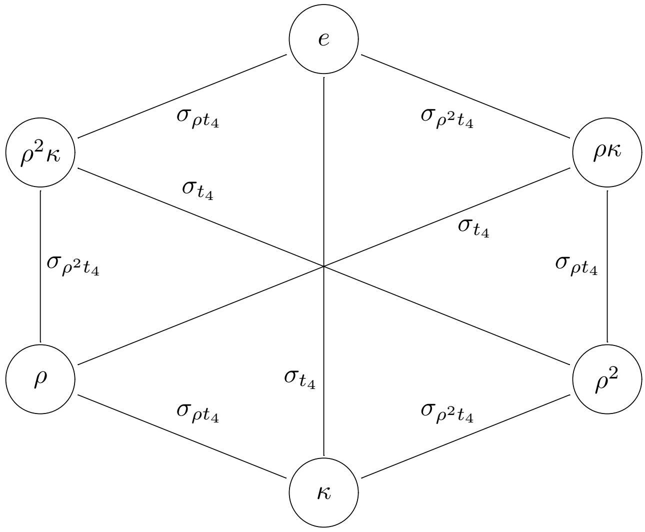
The subgraph induced by $W$ is the GKM graph $\mathcal{G}$ of $\mathrm{Spin}(8)/T$ and it is well understood from [FIM]. Let $\rho^\varepsilon\kappa^\delta\mathcal{G}$ be the GKM subgraph induced by $\rho^\varepsilon\kappa^\delta W$ for $\varepsilon = 0, 1, 2$ and $\delta = 0, 1$. For any $\varepsilon$ and $\delta$, the induced subgraph $\rho^\varepsilon\kappa^\delta\mathcal{G}$ is isomorphic to $\mathcal{G}$ as graphs. Indeed, if an edge $ww'$ in $\mathcal{G}$ satisfies $w' = w\sigma_\alpha$ for a root $\alpha$ of $\mathrm{Spin}(8)$, then $\rho^\varepsilon\kappa^\delta w$ and $\rho^\varepsilon\kappa^\delta w'$ satisfy $\rho^\varepsilon\kappa^\delta w' = \rho^\varepsilon\kappa^\delta w\sigma_\alpha$, and vice versa. Moreover, labels of edges of $\rho^\varepsilon\kappa^\delta\mathcal{G}$ are also determined by $\mathcal{G}$ as follows. When an edge $ww'$ has a root $\pm\beta$ as its label, the label of the edge connecting $\rho^\varepsilon\kappa^\delta w$ and $\rho^\varepsilon\kappa^\delta w'$ is $\pm\rho^\varepsilon\kappa^\delta\beta$. We remark that if an edge $ww'$ in $\rho^\varepsilon\kappa^\delta\mathcal{G}$ satisfies $w' = w\sigma_\alpha$, then $\alpha$ is one of the long roots.

From the above argument, it is sufficient to consider the edges connecting two of the $\rho^\varepsilon\kappa^\delta\mathcal{G}$'s, which correspond to the short roots. Easy calculations show that

$$\sigma_{t_4} = \kappa, \qquad \sigma_{\rho t_4} = \rho^2\kappa, \qquad \sigma_{\rho^2 t_4} = \rho\kappa.$$

Then the GKM graph $\mathcal{F}_4$ has an induced subgraph below, where $e$ denotes the unit element of $W(F_4)$ and an element of $W(F_4)$ in each circle denotes a vertex of $\mathcal{F}_4$. The labels are calculated later.

(3.5)




We will calculate the reflection $\sigma_\alpha$ for a short root $\alpha$ to describe $\mathcal{F}_4$. For example let us consider the short root $\rho t_1$ and the reflection $\sigma_{\rho t_1}$. By (3.3) we have that $\rho t_1 = \frac{1}{2}(t_1 - t_2 - t_3 - t_4) = \sigma_{t_2}\sigma_{t_3}(\rho t_4)$. Then $\sigma_{\rho t_1} = \sigma_{t_2}\sigma_{t_3}\sigma_{\rho t_4}\sigma_{t_3}\sigma_{t_2}$ and $\sigma_{t_2}\sigma_{t_3} \in W$. Since $W$ is a normal subgroup of $W(F_4)$, we have that $W \cdot \rho^2\kappa W = \rho^2\kappa W$ in $W(F_4)/W$. Hence $\sigma_{\rho t_1}$ is also contained in $\rho^2\kappa W$. For any $i$, it is shown similarly that

$$\sigma_{\rho t_i} \in \rho^2\kappa W, \qquad \sigma_{\rho^2 t_i} \in \rho\kappa W,$$

and obviously we have that

$$\sigma_{t_i} \in \kappa W.$$

Hence, for any $0 \le \varepsilon, \varepsilon' \le 2$ and $\delta = 0, 1$, it is independent from the choice of $i$ and $w \in \rho^{\epsilon'}\kappa^{\delta}W$ which coset contains $w\sigma_{\rho^{\varepsilon}t_i}$.

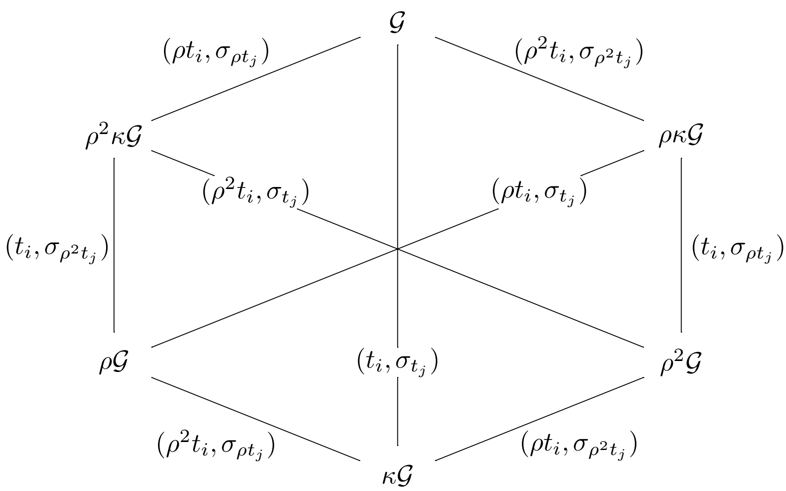
Let us calculate the label of the edge connecting the vertices $\kappa$ and $\rho$ in the GKM subgraph (3.5), which corresponds to a short root $\rho t_4$. The label of the edge turns out to be $\pm\kappa(\rho t_4)$. It follows from (3.4) that

$$\pm\kappa(\rho t_4) = \pm\rho^2\kappa t_4 = \pm\rho^2 t_4.$$

One can make similar calculations of the labels of other edges in the GKM subgraph (3.5). For any $w \in W$, $w$ fixes three sets of short roots $\{\pm t_i\}_{i=1}^4$, $\{\pm\rho t_i\}_{i=1}^4$, and $\{\pm\rho^2 t_i\}_{i=1}^4$ since $w$ permutes $t_i$'s and changes the signs of an even number of $t_i$'s. Hence the label $\pm\rho^{\varepsilon}\kappa^{\delta}w(\alpha)$ is calculated similarly for any short root $\alpha$.

We can now describe a schematic diagram of $\mathcal{F}_4$ as below.

(3.6)




The meaning of this diagram is given as follows. For example, $\mathcal{G}$ and $\rho\mathcal{G}$ are not adjacent in this diagram. It means that, for any vertices $w \in W$ and $w' \in \rho W$, they are not adjacent. On the other hand, $\rho\mathcal{G}$ and $\rho\kappa\mathcal{G}$ are adjacent in this diagram, and a pair $(\rho t_i, \sigma_{t_j})$ is assigned to the edge. The first entry $\rho t_i$ is a root and the second entry $\sigma_{t_j}$ is a reflection. If two vertices $w \in \rho W$ and $w' \in \rho\kappa W$ are adjacent in $\mathcal{F}_4$, then they satisfy $w' = w\sigma_{t_j}$ for some $j$, and the edge $ww'$ is labeled by $\rho t_i$ for some $i$. The label $\pm\rho t_i$ is equal to $\pm w t_j$. Especially each vertex of $\rho\mathcal{G}$ is connected to four vertices of $\rho\kappa\mathcal{G}$ by the edges corresponding to the short roots $t_j$ $(1 \le j \le 4)$, and vice versa. The labels of these edges are $\pm\rho t_i$ $(1 \le i \le 4)$. The $\rho t_i$'s appear as the labels of the edges out of each vertex of $\rho\mathcal{G}$. The situation is the same for any two connected subgraphs in the schematic diagram (3.6).

