An approach to the pseudoprocess driven by the equation $\frac{\partial}{\partial t} = -A\frac{\partial^3}{\partial x^3}$ by a random walk

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Abstract In this paper we study the pseudoprocess driven by $\partial_t = -A\partial_x^3$. Our method is an approximation by the pseudo–random walk. We obtain their joint distribution of the first hitting time and the first hitting place. In addition, this result is provided by the alternate method of Shimoyama.

1. Introduction

The partial equation

$$\frac{\partial u}{\partial t}(t,x) = A\frac{\partial^k u}{\partial x^k}(t,x)$$

has been studied by many authors. Especially in $k = 2$ it has been studied from the point of view of probability theory. Many studies have been conducted on the extension in the case of $k > 2$ such that k is an even number because of the analogy with the case of $k = 2$. We can see some of these results in Funaki [1], Helms [2], Hochberg [3], Krylov [4], Lachel [5], [6], Motoo [7], Nakajima and Sato [8], Nishioka [9], Sato [11], and so on.

Above all, Nishioka [9] studied the first hitting place and obtained the joint distribution of the first hitting time and the first hitting place in $k = 4$ by using Spitzer's equality. His remarkable result is that there exists a linear combination of the distribution of the first hitting place at zero and its differentiation. He explained them as "monopoles" and "dipoles." Sato [11] described Nishioka's explanation as a "random walk."

Nishioka developed his method for when k is even and $k \geq 4$, and Lachel [5], [6] studied it in more detail.

However, when k is odd and $k \geq 3$, there were few results because of its asymmetry. Some authors studied the case when $k = 3$. Orsinger [10] obtained the distribution of the sojourn time in $(0,\infty)$. Nishioka obtained the distribution of the sojourn time in $(0,\infty)$ by using Spitzer's identity and he studied the case

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where $k \geq 3$ and k is odd. Shimoyama [12] studied the first hitting place and obtained the joint distribution of the first hitting time and the first hitting place.

In this paper, we obtained the joint distribution of the first hitting time and the first hitting place by using the method of Sato [11] when $k = 3$. In $k = 3$, we must consider the following by its asymmetry for $A > 0$:

$$\frac{\partial u}{\partial t}(t,x) = -A\frac{\partial^3 u}{\partial x^3}(t,x) \tag{1.1}$$

and

$$\frac{\partial v}{\partial t}(t,x) = A\frac{\partial^3 v}{\partial x^3}(t,x). \tag{1.2}$$

But letting $v(t,x) = u(t,-x)$, we have

$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t,-x)$$

and

$$A\frac{\partial^3}{\partial x^3}v(t,x) = -A\frac{\partial^3 u}{\partial x^3}(t,-x).$$

Thus it suffices to consider (1.1). That is, it is equivalent to considering the behavior of the random walk associated with (1.1) starting at $x < 0$ and considering the behavior of the random walk associated with (1.2) starting at $x > 0$.

The paper is organized as follows.

In Section 2, we define a random walk which has the only finite jumps with signed measure, and its total variation is not one.

In Section 3, we prove that the scaling limit of the random walk which is defined in Section 2 is the fundamental solution of $\partial_t u = -A\partial_x^3 u$.

In Section 4, we study the hitting measure of the joint distribution of the first hitting time and the first hitting place of each of the following cases.

(1) The random walk starts at $x > 0$, and it first hits $\{(t,x) : x < 0\}$.
(2) The random walk starts at $x < 0$, and it first hits $\{(t,x) : x > 0\}$.
(3) The random walk starts at $x > 0$, and it first hits $\{(t,x) : x < 0 \text{ or } x > a\}$.

In Section 5, we study the scaling limit of the hitting measures which are computed in Section 4 and its density in cases (1) and (2). These results correspond to the results of Shimoyama [12].

2. Definition of a pseudoprocess driven by $\partial_t = -A\partial_x^3$

Let $\{Y_i : i = 1, 2, \ldots\}$ be the independent and identically distributed random variables with *signed* distribution defined by

$$P(Y_i = 0) = p, \qquad P(Y_i = 1) = q, \qquad P(Y_i = 2) = r, \qquad P(Y_i = -1) = s,$$

and consider the random walk

$$X_n = X_0 + Y_1 + Y_2 + \cdots + Y_n.$$

Let f be a measurable function, and we shall write

$$E\big[f(X_n)\big] = \sum_k f(k)P[X_n = k]$$

and

$$E_x\big[f(X_n)\big] = E\big[f(X_n) \mid X_0 = x\big].$$

Also, we shall write

$$P_x[X_n = k] = P[X_n = k \mid X_0 = x].$$

By the Taylor expansion, we have

$$\begin{aligned} E_x\big[f(X_1)\big] - f(x) &= (p+q+s+r-1)f(x) + (q-s+2r)f'(x) \\ &\quad + \frac{1}{2}(q+s+4r)f''(x) + \frac{1}{6}(q-s+8r)f'''(x) + \cdots \end{aligned}$$

if we set

$$p+s+q+r-1=0, \qquad q-s+2r=0, \qquad q+s+4r=0;$$

that is,

$$p = 1+3r, \qquad q=-3r, \qquad s=-r,$$

and then

$$E_x\big[f(X_1)\big] - f(x) \simeq r\frac{d^3}{dx^3}f(x).$$

We assume that the number r is negative. Now we compute the characteristic function of Y_i:

$$\begin{aligned} M(\mu) &= E[e^{i\mu Y_i}] \\ &= p + qe^{i\mu} + se^{-i\mu} + re^{2i\mu} \\ &= 1 + 3r - 3re^{i\mu} - re^{-i\mu} + re^{2i\mu}. \end{aligned}$$

Since

$$E_x[e^{i\mu X_n}] = e^{i\mu x}M(\mu)^n,$$

we take any r in $[-1/4, 0)$ for the convergence of this quantity as n goes to infinity.

Avoiding useless confusion, we set $r = -A$ for A in $(0, 1/4]$. We note that

$$\begin{aligned} M(\mu) &= 1 - 3A + 3Ae^{i\mu} + Ae^{-i\mu} - Ae^{2i\mu} \\ &= 1 - Ae^{-i\mu}(e^{i\mu}-1)^3 \\ &= \big\{1 - 2A(1-\cos\mu)^2\big\} + i\big\{2A\sin\mu(1-\cos\mu)^2\big\}. \end{aligned} \tag{2.1}$$

Let $\{\mathcal{F}_n\}$ be the filtration generated by (X_n). Since

$$p + q + s + |r| = 1 + 2A,$$

the total variation of $\mathcal{F}_n$ is less than or equal to $(1+2A)^n$. Since $\mathcal{F}_n$ is essentially a finite set, the mean $E_x[\cdot]$ is defined for any event of $\mathcal{F}_n$. Set

$$p(n,k) = P_0[X_n = k].$$

Then we have

$$\sum_k p(n,k)e^{ik\mu} = M(\mu)^n.$$

Thus $p(n,k)$ is the Fourier coefficient of the right-hand function. Therefore

$$p(n,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} M(\mu)^n e^{-ik\mu}\,d\mu, \tag{2.2}$$

and we have

$$\lim_{n\to\infty} p(n,k) = 0.$$

REMARK 2.1

If $A = 1/4$, then we have

$$p = \frac{1}{4}, \qquad q = \frac{3}{4}, \qquad s = \frac{1}{4}, \qquad r = -\frac{1}{4},$$

and

$$M(\mu) = \frac{1}{4} + \frac{3}{4}e^{i\mu} + \frac{1}{4}e^{-i\mu} - \frac{1}{4}e^{2i\mu}.$$

3. Scaling limit to continuous time and space

The following theorem is the main result in this section.

