Exponential convergence of Markovian semigroups and their spectra on L^p-spaces

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Abstract Markovian semigroups on L^2-space with suitable conditions can be regarded as Markovian semigroups on L^p-spaces for $p \in [1,\infty)$. When we additionally assume the ergodicity of the Markovian semigroups, the rate of convergence on L^p-space for each p is considerable. However, the rate of convergence depends on the norm of the space. The purpose of this paper is to investigate the relation between the rates on L^p-spaces for different p's, to obtain some sufficient condition for the rates to be independent of p, and to give an example for which the rates depend on p. We also consider spectra of Markovian semigroups on L^p-spaces, because the rate of convergence is closely related to the spectra.

1. Introduction

Let $(M,\mathscr{B})$ be a measurable space, let m be a probability measure on $(M,\mathscr{B})$, and let $L^p(m)$ be the L^p-space of $\mathbb{C}$-valued functions with respect to m. We denote the L^p-norm by $\|\cdot\|_p$, $\int f\,dm$ by $\langle f\rangle$ for $f \in L^1(m)$, and the constant function which takes values 1 by $\mathbf{1}$. A semigroup $\{T_t\}$ on $L^2(m)$ is called a *Markovian semigroup* if $0 \le T_t f \le 1$ m-almost everywhere whenever $f \in L^2(m)$ and $0 \le f \le 1$ m-almost everywhere. In this paper, we always assume that $T_t\mathbf{1} = \mathbf{1}$ for all $t \ge 0$. Let $\{T_t\}$ be a strongly continuous Markovian semigroup. We assume that $T_t^*\mathbf{1} = \mathbf{1}$, where T_t^* is the dual operator of T_t on $L^2(m)$. Then, as we will see in Section 2, the semigroup $\{T_t\}$ can be extended or restricted to semigroups on $L^p(m)$ for $p \in [1,\infty]$. Moreover, $\{T_t\}$ is strongly continuous for $p \in [1,\infty]$. Let

$$\gamma_{p\to q} := -\limsup_{t\to\infty} \frac{1}{t} \log \|T_t - m\|_{p\to q}, \tag{1.1}$$

where m means the linear operator $f \mapsto \langle f\rangle\mathbf{1}$ and $\|\cdot\|_{p\to q}$ is the operator norm from $L^p(m)$ to $L^q(m)$ for $p,q \in [1,\infty]$. Consider the case in which $T_t f$ converges to $\langle f\rangle$ for sufficiently many f. In this case $\gamma_{p\to q}$ means the exponential rate of the convergence. Generally, $\gamma_{p\to q}$ depends on $p,q \in [1,\infty]$. In this paper we consider the properties of $\gamma_{p\to q}$, the relations among $\{\gamma_{p\to q}; p,q \in [1,\infty]\}$, and some sufficient conditions for $\gamma_{p\to q}$ to be independent of p and q, and we give some examples in which they depend on p and q. We also consider spectra of

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Markovian semigroups with respect to L^p-spaces, because the rate of convergence is closely related to the spectra.

The organization of this paper is as follows. In Section 2 we consider properties on $\gamma_{p\to q}$ which are obtained by general argument. We also discuss the relation between the spectra of Markovian semigroups and $\gamma_{p\to q}$. In Section 3 we consider properties of hyperbounded Markovian semigroups and the relations between $\gamma_{p\to q}$ for different pairs (p,q). We also consider the cases of hypercontractive Markovian semigroups and ultracontractive Markovian semigroups. In Section 4 we consider a sufficient condition for $\gamma_{p\to p}$ to be independent of p. Precisely speaking, we consider a hyperbounded Markovian semigroup whose generator is a normal operator on L^2-space, and we show the p-independence of the spectra of the generator. In particular, this implies that $\gamma_{p\to p}$ is independent of p. In Section 5 we give a sufficient condition for nonsymmetric Markovian semigroups to be hyperbounded by using the logarithmic Sobolev inequality, and we consider a diffusion process on a manifold as an example. Nonsymmetric diffusion semigroups on manifolds are also considered in [7]. In the paper, equivalent conditions to contractivity conditions are obtained. In Section 6 we consider the relations between the spectra of linear operators which are consistent on L^p-spaces for p. Markovian semigroups and their generators are examples of consistent operators on L^p-spaces. We remark that the self-adjointness of the operator on L^2-space is additionally assumed in Section 6. In Section 7 we give an example of a Markovian semigroup for which $\gamma_{p\to p}$ depends on p. More precisely, we give a generator on the half-line, which is a second-order differential operator with a boundary condition. By investigating the spectra of the generator, we show that $\gamma_{p\to p}$ depends on p.

In the rest of this section, we give some notations used throughout this paper. For $z\in\mathbb{C}$, we denote the complex conjugate of z by $\bar{z}$, and for $p\in[1,\infty]$, we denote by p^* the conjugate exponent, that is, $1/p+1/p^*=1$.

Let (M,m) be a measure space, and let $L^p(m)$ be the L^p-space with respect to m for $p\in[1,\infty]$. For $p\in[1,\infty]$, $f\in L^p(m)$, and $g\in L^{p^*}(m)$, define $\langle f,g\rangle$ by $\int f(x)\overline{g(x)}m(dx)$. This notation is standard for $p=2$, because $\langle\cdot,\cdot\rangle$ is the inner product on $L^2(m)$. On the other hand, the notation may not be standard for $p\neq 2$, because $\langle\cdot,\cdot\rangle$ is not bilinear on $L^p(m)\times L^{p^*}(m)$. In this paper, we consider L^p-spaces and L^2-space at the same time. So, we use the notation $\langle\cdot,\cdot\rangle$ as defined above. Let A_p be a linear operator on $L^p(m)$, and let $\mathrm{Dom}(A_p)$ be the domain of A_p. We define the dual operator $(A_p)^*$ as follows. Let $\mathrm{Dom}((A_p)^*)$ be the total set of $f\in L^{p^*}(m)$ such that there exists $h\in L^{p^*}(m)$ satisfying

$$\langle A_p g,f\rangle=\langle g,h\rangle,\quad g\in\mathrm{Dom}(A_p),\tag{1.2}$$

and for $f\in\mathrm{Dom}((A_p)^*)$ define $(A_p)^*f:=h$ where h is the element of $L^{p^*}(m)$ appearing in (1.2).

We define the point spectra of A_p by the total set of $\lambda\in\mathbb{C}$ such that $\lambda-A_p$ is not injective on $L^p(m)$, and we denote the point spectra of A_p by $\sigma_{\mathrm{p}}(A_p)$. We define the continuous spectra of A_p by the total set of $\lambda\in\mathbb{C}$ such that $\lambda-A_p$ is injective but is not onto, and the range of $\lambda - A_p$ is dense in $L^p(m)$. We denote the continuous spectra of A_p by $\sigma_{\mathrm{c}}(A_p)$. We define the residual spectra of A_p by the total set of $\lambda \in \mathbb{C}$ such that $\lambda - A_p$ is injective but is not onto, and the range of $\lambda - A_p$ is not dense in $L^p(m)$. We denote the residual spectra of A_p by $\sigma_{\mathrm{r}}(A_p)$. Let $\sigma(A_p) := \sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{c}}(A_p) \cup \sigma_{\mathrm{r}}(A_p)$. We define the resolvent set of A_p by the total set of $\lambda \in \mathbb{C}$ such that $\lambda - A_p$ is bijective, and we denote it by $\rho(A_p)$. By the definition, $\sigma_{\mathrm{p}}(A_p)$, $\sigma_{\mathrm{c}}(A_p)$, $\sigma_{\mathrm{r}}(A_p)$, and $\rho(A_p)$ are disjoint sets of $\mathbb{C}$, and their union is equal to $\mathbb{C}$.

In this paper $1/0$ and $1/\infty$ are often regarded as ∞ and 0, respectively.

2. Relation between spectra and the exponential rate of convergence for semigroups

In this section we consider immediate consequences on $\gamma_{p\to q}$ obtained by general theories.

Let (M,m) be a probability space, and let $\{T_t\}$ be a strongly continuous Markovian semigroup on $L^2(m)$. We assume that $T_t^*\mathbf{1} = \mathbf{1}$, where T_t^* is the dual operator of T_t on $L^2(m)$. Then, it is easy to see that m is an invariant measure of both $\{T_t\}$ and $\{T_t^*\}$. By Jensen's inequality, for $p \in [1,\infty)$ we have

$$\int |T_t f|^p \, dm \le \int T_t\big(|f|^p\big) \, dm = \int |f|^p \, dm.$$

This implies that T_t is contractive on $L^p(m)$ for $p \in [1,\infty)$. Since $\{T_t\}$ is positivity preserving on $L^2(m)$ (i.e., $T_t f \ge 0$ if $f \in L^2(m)$ and $f \ge 0$), it is easy to see that T_t is also contractive on $L^\infty(m)$. Hence, $\{T_t\}$ can be extended or restricted to a Markovian semigroup on $L^p(m)$ for $p \in [1,\infty]$. Let $p \in (1,\infty)$. For a given $f \in L^p(m)$ and $\varepsilon > 0$, take a bounded measurable function g such that $\|f - g\|_p < \varepsilon$. Then, by Hölder's inequality

$$\begin{aligned}
\|T_t f - f\|_p &\le \|T_t f - T_t g\|_p + \|T_t g - g\|_p + \|g - f\|_p \\
&\le 2\|f - g\|_p + \Big(\int |T_t g - g| \cdot |T_t g - g|^{p-1} \, dm\Big)^{1/p} \\
&\le 2\varepsilon + \|T_t g - g\|_2^{1/p} \|T_t g - g\|_\infty^{1-1/p} \\
&\le 2\varepsilon + 2\|g\|_\infty^{1-1/p} \|T_t g - g\|_2^{1/p}.
\end{aligned}$$

Hence, $\limsup_{t\to 0} \|T_t f - f\|_p \le 2\varepsilon$. This implies that $\{T_t\}$ is strongly continuous on $L^p(m)$ for $p \in (1,\infty)$. Trivially, $\{T_t\}$ is strongly continuous on $L^1(m)$. Therefore, $\{T_t\}$ is strongly continuous for $p \in [1,\infty)$. Define $\mathfrak{A}_p$ to be the generator of $\{T_t\}$ on $L^p(m)$ for $p \in [1,\infty)$. We regard $\{T_t\}$ as a semigroup on $L^p(m)$ for all $p \in [1,\infty]$. Define $\gamma_{p\to q}$ by (1.1) for $p,q \in [1,\infty]$.

PROPOSITION 2.1

Let $p_1, p_2, q_1, q_2 \in [1,\infty]$. Let r_1 and r_2 be real numbers in $[1,\infty]$ such that there *exists* $\theta \in [0,1]$ *such that*

$$\frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1} \qquad \textit{and} \qquad \frac{1}{r_2} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}.$$

Then,

$$\gamma_{r_1 \to r_2} \geq (1-\theta)\gamma_{p_1 \to p_2} + \theta \tag{2.1}$$

In particular, the function $s \mapsto \gamma_{1/s \to 1/s}$ *on* $[0,1]$ *is concave.*

Proof
By Riesz–Thorin's interpolation theorem (see [2, Theorem 2.2.14]),

$$\|T_t - m\|_{r_1 \to r_2} \leq \|T_t - m\|_{p_1 \to p_2}^{1-\theta} \|T_t - m\|_{q_1 \to q_2}^{\theta}.$$

Hence, by the definition of $\gamma_{p \to q}$ we have the assertion. □

Proposition 2.1 gives us some nice properties on $\gamma_{p \to p}$. We state the properties in the theorems below.