## 4. Proof of the main theorem

There is a fibration sequence

$$F_4/T \longrightarrow ET \times_T F_4/T \longrightarrow BT. \tag{4.1}$$

The cohomology rings of $F_4/T$ and $BT$ are free as $\mathbb{Z}$-modules and have vanishing odd parts. As shown in Section 3, $H^*(BT)$ has five generators $t_1$, $t_2$, $t_3$, $t_4$, and $\gamma$ of degree 2 with one relation of degree 2. According to [TW], $H^*(F_4/T)$ has $\tau_1$, $\tau_2$, $\tau_3$, $\tau_4$, and $\gamma_1$ of degree 2, $\gamma_3$ of degree 6, and $\omega$ of degree 8 as its generators, and $H^*(F_4/T)$ has seven relations of degrees 2, 4, 6, 8, 12, 16, and 24. We can expect that $H^*_T(F_4/T)$ has corresponding generators and relations. It is easy to see that the Poincaré series of $F_4/T$ and $BT$ are

$$(1+x^8+x^{16})\prod_{i=1}^{4}\frac{1-x^{4i}}{1-x^2} \qquad \text{and} \qquad \frac{1}{(1-x^2)^4},$$

respectively. Hence we obtain the following proposition by the Serre spectral sequence for (4.1).

PROPOSITION 4.1

*We have that $H^*_T(F_4/T)$ is free as a $\mathbb{Z}$-module and its Poincaré series is*

$$P\big(H^*(ET\times_T F_4/T),x\big)=\frac{1}{(1-x^2)^4}(1+x^8+x^{16})\prod_{i=1}^{4}\frac{1-x^{4i}}{1-x^2}.$$

By the Serre spectral sequence for the fibration sequence (4.1), we see that generators of $H^*_T(F_4/T)$ come from the cohomology of $F_4/T$ or $BT$. Let us define the corresponding GKM functions $t_i$, $\gamma$, $\tau_i$, $\gamma_1$, and $\gamma_3\in \operatorname{Map}(W(F_4),H^*(BT))$ for $1\le i\le 4$, and let us define GKM functions $\gamma_2$ and $\gamma_4$ to state our results more simply. For any $w\in W(F_4)$,

$$\begin{aligned}
t_i(w)&=t_i \quad (i=1,\dots,4),\\
\gamma(w)&=\gamma,\\
\tau_i(w)&=w(t_i) \quad (i=1,\dots,4),\\
\gamma_j&=\frac{1}{2}\big(e_j(\tau)-e_j(t)\big) \quad (j=1,2,3),
\end{aligned}$$

and

$$\gamma_4(w)=\begin{cases}0, & w\in W\sqcup\rho^2\kappa W,\\ e_4(\rho^2 t), & w\in\rho^2W\sqcup\rho\kappa W,\\ -e_4(t), & w\in\rho W\sqcup\kappa W.\end{cases}$$

Table 1. The value of $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t))$.

| | $j=1$ | $j=2$ | $j=3$ |
|---|---|---|---|
| $\varepsilon=1$ | $-\gamma-t_4$ | $-\gamma^2+t_4^2$ | $t_4\gamma(\gamma-t_4)-t_4(t_1t_2+t_2t_3+t_3t_1)$ |
| $\varepsilon=2$ | $-2\gamma+t_4$ | $(-2\gamma+t_4)t_4$ | $\gamma^3-t_4\gamma^2-\gamma(t_1t_2+t_2t_3+t_3t_1)$ |

Moreover, we define $\omega = e_4(\tau) - e_4(t) - 2\gamma_4$. Then

$$\omega(w) = \begin{cases} 0, & w \in W \sqcup \kappa W, \\ -e_4(\rho^2 t), & w \in \rho W \sqcup \rho\kappa W, \\ e_4(\rho t), & w \in \rho^2 W \sqcup \rho^2 \kappa W. \end{cases} \tag{4.2}$$

Since the $t_i$'s and $\gamma$ are constant functions, they are GKM functions. A straightforward calculation shows that the following relation holds:

$$e_4(t) + e_4(\rho t) + e_4(\rho^2 t) = 0. \tag{4.3}$$

By the schematic diagram (3.6) of $\mathcal{F}_4$, one can see that $\gamma_4$ is a GKM function since $e_4(\rho^\varepsilon t)$ is the product of all $\rho^\varepsilon t_1$, $\rho^\varepsilon t_2$, $\rho^\varepsilon t_3$, and $\rho^\varepsilon t_4$ for $\varepsilon = 0,1,2$. The following calculation shows that the $\tau_i$'s satisfy the GKM condition. For any edge $ww'$ which satisfies $w' = w\sigma_\alpha$, we have that

$$\begin{aligned} \tau_i(w) - \tau_i(w') &= w(t_i) - w'(t_i) \\ &= w\Big(t_i - \Big(t_i - 2\frac{(t_i,\alpha)}{(\alpha,\alpha)}\alpha\Big)\Big) \\ &= 2\frac{(t_i,\alpha)}{(\alpha,\alpha)} w\alpha. \end{aligned}$$

Since GKM functions form a ring, for $j = 1,2,3$, we see that the $\gamma_j$'s are functions from $W(F_4)$ to $H^*(BT) \otimes \mathbb{Z}[\frac{1}{2}]$ which satisfy the GKM condition with $\mathbb{Z}[\frac{1}{2}]$-coefficients; that is, $f(w) - f(w') \in (w\alpha) \subset H^*(BT) \otimes \mathbb{Z}[\frac{1}{2}]$ if $w' = w\sigma_\alpha$. The following calculations show that the $\gamma_j$'s are actually $H^*(BT)$-valued functions. Let us extend $\rho$ to an automorphism of $H^*(BT)$ naturally. For $w \in W \sqcup \kappa W = W(\mathrm{Spin}(9))$ and $\varepsilon = 0,1,2$,

$$\begin{aligned} \gamma_j(\rho^\varepsilon w) &= \frac{1}{2}\big(e_j(\tau) - e_j(t)\big)(\rho^\varepsilon w) \\ &= \frac{1}{2}\big(\rho^\varepsilon e_j\big(w(t)\big) - e_j(t)\big) \\ &= \rho^\varepsilon\Big(\frac{1}{2}\big(e_j\big(w(t)\big) - e_j(t)\big)\Big) + \frac{1}{2}\big(e_j(\rho^\varepsilon t) - e_j(t)\big). \end{aligned}$$

Since $w$ only permutes the $t_i$'s and changes their signs, it is obvious that $\frac{1}{2}(e_j(w(t)) - e_j(t)) \in H^*(BT)$. Then $\rho^\varepsilon(\frac{1}{2}(e_j(w(t)) - e_j(t))) \in H^*(BT)$. On the other hand, one can see that $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t)) \in H^*(BT)$ for $\varepsilon = 0,1,2$ as follows. When $\varepsilon = 0$, $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t)) = 0$ and it is contained in $H^*(BT)$. When $\varepsilon = 1,2$, Table 1 shows the value of $\frac{1}{2}(e_j(\rho^\varepsilon t) - e_j(t))$ for $j = 1,2,3$. Then $\gamma_j$ is an $H^*(BT)$-valued function and then a GKM function.

The following lemma will be proved in Section 5.

LEMMA 4.1 (SEE [FIM, LEMMA 5.4])

*Let $\mathcal{F}_4$ be the GKM graph of $F_4/T$. Then $H^*(\mathcal{F}_4)$ is generated by the GKM functions $t_i$, $\gamma$, $\tau_i$, $\gamma_i$, $\omega$ $(i=1,2,3,4)$ as a ring.*

By the fibration sequence (4.1), we can expect that some relations hold in $H^*(\mathcal{F}_4)$ which come from the relations of $H^*(BT)$ and $H^*(F_4/T)$. Proposition 4.2 claims that the corresponding relations hold in $H^*(\mathcal{F}_4)$.

PROPOSITION 4.2

*The following relations hold in $H^*(\mathcal{F}_4) \subset \mathrm{Map}(W(F_4), H^*(BT))$:*

$$r_1' = e_1(t) - 2\gamma = 0, \tag{4.4}$$

$$R_1 = e_1(\tau) - e_1(t) - 2\gamma_1 = 0, \tag{4.5}$$

$$R_2 = e_2(\tau) - e_2(t) - 2\gamma_2 = 0, \tag{4.6}$$

$$R_3 = e_3(\tau) - e_3(t) - 2\gamma_3 = 0, \tag{4.7}$$

$$R_4 = e_4(\tau) - e_4(t) - 2\gamma_4 - \omega = 0, \tag{4.8}$$

$$r_2 = \sum_{j=1}^{2}(-1)^j\gamma_j\big(\gamma_{2-j} + e_{2-j}(t)\big) = 0, \tag{4.9}$$

$$r_4 = \sum_{j=1}^{4}(-1)^j\gamma_j\big(\gamma_{4-j} + e_{4-j}(t)\big) - \omega = 0, \tag{4.10}$$

$$r_6 = \sum_{j=2}^{4}(-1)^j\gamma_j\big(\gamma_{6-j} + e_{6-j}(t)\big) + (\gamma_2 + \gamma^2)\omega = 0, \tag{4.11}$$

$$r_8 = \gamma_4\big(\gamma_4 + e_4(t)\big) + \omega^2 + \big(\gamma_4 - e_4(\rho t)\big)\omega = 0, \tag{4.12}$$

$$r_{12} = \omega\big(\omega - e_4(\rho t)\big)\big(\omega + e_4(\rho^2 t)\big) = 0. \tag{4.13}$$

Proposition 4.2 is proved in Section 6. The following lemma is proved in Section 7.