THEOREM 3.1

By the scaling $x = k\epsilon$ and $t = n\epsilon^3$ for $\epsilon > 0$, we have

$$q(t,x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}p(n,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iAy^3t - iyx}\,dy,$$

which is the fundamental solution of (1.1).

Proof

Let $\delta(x)$ be Dirac's delta function, and we clearly have

$$q(0,x) = \delta(x).$$

We consider

$$\frac{1}{\epsilon}p(n,k) = \frac{1}{2\pi\epsilon}\int_{-\pi}^{\pi} M(\mu)^{t/\epsilon^3} e^{-ix\mu/\epsilon}\,d\mu = \frac{1}{2\pi}\int_{-\pi/\epsilon}^{\pi/\epsilon} M(\epsilon y)^{t/\epsilon^3} e^{-ixy}\,dy$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\Big(\Big(\int_{\pi/\epsilon-\pi/\epsilon^{1/3}}^{\pi/\epsilon} + \int_{-\pi/\epsilon}^{-(\pi/\epsilon-\pi/\epsilon^{1/3})}\Big) \\
&\quad + \Big(\int_{\pi/\epsilon^{1/3}}^{\pi/\epsilon-\pi/\epsilon^{1/3}} + \int_{-(\pi/\epsilon-\pi/\epsilon^{1/3})}^{-\pi/\epsilon^{1/3}}\Big) + \Big(\int_{-\pi/\epsilon^{1/3}}^{\pi/\epsilon^{1/3}}\Big)\Big) M(\epsilon y)^{t/\epsilon^3} e^{-ixy}\, dy, \\
&= I_1 + I_2 + I_3, \quad \text{say.}
\end{aligned}
$$

By (2.1),

$$
\begin{aligned}
\big|M(\mu)\big|^2 &= \big\{1-2A(1-\cos\mu)^2\big\}^2 + \big\{2A\sin\mu(1-\cos\mu)\big\}^2 \\
&= 1-4A+8A^2+8A(1-3A)\cos\mu - 8A(1-6A)\cos^2\mu - 8A^2\cos^3\mu.
\end{aligned}
$$

We note that $|M(\mu)|$ is an even function.

First, we consider I_1. For $A<1/4$ and sufficiently small ϵ, we have

$$
\begin{aligned}
|I_1| &\le \frac{1}{\pi}\int_{\pi/\epsilon-\pi/\epsilon^{1/3}}^{\pi/\epsilon} \big|M(\epsilon y)\big|^{t/\epsilon^3}\, dy \\
&= \frac{1}{\pi}\int_0^{\pi/\epsilon^{1/3}} \big|M(\pi-\epsilon y)\big|^{t/\epsilon^3}\, dy \\
&= \frac{1}{\pi}\int_0^{\pi/\epsilon^{1/3}} \big|1-4A+8A^2-8A(1-3A)\cos\epsilon y \\
&\quad - 4A(1-6A)\cos^2\epsilon y + 8A^2\cos^3\epsilon y\big|^{t/\epsilon^3}\, dy \\
&\approx \frac{1}{\pi}\int_0^{\pi/\epsilon^{1/3}} \Big|1-4A+8A^2-8A(1-3A)\Big(1-\frac{1}{2}\epsilon^2y^2\Big) \\
&\quad - 4A(1-6A)\Big(1-\frac{1}{2}\epsilon^2y^2\Big)^2 + 8A^2\Big(1-\frac{1}{2}\epsilon^2y^2\Big)^3\Big|^{t/\epsilon^3} \\
&= \frac{1}{\pi}\int_0^{\pi/\epsilon^{1/3}} \big|(1-8A)^2 + 8A(1-6A)\epsilon^2y^2 - A(1-12A)\epsilon^4y^4 - A^2\epsilon^6y^6\big|^{t/\epsilon^3} \\
&\to 0 \quad \text{as } \epsilon\to 0.
\end{aligned}
$$

Second we estimate I_2:

$$
|I_2| \le \frac{1}{\pi}\int_{\pi/\epsilon^{1/3}}^{\pi/\epsilon-\pi/\epsilon^{1/3}} \big|M(\epsilon y)\big|^{t/\epsilon^3}\, dy = \frac{1}{\pi\epsilon}\int_{\epsilon^{2/3}\pi}^{\pi-\epsilon^{2/3}\pi} \big|M(u)\big|^{t/\epsilon^3}\, du.
$$

Since in $u\in[0,\pi]$, $M(u)$ takes maximum at $u=0$ and minimum at u which we take as $\cos u = -1+1/3A$, we have

$$
\begin{aligned}
|I_2| &\le \frac{1}{\epsilon}\max\Big(\big|M(\pi-\epsilon^{2/3}\pi)\big|^{t/\epsilon^3}, \big|M(\epsilon^{2/3}\pi)\big|^{t/\epsilon^3}\Big) \\
&\approx \frac{1}{\epsilon}\max\Big((1-8A)^{t/2\epsilon^3}, (1-A\epsilon^{8/3}\pi^4)^{t/2\epsilon^3}\Big) \\
&\to 0 \quad \text{as } \epsilon\to 0.
\end{aligned}
$$

Finally, in the interval of the integral I_3, we have

$$M(\epsilon y) \approx 1 + iAy^3\epsilon^3.$$

Thus the proof is complete. □

Next we have the following, similar to [11].

THEOREM 3.2

Let f be continuous in $L^2(\mathbf{R})$. Suppose that f is the Fourier transform of a function $\hat{f}$ in $L^2(\mathbf{R})$; that is,

$$f(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{f}(\lambda)\, d\lambda.$$

Set

$$u_\epsilon(t,x) = E_{[\frac{x}{\epsilon}]}\big[f(\epsilon X_{[\frac{t}{\epsilon^3}]})\big],$$

where $[\cdot]$ denotes its integer part. Then there exists the limit

$$u(t,x) = \lim_{\epsilon \to 0} u_\epsilon(t,x)$$

which satisfies (1.1) and

$$u(0,x) = f(x).$$

4. Hitting measure

For each $|s| < 1$, the Green operator of $\{X_n\}_{n \geq 0}$ is defined by

$$G_s f(l) = \sum_{n=0}^{\infty} s^n E_l\big[f(X_n)\big].$$

By Parseval's equality, we have

$$\sum_k p(n,k)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} M(\mu)^{2n}\, d\mu \equiv K_n,$$

where K_n is a positive constant smaller than 1 and tends to zero as n goes to infinity since $|M(\mu)| < 1$ except for $\mu = 0$. Thus

$$\big|E_l\big[f(X_n)\big]\big| = \Big|\sum_k p(n,k-l) f(k)\Big| \leq K_n \|f\|_{l^2}.$$

We obtain the following.

PROPOSITION 4.1

For every f of l^2, $G_s f$ is analytic in $|s| < 1$, and we have

$$\big|G_s f(l)\big| \leq \frac{1}{|1-s|} \|f\|_{l^2}.$$

For l and k in $\mathbf{Z}$, we define

$$g_s(l,k) = G_s 1_k(l) = \sum_{n=0}^{\infty} s^n P_l(X_n = k) = \sum_{n=0}^{\infty} s^n p(n, k-l).$$

For fixed $|s| < 1$, we have the following:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-in\mu}}{1 - sM(\mu)}\, d\mu,$$

by (2.2). We note that $g_s(l,k) = c_{k-l}$.