THEOREM 2.2
The function $p \mapsto \gamma_{p \to p}$ *on* $[1,\infty]$ *is continuous on* $(1,\infty)$. *If* $\gamma_{p \to p} > 0$ *for some* $p \in [1,\infty]$, *then* $\gamma_{p \to p} > 0$ *for all* $p \in (1,\infty)$.

Proof
The equation (2.1) implies that $s \mapsto \gamma_{1/s \to 1/s}$ on $[0,1]$ is concave; hence, $s \mapsto \gamma_{1/s \to 1/s}$ is continuous on $(0,1)$. Hence, the first assertion holds. Since $\|T_t - m\|_{p \to p} \leq 2$ for $p \in [1,\infty]$, $\gamma_{p \to p} \geq 0$ for $p \in [1,\infty]$. This fact and the concavity conclude the second assertion. □

REMARK 2.3
The function $\gamma_{p \to p}$ may not be continuous at $p = 1, \infty$. Indeed, let m be the probability measure with the standard normal distribution, and let $\{T_t\}$ be the Ornstein–Uhlenbeck semigroup. Then, $\gamma_{p \to p} = 1$ for $p \in (1,\infty)$; however, $\gamma_{p \to p} = 0$ for $p = 1, \infty$.

THEOREM 2.4
Assume that $\{T_t\}$ *is self-adjoint on* $L^2(m)$. *Then,* $\gamma_{p \to p} = \gamma_{p^* \to p^*}$ *for* $p \in [1,\infty]$, *and the function* $p \mapsto \gamma_{p \to p}$ *on* $[1,\infty]$ *is nondecreasing on* $[1,2]$ *and nonincreasing on* $[2,\infty]$. *In particular, the maximum is attained at* $p = 2$.

Proof
Let $f(s) := \gamma_{1/s \to 1/s}$ for $s \in [0,1]$. In view of Proposition 2.1 we already know that f is concave on $[0,1]$. On the other hand, the symmetry of $\{T_t\}$ on $L^2(m)$ implies that $\|T_t^* - m\|_{p \to p} = \|T_t - m\|_{p \to p}$. Since the operator norm of the dual operator is equal to that of the original operator, we have $\|T_t - m\|_{p^* \to p^*} = \|T_t - m\|_{p \to p}$.

Hence, $\gamma_{p\to p} = \gamma_{p^*\to p^*}$ for $p \in [1,\infty]$. This fact and the concavity conclude the other assertions. □

REMARK 2.5

In Theorem 2.4 we obtain that $p \mapsto \gamma_{p\to p}$ is nondecreasing on $[1,2]$ and nonincreasing on $[2,\infty]$, and the maximum is attained by $p=2$. This assertion also follows from (2.2) and Remark 6.3 below.

Next we consider the relation between $\gamma_{p\to p}$ and the radius of spectra. When we regard T_t as an operator on $L^p(m)$, we denote $T_t : L^p(m) \to L^p(m)$ by $T_t^{(p)}$. For a bounded linear operator A on a Banach space, define the radius of spectra $\mathrm{Rad}(A)$ by

$$\mathrm{Rad}(A) := \sup\{|\lambda|; \lambda \in \sigma(A)\}.$$

It is well known that the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \|T_t - m\|_{p\to p}$$

exists (see, e.g., [1, Chapter 1, Theorem 1.22]), and of course, the limit equals $-\gamma_{p\to p}$. Moreover, it holds that (see, e.g., [1, Chapter 1, Theorem 1.22] and [2, Theorem 4.1.3])

$$\mathrm{Rad}(T_t^{(p)} - m) = e^{-\gamma_{p\to p} t}. \tag{2.2}$$

Hence, to see $\gamma_{p\to p}$ it is sufficient to see the spectra of $T_t^{(p)}$. There is also some relation between the spectra of semigroups and those of their generators. Let $\mathfrak{A}_p$ be the generator of $\{T_t^{(p)}\}$ for $[1,\infty)$. Then, it is known that

$$e^{t\sigma(\mathfrak{A}_p)\setminus\{0\}} \subset \sigma(T_t^{(p)} - m) \setminus \{0\} \tag{2.3}$$

for $t \in [0,\infty)$ (see, e.g., [1, Chapter 2, Theorem 2.16]). In the general setting, the inclusion cannot be replaced by equality (see [1, Chapter 2, Theorem 2.17]). Sufficient conditions for the inclusion in (2.3) to be replaced by equality are known (see [4, Chapter IV, Corollary 3.12]). For example, if $\{T_t^{(p)}\}$ is an analytic semigroup, then

$$e^{t\sigma(\mathfrak{A}_p)\setminus\{0\}} = \sigma(T_t^{(p)} - m) \setminus \{0\}, \quad t \in [0,\infty). \tag{2.4}$$

On the other hand, in the general setting the two equalities

$$e^{t\sigma_{\mathrm{p}}(\mathfrak{A}_p)\setminus\{0\}} = \sigma_{\mathrm{p}}(T_t^{(p)} - m) \setminus \{0\},$$

$$e^{t\sigma_{\mathrm{r}}(\mathfrak{A}_p)\setminus\{0\}} = \sigma_{\mathrm{r}}(T_t^{(p)} - m) \setminus \{0\}$$

hold for $t \in [0,\infty)$ (see [4, Chapter IV, Theorem 3.7]). Note that the definition of residual spectra in [4] is different from that in this paper. However, it is easy to see that the equality above still holds.

Consider the case in which $\{T_t\}$ is a Markovian semigroup on (M,m) such that $\{T_t^{(2)}\}$ is symmetric on $L^2(m)$. By [10, Chapter III, Section 2, Theorem 1], $\{T_t^{(p)}\}$ is an analytic semigroup on $L^p(m)$ for $p \in (1,\infty)$. Hence, (2.4) holds.

Moreover, by [4, Chapter IV, Corollary 3.12] we obtain

$$\sup\{\operatorname{Re}\lambda; \lambda \in \sigma(\mathfrak{A}_p) \setminus \{0\}\} = \lim_{t\to\infty} \frac{1}{t} \log \|T_t - m\|_{p\to p} \tag{2.5}$$

for $p \in (1,\infty)$. We use this equality in Section 7.

Now we introduce a property of spectra of real operators on a general Banach space. Let B be a complex Banach space, and let A be a linear operator on B. If there exists a bounded linear operator J on B satisfying that

$$\begin{aligned} &J(\alpha x + \beta y) = \bar{\alpha} Jx + \bar{\beta} Jy, \quad \alpha, \beta \in \mathbb{C}, x, y \in B, \\ &J^2 = I, \qquad \|Jx\| = \|x\|, \quad x \in B, \qquad AJ = JA, \end{aligned} \tag{2.6}$$

then A is called a *real operator*. Denote the resolvent operator with respect to $\lambda \in \rho(A)$ by R_λ.

LEMMA 2.6

If A is a real operator, then $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$, $\sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{c}}(A)}$, $\sigma_{\mathrm{r}}(A) = \overline{\sigma_{\mathrm{r}}(A)}$, and $\rho(A) = \overline{\rho(A)}$, where $\overline{\Lambda} := \{\bar{\lambda}; \lambda \in \Lambda\}$ for $\Lambda \subset \mathbb{C}$. Moreover, $R_{\bar{\lambda}} = JR_\lambda J$ for $\lambda \in \rho(A)$.

Proof

If $\lambda x = Ax$ holds for $x \in \mathrm{Dom}(A) \setminus \{0\}$, then $\bar{\lambda} Jx = AJx$ and $Jx \neq 0$. Hence, $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$. If there exists a sequence $\{x_n\} \subset B$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|\lambda x_n - Ax_n\| = 0$, then $\|Jx_n\| = 1$ and $\lim_{n\to\infty} \|\bar{\lambda} Jx_n - AJx_n\| = 0$. This implies that the conjugate of an approximate point spectrum is also an approximate point spectrum. Hence, $\sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{p}}(A)} \cup \overline{\sigma_{\mathrm{c}}(A)}$. Since $\sigma_{\mathrm{p}}(A)$ and $\sigma_{\mathrm{c}}(A)$ are disjoint from each other and $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$, we have $\sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{c}}(A)}$. For $\lambda \in \rho(A)$,

$$JR_\lambda J(\bar{\lambda} - A) = I \quad \text{on } \mathrm{Dom}(A) \qquad \text{and} \qquad (\bar{\lambda} - A)JR_\lambda J = I \quad \text{on } B.$$

This implies that $\bar{\lambda} \in \rho(A)$ and $R_{\bar{\lambda}} = JR_\lambda J$. Since $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$, $\sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{c}}(A)}$, and $\rho(A) = \overline{\rho(A)}$, the disjointness of $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{c}}(A)$, $\sigma_{\mathrm{r}}(A)$, and $\rho(A)$ implies that $\sigma_{\mathrm{r}}(A) = \overline{\sigma_{\mathrm{r}}(A)}$. □

Consider the following property for a linear operator A on a $\mathbb{C}$-valued function space B:

$$\begin{aligned} &\text{if } f \in \mathrm{Dom}(A) \text{ and } f \text{ is a real-valued function,} \\ &\text{then } Af \text{ is also a real-valued function.} \end{aligned} \tag{2.7}$$

It is easy to see that an operator A satisfying (2.7) is a real operator by letting $Jf := \bar{f}$ for B. Since Markovian semigroups are positivity preserving, they satisfy (2.7). Hence, so are the generators of strong continuous Markovian semigroups. Consider $\{T_t\}$ and $\mathfrak{A}_p$ defined in the beginning of this section. Then, $\{T_t\}$ and $\mathfrak{A}_p$ are real operators on $L^p(m)$ for $p \in [1,\infty)$. Hence, by Lemma 2.6 we have that each kind of spectra of $\{T_t\}$ on $L^p(m)$ and $\mathfrak{A}_p$ is symmetric with respect to the real axis.

3. Hyperboundedness and p-independence of $\gamma_{p\to p}$

In this section we discuss the relation between hyperboundedness and $\gamma_{p\to q}$. Hyperboundedness enables us to compare the elements of $\{\gamma_{p\to q}; p,q\in(1,\infty)\}$, and hyperboundedness and $\{\gamma_{p\to q}; p,q\in(1,\infty)\}$ characterize each other. In particular, we obtain the p-independence of $\gamma_{p\to p}$ for $p\in(1,\infty)$ from hyperboundedness. Hence, the results in this section give some sufficient conditions for $\gamma_{p\to p}$ to be p-independent. We also discuss the relation between hypercontractivity and $\gamma_{p\to p}$.

Let (M,m) and $\{T_t\}$ be the same as in Section 2. However, the assumption $T_t^*\mathbf{1}=\mathbf{1}$ is not needed on the results before Proposition 3.3. For $p,q\in(1,\infty)$ such that $p<q$, $\{T_t\}$ is called (p,q)-*hyperbounded* if there exist $K\ge 0$ and $C>0$ such that

$$\|T_K f\|_q \le C\|f\|_p, \quad f\in L^p(m), \tag{3.1}$$

and $\{T_t\}$ is called (p,q)-*hypercontractive* if there exists $K\ge 0$ such that (3.1) holds with $C=1$.