LEMMA 4.2

*We have that $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/(r_1', R_i, r_{2i}, r_{12} \mid 1 \le i \le 4)$ is free as a $\mathbb{Z}$-module, and its Poincaré series coincides with that of $H_T^*(F_4/T)$.*

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1*

Let $I$ denote the ideal $(r_1', R_i, r_{2i}, r_{12} \mid 1 \le i \le 4)$ in the polynomial ring $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]$. We have a surjective ring homomorphism

$$\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4] \to H^*(\mathcal{F}_4)$$

by Lemma 4.1, and it factors through $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4]/I \to H^*(\mathcal{F}_4)$ by Proposition 4.2. It follows from Proposition 4.1 and Lemma 4.2 that $H^*(\mathcal{F}_4)$ and

$\mathbb{Z}[t_i,\gamma,\tau_i,\gamma_i,\omega \mid 1\le i\le 4]/I$ are free as $\mathbb{Z}$-modules. Moreover, Lemma 4.2 claims that $\mathbb{Z}[t_i,\gamma,\tau_i,\gamma_i,\omega \mid 1\le i\le 4]/I$ and $H^*(\mathcal{F}_4)$ have the same rank in each degree. Therefore the ring homomorphism $\mathbb{Z}[t_i,\gamma,\tau_i,\gamma_i,\omega \mid 1\le i\le 4]/I \to H^*(\mathcal{F}_4)$ is an isomorphism and Theorem 1.1 is proved by Theorem 2.1. □

## 5. Proof of Lemma 4.1

First we introduce some notation for the proof of Lemma 4.1. For a positive integer $n$, let $[n]$ and $\pm[n]$ be $\{i\in\mathbb{Z}\mid 1\le i\le n\}$ and $\{\pm i\in\mathbb{Z}\mid 1\le i\le n\}$, respectively. For $1\le n\le 4$, let $I_n$ denote an ordered $n$-tuple $(i_1,\dots,i_n)$ of elements of $[4]$ which does not include the same entries, and let $I'_n$ denote an ordered $n$-tuple $(i'_1,\dots,i'_n)$ of elements of $\pm[4]$ such that $|i'_k|\ne|i'_l|$ for $k\ne l$. We often regard $I_n$, $I'_n$ as the $n$-subsets of $[4]$ by the following maps:

$$(i_1,\dots,i_n)\mapsto\{i_1,\dots,i_n\},\qquad (i'_1,\dots,i'_n)\mapsto\big\{|i'_1|,\dots,|i'_n|\big\}.$$

Let $t_{i'}=\operatorname{sgn}(i')t_{|i'|}$. For $\varepsilon=0,1,2$, we define a subset $\rho^\varepsilon W^{I_n}_{I'_n}$ of $W(F_4)$ as

$$\rho^\varepsilon W^{I_n}_{I'_n}=\big\{w\in W(F_4)\;\big|\; w\in\rho^\varepsilon W\big(\mathrm{Spin}(9)\big), w(t_{i_k})=\rho^\varepsilon t_{i'_k}\ (1\le k\le n)\big\}.$$

We define $I_0$ and $I'_0$ to be the empty set. Note that $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$ includes $\rho^\varepsilon W^{I_n}_{I'_n}$ and decomposes as follows:

$$\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}=\coprod_{i_n\in[4]\setminus I_{n-1}}\rho^\varepsilon W^{(I_{n-1},i_n)}_{(I'_{n-1},i'_n)}\sqcup\coprod_{i_n\in[4]\setminus I_{n-1}}\rho^\varepsilon W^{(I_{n-1},i_n)}_{(I'_{n-1},-i'_n)}. \tag{5.1}$$

For a set $S=\{j_1,\dots,j_k\}$ of natural numbers with $j_1<\dots<j_k$, let $x_S$ denote a sequence $(x_{j_1},\dots,x_{j_k})$ for $x=t,\ \rho t,\ \rho^2t,\ \tau$. For $n\ge 0$, $j\le 4$, and $\varepsilon=0,1,2$, let $\gamma_j^{(\varepsilon)}{}^{I_n}_{I'_n}$ be a function from $\rho^\varepsilon W^{I_n}_{I'_n}$ to $\mathbb{Z}[\frac12][t_1,t_2,t_3,t_4]$ defined as

$$\gamma_j^{(\varepsilon)}{}^{I_n}_{I'_n}=\frac12\big(e_j(\tau_{[4]\setminus I_n})-e_j(\rho^\varepsilon t_{[4]\setminus I'_n})\big),$$

where $I_n$ and $I'_n$ on the right-hand side are regarded as subsets of $[4]$. When $n=0$ we abbreviate $\gamma_j^{(\varepsilon)}{}^{\emptyset}_{\emptyset}$ by $\gamma_j^{(\varepsilon)}$. If $j\le 0$ or $j>4-n$, then we define $\gamma_j^{(\varepsilon)}{}^{I_n}_{I'_n}=0$.

We define a function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ which is useful in the proof of Lemma 4.1 as

$$f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}=\frac12\prod_{k\in[4]\setminus I_{n-1}}(\tau_k-\rho^\varepsilon t_{i'_n}).$$

This function is $H^*(BT)$-valued on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$, since for any $w\in\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$ there exists $k\in[4]\setminus I_{n-1}$ such that $w\in\rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},i'_n)}\sqcup\rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},-i'_n)}$ by the decomposition (5.1), and then $w(t_k)-\rho^\varepsilon t_{i'_n}$ is equal to $0$ or $-2\rho^\varepsilon t_{i'_n}$. Especially we have that

$$\begin{aligned} &f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}(w)\\ &=\begin{cases}0, & w\in\coprod_{k\in[4]\setminus I_{n-1}}\rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},i'_n)},\\ -\rho^\varepsilon t_{i'_n}\prod_{k\in[4]\setminus I'_n}(\rho^\varepsilon t_k-\rho^\varepsilon t_{i'_n}), & w\in\coprod_{k\in[4]\setminus I_{n-1}}\rho^\varepsilon W^{(I_{n-1},k)}_{(I'_{n-1},-i'_n)}.\end{cases}\end{aligned} \tag{5.2}$$

Let $R$ denote the subring of $H^*(\mathcal{F}_4)$ generated by the $t_i$'s, $\gamma$, $\tau_i$'s, and $\gamma_i$'s $(1 \le i \le 4)$. The following proposition claims that this function $f^{(\varepsilon)}{}_{i'_n}^{I_{n-1}}$ can be replaced partly by an element of $R$.

PROPOSITION 5.1

*For $1 \le n \le 4$, there is a polynomial in $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$ over $H^*(BT)$ which coincides with the function $f^{(\varepsilon)}{}_{i'_n}^{I_{n-1}}$ on $\rho^\varepsilon W_{I'_{n-1}}^{I_{n-1}}$.*

Proposition 5.1 is a consequence of Lemmas 5.1 and 5.2 below.