PROPOSITION 4.2

Let $M_1(z) = (z-1)^3 + \frac{1}{A}(\frac{1}{s} - 1)z$, and let $M_2(z) = -(z-1)^3 + \frac{1}{A}(\frac{1}{s} - 1)z^2$. We assume that $|s| < 1$. We have

$$c_n = \begin{cases} \frac{1}{As}\frac{\xi^{-n}}{M_1'(\xi)} & (n \le 0), \\ \frac{1}{As}\left(\frac{\eta_1^{n+1}}{M_2'(\eta_1)} + \frac{\eta_2^{n+1}}{M_2'(\eta_2)}\right), & (n > 0), \end{cases}$$

where a real number ξ is a solution of $M_1(z) = 0$ and complex numbers η_1 and η_2 are conjugate solutions of $M_2(z) = 0$.

Proof

First, we consider that $n \le 0$. We set $z = e^{i\mu}$ and change a variable; then the residue theorem implies

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-in\mu}}{1 - s(1 - Ae^{-i\mu}(e^{i\mu} - 1)^3)}\, d\mu \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{z} \frac{z^{-n}}{1 - s(1 - A\frac{1}{z}(z-1)^3)} \\ &= \frac{1}{2\pi i As}\int_{\mathcal{C}} dz \frac{z^{-n}}{(z-1)^3 + \frac{1}{A}(\frac{1}{s} - 1)z} \\ &= \frac{1}{As}\sum_{|z|<1} \operatorname{Res}\left(\frac{z^{-n}}{(z-1)^3 + \frac{1}{A}(\frac{1}{s} - 1)z}\right), \end{aligned}$$

where $\mathcal{C}$ is the unit circle in the complex plain. We denote by $M_1(z)$ the denominator in the equation above; that is,

$$M_1(z) = (z-1)^3 + \frac{1}{A}\left(\frac{1}{s} - 1\right)z.$$

Second, we consider that $n > 0$. We set $z = e^{-i\mu}$ and in a similar fashion to the case of $n \le 0$ we have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-in\mu}}{1 - s(1 - Ae^{-i\mu}(e^{i\mu} - 1)^3)}\, d\mu \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-in\mu}}{1 - s(1 - Ae^{2i\mu}(1 - e^{-i\mu})^3)}\, d\mu \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{z} \frac{z^n}{1 - s(1 - A\frac{1}{z^2}(1 - z)^3)} \end{aligned}$$

$$= \frac{1}{2\pi i A s} \int_{\mathcal{C}} dz \frac{z^{n+1}}{-(z-1)^3 + \frac{1}{A}(\frac{1}{s}-1)z^2}$$
$$= \frac{1}{As} \sum_{|z|<1} \mathrm{Res}\Big(\frac{z^{n+1}}{-(z-1)^3 + \frac{1}{A}(\frac{1}{s}-1)z^2} \Big),$$

where $\mathcal{C}$ is the unit circle in the complex plain. We denote by $M_2(z)$ the denominator in the equation above; that is,

$$M_2(z) = -(z-1)^3 + \frac{1}{A}\Big(\frac{1}{s}-1\Big)z^2.$$

We consider the zeros of $M_1(z)$ and $M_2(z)$. We note that

$$M_1(z) = 0 \iff M_2(1/z) = 0.$$

So we have to consider the zeros of $M_1(z)$. By $\frac{1}{A}(\frac{1}{s}-1) > 0$, ξ is the one of solutions of $M_1(z) = 0$ which is a real number, and $0 < \xi < 1$ and η_1' and η_2' are the others of solutions of $M_1(z) = 0$ which are complex numbers and $|\eta_i'| > 1$ $(i = 1, 2)$. Taking $\eta_i = 1/\eta_i'$ $(i = 1, 2)$, we get the rest of the claims. □

The behavior of X_n varies at the boundary of an interval according to the starting point; we will study three cases. The first case is the hitting measure to the boundary of $(-\infty, 0)$ when X_n starts at a positive point. The next case is the hitting measure to the boundary of $(0, \infty)$ when X_n starts at a negative point. The last case is the hitting measure to the boundary of $(0, L)$ when X_n starts at l where $0 < l < L$.

4.1. The hitting measure to the boundary of $(-\infty, 0)$ when X_n starts at a positive point

In this section, we assume that X_n starts at l, which is a positive integer.

We define the first hitting time to the set $\mathbf{Z} \cap (-\infty, 0)$ as

$$\sigma^- = \min\{n : X_n < 0\},$$

and we shall write

$$\tilde{p}(n, l, k) = P_l[X_n = k; n < \sigma^-].$$

Since $\mathcal{F}_n$ is essentially finite, note that the quantity is well defined. Then we have

$$\tilde{p}(n, l, k) = P_l[X_n = k] - P_l[X_n = k; \sigma^- \le n]$$
$$= P_l[X_n = k] - \sum_{m=0}^{n} P_l[\sigma^- = m, X_{\sigma^-} = -1] P_{-1}[X_{n-m} = k],$$

which is a strong Markov property. For l and k in $\mathbf{Z}$, define

$$\tilde{g}_s(l, k) = \sum_{n=0}^{\infty} s^n \tilde{p}(n, l, k),$$
$$H_i(s, l) = \sum_{n=0}^{\infty} s^n P_l[\sigma^- = n, X_{\sigma^-} = i].$$

Since the total variation of $\mathcal{F}_n \le (1+2A)^n$, $\tilde{g}_s(l,k)$, and $H_i(s,l)$ are analytic in $|s| < (1+2A)^{-1}$, then we get

$$\tilde{g}_s(l,k) = g_s(l,k) - H_{-1}(s,l)g_s(-1,k).$$

By $l > 0$, we have

$$\tilde{g}_s(l,k) = c_{k-l} - H_{-1}(s,l)c_{k+1}.$$

For $k = -1$, we have

$$0 = c_{-l-1} - c_0 H_{-1}(s,l).$$

By Proposition 4.2, we obtain the next proposition.

PROPOSITION 4.3

For $|s| < 1$ and $l \in \mathbf{Z} \cap (0,\infty)$,

$$H_{-1}(s,l) = \xi^{l+1},$$

where ξ is defined in Proposition 4.2; that is, it is the real-number solution of $(z-1)^3 + \frac{1}{A}(\frac{1}{s} - 1)z = 0$. Moreover $H_{-1}(s,l)$ is analytic in $|s| < 1$.

REMARK 4.4

For the real-number solution of $(z-1)^3 + \frac{1}{A}(\frac{1}{s} - 1)z = 0$ goes to 1 when s goes to 1,

$$\lim_{s\to 1} H_{-1}(s,l) = 1,$$

which is the interpretation of the hitting measure

$$\sum_{m=0}^{\infty} E_l[\sigma^- = m, X_{\sigma^-} = i] = E_l[X_{\sigma^-} = i, \sigma^- < \infty].$$

We note that the latter quantity may not be convergent, because all terms are signed.

4.2. The hitting measure to the boundary of $(0,\infty)$ when X_n starts at a negative point

In this section, we assume that X_n starts at $-l$, which is a negative integer.

We note that the following quantities are almost well defined by the same reasons given in Section 4.1.