First we prepare the following lemma.

LEMMA 3.1

Let $p,q\in(1,\infty)$ such that $p<q$. If there exist nonnegative constants K and C such that $\|T_Kf\|_q\le C\|f\|_p$ for $f\in L^p(m)$, then, for $n_1,n_2\in\mathbb{N}$ such that $q^{-n_1}/p^{-n_1-1}>1$,

$$\|T_{(n_1+n_2)K}f\|_{q^{n_2}/p^{n_2-1}} \le C^{\alpha(n_1,n_2)}\|f\|_{q^{-n_1}/p^{-n_1-1}}, \quad f\in L^{q^{-n_1}/p^{-n_1-1}}(m),$$

where $\alpha(n_1,n_2)=\sum_{k=-n_1}^{n_2-1}p^k/q^k$.

Proof

Let $f\in L^{q^{-n_1}/p^{-n_1-1}}(m)$. By the positivity of $\{T_t\}$, Jensen's inequality, and the assumption, for $n\in\mathbb{N}$ and $m\in\mathbb{Z}$ such that $q^{m-1}/p^{m-2}>1$ we have that

$$\begin{aligned}
\|T_{nK}f\|_{q^m/p^{m-1}} &\le \Big[\int \big(T_K\big(|T_{(n-1)K}f|^{q^{m-1}/p^{m-1}}\big)\big)^q\,dm\Big]^{p^{m-1}/q^m}\\
&= \big\|T_K\big(|T_{(n-1)K}f|^{q^{m-1}/p^{m-1}}\big)\big\|_q^{p^{m-1}/q^{m-1}}\\
&\le C^{p^{m-1}/q^{m-1}}\big\||T_{(n-1)K}f|^{q^{m-1}/p^{m-1}}\big\|_p^{p^{m-1}/q^{m-1}}\\
&= C^{p^{m-1}/q^{m-1}}\|T_{(n-1)K}f\|_{q^{m-1}/p^{m-2}}.
\end{aligned}$$

Iterating this calculation, we have the conclusion. □

Next we give the following theorem on hyperboundedness and hypercontractivity.

THEOREM 3.2

If $\{T_t\}$ is (p,q)-hyperbounded for some $p,q\in(1,\infty)$ such that $p<q$, then $\{T_t\}$ is (p,q)-hyperbounded for any $p,q\in(1,\infty)$ such that $p<q$. Moreover, if $\{T_t\}$ is

(p,q)-hypercontractive for some $p,q \in (1,\infty)$ such that $p<q$, then $\{T_t\}$ is (p,q)-hypercontractive for any $p,q \in (1,\infty)$ such that $p<q$.

Proof
Assume that $\{T_t\}$ is (p_1,q_1)-hyperbounded for $p_1<q_1$. It is easy to see that $\{T_t\}$ is (p_2,q_2)-hyperbounded for $p_1 \le p_2 < q_2 \le q_1$. Let $p,q \in (1,\infty)$ such that $p<q$. Choose p_2 and q_2 so that $p_1 \le p_2 < q_2 \le q_1$ and so that $1 < {p_2}^{n_1+1}/{q_2}^{n_1} < p$ with some $n_1 \in \mathbb{N}$. Take $n_2 \in \mathbb{N}$ such that ${q_2}^{n_2}/{p_2}^{n_2-1} > q$. Then, by applying Lemma 3.1 we have that $\{T_t\}$ is $({p_2}^{n_1+1}/{q_2}^{n_1}, {q_2}^{n_2}/{p_2}^{n_2-1})$-hyperbounded, and therefore, $\{T_t\}$ is (p,q)-hyperbounded. Similarly, we obtain the second assertion. □

This theorem says that (p,q)-hyperboundedness for some $p,q \in (1,\infty)$ such that $p<q$ implies (p,q)-hyperboundedness for all $p,q \in (1,\infty)$ such that $p<q$, and the same assertion holds for hypercontractivity. Hence, we simply say that $\{T_t\}$ is hyperbounded and hypercontractive instead of saying that $\{T_t\}$ is (p,q)-hyperbounded and (p,q)-hypercontractive, respectively.

In the rest of this section we consider the relation between hypercontractivity (or hyperboundedness) and the exponential rate of convergence $\gamma_{p\to p}$. Note that the assumption $T_t^*\mathbf{1}=\mathbf{1}$ is needed from here on. First we show the following proposition, which is an extension of the first assertion of [3, Lemma 6.1.5].

PROPOSITION 3.3
Assume that

$$\|T_K f\|_r \le \|f\|_2, \quad f \in L^2(m), \tag{3.2}$$

for some $K>0$ and $r>2$. Then, we have that

$$\big\|T_K f - \langle f\rangle\big\|_2 \le (r-1)^{-1/2}\|f\|_2, \quad f \in L^2(m), \tag{3.3}$$

and

$$\big\|T_t f - \langle f\rangle\big\|_2 \le \sqrt{r-1}\exp\Big\{-\frac{t}{K}\log\sqrt{r-1}\Big\}\|f\|_2, \quad f \in L^2(m), t \in [0,\infty). \tag{3.4}$$

Proof
Let $f \in L^\infty(m)$ such that $\langle f\rangle = 0$ and $\|f\|_\infty \le a_0$ with a nonnegative constant a_0, and let a be a positive constant such that $a>a_0$. From (3.2) we have

$$\big(a^2+\|f\|_2^2\big)^{r/2} = \|a+f\|_2^r \ge \big\|T_K(a+f)\big\|_r^r = \int \big|a+T_K f(x)\big|^r m(dx). \tag{3.5}$$

By the Taylor theorem there exists $\theta \in [0,1]$ such that

$$\big(a^2+\|f\|_2^2\big)^{r/2} = a^r + \frac{r}{2}a^{r-2}\|f\|_2^2 + \frac{1}{2}\frac{r(r-2)}{4}\big(a^2+\theta\|f\|_2^2\big)^{r/2-2}\|f\|_2^4. \tag{3.6}$$

Since $\{T_t\}$ is a Markovian semigroup, $\|T_K f\|_\infty \le a_0$. Hence, by the Taylor theorem again, for each x there exists $\eta_x \in [0,1]$ such that

$$(a+T_Kf)^r(x) = a^r + ra^{r-1}T_Kf(x) + \frac{r(r-1)}{2}a^{r-2}(T_Kf)^2(x) + \frac{r(r-1)(r-2)}{6}(a+\eta_x T_Kf)^{r-3}(x)(T_Kf)^3(x).$$

By integrating both sides we have

$$\int (a+T_Kf)^r\,dm = a^r + \frac{r(r-1)}{2}a^{r-2}\|T_Kf\|_2^2 + \frac{r(r-1)(r-2)}{6}\int (a+\eta_x T_Kf)^{r-3}(T_Kf)^3\,dm. \tag{3.7}$$

From (3.5), (3.6), and (3.7),

$$\begin{aligned}&\frac{r}{2}a^{r-2}\|f\|_2^2 + \frac{1}{2}\frac{r(r-2)}{4}\big(a^2+\theta\|f\|_2^2\big)^{r/2-2}\|f\|_2^4\\ &\geq \frac{r(r-1)}{2}a^{r-2}\|T_Kf\|_2^2\\ &\quad + \frac{r(r-1)(r-2)}{6}\int (a+\eta_x T_Kf)^{r-3}(x)(T_Kf)^3(x)m(dx).\end{aligned}$$

Dividing both sides by a^{r-2} and taking the limit as $a\to\infty$, we have

$$\frac{r}{2}\|f\|_2^2 \geq \frac{r(r-1)}{2}\|T_Kf\|_2^2.$$

Hence, (3.3) follows.

To show (3.4), for a given $t\geq 0$ take $n\in\mathbb{N}\cup\{0\}$ and $\rho\in[0,K)$ such that $t=nK+\rho$. Then, by (3.3)

$$\begin{aligned}\big\|T_tf-\langle f\rangle\big\|_2 &= \big\|T_{nK}T_\rho f-\langle T_\rho f\rangle\big\|_2 \leq (r-1)^{-n/2}\|T_\rho f\|_2\\ &\leq (r-1)^{-(1/2)(t/K-1)}\|f\|_2 \leq \sqrt{r-1}\exp\Big\{-\frac{t}{K}\log\sqrt{r-1}\Big\}\|f\|_2.\end{aligned}$$

Hence, we have (3.4). □

Next we show the following theorem, which tells us the relation between hyperboundedness and $\gamma_{p\to q}$.

THEOREM 3.4

The following conditions are equivalent:

(i) $\{T_t\}$ *is hyperbounded.*
(ii) $\gamma_{p\to q}\geq 0$ *for some* $1<p<q<\infty$.
(iii) $\gamma_{p\to q}=\gamma_{2\to 2}$ *for all* $p,q\in(1,\infty)$.

Proof

First we show that (ii) implies (i). By the definition of $\gamma_{p\to q}$ there exists $K>0$ such that $\|T_K-m\|_{p\to q}<\infty$. Hence, $\|T_K\|_{p\to q}<\infty$. Therefore, we obtain (i) by Theorem 3.2. Immediately (ii) follows from (iii), since $\gamma_{2\to 2}\geq 0$.

Finally we show that (i) implies (iii). For given $p,q,r,s\in(1,\infty)$ take $K>0$ and $C>0$ such that $\|T_K\|_{p\to r}\le C$ and $\|T_K\|_{s\to q}\le C$. Then, it is easy to see that

$$\|T_K-m\|_{p\to r}\le C+1 \qquad \text{and} \qquad \|T_K-m\|_{s\to q}\le C+1. \tag{3.8}$$

Since

$$\|T_{t+2K}-m\|_{p\to q}\le \|T_K-m\|_{p\to r}\|T_t-m\|_{r\to s}\|T_K-m\|_{s\to q},$$

we have that

$$\begin{aligned}&-\frac{1}{t}\log\|T_{t+2K}-m\|_{p\to q}\\ &\ge -\frac{1}{t}\log\|T_K-m\|_{p\to r}-\frac{1}{t}\log\|T_t-m\|_{r\to s}-\frac{1}{t}\log\|T_K-m\|_{s\to q}.\end{aligned}$$

In view of (3.8), letting $t\to\infty$, we obtain $\gamma_{p\to q}\ge\gamma_{r\to s}$. Since $p,q,r,s\in(1,\infty)$ are arbitrary, (iii) follows. □

Finally, we show the following theorem, which tells us the relation between hypercontractivity and $\gamma_{p\to q}$ and also gives some criteria for $\{T_t\}$ to be hypercontractive.

THEOREM 3.5

The following conditions are equivalent:

(i) $\{T_t\}$ *is hypercontractive.*
(ii) $\gamma_{p\to q}>0$ *for some* $1<p<q<\infty$.
(iii) $\gamma_{p\to q}=\gamma_{2\to 2}$ *for all* $p,q\in(1,\infty)$ *and* $\gamma_{2\to 2}>0$.
(iv) *There exist* $K>0$ *and* $r>0$ *such that*

$$\|T_K\|_{2\to r}<\infty \qquad \textit{and} \qquad \|T_K-m\|_{2\to 2}<1.$$

Proof

By Theorem 3.4 we have that (ii) implies (iii). Trivially, (ii) follows from (iii).