LEMMA 5.1

*For $1 \le n \le 4$, there is a polynomial in the $\gamma_j^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}}$'s $(1 \le j \le 4-(n-1))$ over $H^*(BT)$ which coincides with $f^{(\varepsilon)}{}_{i'_n}^{I_{n-1}}$ on $\rho^\varepsilon W_{I'_{n-1}}^{I_{n-1}}$.*

LEMMA 5.2 ([FIM, LEMMA 5.3])

*For $1 \le n \le 4$ and $1 \le j \le 4-n$, there is a polynomial in $\gamma_1^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}}, \dots, \gamma_{4-n}^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}}$ over $H^*(BT)$ which coincides with $\gamma_j^{(\varepsilon)}{}_{(I'_{n-1},i'_n)}^{(I_{n-1},i_n)}$ on $\rho^\varepsilon W_{(I'_{n-1},i'_n)}^{(I_{n-1},i_n)}$. More explicitly,*

$$\gamma_j^{(\varepsilon)}{}_{I'_n}^{I_n} = \begin{cases} \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}} (-\rho^\varepsilon t_{i'_n})^k, & \operatorname{sgn} i'_n = 1, \\ \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}} (-\rho^\varepsilon t_{i'_n})^k & \\ \quad + \sum_{k=1}^{j} e_{j-k}(\rho^\varepsilon t_{[4]\setminus I'_n})(-\rho^\varepsilon t_{i'_n})^k, & \operatorname{sgn} i'_n = -1. \end{cases}$$

*Proof of Proposition 5.1*

By Lemma 5.1, there is a polynomial in the $\gamma_j^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}}$'s $(1 \le j \le 4-(n-1))$ over $H^*(BT)$ which coincides with $f^{(\varepsilon)}{}_{i'_n}^{I_{n-1}}$ on $\rho^\varepsilon W_{I'_{n-1}}^{I_{n-1}}$ for $\varepsilon = 0,1,2$. Then by Lemma 5.2 $\gamma_j^{(\varepsilon)}{}_{(I'_{n-1},i'_n)}^{(I_{n-1},i_n)}$ can be replaced by some polynomial in $\gamma_1^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}}, \dots, \gamma_{4-n}^{(\varepsilon)}{}_{I'_{n-1}}^{I_{n-1}}$ over $H^*(BT)$. By a descending induction on $n$ we reach a polynomial in $\gamma_1^{(\varepsilon)}$, $\gamma_2^{(\varepsilon)}$, $\gamma_3^{(\varepsilon)}$, $\gamma_4^{(\varepsilon)}$ over $H^*(BT)$ which coincides with $f^{(\varepsilon)}{}_{i'_n}^{I_{n-1}}$ on $\rho^\varepsilon W_{I'_{n-1}}^{I_{n-1}}$ for $\varepsilon = 0,1,2$. Next we need to show that $\gamma_j - \gamma_j^{(\varepsilon)} \in H^*(BT)$ on $\rho^\varepsilon W(\mathrm{Spin}(9))$ for $1 \le j \le 4$ and $\varepsilon = 0,1,2$ to complete the proof of Proposition 5.1. By definition we have that

$$\gamma_j^{(\varepsilon)} = \gamma_j + \frac{1}{2}\bigl(e_j(t) - e_j(\rho^\varepsilon t)\bigr) \quad (j = 1,2,3).$$

For $\varepsilon = 0,1,2$ and $j = 1,2,3$, Table 1 shows that $(e_j(t) - e_j(\rho^\varepsilon t))/2 \in H^*(BT)$ and then $\gamma_j - \gamma_j^{(\varepsilon)} \in H^*(BT)$ on $\rho^\varepsilon W(\mathrm{Spin}(9))$. By the definition of $\gamma_4$ and (4.3), we have that

$$\begin{aligned} \gamma_4^{(0)} &= \gamma_4 \quad \text{on } W\bigl(\mathrm{Spin}(9)\bigr), \\ \gamma_4^{(1)} &= \gamma_4 + e_4(t) \quad \text{on } \rho W\bigl(\mathrm{Spin}(9)\bigr), \\ \gamma_4^{(2)} &= \gamma_4 - e_4(\rho^2 t) \quad \text{on } \rho^2 W\bigl(\mathrm{Spin}(9)\bigr). \end{aligned}$$

Therefore there is a polynomial in $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$ over $H^*(BT)$ which coincides with the function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ on $\rho^\varepsilon W^{I_{n-1}}_{I'_{n-1}}$. □

*Proof of Lemma 5.1*

Without loss of generality, we may suppose that $I_{n-1}=(1,\dots,n-1)$. Note that $e_j(x_S)=0$ for $j>\#S$ or $j<0$, and note that we have

$$e_j(x_1,\dots,x_{m-1},x_m)=e_j(x_1,\dots,x_{m-1})+e_{j-1}(x_1,\dots,x_{m-1})x_m. \tag{5.3}$$

By the definition of $\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}$ we can expand the GKM function $f^{(\varepsilon)}{}^{I_{n-1}}_{i'_n}$ as follows:

$$\begin{aligned}\frac{1}{2}\prod_{l=0}^{4-n}(\tau_{n+l}-\rho^\varepsilon t_{i'_n}) &= \frac{1}{2}\sum_{j=0}^{5-n} e_j(\tau_{[4]\setminus I_{n-1}})(-\rho^\varepsilon t_{i'_n})^{5-n-j}\\ &= \frac{1}{2}\sum_{j=0}^{5-n}\big(2\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}+e_j(\rho^\varepsilon t_{[4]\setminus I'_{n-1}})\big)(-\rho^\varepsilon t_{i'_n})^{5-n-j}.\end{aligned}$$

Pay attention to the sign of $i'_n$, and recall that $[4]\setminus I'_{n-1}=\{i\in[4]\mid \pm i\notin I'_{n-1}\}$. By (5.3), the above statement is equal to

$$\begin{aligned}&\sum_{j=0}^{5-n}\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}(-\rho^\varepsilon t_{i'_n})^{5-n-j}\\ &\qquad+\frac{1}{2}\sum_{j=0}^{5-n}\big(e_j(\rho^\varepsilon t_{[4]\setminus I'_n})+e_{j-1}(\rho^\varepsilon t_{[4]\setminus I'_n})\rho^\varepsilon t_{|i'_n|}\big)(-\rho^\varepsilon t_{i'_n})^{5-n-j}\\ &=\begin{cases}\sum_{j=0}^{5-n}\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}(-\rho^\varepsilon t_{i'_n})^{5-n-j}, & \operatorname{sgn} i'_n=1,\\ \sum_{j=0}^{5-n}\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}(-\rho^\varepsilon t_{i'_n})^{5-n-j} & \\ \quad+\sum_{j=0}^{4-n}e_j(\rho^\varepsilon t_{[4]\setminus I'_n})(-\rho^\varepsilon t_{i'_n})^{5-n-j}, & \operatorname{sgn} i'_n=-1.\end{cases}\end{aligned}$$

□

*Proof of Lemma 5.2*

The relation $\tau_{i_n}=\rho^\varepsilon t_{i'_n}$ holds on $\rho^\varepsilon W^{(I_{n-1},i_n)}_{(I'_{n-1},i'_n)}$. Then we have that

$$\begin{aligned}&\gamma_j^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}}-\gamma_j^{(\varepsilon)}{}^{(I_{n-1},i_n)}_{(I'_{n-1},i'_n)}\\ &=\frac{1}{2}\big(e_j(\tau_{i\in[4]\setminus I_{n-1}})-e_j(\rho^\varepsilon t_{i'\in[4]\setminus I'_{n-1}})\big)-\frac{1}{2}\big(e_j(\tau_{i\in[4]\setminus I_n})-e_j(\rho^\varepsilon t_{i'\in[4]\setminus I'_n})\big)\\ &=\frac{1}{2}\big(e_{j-1}(\tau_{i\in[4]\setminus I_n})\tau_{i_n}-e_{j-1}(\rho^\varepsilon t_{i'\in[4]\setminus I'_n})\rho^\varepsilon t_{|i'_n|}\big)\\ &=\begin{cases}\gamma_{j-1}^{(\varepsilon)}{}^{I_n}_{I'_n}\rho^\varepsilon t_{i'_n}, & \operatorname{sgn} i'_n=1,\\ \gamma_{j-1}^{(\varepsilon)}{}^{I_n}_{I'_n}\rho^\varepsilon t_{i'_n}+e_{j-1}(\rho^\varepsilon t_{i'\in[4]\setminus I'_n})\rho^\varepsilon t_{i'_n}, & \operatorname{sgn} i'_n=-1.\end{cases}\end{aligned}$$

Iterated use of this equation shows that

$$\gamma_j^{(\varepsilon)\,(I_{n-1},i_n)}{}_{(I'_{n-1},i'_n)} = \begin{cases} \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}} (-\rho^\varepsilon t_{i'_n})^k, & \operatorname{sgn} i'_n = 1, \\ \sum_{k=0}^{j-1} \gamma_{j-k}^{(\varepsilon)}{}^{I_{n-1}}_{I'_{n-1}} (-\rho^\varepsilon t_{i'_n})^k & \\ \quad + \sum_{k=1}^{j} e_{j-k}(\rho^\varepsilon t_{[4]\setminus I'_n})(-\rho^\varepsilon t_{i'_n})^k, & \operatorname{sgn} i'_n = -1. \end{cases} \qquad \square$$

Now we are ready to prove Lemma 4.1.