We define the first hitting time to the set $\mathbf{Z} \cap (0,\infty)$ as

$$\sigma^+ = \min\{n : X_n > 0\},$$

and we shall write

$$\tilde{p}(n,-l,k) = P_{-l}[X_n = k; n < \sigma^+].$$

Then we have

$$\begin{aligned}\tilde{p}(n,-l,k) &= P_{-l}[X_n = k] - P_{-l}[X_n = k; \sigma^+ \le n] \\ &= P_{-l}[X_n = k] - \sum_{i=1,2}\sum_{m=0}^{n} P_{-l}[\sigma^+ = m, X_{\sigma^+} = i]P_i[X_{n-m} = k].\end{aligned}$$

For l and k in $\mathbf{Z}$, define

$$\tilde{g}_s(-l,k) = \sum_{n=0}^{\infty} s^n \tilde{p}(n,-l,k),$$

$$H_i(s,-l) = \sum_{n=0}^{\infty} s^n P_{-l}[\sigma^+ = n, X_{\sigma^+} = i].$$

We get

$$\begin{aligned}\tilde{g}_s(-l,k) &= g_s(-l,k) - \sum_{i=1,2} H_i(s,-l) g_s(i,k) \\ &= c_{k+l} - \sum_{i=1,2} H_i(s,-l) c_{k-i}.\end{aligned}$$

Especially, when $k=1$ and 2, we have

$$0 = c_{l+1} - c_0 H_1(s,-l) - c_{-1} H_2(s,-l), \tag{4.1}$$

$$0 = c_{l+2} - c_1 H_1(s,-l) - c_0 H_2(s,-l). \tag{4.2}$$

Thus we obtain the next proposition.

PROPOSITION 4.5

For $|s|<1$ ($s \in \mathbf{R}$) and $l \in \mathbf{Z} \cap (0,\infty)$,

$$H_1(s,-l) = \frac{c_0 c_{l+1} - c_{-1} c_{l+2}}{c_0^2 - c_1 c_{-1}},$$

$$H_2(s,-l) = \frac{c_0 c_{l+2} - c_1 c_{l+1}}{c_0^2 - c_1 c_{-1}}.$$

Moreover $H_i(s,-l)$ ($i=1,2$) are analytic in $|s|<1$ ($s \in \mathbf{R}$).

Proof

To show the proposition, it suffices to confirm that the denominator $c_0^2 - c_1 c_{-1} \neq 0$ in $|s|<1$ ($s \in \mathbf{R}$).

By $s \in \mathbf{R}$, we note that $\Re\{M(\mu)\}$ is an even function and $\Im\{M(\mu)\}$ is an odd function. So $|1 - sM(\mu)|^2$ is an even function with respect to μ.

For $s=0$, we have

$$c_n = \begin{cases} 1, & n=0, \\ 0, & n \neq 0. \end{cases}$$

So $c_0^2 - c_1 c_{-1} = 1$. Thus we consider $c_0^2 - c_1 c_{-1}$ in the case $s \neq 0$.

Then we have to prove the claim in the case when $s \in \mathbf{R}, s \neq 0$, and $|s|<1$. We will divide this proof into some steps.

Step 1. We shall prove $\Re\{c_0\} > 0$ for $|s|<1$ ($s \in \mathcal{C}$).

We set $s = \rho e^{i\theta}$ and $M(\mu) = |M(\mu)| e^{i\lambda(\mu)}$, where $0<\rho<1$, $\pi<\theta<-\pi$, and $\lambda(\mu)$ is the argument of $|M(\mu)|$:

$$\begin{aligned}\Re\{c_0\} &= \Re\Big\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1-sM(\mu)}\,d\mu\Big\}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\Re\{1-\bar{s}\bar{M}(\mu)\}}{|1-sM(\mu)|^2}\,d\mu\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\rho|M(\mu)|\cos(\theta+\lambda(\mu))}{|1-sM(\mu)|^2}\,d\mu.\end{aligned}$$

Since $|M(\mu)|<1$, we have $1-\rho|M(\mu)|\cos(\theta+\lambda(\mu))>0$. Thus $\Re\{c_0\}>0$.

In the following steps we assume that $s\in\mathbf{R}$; then we note that c_n is a real number.

Step 2. We shall prove $c_0+c_{-1}>0$ for $0<|s|<1$.

By

$$\begin{aligned}c_0+c_{-1} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+e^{i\mu}}{1-sM(\mu)}\,d\mu\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1+e^{i\mu})(1-s\bar{M}(\mu))}{|1-sM(\mu)|^2}\,d\mu,\end{aligned}$$

it suffices to consider the real part of a numerator of the integrand:

$$\begin{aligned}&\Re\big\{(1+e^{i\mu})\big(1-s\bar{M}(\mu)\big)\big\}\\ &\quad= (1+\cos\mu)\big(1-s\Re\big\{M(\mu)\big\}\big)-s\sin\mu\,\Im\big\{M(\mu)\big\}\\ &\quad= (1+\cos\mu)\big(1-s+2As(1-\cos\mu)^2\big)-2As\sin^2\mu(1-\cos\mu)\\ &\quad= (1+\cos\mu)(1-s).\end{aligned}$$

Then it is positive for $0<|s|<1$.

Step 3. We shall prove $c_0-c_{-1}>0$ for $0<|s|<1$.

By

$$c_0-c_{-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-e^{i\mu})(1-s\bar{M}(\mu))}{|1-sM(\mu)|^2}\,d\mu,$$

it suffices to consider the real part of a numerator of the integrand,

$$\begin{aligned}&\Re\big\{(1-e^{i\mu})\big(1-s\bar{M}(\mu)\big)\big\}\\ &\quad= (1-\cos\mu)\big(1-s+2As(1-\cos\mu)^2\big)+2As\sin^2\mu(1-\cos\mu)\\ &\quad= (1-\cos\mu)\big(1-s+4As(1-\cos\mu)\big).\end{aligned}$$

For $1>s>0$, it is clear that $c_0-c_{-1}>0$.

Now we consider $-1<s<0$. By $0<A\le 1/4$,

$$\begin{aligned}&(1-\cos\mu)\big(1-s+4As(1-\cos\mu)\big)\\ &\quad\ge (1-\cos\mu)(1+s\cos\mu)\\ &\quad> 0.\end{aligned}$$

Then it is positive for $0<|s|<1$.

Step 4. We shall prove $c_{-1} - c_1 > 0$ if $-1 < s < 0$ and $c_{-1} - c_1 < 0$ if $1 > s > 0$.
By

$$c_{-1} - c_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(e^{i\mu} - e^{-i\mu})(1 - s\bar{M}(\mu))}{|1 - sM(\mu)|^2}\, d\mu,$$

it suffices to consider the real part of a numerator of the integrand,

$$\begin{aligned} &\Re\big\{(e^{i\mu} - e^{-i\mu})\big(1 - s\bar{M}(\mu)\big)\big\} \\ &= -2s \sin^2 \mu \Im\big(M(\mu)\big) \\ &= -4As \sin^2 \mu (1 - \cos \mu). \end{aligned}$$

Then we get the claim.

Step 5. We shall prove $c_{-1} < 0$ if $1 > s > 0$.
By

$$\begin{aligned} c_{-1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\mu}(1 - s\bar{M}(\mu))}{|1 - sM(\mu)|^2}\, d\mu \\ &= \frac{2}{2\pi} \int_{0}^{\pi} \frac{\Re[e^{i\mu}(1 - s\bar{M}(\mu))]}{|1 - sM(\mu)|^2}\, d\mu, \end{aligned}$$

it suffices to consider the real part of a numerator of the integrand:

$$\Re\big\{e^{i\mu}\big(1 - s\bar{M}(\mu)\big)\big\} = -2As \cos^2 \mu + (1 - s + 4As) \cos \mu - 2As.$$

Setting $t = \cos \mu$, we denote by $F(t)$ the above equation, or

$$F(t) = -2Ast^2 \mu + (1 - s + 4As)t - 2As.$$

We denote by α and β ($|\alpha| < |\beta|$) solutions of the equation $F(t) = 0$.