By Theorem 3.4, (i) implies that $\gamma_{p\to q}=\gamma_{2\to 2}$ for all $p,q\in(1,\infty)$. On the other hand, by Proposition 3.3 we obtain from (i) that $\gamma_{2\to 2}>0$. Hence, (i) implies (iii). Theorem 3.2 and [3, Lemma 6.1.5] give that (iv) implies (i).

To finish the proof, it is sufficient to prove that (iii) implies (iv). Assume (iii). As we have seen in Theorem 3.4, there exist $K>0$ and $r>0$ such that $\|T_K\|_{2\to r}<\infty$. Since $\gamma_{2\to 2}>0$, by the definition of $\gamma_{p\to q}$ it holds that there exists $K>0$ such that $\|T_K-m\|_{2\to 2}<1$. Thus, we obtain (iv). □

REMARK 3.6

We introduce the defective logarithmic Sobolev inequality and the logarithmic Sobolev inequality in Section 5 below. It is known that hyperboundedness and hypercontractivity are equivalent to the defective logarithmic Sobolev inequality and the logarithmic Sobolev inequality, respectively (see [3, Theorem 6.1.14]).

4. Sufficient conditions for spectra to be p-independent

In Section 3 we showed that when hyperboundedness holds, the exponential rates of convergence $\{\gamma_{p\to p}; p\in(1,\infty)\}$ are independent of p. However, hyperboundedness gives us the further information that the spectra of $\{-\mathfrak{A}_p; p\in(1,\infty)\}$ are independent of p. Recall that $-\mathfrak{A}_p$ and $\gamma_{p\to p}$ are closely related to each other (see Section 2). In this section we show the assertion.

Let (M,m) and $\{T_t\}$ be the same as in Section 2. Let $p\in(2,\infty)$, and fix p. Assume that there exist positive constants K and C such that

$$\|T_K f\|_p \le C\|f\|_2, \quad f\in L^2(m). \tag{4.1}$$

By Theorem 3.2 this assumption is equivalent to hyperboundedness on $\{T_t\}$. Hence, by taking another pair (K,C), both (4.1) and

$$\|T_K f\|_2 \le C\|f\|_{p^*}, \quad f\in L^{p^*}(m), \tag{4.2}$$

hold. We choose a pair (K,C) such that both (4.1) and (4.2) hold and fix it. Let $\mathfrak{A}_p$ be the generator of $\{T_t\}$ on $L^p(m)$ for $p\in[1,\infty)$, and assume that $\mathfrak{A}_2$ is a *normal* operator, that is, $(\mathfrak{A}_2)^*\mathfrak{A}_2=\mathfrak{A}_2(\mathfrak{A}_2)^*$. Then, we can consider the spectral decomposition of $-\mathfrak{A}_2$ (see [8]) as follows:

$$-\mathfrak{A}_2=\int_{\mathbb{C}}\lambda\, dE_\lambda.$$

For a bounded $\mathbb{C}$-valued measurable function ϕ on $\mathbb{C}$, define an operator $\phi(-\mathfrak{A}_2)$ on $L^2(m)$ by

$$\phi(-\mathfrak{A}_2)=\int_{\mathbb{C}}\phi(\lambda)\, dE_\lambda.$$

Note that it is sufficient that ϕ is defined only on $\sigma(-\mathfrak{A}_2)$. Since $L^p(m)\subset L^2(m)$ and $L^2(m)$ is dense in $L^{p^*}(m)$ in our setting, $\phi(-\mathfrak{A}_2)$ can be regarded as a linear operator on $L^p(m)$ and on $L^{p^*}(m)$. So, we denote $\phi(-\mathfrak{A}_2)$ by $\phi(-\mathfrak{A})$ simply and regard $\phi(-\mathfrak{A})$ as a linear operator on $L^2(m)$, on $L^p(m)$, and on $L^{p^*}(m)$.

It is easy to see that $\phi(-\mathfrak{A})$ is a bounded operator on $L^2(m)$ if and only if ϕ is bounded on $\sigma(-\mathfrak{A}_2)$. However, it is not easy to obtain sufficient conditions for $\phi(-\mathfrak{A})$ to be a bounded operator on $L^p(m)$ and on $L^{p^*}(m)$. Now we consider a sufficient condition for the boundedness of $\phi(-\mathfrak{A})$ on $L^p(m)$ and on $L^{p^*}(m)$ under the assumption (4.1). Define a function χ on $\mathbb{C}$ by

$$\chi(\lambda):=\begin{cases}0 & \operatorname{Re}\lambda<0,\\ 1 & \operatorname{Re}\lambda\ge 0,\end{cases}$$

and let $\chi_n(\lambda):=\chi(\lambda-n)$.

PROPOSITION 4.1

The following hold.

(i) *If ϕ is bounded and the real part of the support of ϕ is bounded, then $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$ and also on $L^{p^*}(m)$.*

(ii) *There exists a positive constant* $c = c(p,n)$ *satisfying*

$$\big\|T_t\chi_n(-\mathfrak{A})\big\|_{p\to p} \le ce^{-nt}, \tag{4.3}$$

$$\big\|T_t\chi_n(-\mathfrak{A})\big\|_{p^*\to p^*} \le ce^{-nt}, \tag{4.4}$$

for $t \in [0,\infty)$.

Proof
To show (i), let $\psi(\lambda) := \phi(\lambda)e^{K\lambda}$, where K is the constant which appeared in (4.1). Since the real part of the support of ϕ is bounded, $\psi(-\mathfrak{A})$ is a bounded operator on $L^2(m)$. By using the fact that $\phi(-\mathfrak{A}) = T_K\psi(-\mathfrak{A})$ and (4.1), we have that

$$\big\|\phi(-\mathfrak{A})\big\|_{2\to p} \le \|T_K\|_{2\to p}\big\|\psi(-\mathfrak{A})\big\|_{2\to 2} \le C\big\|\psi(-\mathfrak{A})\big\|_{2\to 2}.$$

Hence, by the continuity of the embedding $L^p(m) \hookrightarrow L^2(m)$ we have that $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. A similar argument is available to estimate $\|\phi(-\mathfrak{A})\|_{p^*\to 2}$, and we have that $\phi(-\mathfrak{A})$ is a bounded operator on $L^{p^*}(m)$. Thus, we obtain (i).

Next we show (ii). Since $\sup_{\mathrm{Re}\,\lambda\ge 0}|e^{-t\lambda}|\chi_n(\lambda) \le e^{-nt}$, we have that

$$\big\|T_t\chi_n(-\mathfrak{A})\big\|_{2\to 2} \le e^{-nt}.$$

Hence, by (4.1), for $t \ge 0$

$$\big\|T_{t+K}\chi_n(-\mathfrak{A})\big\|_{2\to p} \le \|T_K\|_{2\to p}\big\|T_t\chi_n(-\mathfrak{A})\big\|_{2\to 2} \le Ce^{-nt} \le Ce^{nK}e^{-n(t+K)}.$$

Therefore, choosing $c \ge Ce^{nK}$, (4.3) holds for $t \ge K$.

Since $\mathbb{I}_{\{\mathrm{Re}\,\lambda\ge 0\}} - \chi_n$ is bounded and the real part of its support is bounded, (i) implies that $I - \chi_n(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. Here, note that $\sigma(-\mathfrak{A}_2) \subset \{z \in \mathbb{C}; \mathrm{Re}\,z \ge 0\}$. Thus, for $t \in [0,K]$

$$\begin{aligned}
\big\|T_t\chi_n(-\mathfrak{A})\big\|_{p\to p} &= \big\|T_t\big(I - I + \chi_n(-\mathfrak{A})\big)\big\|_{p\to p} \\
&\le \|T_t\|_{p\to p} + \big\|T_t\big(I - \chi_n(-\mathfrak{A})\big)\big\|_{p\to p} \\
&\le 1 + \big\|\big(I - \chi_n(-\mathfrak{A})\big)\big\|_{p\to p} \\
&\le \big(1 + \big\|\big(I - \chi_n(-\mathfrak{A})\big)\big\|_{p\to p}\big)e^{nK}e^{-nt}.
\end{aligned}$$

Therefore, by taking $c \ge (1 + \|(I - \chi_n(-\mathfrak{A}))\|_{p\to p})e^{nK}$, (4.3) holds for $t \in [0,K]$. Consequently, if we let $c = \max\{Ce^{nK}, (1 + \|(I - \chi_n(-\mathfrak{A}))\|_{p\to p})e^{nK}\}$, then (4.3) holds for $t \in [0,\infty)$.

We are able to prove (4.4) in a similar way. Hence, we omit the proof. □

By using Proposition 4.1 we can show a sufficient condition for $\phi(-\mathfrak{A})$ to be a bounded linear operator on $L^p(m)$ and on $L^{p^*}(m)$. The following theorem is an extension of the result by Meyer [5, Chapter IV, Section 3].

THEOREM 4.2
Assume (4.1). Let h *be a* $\mathbb{C}$*-valued bounded measurable function on* $\mathbb{C}$ *which is*

analytic on the neighborhood around 0*, and define a* $\mathbb{C}$*-valued bounded function* ϕ *on* $\mathbb{C}$ *by* $\phi(\lambda) = h(1/\lambda)$*. Then,* $\phi(-\mathfrak{A})$ *is a bounded operator on* $L^p(m)$ *and also on* $L^{p^*}(m)$*.*

Proof
The proofs for the boundedness of $\phi(-\mathfrak{A})$ on $L^p(m)$ and for that on $L^{p^*}(m)$ are the same. So, we only prove that $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. Choose $n \in \mathbb{N}$ such that h is analytic on $\{z \in \mathbb{C}; |z| \le 1/n\}$, and let

$$\phi^{(1)} := \phi(1-\chi_n) \qquad \text{and} \qquad \phi^{(2)} := \phi\chi_n.$$

Then, ϕ is decomposed as

$$\phi = \phi^{(1)} + \phi^{(2)}.$$

Since $\sigma(-\mathfrak{A}_2) \subset \{z \in \mathbb{C}; \operatorname{Re} z \ge 0\}$, Proposition 4.1(i) implies that $\phi^{(1)}(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. Hence, it is sufficient to show that $\phi^{(2)}(-\mathfrak{A})$ is a bounded operator on $L^p(m)$.

Let

$$R := \int_0^\infty T_t \chi_n(-\mathfrak{A})\, dt.$$

Since for $k \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} R^k &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty T_{t_1}\chi_n(-\mathfrak{A}) T_{t_2}\chi_n(-\mathfrak{A}) \cdots T_{t_k}\chi_n(-\mathfrak{A})\, dt_1\, dt_2 \cdots dt_k \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty T_{t_1+t_2+\cdots+t_k}\chi_n(-\mathfrak{A})\, dt_1\, dt_2 \cdots dt_k, \end{aligned}$$

by Proposition 4.1(ii) we have

$$\|R^k\|_{p\to p} \le cn^{-k}, \quad k \in \mathbb{N} \cup \{0\}. \tag{4.5}$$

By using the spectral argument on L^2-space,

$$R = \int_0^\infty \int_{\{\operatorname{Re}\lambda \ge n\}} e^{-\lambda t}\, dE_\lambda\, dt = \int_{\{\operatorname{Re}\lambda \ge n\}} \lambda^{-1}\, dE_\lambda,$$

and hence

$$R^k = \int_{\{\operatorname{Re}\lambda \ge n\}} \lambda^{-k}\, dE_\lambda. \tag{4.6}$$

On the other hand, since h is analytic on $\{z \in \mathbb{C}; |z| \le 1/n\}$, by using Taylor expansion we have that

$$h(z) = \sum_{k=0}^\infty a_k z^k, \quad |z| \le \frac{1}{n}.$$

Note that $\sum_{k=0}^\infty |a_k| n^{-k} < \infty$. Hence, by (4.6) we obtain that

$$\phi^{(2)}(-\mathfrak{A}) = \int_{\{\operatorname{Re}\lambda \ge n\}} h(\lambda^{-1})\, dE_\lambda = \sum_{k=0}^\infty a_k \int_{\{\operatorname{Re}\lambda \ge n\}} \lambda^{-k}\, dE_\lambda = \sum_{k=0}^\infty a_k R^k.$$

Therefore, (4.5) implies that $\phi^{(2)}(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. $\square$

Theorem 4.2 enables us to show that the spectra of $\mathfrak{A}_p$ are independent of p under the condition (4.1) as follows.