*Proof of Lemma 4.1*
We show that any GKM function $h \in H^*(\mathcal{F}_4)$ belongs to the subring $R$ generated by the $t_i$'s, $\gamma$, $\tau_i$'s, $\gamma_i$'s, and $\omega$ $(1 \le i \le 4)$. By the definition of $\rho$, the set of all vertices $W(F_4)$ of $\mathcal{F}_4$ decomposes as

$$W(F_4) = W\big(\mathrm{Spin}(9)\big) \sqcup \rho W\big(\mathrm{Spin}(9)\big) \sqcup \rho^2 W\big(\mathrm{Spin}(9)\big).$$

For each $\varepsilon = 0, 1, 2$, $\rho^\varepsilon W(\mathrm{Spin}(9))$ has a filtration

$$\rho^\varepsilon W_{I'_4}^{I_4} \subset \cdots \subset \rho^\varepsilon W_{I'_n}^{I_n} \subset \rho^\varepsilon W_{I'_{n-1}}^{I_{n-1}} \subset \cdots \subset \rho^\varepsilon W_{I'_0}^{I_0} = \rho^\varepsilon W\big(\mathrm{Spin}(9)\big).$$

By descending induction on $n$, we will show that any GKM function $h$ can be modified to be 0 on $\rho^\varepsilon W_{I'_n}^{I_n}$ by subtracting some GKM function in $R$. Moreover, in the induction step on $n$, we give an induction to fill the decomposition (5.1) of $\rho^\varepsilon W_{I'_{n-1}}^{I_{n-1}}$.

Let $0 \le n \le 4$. The following claim in the case where $n = 0$ shows that $h$ can be modified to be 0 on $W(\mathrm{Spin}(9))$.

CLAIM 1 ($n$)
*For any ordered $n$-tuples $I_n$, $I'_n$ and any function $h$ from $W_{I'_n}^{I_n}$ to $H^*(BT)$ which satisfies the GKM condition on $W_{I'_n}^{I_n}$, there is a GKM function $G \in R$ which coincides with $h$ on $W_{I'_n}^{I_n}$.*

We show this claim by descending induction on $n$. For $n = 4$, since $W_{I'_4}^{I_4}$ is a one-point set, the claim holds obviously. Assume that Claim 1 ($n$) holds, and fix $I_n = (i_1, \dots, i_n)$ and $I'_n = (i'_1, \dots, i'_n)$. Then we have a GKM function which coincides with $h$ on $W_{I'_n}^{I_n}$. Subtracting this GKM function from $h$, we may assume that $h$ vanishes on $W_{I'_n}^{I_n}$. We give an induction to fill the decomposition (5.1) of $W_{I'_{n-1}}^{I_{n-1}}$ as follows. For any $k \in [4] \setminus I_n$, let $\sigma_k$ denote the reflection associated with $t_k - t_{i_n}$. Then $\sigma_k$ interchanges $t_k$ and $t_{i_n}$, and for any $w \in W_{(I'_{n-1},i'_n)}^{(I_{n-1},k)}$, $w\sigma_k$ is contained in $W_{(I'_{n-1},i'_n)}^{(I_{n-1},i_n)}$. By the GKM condition, $h(w) - h(w\sigma_k) = h(w)$ belongs to the ideal generated by $w(t_{i_n} - t_k) = w(t_{i_n}) - t_{i'_n} = \tau_{i_n}(w) - t_{i'_n}$. Put $k_0, \dots, k_{4-n} \in [4] \setminus I_{n-1}$ as $k_0 = i_n$, $k_s < k_t$ for $1 \le s < t$, and $\{k_0, \dots, k_{4-n}\} \cup I_{n-1} = [4]$.

CLAIM 2 ($t$)

*If $h$ is a GKM function which vanishes on $\coprod_{s<t} W^{(I_{n-1},k_s)}_{(I'_{n-1},i'_n)}$, then there is a GKM function $G \in R$ such that $h$ coincides with $\prod_{s<t}(\tau_{k_s} - t_{i'_n})G$ on $W^{(I_{n-1},k_t)}_{(I'_{n-1},i'_n)}$.*

We show this claim by induction on $t$ ($0 \le t \le 4-n$). Without loss of generality, we may suppose that $I_{n-1} = (1,\dots,n-1)$ and $k_0 = i_n = n$, $k_1 = n+1,\dots,$ $k_{4-n} = 4$. We rephrase Claim 2 ($t$) as follows.

CLAIM 2 ($k$)

*If $h$ vanishes on $\coprod_{0\le l<k} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)}$, then there is a GKM function $G \in R$ such that $h$ coincides with $\prod_{0\le l<k}(\tau_{n+l} - t_{i'_n})G$ on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$.*

Obviously $\prod_{0\le l<k}(\tau_{n+l} - t_{i'_n})$ vanishes on $\coprod_{0\le l<k} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)}$. For $w \in W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$, by the GKM condition, there is an element $g_w \in H^*(BT)$ such that

$$h(w) = \Big(\prod_{0\le l<k}(\tau_{n+l} - t_{i'_n})(w)\Big)g_w.$$

One can verify that a function $G'\colon W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)} \to H^*(BT)$ given by

$$G'(w) = g_w$$

satisfies the GKM condition on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$ as follows. Assume that two vertices $w,\ w' \in W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$ of $\mathcal{F}_4$ satisfy $w' = w\sigma_\alpha$ for some positive root $\alpha$. Then $\alpha = t_i - t_j$, where $i<j$ and $i,\ j \in \{m \in \mathbb{Z} \mid n \le m \le n+k-1 \text{ or } n+k+1 \le m \le 4\}$. When $i<j<n+k$ or $n+k<i<j$, the GKM condition says that

$$\begin{aligned} &h(w) - h(w') \\ &\quad= \Big(\prod_{0\le l<k}\big(w(t_{n+l}) - t_{i'_n}\big)\Big)G'(w) - \Big(\prod_{0\le l<k}\big(w\sigma_{t_i-t_j}(t_{n+l}) - t_{i'_n}\big)\Big)G'(w') \\ &\quad= \Big(\prod_{0\le l<k}\big(w(t_{n+l}) - t_{i'_n}\big)\Big)\big(G'(w) - G'(w')\big) \end{aligned}$$

belongs to the ideal $(w(t_i - t_j))$. Since $w(t_{n+l}) - t_{i'_n}$ and $w(t_i - t_j)$ are relatively prime, $G'(w) - G'(w')$ also belongs to the ideal $(w(t_i - t_j))$. When $i < n+k < j$, the GKM condition says that

$$\begin{aligned} &h(w) - h(w') \\ &\quad= \Big(\prod_{0\le l<k}\big(w(t_{n+l}) - t_{i'_n}\big)\Big)G'(w) - \Big(\prod_{0\le l<k}\big(w\sigma_{t_i-t_j}(t_{n+l}) - t_{i'_n}\big)\Big)G'(w') \\ &\quad= \Big(\prod_{0\le l<k, l\ne i}\big(w(t_{n+l}) - t_{i'_n}\big)\Big) \\ &\qquad\times \Big(\big(w(t_i) - t_{i'_n}\big)\big(G'(w) - G'(w')\big) + \big(w(t_i) - w(t_j)\big)G'(w')\Big) \end{aligned}$$

belongs to the ideal $(w(t_i - t_j))$. Since $w(t_{n+l}) - t_{i'_n}$ and $w(t_i - t_j)$ are relatively prime, $G'(w) - G'(w')$ also belongs to the ideal $(w(t_i - t_j))$. Hence the function $G'$ satisfies the GKM condition on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$.

By (descending) induction on $n$ there is a GKM function $G \in R$ such that $G$ and $G'$ coincide on $W^{(I_{n-1},n+k)}_{(I'_{n-1},i'_n)}$. Then

$$h - \Big(\prod_{0\le l<k}(\tau_{n+l} - t_{i'_n})\Big)G = 0 \quad \text{on} \quad \coprod_{0\le l\le k} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)}.$$

Therefore the induction on $k$ proceeds.

Next we fill the other half of the decomposition (5.1). Note that, when $I_{n-1} = (1,\dots,n-1)$,

$$f^{(0)}{}^{I_{n-1}}_{i'_n} = \frac{1}{2}\prod_{0\le l\le 4-n}(\tau_{n+l} - t_{i'_n}).$$

Let $0 \le k' \le 4-n$.