By the discriminant of $F(t) = 0$, α and β are real numbers. Moreover by Vièta's formula, $0 < \alpha < 1 < \beta$.

Meanwhile, we have $F(1) > 0$, $F(-1) < 0$, and $F(1) < |F(-1)|$. So $|(-1, \alpha)| > |(\alpha, 1)|$, and $F(t)$ is monotone function in $(-1, 1)$.

Thus we get $c_{-1} < 0$ if $1 > s > 0$.

Step 6. We shall prove $c_{-1} > 0$ if $-1 < s < 0$.

We use the notation of step 5.

Then the discriminant of $F(t) = 0$ is $(1 - s)(1 - (1 - 8A)s)$. In the case when the discriminant of $F(t) = 0$ is negative, that is, $(1 - (1 - 8A)s) < 0$, it is clear that $c_{-1} > 0$ if $s < 0$ by $F(t) > 0$ for $|t| < 1$.

We shall consider the case when the discriminant of $F(t) = 0$ is nonnegative; that is, $(1 - (1 - 8A)s) \geq 0$.

By the relation of solutions and coefficients about $F(t) = 0$, $\beta < -1 < \alpha < 0$.

Meanwhile, we have $F(1) > 0$, $F(-1) < 0$, and $F(1) > |F(-1)|$. So $|(-1, \alpha)| < |(\alpha, 1)|$ and $F(t)$ is a monotone function in $(-1, 1)$.

Thus we get $c_{-1} > 0$ if $-1 < s < 0$.

Pulling together the above arguments, we can get the claim of the proposition as follows.

For $s = 0$, $c_0^2 - c_1 c_{-1} = 1 > 0$.

For $1 > s > 0$, by step 4 we have $c_{-1} < c_1$, and then by step 5, $-c_{-1}^2 < -c_1 c_{-1}$. So by steps 2 and 3,

$$\begin{aligned} c_0^2 - c_1 c_{-1} &> c_0^2 - c_{-1}^2 \\ &= (c_0 - c_{-1})(c_0 + c_{-1}) \\ &> 0. \end{aligned}$$

For $-1 < s < 0$, by step 4 we have $c_{-1} > c_1$, and then by step 6, $c_{-1}^2 > c_1 c_{-1}$. Now by step 3, $c_0 > c_{-1}$, and then by step 1,

$$\begin{aligned} c_0^2 &> c_0 c_{-1} \\ &> c_{-1}^2 > c_1 c_{-1}. \end{aligned}$$

Therefore, $c_0^2 - c_1 c_{-1} > 0$. □

Next we will compute

$$\lim_{s\to 1} H_i(s, -l), \quad i = 1, 2.$$

At first, we compute c_n. By Proposition 4.2 and its proof, we consider sufficient zero points of $M_1(z)$. We set

$$v^3 = \frac{1}{A}\Big(\frac{1}{s} - 1\Big) \tag{4.3}$$

and

$$z = 1 - vz'.$$

Then we have

$$M_1(1 - vz') = -v^3 z'^3 + v^3(1 - vz') = 0.$$

By Cardano's formula, we obtain that

$$\begin{aligned} z_1' &= x + y, \\ z_2' &= \omega x + \omega^2 y, \end{aligned}$$

and

$$z_3' = \omega^2 x + \omega y$$

are zero points of $M_1(1 - vz')$, where

$$x = \sqrt[3]{\frac{1 + \sqrt{1 + \frac{4}{27}v^3}}{2}},$$

$$y = \sqrt[3]{\frac{1 - \sqrt{1 + \frac{4}{27}v^3}}{2}},$$

and ω is an imaginary cubic root. Thus we set

$$\xi = 1 - vz_1',$$
$$\eta_1 = \frac{1}{1 - vz_2'},$$

and

$$\eta_2 = \frac{1}{1 - vz_3'}.$$

We note that ξ is a zero of $M_1(z)$, which satisfies $|\xi| < 1$, and η_1 and η_2 are zeros of $M_2(z)$, which satisfy $|\eta_i| < 1$ $(i = 1, 2)$.

By Taylor expansion, we have the following:

$$\xi = 1 - v + \frac{1}{3}v^2 - \frac{1}{81}v^4 + O(v^5),$$
$$\eta_1 = 1 + \frac{-1 + i\sqrt{3}}{2}v + \frac{-1 - i\sqrt{3}}{3}v^2 + \frac{1}{3}v^3 + \frac{5}{81}(-1 + i\sqrt{3})v^4 + O(v^5),$$
$$\eta_2 = 1 + \frac{-1 - i\sqrt{3}}{2}v + \frac{-1 + i\sqrt{3}}{3}v^2 + \frac{1}{3}v^3 + \frac{5}{81}(-1 - i\sqrt{3})v^4 + O(v^5).$$

So, by Proposition 4.2 we get

$$c_n = \begin{cases} \frac{1}{3As}\frac{1}{v^2} + \frac{3n+1}{9As}\frac{1}{v} - \frac{n^2+n}{6As} + \frac{27n^3+54n^2+9n-10}{486As}v \\ \quad + \frac{81n^4+270n^3+135n^2-150n-56}{5832As}v^2 + O(v^3) & (n \leq 0), \\ \frac{1}{3As}\frac{1}{v^2} + \frac{3n+1}{9As}\frac{1}{v} - \frac{n^2+n}{3As} + \frac{27n^3+54n^2+9n-10}{486As}v \\ \quad + \frac{81n^4+270n^3+135n^2-150n-56}{5832As}v^2 + O(v^3) & (n > 0). \end{cases} \tag{4.4}$$

From Proposition 4.5, we get

$$H_1(s, -l) = (2 + l) - (l^2 + 3l + 2)\frac{1}{2}v + O(v^2),$$
$$H_2(s, -l) = (-1 - l) + (l^2 + 3l + 2)\frac{1}{2}v + O(v^2).$$

If s goes to one, then v goes to zero. Therefore we obtain the next proposition.

PROPOSITION 4.6

We have

$$\lim_{s \to 1} H_1(s, -l) = 2 + l,$$
$$\lim_{s \to 1} H_2(s, -l) = -1 - l.$$

4.3. The hitting measure to the boundary of $(0, L)$ when X_n starts at l $(0 < l < L)$

In this section, we assume that X_n starts at l, which is integer and $0 < l < L$.

We note that the following quantities are almost well defined for the same reasons given in Sections 4.2 and 4.3.