THEOREM 4.3

Assume that (4.1) holds for some $p\in(2,\infty)$ and positive numbers K and C. Then, $\sigma(-\mathfrak{A}_q)=\sigma(-\mathfrak{A}_2)$ for $q\in(1,\infty)$.

Proof

As mentioned in the beginning of this section, in view of Theorem 3.2 the assumption that (4.1) holds for some $p\in(2,\infty)$, $K>0$, and $C>0$ implies that for any $p\in(2,\infty)$ there exist $K>0$ and $C>0$ such that (4.1) and (4.2) hold.

First we show that $\sigma(-\mathfrak{A}_q)\supset\sigma(-\mathfrak{A}_2)$ for $q\in(1,\infty)$. For given $p\in(2,\infty)$, take positive numbers K and C such that (4.1) and (4.2) hold and fix them. Let $\alpha\in\sigma(-\mathfrak{A}_2)$. For $n\in\mathbb{N}$, define $U_n:=\{z\in\mathbb{C};|z-\alpha|\le 1/n\}$, and define $S_n:=\{\int_{U_n}dE_\lambda f; f\in L^2(m)\}$. Then, S_n is a closed linear subspace of $L^2(m)$ and $S_n\neq\{0\}$ for $n\in\mathbb{N}$. Take $f_n\in S_n$ such that $\|f_n\|_2=1$. Then, it is easy to see that $\lim_{n\to\infty}\|\mathfrak{A}f_n+\alpha f_n\|_2=0$. Since

$$\begin{aligned}\mathfrak{A}f_n+\alpha f_n&=-\int_{U_n}\lambda\,dE_\lambda f_n+\alpha f_n\\&=-\int_{U_n}e^{-K\lambda}e^{K\lambda}\lambda\,dE_\lambda f_n+\int_{U_n}e^{-K\lambda}e^{K\lambda}\alpha\,dE_\lambda f_n\\&=\Big(\int_{U_n}e^{-K\lambda}\,dE_\lambda\Big)\Big(\int_{U_n}e^{K\lambda}(\alpha-\lambda)\,dE_\lambda f_n\Big)\\&=T_K\int_{U_n}e^{K\lambda}(\alpha-\lambda)\,dE_\lambda f_n,\end{aligned}$$

by (4.1) we have that

$$\|\mathfrak{A}f_n+\alpha f_n\|_p\le C\Big\|\int_{U_n}e^{K\lambda}(\alpha-\lambda)\,dE_\lambda f_n\Big\|_2\le\frac{C}{n}e^{K(\mathrm{Re}\,\alpha+1/n)}\|f_n\|_2.$$

Hence, $\lim_{n\to\infty}\|\mathfrak{A}f_n+\alpha f_n\|_p=0$. On the other hand, $\|f_n\|_p\ge\|f_n\|_2=1$. These yield that $\alpha\in\sigma(-\mathfrak{A}_p)$. Similar to the argument above,

$$\begin{aligned}\mathfrak{A}f_n+\alpha f_n&=-\int_{U_n}e^{K\lambda}e^{-K\lambda}\lambda\,dE_\lambda f_n+\int_{U_n}e^{K\lambda}e^{-K\lambda}\alpha\,dE_\lambda f_n\\&=\Big(\int_{U_n}e^{K\lambda}(\alpha-\lambda)\,dE_\lambda\Big)\Big(\int_{U_n}e^{-K\lambda}\,dE_\lambda f_n\Big)\\&=\int_{U_n}e^{K\lambda}(\alpha-\lambda)\,dE_\lambda(T_Kf_n).\end{aligned}$$

Hence, by (4.2) we have

$$\begin{aligned}\|\mathfrak{A}f_n+\alpha f_n\|_{p^*}&\le\Big\|\int_{U_n}e^{K\lambda}(\alpha-\lambda)\,dE_\lambda(T_Kf_n)\Big\|_2\\&\le\frac{1}{n}e^{K(\mathrm{Re}\,\alpha+1/n)}\|T_Kf_n\|_2\le\frac{C}{n}e^{K(\mathrm{Re}\,\alpha+1/n)}\|f_n\|_{p^*}.\end{aligned}$$

Letting $\tilde{f}_n := f_n/\|f_n\|_{p^*}$, we have $\|\tilde{f}_n\|_{p^*} = 1$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\mathfrak{A}\tilde{f}_n + \alpha\tilde{f}_n\|_{p^*} = 0$. This yields that $\alpha \in \sigma(-\mathfrak{A}_{p^*})$. Thus, we have $\sigma(-\mathfrak{A}_2) \subset \sigma(-\mathfrak{A}_q)$ for $q \in (1,\infty)$.

Next we show that $\sigma(-\mathfrak{A}_q) \subset \sigma(-\mathfrak{A}_2)$ for $q \in (1,\infty)$. It is sufficient to show that $\rho(-\mathfrak{A}_q) \supset \rho(-\mathfrak{A}_2)$ for $q \in (1,\infty)$. For given $p \in (2,\infty)$, take positive numbers K and C such that (4.1) and (4.2) hold and fix them. Let $\alpha \in \rho(-\mathfrak{A}_2)$, and let $\phi(z) := 1/(\alpha + z)$. Then,

$$(\alpha - \mathfrak{A})^{-1} = \int_{\mathbb{C}} \phi(\lambda)\, dE_\lambda, \tag{4.7}$$

$$\phi\Big(\frac{1}{z}\Big) = \frac{z}{\alpha z + 1}. \tag{4.8}$$

The equality (4.8) implies that $\phi(1/z)$ is analytic on a neighborhood around $z = 0$. Since $\alpha \in \rho(-\mathfrak{A}_2)$, the integral on the right-hand side of (4.7) is not changed by replacing $\phi(\lambda)$ by 0 on a neighborhood around $\lambda = -\alpha$. This implies that we can regard ϕ as a bounded function. Hence, applying Theorem 4.2, we have that $(\alpha - \mathfrak{A})^{-1}$ is a bounded operator on $L^p(m)$. Therefore, $\alpha \in \rho(-\mathfrak{A}_p)$. We also have $\alpha \in \rho(-\mathfrak{A}_{p^*})$ in the same manner. Thus, we have $\rho(-\mathfrak{A}_2) \subset \rho(-\mathfrak{A}_q)$ for $q \in (1,\infty)$. □

By using Theorem 4.3, we are able to know a little more information on the spectra of $\{T_t\}$ satisfying hyperboundedness.

THEOREM 4.4
If $\{T_t\}$ is hyperbounded, then $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$, and $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$ for $p \in (1,\infty)$.

Proof
Let $p, q \in (1,\infty)$. Let $\alpha \in \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$. Then, there exists $f \in \mathrm{Dom}(-\mathfrak{A}_p) \setminus \{0\}$ such that $\alpha f + \mathfrak{A} f = 0$. Hence, $\alpha T_t f + \mathfrak{A} T_t f = 0$ for $t \in [0,\infty)$. Since $\{T_t\}$ is hyperbounded, there exists a sufficiently large $t \in [0,\infty)$ such that $T_t f \in \mathrm{Dom}(-\mathfrak{A}_q) \setminus \{0\}$. This implies that $\alpha \in \sigma_{\mathrm{p}}(-\mathfrak{A}_q)$ and $T_t f$ is an eigenfunction with respect to α. Hence, $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \subset \sigma_{\mathrm{p}}(-\mathfrak{A}_q)$. Since this holds for arbitrary $p, q \in (1,\infty)$, we have $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$ for $p \in (1,\infty)$.

Let $p, q \in (1,\infty)$ such that $p < q$. By using a dual argument we have that

$$\|T_t\|_{p\to q} = \|T_t^*\|_{q^*\to p^*}, \quad t \in [0,\infty).$$

Note that T_t^* is also a normal operator, the generator of T_t^* on $L^{p^*}(m)$ is $(\mathfrak{A}_p)^*$, and $q^* < p^*$. In view of Theorem 3.2, the hyperboundedness of $\{T_t\}$ implies that of $\{T_t^*\}$. Applying the argument above to $\{T_t^*\}$, we have that

$$\sigma_{\mathrm{p}}\big(-(\mathfrak{A}_2)^*\big) = \sigma_{\mathrm{p}}\big(-(\mathfrak{A}_p)^*\big), \quad p \in (1,\infty). \tag{4.9}$$

Now assume that $\alpha \in \sigma_{\mathrm{r}}(-\mathfrak{A}_p)$ for some $p \in (1,\infty)$, and we make contradiction. Since there exists $f \in L^{p^*}(m)$ such that $\langle(\alpha + \mathfrak{A}_p)g, f\rangle = 0$ for $g \in \mathrm{Dom}(\mathfrak{A}_p)$, $f \in \mathrm{Dom}((\mathfrak{A}_p)^*)$ and $-(\mathfrak{A}_p)^* f = \bar{\alpha} f$. Hence, $\alpha \in \overline{\sigma_{\mathrm{p}}(-(\mathfrak{A}_p)^*)}$. Since $\mathfrak{A}_2$ is a normal operator, it is easy to see that $\|(z+\mathfrak{A}_2)f\|_2 = \|(\bar{z}+(\mathfrak{A}_2)^*)f\|_2$ for $f \in \mathrm{Dom}(\mathfrak{A}_2)$ and $z \in \mathbb{C}$. In particular, $\overline{\sigma_{\mathrm{p}}(-(\mathfrak{A}_2)^*)} = \sigma_{\mathrm{p}}(-\mathfrak{A}_2)$. Hence, by (4.9) we have that

$$\overline{\sigma_{\mathrm{p}}\big(-(\mathfrak{A}_p)^*\big)} = \overline{\sigma_{\mathrm{p}}\big(-(\mathfrak{A}_2)^*\big)} = \sigma_{\mathrm{p}}(-\mathfrak{A}_2).$$

Since $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, we have that $\alpha \in \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$. However, this conflicts with the disjointness of $\sigma_{\mathrm{r}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$. Hence, $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$.

By Theorem 4.3 and the disjointness of $\sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, we have $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ for $p \in (1,\infty)$. □

In Section 5 we consider a sufficient condition for hyperboundedness via logarithmic Sobolev inequalities. It is to be obtained that spectra are the same for $p \in (1,\infty)$ if generators are normal (not necessarily symmetric) and the assumptions hold in Theorem 5.1.