CLAIM 3 ($k'$)
*If $h$ vanishes on $\coprod_{0\le l\le 4-n} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)} \sqcup \coprod_{0\le l<k'} W^{(I_{n-1},n+l)}_{(I'_{n-1},-i'_n)}$, then there is a GKM function $G \in R$ such that $h$ coincides with $f^{(0)}{}^{I_{n-1}}_{i'_n}\prod_{0\le l<k'}(\tau_{n+l} + t_{i'_n})G$ on $W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$.*

We show this claim by induction on $k'$. For $w \in W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$, by the GKM condition, $h(w)$ belongs to the ideal generated by the product of the following elements of $H^*(BT)$:

$$\begin{aligned} w(t_{n+l} - t_{n+k'}) &= w(t_{n+l}) + t_{i'_n} \qquad \text{for } 0\le l<k',\\ w(t_{n+l} + t_{n+k'}) &= w(t_{n+l}) - t_{i'_n} \qquad \text{for } 0\le l\le 4-n, l\ne k',\\ w(t_{n+k'}) &= -t_{i'_n}. \end{aligned}$$

For $w \in W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$, by (5.2), there is an element $g_w \in H^*(BT)$ such that

$$h(w) = f^{(0)}{}^{I_{n-1}}_{i'_n}(w)\Big(\prod_{0\le l<k'}(\tau_{n+l} + t_{i'_n})(w)\Big)g_w.$$

One can verify that a function $G'$ given by $G'(w) = g_w$ satisfies the GKM condition on $W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$ as above. By (descending) induction on $n$ there is a GKM function $G \in R$ such that $G$ and $G'$ coincide on $W^{(I_{n-1},n+k')}_{(I'_{n-1},-i'_n)}$. Then

$$h - f^{(0)}{}^{I_{n-1}}_{i'_n}\Big(\prod_{0\le l<k'}(\tau_{n+l} + t_{i'_n})\Big)G = 0$$

$$\text{on} \quad \coprod_{0\le l\le 4-n} W^{(I_{n-1},n+l)}_{(I'_{n-1},i'_n)} \sqcup \coprod_{0\le l\le k'} W^{(I_{n-1},n+l)}_{(I'_{n-1},-i'_n)}.$$

Therefore the induction on $k'$ proceeds. By Proposition 5.1, the function

$$f^{(0)}{}^{I_{n-1}}_{i'_n} = \frac{1}{2}\prod_{0\le l\le 4-n}(\tau_{n+l}-t_{i'_n})$$

can be replaced by a polynomial in the $\gamma_j$'s $(1\le j\le 4)$ over $H^*(BT)$. Therefore the (descending) induction on $n$ proceeds, and we may assume that $h$ vanishes on $W(\mathrm{Spin}(9)) = W\sqcup \kappa W$.

Next we show that, for a GKM function $h$ which vanishes on $W(\mathrm{Spin}(9))$, there is a GKM function $G\in R$ such that $h-\omega G=0$ on $W(\mathrm{Spin}(9))\sqcup\rho W(\mathrm{Spin}(9))$, where $\omega$ vanishes on $W(\mathrm{Spin}(9))$. Recall that the schematic diagram (3.6) says that each $w\in\rho W\sqcup\rho\kappa W$ is adjacent to four vertices of $W\sqcup\kappa W$, and the labels of these edges are $\rho^2 t_i$ $(1\le i\le 4)$ and different from each other. The GKM condition says that, for $w\in\rho W(\mathrm{Spin}(9))$, $h(w)$ belongs to the ideal $(\prod_{i=1}^4\rho^2 t_i)$. For $w\in\rho W(\mathrm{Spin}(9))$, there is an element $g_w\in H^*(BT)$ such that

$$h(w) = -e_4(\rho^2 t)g_w = \omega(w)g_w.$$

It is obvious that a function $G'$ given by $G'(w)=g_w$ satisfies the GKM condition on $\rho W(\mathrm{Spin}(9))$, since the edges in the GKM subgraph induced by $\rho W(\mathrm{Spin}(9))$ have the long roots or $\rho t_i$ as their labels and all the positive roots of $F_4$ are relatively prime in $H^*(BT)$. Then we claim that there is a GKM function $G$ such that $G=G'$ on $\rho W(\mathrm{Spin}(9))$. This claim is proved as above, changing $W^{I_n}_{I'_n}$ to $\rho W^{I_n}_{I'_n}$, $\tau_{k_s}-t_{i'_n}$ to $\tau_{k_s}-\rho t_{i'_n}$, and $f^{(0)}{}^{I_{n-1}}_{i'_n}$ to $f^{(1)}{}^{I_{n-1}}_{i'_n}$.

Finally we show that, for a GKM function $h$ which vanishes on $W(\mathrm{Spin}(9))\sqcup\rho W(\mathrm{Spin}(9))$, there is a GKM function $G\in R$ such that $h-\omega(\omega+e_4(\rho^2 t))G=0$ as a GKM function on the whole $W(F_4)$, where $\omega+e_4(\rho^2 t)$ vanishes on $\rho W(\mathrm{Spin}(9))$. It is proved as above that, for $w\in\rho^2 W(\mathrm{Spin}(9))$, $h(w)$ belongs to the ideal $(\prod_{i=1}^4 t_i\prod_{i=1}^4\rho t_i)$. For $w\in\rho^2 W(\mathrm{Spin}(9))$, there is an element $g_w\in H^*(BT)$ such that

$$h(w) = -e_4(\rho t)e_4(t)g_w = \omega(w)\big(\omega(w)+e_4(\rho^2 t)\big)g_w,$$

where the latter equality is due to (4.3). Then we claim that a function $G'$ given by $G'(w)=g_w$ satisfies the GKM condition on $\rho^2 W(\mathrm{Spin}(9))$, and that there is a GKM function $G$ such that $G=G'$ on $\rho^2 W(\mathrm{Spin}(9))$. This claim is proved as above, changing $W^{I_n}_{I'_n}$ to $\rho^2 W^{I_n}_{I'_n}$, $\tau_{k_s}-t_{i'_n}$ to $\tau_{k_s}-\rho^2 t_{i'_n}$, and $f^{(0)}{}^{I_{n-1}}_{i'_n}$ to $f^{(2)}{}^{I_{n-1}}_{i'_n}$. The proof is completed. □

## 6. Proof of Proposition 4.2

We prove Proposition 4.2 in a way similar to that of [FIM, Lemma 5.5].

*Proof of Proposition 4.2*

The relations (4.4), (4.5), (4.6), (4.7), and (4.8) hold obviously by definition, and the relation (4.13) holds by (4.2). To show that (4.9), (4.10), (4.11), and (4.12) hold, we claim that the following relations hold in $H_T^*(\mathcal{F}_4)$:

(6.1) $$e_1(\tau^2)-e_1(t^2)=0,$$

$$(6.2)\qquad e_2(\tau^2) - e_2(t^2) - 6\omega = 0,$$

$$(6.3)\qquad e_3(\tau^2) - e_3(t^2) - e_1(t^2)\omega = 0,$$

$$(6.4)\qquad e_4(\tau^2) - e_4(t^2) + 3\omega^2 - 2\big(e_4(\rho t) - e_4(\rho^2 t)\big)\omega = 0.$$

The left-hand side functions of these equations are constant on each $\rho^\varepsilon W(\mathrm{Spin}(9))$ for $\varepsilon = 0, 1, 2$. Calculations of each value on $\rho^\varepsilon W(\mathrm{Spin}(9))$ with (4.3) show that (6.1), (6.2), (6.3), and (6.4) hold.

We show that (6.1), (6.2), (6.3), and (6.4) are divisible by 4 to deduce (4.9), (4.10), (4.11), and (4.12). Let $x$ be an indeterminate, and put $X = -6\omega x^4 + e_1(t^2)\omega x^6 + (3\omega^2 - 2(e_4(\rho t) - e_4(\rho^2 t))\omega)x^8$. It follows from (6.1), (6.2), (6.3), and (6.4) that

$$\begin{aligned} 0 &= \prod_{i=1}^{4}(1 - \tau_i^2 x^2) - \prod_{i=1}^{4}(1 - t_i^2 x^2) + X \\ &= \sum_{k=0}^{4}\big(1 + (-1)^k e_k(\tau) x^k\big) \sum_{k=0}^{4}\big(1 + e_k(\tau) x^k\big) \\ &\quad - \sum_{k=0}^{4}\big(1 + (-1)^k e_k(t) x^k\big) \sum_{k=0}^{4}\big(1 + e_k(t) x^k\big) + X. \end{aligned}$$