We define the first hitting time to the set $\mathbf{Z} \cap (0, L)$ as

$$\sigma = \min\{n : X_n < 0 \text{ or } X_n > L\},$$

and we shall write

$$\tilde{p}(n, l, k) = P_l[X_n = k; n < \sigma].$$

Then we have

$$\begin{aligned}\tilde{p}(n, l, k) &= P_l[X_n = k] - P_{-l}[X_n = k; \sigma \le n] \\ &= P_l[X_n = k] - \sum_{i=-1,L+1,L+2} \sum_{m=0}^{n} P_l[\sigma = m, X_\sigma = i] P_i[X_{n-m} = k].\end{aligned}$$

For l and k in $\mathbf{Z}$, define

$$\tilde{g}_s(l, k) = \sum_{n=0}^{\infty} s^n \tilde{p}(n, l, k),$$

$$H_i(s, l) = \sum_{n=0}^{\infty} s^n P_l[\sigma = n, X_\sigma = i].$$

We get

$$\begin{aligned}\tilde{g}_s(l, k) &= g_s(l, k) - \sum_{i=-1,L+1,L+2} H_i(s, l) g_s(i, k) \\ &= c_{k-l} - \sum_{i=-1,L+1,L+2} H_i(s, l) c_{k-i}.\end{aligned}$$

Especially, when $k = -1$, $L + 1$ and $L + 2$, we have

$$\text{(4.5)} \qquad \begin{cases} 0 = c_{-l-1} - c_0 H_{-1}(s, l) - c_{-L-2} H_{L+1}(s, l) - c_{-L-3} H_{L+2}(s, l), \\ 0 = c_{L+1-l} - c_{L+2} H_{-1}(s, l) - c_0 H_{L+1}(s, l) - c_{-1} H_{L+2}(s, l), \\ 0 = c_{L+2-l} - c_{L+3} H_{-1}(s, l) - c_1 H_{L+1}(s, l) - c_0 H_{L+2}(s, l). \end{cases}$$

So we have to solve these equations about $H_{-1}(s, l)$, $H_{L+1}(s, l)$, and $H_{L+2}(s, l)$. But we are interested in the behavior of $H_{-1}(s, l)$, $H_{L+1}(s, l)$, and $H_{L+2}(s, l)$ when s goes to one. Then we consider that $s \approx 1$.

First we will show that the determinant

$$\begin{aligned}|\Lambda| &= \begin{vmatrix} c_0 & c_{-L-2} & c_{-L-3} \\ c_{L+1} & c_0 & c_{-1} \\ c_{L+3} & c_1 & c_0 \end{vmatrix} \\ &= c_0 \begin{vmatrix} c_0 & c_{-1} \\ c_1 & c_0 \end{vmatrix} - c_{-L-2} \begin{vmatrix} c_{L+1} & c_{-1} \\ c_{L+3} & c_0 \end{vmatrix} + c_{-L-3} \begin{vmatrix} c_{L+1} & c_0 \\ c_{L+3} & c_1 \end{vmatrix}\end{aligned}$$

is not zero.

Meanwhile, for

$$c_n = p(0, n) + p(1, n)s + p(2, n)s^2 + p(3, n)s^3 + O(s^4)$$

we have

$$c_{-L-2}\begin{vmatrix} c_{L+1} & c_{-1} \\ c_{L+3} & c_0 \end{vmatrix} = O(s^4)$$

and

$$c_{-L-3}\begin{vmatrix} c_{L+1} & c_0 \\ c_{L+3} & c_1 \end{vmatrix} = O(s^4).$$

By the proof of Proposition 4.5 we get that the determinant $|\Lambda|$ is not zero.

Second, we will compute the limit

$$\lim_{s\to 1} H_i(s,l)$$

for $i = -1, L+1, L+2$.

From (4.4) and (4.5), we have

$$H_{-1}(s,l) = \frac{2+l^2+3L+L^2-l(3+2L)}{6+5L+L^2} + O(v^2),$$
$$H_{L+1}(s,l) = \frac{2+l^2-l^2+L+lL}{2+L} + O(v^2),$$
$$H_{L+2}(s,l) = \frac{(1+l)(-1+l+L)}{3+L} + O(v^2),$$

where v is defined in (4.3). Thus we get the following.

PROPOSITION 4.7

We have

$$\lim_{s\to 1} H_{-1}(s,l) = \frac{L^2-(2l-3)L+(l-1)(l-2)}{(L+2)(L+3)},$$
$$\lim_{s\to 1} H_{L+1}(s,l) = \frac{(l+1)(L-l+2)}{L+2},$$
$$\lim_{s\to 1} H_{L+2}(s,l) = \frac{(l+1)(L+l-1)}{L+3}.$$

5. Scaling limit of the first hitting time and place

Let $s = e^{-\epsilon^3\lambda}$, and let $y = n\epsilon$. We have

$$v = \left(\frac{1}{A}(e^{\epsilon^3\lambda}-1)\right)^{1/3} \sim \epsilon\left(\frac{\lambda}{A}\right)^{1/3}.$$

For simplicity, we set

$$\nu = \left(\frac{\lambda}{A}\right)^{1/3}.$$

First, we consider the behavior of ξ, η_1, and η_2 in Proposition 4.2 when ϵ goes to zero. By Taylor expansion, we have

$$\log \xi = -v - \frac{1}{6}v^2 + \frac{5}{324}v^4 + O(v^5),$$
$$\log \eta_1 = \frac{(-1+i\sqrt{3})}{2}v + \frac{(-1-i\sqrt{3})}{12}v^2 + \frac{5(1-i\sqrt{3})}{648}v^4 + O(v^5),$$
$$\log \eta_2 = \frac{(-1-i\sqrt{3})}{2}v + \frac{(-1+i\sqrt{3})}{12}v^2 + \frac{5(1+i\sqrt{3})}{648}v^4 + O(v^5).$$

Then we have

$$\xi^n \sim \exp\Big(n\Big(-v - \frac{1}{6}v^2 + \frac{5}{324}v^4\Big)\Big),$$
$$\eta_1^n \sim \exp\Big(n\Big(-\frac{1}{2}v - \frac{1}{12}v^2 + \frac{5}{648}v^4\Big)\Big)$$
$$\times\Big(\cos\Big(n\frac{\sqrt{3}(-324v+54v^2+5v^4)}{648}\Big) - i\sin\Big(n\frac{\sqrt{3}(-324v+54v^2+5v^4)}{648}\Big)\Big),$$
$$\eta_2^n \sim \exp\Big(n\Big(-\frac{1}{2}v - \frac{1}{12}v^2 + \frac{5}{648}v^4\Big)\Big)$$
$$\times\Big(\cos\Big(n\frac{\sqrt{3}(-324v+54v^2+5v^4)}{648}\Big) + i\sin\Big(n\frac{\sqrt{3}(-324v+54v^2+5v^4)}{648}\Big)\Big).$$

Thus we get the next proposition.

PROPOSITION 5.1

We set $\nu = (\lambda/A)^{1/3}$. Let ϵ go to zero; then we have

$$\xi^{\frac{y}{\epsilon}} \to e^{-y\nu},$$
$$\eta_1^{\frac{y}{\epsilon}} \to e^{-\frac{y\nu}{2}}\Big(\cos\Big(\frac{\sqrt{3}}{2}y\nu\Big) + i\sin\Big(\frac{\sqrt{3}}{2}y\nu\Big)\Big),$$
$$\eta_2^{\frac{y}{\epsilon}} \to e^{-\frac{y\nu}{2}}\Big(\cos\Big(\frac{\sqrt{3}}{2}y\nu\Big) - i\sin\Big(\frac{\sqrt{3}}{2}y\nu\Big)\Big),$$

where ξ, η_1, and η_2 are defined in Proposition 4.2.

Let $f(x)$ be a differentiable function, and let $x>0$ in the following subsections.