Now we consider the relation between ultracontractivity and $\{\gamma_{p\to p}; p \in [1,\infty]\}$. If there exist positive constants K and C such that

$$\|T_K f\|_\infty \le C\|f\|_1, \quad f \in L^1(m),$$

then $\{T_t\}$ is called *ultracontractive*. In the case in which $\{T_t\}$ is symmetric, we have the following proposition.

PROPOSITION 4.5

If $\{T_t\}$ is symmetric on $L^2(m)$, then $\{T_t\}$ is ultracontractive if and only if there exists $q \in [1,\infty)$ such that

$$\|T_K f\|_\infty \le C\|f\|_q, \quad f \in L^q(m) \tag{4.10}$$

with some positive constants K and C.

Proof

It is sufficient to show that ultracontractivity holds if (4.10) holds for some q, K, and C. It is immediately obtained that $\{T_t\}$ is (p,q)-hyperbounded for any $p \in (1,\infty)$. Hence, by Theorem 3.2 there exists $K' > 0$ such that $\|T_{K'}\|_{q^*\to q} < \infty$. The symmetry of $\{T_t\}$ on $L^2(m)$ implies that $\|T_t\|_{1\to q^*} = \|T_t^*\|_{1\to q^*}$. On the other hand, by the duality we have that $\|T_t^*\|_{1\to q^*} = \|T_t\|_{q\to\infty}$. Hence, (4.10) implies that $\|T_K\|_{1\to q^*} = \|T_K\|_{q\to\infty} < \infty$. Thus, we have that

$$\|T_{2K+K'}\|_{1\to\infty} \le \|T_K\|_{1\to q^*}\|T_{K'}\|_{q^*\to q}\|T_K\|_{q\to\infty} < \infty.$$ □

When $\{T_t\}$ is ultracontractive, we can discuss the p-independence of the spectra of the generator of $\{T_t\}$ for $p \in [1,\infty)$ in the same way as in the case of hyperbounded Markovian semigroups.

THEOREM 4.6

Assume that $\{T_t\}$ is ultracontractive, and assume that $\mathfrak{A}_2$ is a normal operator. Then, $\sigma(-\mathfrak{A}_p) = \sigma(-\mathfrak{A}_2)$ for $p \in [1,\infty)$. Moreover, $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$, and $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$ for $p \in [1,\infty)$.

Note that $\{T_t\}$ is not necessarily symmetric (or, equivalently, $\mathfrak{A}_2$ is not) in Theorem 4.6.

REMARK 4.7
If $\{T_t\}$ is symmetric on $L^2(m)$ and ultracontractive, the compactness of T_t on $L^p(m)$ for $p \in (1,\infty)$ and $t \geq K$ is to be obtained (see [2, Theorem 13.4.2]).

REMARK 4.8
When $T_t f(x) = \int f(y) p_t(x,y) m(dy)$ and

$$\int\int \left|p_K(x,y)\right|^2 m(dy)m(dx) < \infty$$

holds for some $K > 0$, we have the compactness of T_K on $L^2(m)$ by [2, Theorem 4.2.16]. Therefore, the p-independence of spectra is obtained (see Remark 6.8).

5. Nonsymmetric Markovian semigroups and logarithmic Sobolev inequality

In Section 4 we obtain some sufficient conditions for the spectra of a Markovian semigroup $\{T_t\}$ on $L^p(m)$ to be independent of $p \in (1,\infty)$. In this section we consider a sufficient condition for nonsymmetric Markovian semigroups to satisfy hyperboundedness.

Let (M,m) and $\{T_t\}$ be the same as in Section 2. However, in this section, the finiteness of m is not needed. Let $\mathfrak{A}_p$ be the generator of $\{T_t\}$ on $L^p(m)$. We often denote $\mathfrak{A}_p$ by $\mathfrak{A}$ simply. Let $\{R_\alpha\}$ be the resolvent operator of $\{T_t\}$ on $L^2(m)$, and define

$$\mathscr{D} := R_1\bigl(L^1(m) \cap L^\infty(m)\bigr).$$

Then, $\mathscr{D} \subset \mathrm{Dom}(\mathfrak{A}_p)$ for $p \in [1,\infty]$ and $\mathscr{D} \subset L^1(m) \cap L^\infty(m)$.

We prepare another supplementary symmetric semigroup $\{S_t\}$ on $L^2(m)$. Let $\mathscr{E}$ be the Dirichlet form associated with $\{S_t\}$. Let $\alpha \in (0,\infty)$, let $\beta \in [0,\infty)$, and assume that

$$\text{(5.1)} \qquad \int \left|f(x)\right|^2 \log\bigl(\left|f(x)\right|^2/\|f\|_2^2\bigr) m(dx) \leq \alpha\mathscr{E}(f,f) + \beta\|f\|_2^2, \quad f \in L^2(m).$$

This inequality is called a *defective logarithmic Sobolev inequality*. In the case in which $\alpha > 0$ and $\beta = 0$, (5.1) is called a *logarithmic Sobolev inequality*. Additionally assume the following.

$$\text{(5.2)} \qquad \begin{gathered} \text{For } p > 1 \text{ and } f \in \mathscr{D}, |f|^{p/2} \in \mathrm{Dom}(\mathscr{E}) \text{ and} \\ \frac{4(p-1)}{p^2}\mathscr{E}\bigl(|f|^{p/2}, |f|^{p/2}\bigr) \leq -\bigl(\mathfrak{A}f, |f|^{p-1}\operatorname{sgn}(f)\bigr). \end{gathered}$$

When T_t is symmetric on $L^2(m)$, by letting $S_t := T_t$ we have (5.2) (see [3, proof of Theorem 6.1.14]).

THEOREM 5.1

Assume (5.1) and (5.2). Then, we have that

$$\|T_t\|_{p\to q} \le \exp\Big\{\beta\Big(\frac{1}{p}-\frac{1}{q}\Big)\Big\}$$

for $t>0$ *and* $1<p\le q<\infty$ *such that* $e^{4t/\alpha}\ge (q-1)/(p-1)$. *Hence,* $\{T_t\}$ *is hyperbounded. Moreover,* $\{T_t\}$ *is hypercontractive if* $\beta=0$.

Proof

The proof is just the same as [3, proof of Theorem 6.1.14]. Let $f\in\mathscr{D}$, and denote T_tf by f_t. Let $q(t):=1+(p-1)e^{4t/\alpha}$. By following [3, proof of Theorem 6.1.14] we have that

$$\begin{aligned}&\|f_t\|_{q(t)}^{q(t)-1}\frac{d}{dt}\|f_t\|_{q(t)}\\ &\quad=\int |f_t|^{q(t)-1}\operatorname{sgn}(f_t)\mathfrak{A}f_t\,dm+\frac{q'(t)}{q(t)^2}\int |f_t|^{q(t)}\log\big(|f_t|^{q(t)}/\|f_t\|_{q(t)}^{q(t)}\big)\,dm.\end{aligned}$$

By (5.2) we obtain that

$$\begin{aligned}&\|f_t\|_{q(t)}^{q(t)-1}\frac{d}{dt}\|f_t\|_{q(t)}\\ &\quad\le -\frac{4(q(t)-1)}{q(t)^2}\mathscr{E}\big(|f_t|^{q(t)/2},|f_t|^{q(t)/2}\big)+\frac{q'(t)}{q(t)^2}\int |f_t|^{q(t)}\log\big(|f_t|^{q(t)}/\|f_t\|_{q(t)}^{q(t)}\big)\,dm.\end{aligned}$$

Hence, we can continue our proof in the same way as [3, proof of Theorem 6.1.14] and obtain the conclusion. □

In Theorem 5.1 we assumed (5.1) and (5.2). Now, we give an example of a non-symmetric Markovian semigroup $\{T_t\}$ satisfying (5.1) and (5.2).

Let M be a complete Riemannian manifold, and let m be the volume measure on M. Denote the total set of vector fields on M by D. We define the basis measure ν on M by $\nu:=e^{-U}m$ where U is a C^∞-function on M such that $\int_M e^{-U}\,dm=1$. Let ∇ be an affine connection. Then, the dual ∇_ν^* of ∇ on $L^2(\nu)$ is characterized by $\nabla_\nu^*\theta=\nabla^*\theta+(\nabla U,\theta)$ for $\theta\in D$, where ∇^* is the dual of ∇ on $L^2(m)$.

Let $b\in D$, and consider the generator $\mathfrak{A}$ defined by

$$\mathfrak{A}=-\frac{1}{2}\nabla_\nu^*\nabla+b. \tag{5.3}$$

Then, the dual $\mathfrak{A}_\nu^*$ of $\mathfrak{A}$ on $L^2(\nu)$ satisfies

$$\mathfrak{A}_\nu^*=-\frac{1}{2}\nabla_\nu^*\nabla-b-\operatorname{div}_\nu b,$$

where div_ν is the divergence on $L^2(\nu)$, that is, div_ν is the linear operator on D which is characterized by

$$\int Xf\,d\nu=-\int f\operatorname{div}_\nu X\,d\nu,\quad f\in C_0^1(M).$$

Let $\mathfrak{B} := -\frac{1}{2}\nabla_\nu^*\nabla$, and let $\mathscr{E}$ be the Dirichlet form associated with $\mathfrak{B}$. Then,

$$\mathscr{E}(f,g) = -\frac{1}{2}\int (\operatorname{grad} f, \operatorname{grad} g)\, d\nu, \quad f,g \in C_0^\infty(M),$$

where $\operatorname{grad} f$ is the gradient of $f \in C^\infty(M)$. For $\mathfrak{B}$ to be a generator of a Markovian semigroup, we assume that the closure of $\mathfrak{B}$ defined on $C_0^\infty(M)$ is m-dissipative on $L^p(m)$ for $p \in [0,\infty]$. Sufficient conditions for the assumption are found in [9]. Additionally, we assume that

$$\operatorname{div}_\nu b \geq 0. \tag{5.4}$$

Under these assumptions we show (5.2). Since $\mathfrak{B}$ is symmetric on $L^2(\nu)$, (5.2) holds for $\mathfrak{B}$ and $\mathscr{E}$ (see the remark just after (5.2)). Hence, letting $\{G_\alpha\}$ be the resolvent associated with $\mathfrak{B}$, we have for $f \in G_1(L^1 \cap L^\infty)$ that

$$\frac{4(p-1)}{p^2}\mathscr{E}\big(|f|^{p/2}, |f|^{p/2}\big) \leq \frac{1}{2}\big(\nabla_\nu^*\nabla f, |f|^{p-1}\operatorname{sgn}(f)\big). \tag{5.5}$$

In particular, since $C_0^\infty(M) \subset G_1(L^1 \cap L^\infty)$, (5.5) holds for $f \in C_0^\infty(M)$. For $f \in C_0^\infty(M)$ we have that

$$\begin{aligned} -\big(\mathfrak{A}f, |f|^{p-1}\operatorname{sgn}(f)\big) &= \int \Big(\frac{1}{2}\nabla_\nu^*\nabla f - bf\Big)|f|^{p-1}\operatorname{sgn}(f)\, d\nu \\ &= \frac{1}{2}\int (\nabla_\nu^*\nabla f)|f|^{p-1}\operatorname{sgn}(f)\, d\nu - \int (bf)|f|^{p-1}\operatorname{sgn}(f)\, d\nu. \end{aligned}$$