We can erase $e_k(\tau)$ by (4.5), (4.6), (4.7), and (4.8), and obtain

$$\begin{aligned} &4\sum_{k=1}^{3}(-1)^k \gamma_k^2 x^{2k} - 8\gamma_1\gamma_3 x^4 + 4\sum_{k=1}^{3}\sum_{i=n_k}^{m_k}(-1)^i \gamma_i e_{2k-i}(t) x^{2k} \\ &\quad + 2(2\gamma_4 + \omega)x^4 + 4\gamma_2\big(e_4(t) + 2\gamma_4 + \omega\big)x^6 + 2e_2(t)(2\gamma_4 + \omega)x^6 \\ &\quad + \big(2e_4(t) + 2\gamma_4 + \omega\big)(2\gamma_4 + \omega)x^8 + X, \end{aligned}$$

where $n_k = \max\{1, 2k-3\}$ and $m_k = \min\{3, 2k\}$. This calculation is similar to the calculation in [FIM, proof of Lemma 5.5], but note that $\gamma_4 \neq \frac{1}{2}(e_4(\tau) - e_4(t))$. Then comparing the coefficients, we obtain

$$\begin{aligned} 0 &= -4\gamma_1^2 + 4\big(-\gamma_1 e_1(t) + \gamma_2\big) = 4\sum_{j=1}^{2}(-1)^j \gamma_j\big(\gamma_{2-j} + e_{2-j}(t)\big), \\ 0 &= 4\gamma_2^2 - 8\gamma_1\gamma_3 + 4\big(-\gamma_1 e_3(t) + \gamma_2 e_2(t) - \gamma_3 e_1(t)\big) + 4\gamma_4 - 4\omega \\ &= 4\Big(\sum_{j=1}^{4}(-1)^j \gamma_j\big(\gamma_{4-j} + e_{4-j}(t)\big) - \omega\Big), \\ 0 &= -4\gamma_3^2 - 4\gamma_3 e_3(t) + 4\gamma_2\big(e_4(t) + 2\gamma_4 + \omega\big) + 2e_2(t)(2\gamma_4 + \omega) \\ &\quad + \big(e_1(t)^2 - 2e_2(t)\big)\omega \\ &= 4\Big(\sum_{j=2}^{4}(-1)^j \gamma_j\big(\gamma_{6-j} + e_{6-j}(t)\big) + (\gamma_2 + \gamma^2)\omega\Big), \end{aligned}$$

$$0 = \big(2e_4(t) + 2\gamma_4 + \omega\big)(2\gamma_4 + \omega) + \big(3\omega^2 - 2\big(e_4(\rho t) - e_4(\rho^2 t)\big)\omega\big)$$
$$= 4\big(\gamma_4\big(\gamma_4 + e_4(t)\big) + \omega^2 + \big(\gamma_4 - e_4(\rho t)\big)\omega\big).$$

Regarding GKM functions as elements of $\mathrm{Map}(W(G), H^*(BT)\otimes\mathbb{Q})$, we can divide them by 4 to obtain

$$0 = \sum_{j=1}^{2}(-1)^j\gamma_j\big(\gamma_{2-j} + e_{2-j}(t)\big), \qquad 0 = \sum_{j=1}^{4}(-1)^j\gamma_j\big(\gamma_{4-j} + e_{4-j}(t)\big) - \omega,$$
$$0 = \sum_{j=2}^{4}(-1)^j\gamma_j\big(\gamma_{6-j} + e_{6-j}(t)\big) + (\gamma_2 + \gamma^2)\omega,$$
$$0 = \gamma_4\big(\gamma_4 + e_4(t)\big) + \omega^2 + \big(\gamma_4 - e_4(\rho t)\big)\omega.$$

Since the right-hand sides of these equations remain polynomials in $H^*(BT)$-valued GKM functions over $\mathbb{Z}$, these equations hold in $H^*(\mathcal{F}_4) \subset \mathrm{Map}(W(G), H^*(BT))$. □

## 7. Proof of Lemma 4.2

We will prove Lemma 4.2 by the argument of regular sequences.

DEFINITION 7.1

A sequence $a_1, \ldots, a_n$ of elements of a ring $R$ is called *regular* if, for any $i$, $a_i$ is not a zero divisor in $R/(a_1, \ldots, a_{i-1})$.

The following theorems and propositions are useful. Propositions 7.1 and 7.2 are obvious by definition.

PROPOSITION 7.1

*If $a_1, \ldots, a_n$ is a regular sequence, then so is $a_1, \ldots, a_{i-1}, a_i + b, a_{i+1}, \ldots, a_n$ for $1 \le i \le n$ and any $b \in (a_1, \ldots, a_{i-1})$.*

PROPOSITION 7.2

*If $a_1, \ldots, a_n$ is a regular sequence, then so is $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ for $1 \le i \le n$.*

THEOREM 7.1 ([M, THEOREM 16.1])

*If $a_1, \ldots, a_n$ is a regular sequence, then so is $a_1^{v_1}, \ldots, a_n^{v_n}$ for any positive integers $v_1, \ldots, v_n$.*

THEOREM 7.2 ([M, COROLLARY OF THEOREM 16.3])

*Let $A$ be a Noetherian ring and nonnegatively graded. If $a_1, \ldots, a_n$ is a regular* *sequence in $A$ and each $a_i$ is homogeneous of positive degree, then any permutation of $a_1, \dots, a_n$ is again a regular sequence.*

THEOREM 7.3 ([NS, THEOREM 5.5.1])
*Let $F$ be a field, and let $R = F[g_i \mid 1 \le i \le m]$ be a nonnegatively graded polynomial ring with $|g_i| > 0$ for any $1 \le i \le m$. Assume that $a_1, \dots, a_n$ is a regular sequence in $R$ which consists of homogeneous elements of positive degree. Then the Poincaré series of $R/(a_i \mid 1 \le i \le n)$ is given as*

$$\frac{\prod_{i=1}^n (1 - x^{|a_i|})}{\prod_{i=1}^m (1 - x^{|g_i|})}.$$

*Proof*
For a nonnegatively graded $F$-module $M$ of finite type, let $P(M,x)$ denote the Poincaré series of $M$, namely,

$$P(M,x) = \sum_{n=0}^{\infty} (\dim_F M_n) x^n,$$

where $M_n$ denotes the degree $n$ part of $M$. Then obviously we have that

$$P(R,x) = \frac{1}{\prod_{i=1}^m (1 - x^{|g_i|})}.$$

Since $a_1, \dots, a_n$ is a regular sequence, the multiplication by $a_i$ induces an injection on a graded $F$-module $R/(a_1, \dots, a_{i-1})$. Therefore

$$P\bigl(R/(a_1, \dots, a_i), x\bigr) = (1 - x^{|a_i|}) P\bigl(R/(a_1, \dots, a_{i-1}), x\bigr).$$

The induction on $i$ completes the proof. □

*Proof of Lemma 4.2*
Let $p$ be a prime number, and let

$$M = \bigl(\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \le i \le 4] / \{r_1', R_i, r_{2i}, r_{12} \mid 1 \le i \le 4\}\bigr),$$

where $|t_i| = 2$, $|\gamma_i| = 2i$, and $|\omega| = 8$. We will show that the Poincaré series of $M \otimes (\mathbb{Z}/p\mathbb{Z})$ does not depend on $p$. Then the graded $\mathbb{Z}$-module $M$ of finite type must be free. The relations (4.9) and (4.10) say that

$$\gamma_2 = \gamma_1 \bigl(\gamma_1 + e_1(t)\bigr), \qquad \gamma_4 = -\Bigl(\sum_{j=1}^{3} (-1)^j \gamma_j \bigl(\gamma_{4-j} + e_{4-j}(t)\bigr) - \omega\Bigr),$$

and then we can erase $\gamma_2$ and $\gamma_4$. Let $R$ denote the polynomial ring $\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_1, \gamma_3, \omega \mid 1 \le i \le 4]$, let $r_1'$, $R_i$'s, and $r_i$'s also denote the corresponding elements of $R$, and let $I$ denote the ideal generated by $\{r_1', R_i, r_6, r_8, r_{12} \mid 1 \le i \le 4\}$ in $R$. Since $M \cong R/I$, it is sufficient to compute the Poincaré series of $(R/I) \otimes (\mathbb{Z}/p\mathbb{Z})$.