5.1. The boundary of $(-\infty,0)$ when X_t starts at a positive point

Let $x>0$ and

$$\tau^- = \inf\{t : X_t < 0\}.$$

We have

$$\begin{aligned} E_x\big[e^{-\lambda\tau^-}f(X_{\tau^-})\big] &\sim E_l\big[e^{-\epsilon^3\lambda\sigma^-}f(\epsilon X_{\sigma^-})\big] \\ &= f(-\epsilon)E_l[e^{-\epsilon^3\lambda\sigma^-}; X_{\sigma^-} = -1] \\ &= f(-\epsilon)H_{-1}(e^{-\epsilon^3\lambda}, l). \end{aligned}$$

Then we obtain the next theorem by Propositions 4.3 and 5.1.

THEOREM 5.2

For every differentiable function f and $x > 0$, we have

$$E_x\big[e^{-\lambda\tau^-} f(X_{\tau^-})\big] = \exp\{-x(\lambda/A)^{1/3}\} f(0). \tag{5.1}$$

From the Laplace inversion formula, a direct calculation shows that

$$\mathcal{L}^{-1}\big(\exp\{-x(\lambda/A)^{1/3}\}\big) = \frac{3A}{\pi}\int_0^\infty a^2 \exp\Big\{-a^3 tA - \frac{x}{2}a\Big\} \sin\Big(\frac{\sqrt{3}}{2}xa\Big)\, da.$$

Then we get the density function of the first hitting time τ^- and the hitting place:

$$\begin{aligned} P_x\{\tau^- \in dt, X_{\tau^-} \in dy\} &= \rho(t,x)\, dt\delta_0(dy) \\ &= \frac{3A}{\pi}\int_0^\infty a^2 \exp\Big\{-a^3 tA - \frac{x}{2}a\Big\} \sin\Big(\frac{\sqrt{3}}{2}xa\Big)\, da\, dt\delta_0(dy). \end{aligned}$$

Moreover we integrate the above with respect to t from zero to t; we have the next theorem by Shimoyama [12].

THEOREM 5.3

We have

$$\begin{aligned} &P_x^-\big\{\tau^- \in [0,t], X_{\tau^-} \in dy\big\} \\ &\quad = 1 - \frac{3A}{\pi}\int_0^\infty \frac{1}{a} \exp\Big\{-a^3 tA - \frac{x}{2}a\Big\} \sin\Big(\frac{\sqrt{3}}{2}xa\Big)\, da\delta_0(dy). \end{aligned}$$

5.2. The boundary of $(0,\infty)$ when X_t starts at a negative point

Let

$$\tau^+ = \inf\{t : X_t > 0\}.$$

Consider

$$\begin{aligned} &E_{-x}\big[e^{-\lambda\tau^+} f(X_{\tau^+})\big] \\ &\quad \sim E_{-l}\big[e^{-\epsilon^3\lambda\sigma^+} f(\epsilon X_{\sigma^+})\big] \\ &\quad = f(\epsilon)E_{-l}[e^{-\epsilon^3\lambda\sigma^+}; X_{\sigma^+} = 1] + f(2\epsilon)E_{-l}[e^{-\epsilon^3\lambda\sigma^+}; X_{\sigma^+} = 2] \\ &\quad = f(\epsilon)H_1(e^{-\epsilon^3\lambda}, -l) + f(2\epsilon)H_2(e^{-\epsilon^3\lambda}, -l) \\ &\quad = \frac{1}{2}\big(f(\epsilon) + f(2\epsilon)\big)\big(H_1(e^{-\epsilon^3\lambda}, -l) + H_2(e^{-\epsilon^3\lambda}, -l)\big) \\ &\qquad + \frac{1}{2}\big(f(\epsilon) - f(2\epsilon)\big)\big(H_1(e^{-\epsilon^3\lambda}, -l) - H_2(e^{-\epsilon^3\lambda}, -l)\big). \end{aligned}$$

By Propositions 4.5 and 5.1 we get the next theorem.

THEOREM 5.4

For every differentiable function f and $x > 0$, we have

$$E_{-x}\big[e^{-\lambda\tau^+} f(X_{\tau^+})\big] = k(\lambda,x)f(0) - j(\lambda,x)f'(0),$$

where

$$k(\lambda, x) = \frac{2}{\sqrt{3}} e^{-\frac{x}{2}(\lambda/A)^{1/3}} \sin\Big(\frac{\sqrt{3}}{2}\Big(\frac{\lambda}{A}\Big)^{1/3} x + \frac{\pi}{3}\Big),$$

$$j(\lambda, x) = \frac{2}{\sqrt{3}} \Big(\frac{A}{\lambda}\Big)^{1/3} e^{-\frac{x}{2}(\lambda/A)^{1/3}} \sin\Big(\frac{\sqrt{3}}{2}\Big(\frac{\lambda}{A}\Big)^{1/3} x\Big).$$

From the Laplace inversion formula, we get the density function of the first hitting time τ^- and the hitting place

$$P_{-x}\{\tau^+ \in dt, X_{\tau^+} \in dy\} = \mu_k(t,x)\delta_0(dy) - \mu_j(t,x)\delta_0'(dy),$$

where

$$\mu_k(t,x) = \frac{\sqrt{3}}{2}\frac{A}{\pi}\int_0^\infty a^2 e^{-a^3 tA}\Big(e^{-xa} + 2e^{\frac{x}{2}a}\sin\Big(\frac{\sqrt{3}}{2}xa - \frac{\pi}{6}\Big)\Big)\,da$$

and

$$\mu_j(t,x) = \frac{\sqrt{3}}{2}\frac{A}{\pi}\int_0^\infty a e^{-a^3 tA}\Big(e^{-xa} - 2e^{\frac{x}{2}a}\cos\Big(\frac{\sqrt{3}}{2}xa + \frac{\pi}{3}\Big)\Big)\,da.$$

REMARK 5.5

By direct calculus, we get the relation of $\mu_k(t,x)$, $\mu_j(t,x)$, and $\rho(t,x)$ as follows:

$$\mu_k(t,x) + \frac{\partial}{\partial x}\mu_j(t,x) = \rho(t,-x).$$

Then we have

$$P_{-x}\{\tau^+ \in dt, X_{\tau^+} \in dy\} = \frac{\partial}{\partial z}\{-\mu_j(t,x+z)\delta_0(dy+z)\}\Big|_{z=0} dt + \rho(t,-x)\,dt\delta_0(dy).$$

Moreover, letting

$$P_{-x}\big\{\tau^+ \in [0,t], X_{\tau^+} \in dy\big\} = \mu_K(t,x)\delta_0(dy) - \mu_J(t,x)\delta_0'(dy),$$

we will compute $\mu_K(t,x)$ and $\mu_J(t,x)$.