By using (5.4),

$$-\int (bf)|f|^{p-1}\operatorname{sgn}(f)\, d\nu = -\frac{1}{p}\int b\big(|f|^p\big)\, d\nu = \frac{1}{p}\int (\operatorname{div}_\nu b)|f|^p\, d\nu \geq 0.$$

Hence, by (5.5) we obtain that

$$-\big(\mathfrak{A}f, |f|^{p-1}\operatorname{sgn}(f)\big) \geq \frac{4(p-1)}{p^2}\mathscr{E}\big(|f|^{p/2}, |f|^{p/2}\big), \quad f \in C_0^\infty(M). \tag{5.6}$$

Since each function f which belongs to $\operatorname{Dom}(\mathfrak{A}_p)$ can be approximated by a sequence $\{f_n\}$ in $C_0^\infty(M)$ with respect to the graph norm of $\mathfrak{A}_p$, (5.6) implies that $\sup_n \mathscr{E}(|f_n|^{p/2}, |f_n|^{p/2}) < \infty$. Hence, there exists a subsequence of $\{f_n\}$ which converges weakly with respect to the norm given by the inner product $\mathscr{E}_1(\cdot,\cdot) := (\cdot,\cdot) + \mathscr{E}(\cdot,\cdot)$. Denote the subsequence by $\{f_n\}$ again. Clearly, the limit of $\{f_n\}$ is f. By (5.6) we have that

$$\begin{aligned} \frac{4(p-1)}{p^2}\mathscr{E}\big(|f|^{p/2}, |f|^{p/2}\big) &\leq \liminf_{n\to\infty} \frac{4(p-1)}{p^2}\mathscr{E}\big(|f_n|^{p/2}, |f_n|^{p/2}\big) \\ &\leq -\limsup_{n\to\infty}\big(\mathfrak{A}f_n, |f_n|^{p-1}\operatorname{sgn}(f_n)\big) \\ &\leq -\big(\mathfrak{A}f, |f|^{p-1}\operatorname{sgn}(f)\big). \end{aligned}$$

Therefore, (5.2) holds.

For (5.1) we additionally assume that

$$\operatorname{Ric} + \operatorname{Hess} U \geq \varepsilon I$$

for some $\varepsilon > 0$. Then it is known that the logarithmic Sobolev inequality holds for $\mathfrak{B}$ (see [3, Theorem 6.2.42]). Hence, (5.1) holds.

By Theorem 5.1, the hyperboundedness holds. Furthermore, when we apply the results in Section 4, we need the conditions that ν is the invariant measure with respect to the semigroup generated by $\mathfrak{A}$ and that $\mathfrak{A}$ is normal on $L^2(\nu)$.

EXAMPLE 5.2

Let $M := \mathbb{R}^2$, let $\nu(dx) := (1/2\pi)e^{-|x|^2/2}\,dx$, and let

$$b = b_1(x)\frac{\partial}{\partial x_1} + b_2(x)\frac{\partial}{\partial x_2} := -cx_2\frac{\partial}{\partial x_1} + cx_1\frac{\partial}{\partial x_2},$$

where c is a positive constant. Then,

$$\mathfrak{A} = -\frac{1}{2}\nabla_\nu^*\nabla + b = \frac{1}{2}\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big) - x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + b.$$

Hence, the diffusion associated with $\mathfrak{A}$ is the Ornstein–Uhlenbeck diffusion with rotation. In this case, by explicit calculation we have that ν is the invariant measure and that $\mathfrak{A}$ is normal on $L^2(\nu)$.

6. Properties on spectra of operators on L^p-spaces

In this section we consider consistent linear operators on L^p-spaces and discuss their spectra with respect to L^p-spaces. Let (M, m) be a probability space, and let $L^p(m)$ be the L^p-space of $\mathbb{C}$-valued functions with respect to m. For $p \in [1,\infty)$ let A_p be a densely defined closed linear operator on $L^p(m)$, and assume that $\{A_p; p \in [1,\infty)\}$ are consistent; that is, if $p > q$, then $\mathrm{Dom}(A_p) \subset \mathrm{Dom}(A_q)$ and $A_p f = A_q f$ for $f \in \mathrm{Dom}(A_p)$. Moreover, assume that A_p is a real operator for some $p \in [1,\infty)$. Note that A_p is a real operator for all $p \in [1,\infty)$ by this assumption. A Markovian semigroup $\{T_t\}$ and its generators $\{\mathfrak{A}_p; p \in [1,\infty)\}$ defined in Section 2 satisfy the assumption on $\{A_p; p \in [1,\infty)\}$. Since the argument below is applicable to both $\{T_t\}$ and $\{\mathfrak{A}_p; p \in [1,\infty)\}$, we prepare $\{A_p; p \in [1,\infty)\}$ as a unified notation. Also note that, when we consider a Markovian semigroup $\{T_t\}$ as $\{A_p\}$, the results below include the case in which $p = \infty$.

In this section, we additionally assume that A_2 is self-adjoint on $L^2(m)$, that is, that $A_2 = A_2^*$. By using consistency it is easy to see that $(A_p)^* = A_{p^*}$ for $p \in [1,\infty)$. We denote A_p by simply A when confusion does not occur.

LEMMA 6.1

We have that $\sigma_{\mathrm{r}}(A_p) = \emptyset$ for $p \le 2$.

Proof

Assuming that there exists $\lambda \in \sigma_{\mathrm{r}}(A_p)$, we will make a contradiction. Then, there exists $f \in L^{p^*}(m) \setminus \{0\}$ such that $\langle (\lambda - A)g, f\rangle = 0$ for $g \in \mathrm{Dom}(A_p)$. Since $g \mapsto \langle Ag, f\rangle = \langle g, \bar{\lambda} f\rangle$ is a bounded linear functional on $\mathrm{Dom}(A_p)$, $f \in \mathrm{Dom}((A_p)^*) = \mathrm{Dom}(A_{p^*})$ and $Af = \bar{\lambda} f$. On the other hand, $f \in \mathrm{Dom}(A_{p^*}) \setminus \{0\} \subset \mathrm{Dom}(A_p) \setminus \{0\}$. This implies that f is an eigenfunction of A_p with respect to the eigenvalue $\bar{\lambda}$.

By Lemma 2.6 we have that $\lambda \in \sigma_{\mathrm{p}}(A_p)$. This conflicts with the disjointness of $\sigma_{\mathrm{p}}(A_p)$ and $\sigma_{\mathrm{r}}(A_p)$. □

PROPOSITION 6.2

We have the following.

(i) $\sigma_{\mathrm{p}}(A_p) \subset \sigma_{\mathrm{p}}(A_q)$ *for* $q \le p$.
(ii) $\sigma_{\mathrm{r}}(A_q) \subset \sigma_{\mathrm{r}}(A_p)$ *for* $q \le p$.
(iii) $\sigma_{\mathrm{c}}(A_p) \subset \sigma_{\mathrm{c}}(A_q) \cup \sigma_{\mathrm{p}}(A_q)$ *for* $q \le p \le 2$.
(iv) $\rho(A_q) \subset \rho(A_p)$ *for* $q \le p \le 2$.

Proof

Let $\lambda \in \sigma_{\mathrm{p}}(A_p)$. Then there exists $f \in \mathrm{Dom}(A_p) \setminus \{0\}$ such that $\lambda f = Af$. This implies that $\lambda \in \sigma_{\mathrm{p}}(A_q)$, because $f \in \mathrm{Dom}(A_p) \setminus \{0\} \subset \mathrm{Dom}(A_q) \setminus \{0\}$. Therefore, we have (i).

Next we prove (ii). Let $\lambda \in \sigma_{\mathrm{r}}(A_q)$. If $\lambda \in \sigma_{\mathrm{p}}(A_p)$, by (i) we have that $\lambda \in \sigma_{\mathrm{p}}(A_q)$. This conflicts with the fact that $\sigma_{\mathrm{p}}(A_q)$ and $\sigma_{\mathrm{r}}(A_q)$ are disjoint from each other. Thus, $\lambda \notin \sigma_{\mathrm{p}}(A_p)$. Since $\lambda \in \sigma_{\mathrm{r}}(A_q)$, there exists $f \in L^{q^*}(m) \setminus \{0\}$ and $\langle (\lambda - A)g, f\rangle = 0$ for $g \in \mathrm{Dom}(A_q)$. Noting that $q^* \ge p^*$, we have that $f \in L^{p^*}(m) \setminus \{0\}$ and that $\langle (\lambda - A)g, f\rangle = 0$ for $g \in \mathrm{Dom}(A_p)$. Hence, $\lambda \in \sigma_{\mathrm{r}}(A_p)$. Thus, (ii) follows.

Now we show (iv). Let $q \le p \le 2$. Let $\lambda \in \rho(A_q)$. Note that $\rho(A_q) = \rho(A_{q^*})$. Let $(\lambda - A_q)^{-1}$ and $(\lambda - A_{q^*})^{-1}$ be the resolvent operators of A_q and A_{q^*} with respect to λ, respectively. Define a linear operator $R_\lambda^{(p)}$ on $L^p(m)$ by $R_\lambda^{(p)} f := (\lambda - A_q)^{-1} f$ for $f \in \mathrm{Dom}(R_\lambda^{(p)})$, where $\mathrm{Dom}(R_\lambda^{(p)}) := \{f \in L^p(m); (\lambda - A_q)^{-1} f \in L^p(m)\}$. Then, $R_\lambda^{(p)}$, $(\lambda - A_q)^{-1}$, and $(\lambda - A_{q^*})^{-1}$ are consistent. Hence, $L^{q^*}(m) \subset \mathrm{Dom}(R_\lambda^{(p)})$, and $\mathrm{Dom}(R_\lambda^{(p)})$ is dense in $L^p(m)$. By the Riesz–Thorin theorem we have that

$$\|R_\lambda^{(p)}\|_{p\to p} \le \big\|(\lambda - A_q)^{-1}\big\|_{q\to q}^{1-\theta} \big\|(\lambda - A_{q^*})^{-1}\big\|_{q^*\to q^*}^{\theta},$$

where $\theta \in [0,1]$ satisfies $1/p = (1-\theta)/q + \theta/q^*$. This implies that $\|R_\lambda^{(p)}\|_{p\to p} < \infty$. By the definition of $R_\lambda^{(p)}$ we have that

$$(\lambda - A_p) R_\lambda^{(p)} = I, \quad \text{on } \mathrm{Dom}(R_\lambda^{(p)}),$$
$$R_\lambda^{(p)}(\lambda - A_p) = I, \quad \text{on } \mathrm{Dom}(A_p),$$

and therefore the closure of $R_\lambda^{(p)}$ is the resolvent operator of A_p with respect to λ. Hence, $\lambda \in \rho(A_p)$ and we have (iv).

We obtain (iii) by (iv) and Lemma 6.1. □

REMARK 6.3

By Proposition 6.2(iv) we have that $\sigma(A_p)$ is decreasing for $p \in [1,2]$ and increasing for $p \in [2,\infty)$.

COROLLARY 6.4

Let $p \in [2,\infty)$. Then the following hold.

(i) $\sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{r}}(A_p) = \sigma_{\mathrm{p}}(A_{p^*})$.
(ii) $\sigma_{\mathrm{c}}(A_p) = \sigma_{\mathrm{c}}(A_{p^*})$.