When $p = 2$, we show that the sequence

$$r_1', r_{12}, R_4, R_3, R_2, R_1, r_6, r_8$$

is regular and compute the Poincaré series from this sequence. In $(R/I)\otimes(\mathbb{Z}/2\mathbb{Z})$, we have that

$$\begin{aligned}
r_1' &= e_1(t),\\
R_1 &= -\bigl(e_1(\tau)-e_1(t)\bigr),\\
R_2 &= -\bigl(e_2(\tau)-e_2(t)\bigr)+\bigl(e_1(\tau)-e_1(t)\bigr)e_1(t),\\
R_3 &= -\bigl(e_3(\tau)-e_3(t)\bigr),\\
R_4 &= e_4(\tau)-e_4(t)-\omega,\\
r_6 &\equiv \gamma_3^2 \quad \bigl(\mathrm{mod}(\gamma, e_i(t), \omega \bigm| 1\le i\le 4)\bigr),\\
r_8 &\equiv \gamma_4^2 \equiv \gamma_2^2 \equiv \gamma_1^8 \quad \bigl(\mathrm{mod}(\gamma, e_i(t), \omega \bigm| 1\le i\le 4)\bigr).
\end{aligned}$$

It is well known that the sequence of the elementary symmetric polynomials

$$e_1(x), e_2(x), \ldots, e_n(x),$$

that is, the sequence of the Chern classes, is regular in $(\mathbb{Z}/p\mathbb{Z})[x_i \mid 1\le i\le n]$ for any prime $p$. Since a polynomial ring over a field is Noetherian, by Theorem 7.2, the sequence

$$\gamma, e_1(t), e_2(t), e_3(t), e_4(t), \omega, e_4(\tau), e_3(\tau), e_2(\tau), e_1(\tau), \gamma_3^2, \gamma_1^8$$

is regular in $R\otimes(\mathbb{Z}/2\mathbb{Z})$. We modify this sequence by Theorem 7.1 and Proposition 7.1 to obtain the following regular sequence:

$$\gamma, r_1', e_2(t), e_3(t), e_4(t), \omega^3, R_4, R_3, R_2, R_1, r_6, r_8.$$

Since $\rho^2 t_4 = -\gamma$ and $e_4(\rho t) = -e_4(t) - e_4(\rho^2 t) \equiv 0 \pmod{\gamma, e_4(t)}$, by Proposition 7.1

$$\gamma, r_1', e_2(t), e_3(t), e_4(t), r_{12}, R_4, R_3, R_2, R_1, r_6, r_8$$

is a regular sequence. Hence

$$r_1', r_{12}, R_4, R_3, R_2, R_1, r_6, r_8$$

is a regular sequence by Proposition 7.2. Finally, the Poincaré series of $(R/I)\otimes(\mathbb{Z}/2\mathbb{Z})$ is calculated from the degrees of the generators and the relations by Theorem 7.3, and we have that

$$P\bigl(M\otimes(\mathbb{Z}/2\mathbb{Z}), x\bigr) = \frac{1}{(1-x^2)^4}(1+x^8+x^{16})\prod_{i=1}^{4}\frac{1-x^{4i}}{1-x^2}.$$

Next let us consider the case where $p\ge 3$. Let $e_1$, $e_2$, $e_3$, and $e_4$ be the left-hand sides of (6.1), (6.2), (6.3), and (6.4), respectively, namely,

$$\begin{aligned}
e_1 &= e_1(\tau^2) - e_2(t^2), \qquad e_2 = e_2(\tau^2) - e_2(t^2) - 6\omega,\\
e_3 &= e_3(\tau^2) - e_2(t^2) - e_1(t^2)\omega,\\
e_4 &= e_4(\tau^2) - e_4(t^2) + 3\omega^2 - 2\bigl(e_4(\rho t) - e_4(\rho^2 t)\bigr)\omega.
\end{aligned}$$

Recall that $e_1$, $e_2$, $e_3$, and $e_4$ are divided by 4 to yield $r_2$, $r_4$, $r_6$, and $r_8$, respectively. We have that

$$\begin{aligned} M \otimes (\mathbb{Z}/p\mathbb{Z}) &\cong \big(\mathbb{Z}[t_i, \gamma, \tau_i, \gamma_i, \omega \mid 1 \leq i \leq 4]/(r_1', R_i, e_{2i}, r_{12} \mid 1 \leq i \leq 4)\big) \otimes (\mathbb{Z}/p\mathbb{Z}) \\ &\cong \big(\mathbb{Z}[t_i, \gamma, \tau_i, \omega \mid 1 \leq i \leq 4]/(r_1', e_{2i}, r_{12} \mid 1 \leq i \leq 4)\big) \otimes (\mathbb{Z}/p\mathbb{Z}), \end{aligned}$$

since 2 is invertible in $\mathbb{Z}/p\mathbb{Z}$. We will show that the sequence

$$r_1', r_{12}, e_8, e_6, e_4, e_2$$

is a regular sequence. It is well known that the sequence of elementary symmetric polynomials in $\{x_i^2\}_{i=1}^n$

$$e_1(x^2), e_2(x^2), \ldots, e_n(x^2),$$

that is, the sequence of the Pontryagin classes, is regular in $(\mathbb{Z}/p\mathbb{Z})[x_i \mid 1 \leq i \leq n]$ for any prime $p$. By Theorem 7.2, the sequence

$$\gamma, e_1(t), e_2(t), e_3(t), e_4(t), \omega, e_4(\tau^2), e_3(\tau^2), e_2(\tau^2), e_1(\tau^2)$$

is regular in $(\mathbb{Z}/p\mathbb{Z})[t_i, \gamma, \tau_i, \omega \mid 1 \leq i \leq 4]$. We modify this sequence by Theorem 7.1 and Proposition 7.1 to obtain the following regular sequence:

$$\gamma, r_1', e_2(t), e_3(t), e_4(t), r_{12}, e_8, e_6, e_4, e_2.$$

Hence

$$r_1', r_{12}, e_8, e_6, e_4, e_2$$

is a regular sequence by Proposition 7.2. Therefore, by Theorem 7.3, we have that

$$P\big(M \otimes (\mathbb{Z}/p\mathbb{Z}), x\big) = \frac{1}{(1-x^2)^4}(1 + x^8 + x^{16}) \prod_{i=1}^{4} \frac{1-x^{4i}}{1-x^2}. \qquad \square$$

## 8. Proof of Corollary 1.1

*Proof of Corollary 1.1*
By the argument in Section 1 we have the isomorphisms

$$\begin{aligned} H^*(F_4/T) &\cong H_T^*(F_4/T)/(t_1, t_2, t_3, t_4, \gamma) \\ &\cong \mathbb{Z}[\tau_i, \gamma_i, \omega \mid 1 \leq i \leq 4]/(Q_i, q_{2i}, q_{12} \mid 1 \leq i \leq 4), \end{aligned}$$

where

$$\begin{aligned} &Q_i = e_i(\tau) - 2\gamma_i \quad (i = 1, 2, 3), \qquad Q_4 = e_4(\tau) - 2\gamma_4 - \omega, \\ &q_2 = \gamma_2 - \gamma_1^2, \qquad q_4 = \gamma_4 - 2\gamma_1\gamma_3 + \gamma_2^2 - \omega, \\ &q_6 = 2\gamma_2\gamma_4 - \gamma_3^2 + \gamma_2\omega, \qquad q_8 = \gamma_4^2 + \gamma_4\omega + \omega^2, \\ &q_{12} = \omega^3. \end{aligned}$$

We can regard $\gamma_2$ and $\gamma_4$ as dependent variables by the relations $q_2$ and $q_4$. Let $R$ be the polynomial ring $\mathbb{Z}[\tau_i, \gamma_1, \gamma_3, \omega \mid 1 \le i \le 4]$. Then

$$H^*(F_4/T) \cong R/(Q_i, q_6, q_8, q_{12} \mid 1 \le i \le 4).$$

Obviously we have that

$$Q_i = -\overline{r}_i \quad (i = 1, 2, 3), \qquad Q_4 \equiv \overline{r}_4 \pmod{Q_3},$$
$$q_6 \equiv \gamma_2 e_4(\tau) - \gamma_3^2 = -\overline{r}_6 \pmod{Q_4}, \qquad q_{12} = \overline{r}_{12}.$$

Moreover, we have that

$$\begin{aligned} q_8 &= 4\gamma_1^2\gamma_3^2 - 4\gamma_1^5\gamma_3 + \gamma_1^8 + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega \\ &\equiv 8\gamma_1^4\gamma_4 + 4\gamma_1^4\omega - 4\gamma_1^5\gamma_3 + \gamma_1^8 + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega \pmod{q_6} \\ &= 12\gamma_1^5\gamma_3 - 7\gamma_1^8 + 9\gamma_1^4\omega + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega. \end{aligned}$$

On the other hand,

$$\begin{aligned} \overline{r}_8 &= 3e_4(\tau)\gamma_1^4 - \gamma_1^8 + 3\omega\big(\omega + e_3(\tau)\gamma_1\big) \\ &\equiv 3(2\gamma_4 + \omega)\gamma_1^4 - \gamma_1^8 + 3\omega(\omega + 2\gamma_1\gamma_3) \pmod{Q_3, Q_4} \\ &= 12\gamma_1^5\gamma_3 - 7\gamma_1^8 + 9\gamma_1^4\omega + 3\omega(\omega + 2\gamma_1\gamma_3) - 3\gamma_1^4\omega. \end{aligned}$$

Hence $q_8 \equiv \overline{r}_8 \pmod{q_6, Q_3, Q_4}$. Therefore

$$H^*(F_4/T) \cong R/(Q_i, q_6, q_8, q_{12} \mid 1 \le i \le 4) \cong R/(\overline{r}_1, \overline{r}_2, \overline{r}_3, \overline{r}_4, \overline{r}_6, \overline{r}_8, \overline{r}_{12}). \qquad \square$$

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