First we consider $\mu_K(t,x)$. By the definition of $\mu_k(t,x)$, $\mathcal{L}(\mu_k(\cdot,x))(\lambda) = k(\lambda,x)$. Noting that $\mu_K(t,x) = \int_0^t \mu_k(u,x)\,du$, we have

$$\begin{aligned}\lim_{t\to\infty}\mu_K(t,x) &= \lim_{\lambda\to 0}\lambda\mathcal{L}\big(\mu_k(\cdot,x)\big)(\lambda)\\ &= \lim_{\lambda\to 0} k(\lambda,x) = 1\end{aligned}$$

by the final-value theorem of the theory of Laplace transforms. Thus we get

$$\begin{aligned}\mu_K(t,x) &= \int_0^\infty \mu_k(u,x)\,du - \int_t^\infty \mu_k(u,x)\,du\\ &= 1 - \int_t^\infty \mu_k(u,x)\,du\\ &= 1 - \frac{\sqrt{3}}{2}\frac{A}{\pi}\int_0^\infty \frac{e^{-a^3 tA}}{a}\Big(e^{-xa} + 2e^{-\frac{x}{2}a}\sin\Big(\frac{\sqrt{3}}{2}xa - \frac{\pi}{6}\Big)\Big)\,da.\end{aligned}$$

Next we consider $\mu_J(t,x)$. We set $\mathcal{L}(\mu_j(\cdot,x))(\lambda) = j(\lambda,x)$ and $\mu_J(t,x) = \int_0^t \mu_j(u,x)\,du$. Noting that

$$\begin{aligned}\lim_{t\to\infty}\mu_J(t,x) &= \lim_{\lambda\to 0}\lambda\mathcal{L}\big(\mu_j(\cdot,x)\big)(\lambda)\\ &= \lim_{\lambda\to 0} j(\lambda,x) = x\end{aligned}$$

we get

$$\begin{aligned}\mu_J(t,x) &= \int_0^\infty \mu_j(u,x)\,du - \int_t^\infty \mu_j(u,x)\,du\\ &= x - \int_t^\infty \mu_j(u,x)\,du\\ &= x - \frac{\sqrt{3}}{2}\frac{A}{\pi}\int_0^\infty \frac{e^{-a^3tA}}{a}\Big(e^{-xa} - 2e^{\frac{x}{2}a}\sin\Big(\frac{\sqrt{3}}{2}xa + \frac{\pi}{3}\Big)\Big)\,da\end{aligned}$$

in a way similar to that used above. We have the next theorem by Shimoyama [12].

THEOREM 5.6

We have

$$P_{-x}\big\{\tau^+ \in [0,t], X_{\tau^+} \in dy\big\} = \mu_K(t,x)\delta_0(dy) - \mu_J(t,x)\delta_0'(dy),$$

where

$$\mu_K(t,x) = 1 - \frac{\sqrt{3}}{2}\frac{A}{\pi}\int_0^\infty \frac{e^{-a^3tA}}{a}\Big(e^{-xa} + 2e^{-\frac{x}{2}a}\sin\Big(\frac{\sqrt{3}}{2}xa - \frac{\pi}{6}\Big)\Big)\,da$$

and

$$\mu_J(t,x) = x - \frac{\sqrt{3}}{2}\frac{A}{\pi}\int_0^\infty \frac{e^{-a^3tA}}{a}\Big(e^{-xa} - 2e^{\frac{x}{2}a}\sin\Big(\frac{\sqrt{3}}{2}xa + \frac{\pi}{3}\Big)\Big)\,da.$$

5.3. The boundary of $(0,a)$ when X_t starts at x ($0 < x < a$)

Let

$$\tau = \inf\{t : X_t < 0 \text{ or } X_t > a\}.$$

Consider

$$\begin{aligned}&E_x\big[e^{-\lambda\tau}f(X_\tau)\big]\\ &\quad\sim E_l\big[e^{-\epsilon^3\lambda\sigma}f(\epsilon X_\sigma)\big]\\ &\quad= f(-\epsilon)E_l[e^{-\epsilon^3\lambda\sigma}; X_\sigma = -1]\\ &\qquad + f(a+\epsilon)E_l[e^{-\epsilon^3\lambda\sigma}; X_\sigma = L+1] + f(a+2\epsilon)E_l[e^{-\epsilon^3\lambda\sigma}; X_\sigma = L+2]\\ &\quad= f(-\epsilon)H_{-1}(e^{-\epsilon^3\lambda}, l) + f(a+\epsilon)H_{L+1}(e^{-\epsilon^3\lambda}, l) + f(a+2\epsilon)H_{L+2}(e^{-\epsilon^3\lambda}, l)\\ &\quad= f(-\epsilon)H_{-1}(e^{-\epsilon^3\lambda}, l) + f(a+\epsilon)H_{L+1}(e^{-\epsilon^3\lambda}, l) + f(a+2\epsilon)H_{L+2}(e^{-\epsilon^3\lambda}, l)\\ &\quad= f(-\epsilon)H_{-1}(e^{-\epsilon^3\lambda}, l)\end{aligned}$$

$$+\frac{1}{2}\big(f(a+\epsilon)+f(a+2\epsilon)\big)\big(H_{L+1}(e^{-\epsilon^3\lambda},l)+H_{L+2}(e^{-\epsilon^3\lambda},l)\big)$$
$$+\frac{1}{2}\big(f(a+\epsilon)-f(a+2\epsilon)\big)\big(H_{L+1}(e^{-\epsilon^3\lambda},l)-H_{L+2}(e^{-\epsilon^3\lambda},l)\big).$$

Then by (4.5) and Proposition 5.1 we get the following.

THEOREM 5.7

Let $\nu=(\lambda/A)^{1/3}$. For every differentiable function f and $x>0$, we have

$$(5.2)\quad \begin{aligned} &E_x\big[e^{-\lambda\tau}f(X_\tau)\big] \\ &\quad=\frac{-e^{(a-x)\nu}+2e^{-\frac{1}{2}(a-x)\nu}\sin(\frac{\sqrt{3}}{2}(a-x)\nu+\frac{\pi}{6})}{-e^{a\nu}+2e^{-\frac{1}{2}a\nu}\sin(\frac{\sqrt{3}}{2}a\nu+\frac{\pi}{6})}f(0) \\ &\qquad+\frac{2}{\sqrt{3}}\Big(\Big(e^{(\frac{a}{2}-x)\nu}\sin\Big(\frac{\sqrt{3}}{2}a\nu+\frac{\pi}{3}\Big)-e^{\frac{1}{2}(a+x)\nu}\sin\Big(\frac{\sqrt{3}}{2}(a-x)\nu+\frac{\pi}{3}\Big) \\ &\qquad+e^{-\frac{1}{2}(2a-x)\nu}\sin\Big(\frac{\sqrt{3}}{2}x\nu\Big)\Big)\Big/\Big(-e^{a\nu}+2e^{-\frac{1}{2}a\nu}\sin\Big(\frac{\sqrt{3}}{2}a\nu+\frac{\pi}{6}\Big)\Big)\Big)f(a) \\ &\qquad-\frac{2}{\sqrt{3}\nu}\Big(\Big(e^{(\frac{a}{2}-x)\nu}\sin\Big(\frac{\sqrt{3}}{2}a\nu\Big)-e^{\frac{1}{2}(a+x)\nu}\sin\Big(\frac{\sqrt{3}}{2}(a-x)\nu\Big) \\ &\qquad+e^{-\frac{1}{2}(2a-x)\nu}\sin\Big(\frac{\sqrt{3}}{2}x\nu\Big)\Big)\Big/\Big(-e^{a\nu}+2e^{-\frac{1}{2}a\nu}\sin\Big(\frac{\sqrt{3}}{2}a\nu+\frac{\pi}{6}\Big)\Big)\Big)f'(a). \end{aligned}$$

REMARK 5.8

As λ goes to zero in (5.2), we obtain

$$u(x)=E_x\big[f(X_\tau)\big]=\frac{(a-x)^2}{a^2}f(0)+\frac{x(2a-x)}{a^2}f(a)+\frac{x(x-a)}{a}f'(a),$$

which satisfies

$$u'''=0,\qquad u(0)=f(0),\qquad u(a)=f(a),\qquad u'(a)=f'(a).$$

REMARK 5.9

As a goes to infinity in (5.2), we obtain (5.1).

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