Proof

By Proposition 6.2(i), we have that $\sigma_{\mathrm{p}}(A_p) \subset \sigma_{\mathrm{p}}(A_{p^*})$. By an argument similar to that in the proof of Lemma 6.1, it holds that $\sigma_{\mathrm{r}}(A_p) \subset \sigma_{\mathrm{p}}(A_{p^*})$. Hence, we have that

$$\sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{r}}(A_p) \subset \sigma_{\mathrm{p}}(A_{p^*}). \tag{6.1}$$

Let $\lambda \in \sigma_{\mathrm{p}}(A_{p^*})$, and let S be the total set of $f \in \mathrm{Dom}(A_{p^*})$ such that $\lambda f = Af$. Since $\lambda \in \sigma_{\mathrm{p}}(A_{p^*})$, $S \neq \{0\}$. If $L^p(m) \cap S \neq \{0\}$, then $\lambda \in \sigma_{\mathrm{p}}(A_p)$. Consider the case in which $L^p(m) \cap S = \{0\}$. Then, $\lambda \notin \sigma_{\mathrm{p}}(A_p)$. Take $f \in S \setminus \{0\}$. Then, it holds that $\langle \lambda f, g\rangle = \langle Af, g\rangle$ for $g \in L^p(m)$. Hence, by the symmetry of A we have that $\langle f, \bar{\lambda} g\rangle = \langle f, Ag\rangle$ for $g \in \mathrm{Dom}(A_p)$. Here, note the definition of $\langle \cdot,\cdot\rangle$ in Section 1. On the other hand, since $\lambda \notin \sigma_{\mathrm{p}}(A_p)$, we have that $\bar{\lambda} \notin \sigma_{\mathrm{p}}(A_p)$ by Lemma 2.6. These facts imply that $\bar{\lambda} \in \sigma_{\mathrm{r}}(A_p)$. By Lemma 2.6 again, we have that $\lambda \in \sigma_{\mathrm{r}}(A_p)$. Thus,

$$\sigma_{\mathrm{p}}(A_{p^*}) \subset \sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{r}}(A_p). \tag{6.2}$$

By (6.1) and (6.2) we have (i).

Since $\sigma(A_p) = \sigma(A_{p^*})$, we have (ii). □

COROLLARY 6.5

We have that $\sigma_{\mathrm{p}}(A_p) \subset \mathbb{R}$ for $p \in [2,\infty)$.

Proof

The assertion immediately follows by Proposition 6.2(i) and $\sigma(A_2) \subset \mathbb{R}$. □

REMARK 6.6

Since A_2 is a self-adjoint operator, by using the general theory of self-adjoint operators on Hilbert spaces it is obtained that $\sigma(A_2) \subset \mathbb{R}$. However, when $p \neq 2$, it does not always hold. An example that $\sigma(A_p) \not\subset \mathbb{R}$ when $p \neq 2$ is given in Section 7.

Let $\lambda_p^{\min} := \min\{|\lambda|; \lambda \in \sigma(A_p)\}$, and let $\lambda_p^{\max} := \max\{|\lambda|; \lambda \in \sigma(A_p)\}$ for $p \in [1,\infty)$. Note that the minimum and the maximum above exist in $[0,\infty]$, because $\sigma(A_p)$ is closed set in $\mathbb{C}$. The following corollary follows immediately from Proposition 6.2(iv).

COROLLARY 6.7

We have that $\lambda_q^{\min} \geq \lambda_p^{\min}$ and $\lambda_q^{\max} \geq \lambda_p^{\max}$ for $q \in [1, \min\{p,p^*\}] \cup [\max\{p,p^*\}, \infty)$.

This corollary gives the relation of the exponential rate of convergence for Markovian semigroups. For example, let $A_p = \mathfrak{A}_p$, where $\mathfrak{A}_p$ is the generator of the Markovian semigroup on $L^p(m)$ defined in Section 2. Then, $\lambda_p^{\min}$ is the distance between 0 and $\sigma(\mathfrak{A}_p)$. For another example, let A_p be $T_t^{(p)} - m$ for some $t > 0$, where $T_t^{(p)}$ is the Markovian semigroup on $L^p(m)$ defined in Section 2. Then, $\lambda_p^{\max} = \mathrm{Rad}(T_t^{(p)} - m)$. As mentioned in Section 2, these are related to the rate of convergence of the Markovian semigroups.

REMARK 6.8

In [2, Chapter 4] spectra of consistent bounded operators are considered. When we additionally assume that A_p is bounded for any $p \in [1, \infty)$ and that A_p is compact for some $p \in [1, \infty)$, then the p-independence of spectra of A_p is obtained by using Schauder's theorem (see [2, Theorem 4.2.13]) and [2, Theorem 4.2.14].

7. Example in which $\gamma_{p\to p}$ depends on p

In Section 4 we give a sufficient condition for the spectra of a Markovian semigroup as an operator on $L^p(m)$ to be independent of p. However, generally the spectra depend on p. We give an example so that the spectra depend on p in this section.

Let $p \in [1, \infty)$. Define a measure ν on $[0, \infty)$ by $\nu(dx) := e^{-x}\,dx$, and define a differential operator $\mathfrak{A}_p^\circ$ with its domain $\mathrm{Dom}(\mathfrak{A}_p^\circ)$ by

$$\mathrm{Dom}(\mathfrak{A}_p^\circ) := \big\{ f \in C_0^2\big([0,\infty);\mathbb{C}\big); f'(0) = 0 \big\},$$
$$\mathfrak{A}_p^\circ := \frac{d^2}{dx^2} - \frac{d}{dx}.$$

Consider a generator $\mathfrak{A}_p$ by the closed extension of $\mathfrak{A}_p^\circ$ on $L^p(\nu)$. Note that $\mathfrak{A}_2$ is a self-adjoint operator on $L^2(\nu)$. This is an example that the spectra $\sigma(\mathfrak{A}_p)$ depend on p and $\gamma_{q\to q} < \gamma_{p\to p}$ for $q < p \le 2$. Now, we show them by investigating $\sigma(\mathfrak{A}_p)$ explicitly.

Let $p \in [1, 2]$. Consider the linear transformation I defined by

$$(If)(x) := e^{-x/2} f(x). \tag{7.1}$$

Then, we have that

$$\int_0^\infty \big|If(x)\big|^p e^{(p/2-1)x}\,dx = \int_0^\infty \big|f(x)\big|^p \nu(dx),$$

and $f'(0) = 0$ if and only if $\frac{1}{2}(If)(0) + (If)'(0) = 0$ for $f \in C^1([0,\infty);\mathbb{C})$. Hence, I is an isometric transformation from $L^p(\nu)$ to $L^p(\tilde{\nu}_p)$, where $\tilde{\nu}_p := e^{(p/2-1)x}\,dx$. Define a linear operator $\tilde{\mathfrak{A}}_p$ on $L^p(\tilde{\nu}_p)$ by

$$\begin{aligned} \mathrm{Dom}(\tilde{\mathfrak{A}}_p) &:= \Big\{ \tilde{f} \in W^{2,p}(\tilde{\nu}_p); \frac{1}{2}\tilde{f}(0) + \tilde{f}'(0) = 0 \Big\}, \\ \tilde{\mathfrak{A}}_p \tilde{f} &:= \frac{d^2}{dx^2}\tilde{f} - \frac{1}{4}\tilde{f}. \end{aligned} \tag{7.2}$$

Then, we have for $\tilde{f} \in C_0^\infty([0,\infty);\mathbb{C})$

$$\begin{aligned}(I \circ \mathfrak{A}_p \circ I^{-1})\tilde{f}(x) &= e^{-x/2}\Big(\frac{d^2}{dx^2} - \frac{d}{dx}\Big)e^{x/2}\tilde{f}(x) \\ &= \tilde{f}''(x) + \tilde{f}'(x) + \frac{1}{4}\tilde{f}(x) - \tilde{f}'(x) - \frac{1}{2}\tilde{f}(x) \\ &= \tilde{f}''(x) - \frac{1}{4}\tilde{f}(x).\end{aligned}$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} L^p(\nu) & \xrightarrow{\mathfrak{A}_p} & L^p(\nu) \\ I\downarrow & & \downarrow I \\ L^p(\tilde{\nu}_p) & \xrightarrow{\tilde{\mathfrak{A}}_p} & L^p(\tilde{\nu}_p) \end{array}$$

By this diagram we have

$$\sigma_{\mathrm{p}}(\mathfrak{A}_p) = \sigma_{\mathrm{p}}(\tilde{\mathfrak{A}}_p), \qquad \sigma_{\mathrm{c}}(\mathfrak{A}_p) = \sigma_{\mathrm{c}}(\tilde{\mathfrak{A}}_p), \qquad \text{and} \qquad \sigma_{\mathrm{r}}(\mathfrak{A}_p) = \sigma_{\mathrm{r}}(\tilde{\mathfrak{A}}_p). \tag{7.3}$$

Hence, to see the spectra of $\mathfrak{A}_p$, it is sufficient to see the spectra of $\tilde{\mathfrak{A}}_p$.

From here on we cannot discuss the cases in which $1 \le p < 2$ and $p = 2$ in the same way. First we consider the case in which $1 \le p < 2$. Let $\sqrt{z} := \sqrt{r}e^{i\theta/2}$ for $z \in \mathbb{C}$ where $z = re^{i\theta}$ such that $r \ge 0$ and $\theta \in (-\pi,\pi]$.

LEMMA 7.1

If $1 \le p < 2$, then

$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\} \cup \Big\{x + iy; x, y \in \mathbb{R}, x > \frac{p-1}{p^2}, |y| < \Big(\frac{2}{p} - 1\Big)\sqrt{x - \frac{p-1}{p^2}}\Big\}.$$

Proof

Let $u(x) = x - 2$ for $x \in [0,\infty)$. Then, $u \in L^p(\tilde{\nu}_p)$,

$$-\frac{d^2}{dx^2}u + \frac{1}{4}u = \frac{1}{4}u, \qquad \text{and} \qquad \frac{1}{2}u(0) + u'(0) = 0.$$

Hence,

$$\frac{1}{4} \in \sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p). \tag{7.4}$$

Let $\lambda \in \mathbb{C} \setminus \{\frac{1}{4}\}$. Consider the differential equation

$$-\frac{d^2}{dx^2}u + \frac{1}{4}u = \lambda u, \tag{7.5}$$

where $u : [0,\infty) \to \mathbb{C}$. Then, u is the solution of (7.5) if and only if

$$u(x) = C_1 e^{x\sqrt{-\lambda+1/4}} + C_2 e^{-x\sqrt{-\lambda+1/4}},$$

where C_1, C_2 are constants in $\mathbb{C}$. Note that $\frac{1}{2}u(0) + u'(0) = 0$ if and only if $C_1(1/2 + \sqrt{-\lambda + 1/4}) + C_2(1/2 - \sqrt{-\lambda + 1/4}) = 0$. Hence, u is the solution of the following boundary value problem on $[0,\infty)$:

$$\begin{cases} -\frac{d^2}{dx^2}u + \frac{1}{4}u = \lambda u, \\ \frac{1}{2}u(0) + u'(0) = 0, \end{cases}$$

if and only if

$$\begin{cases} u(x) = C_1 e^{x\sqrt{-\lambda+1/4}} + C_2 e^{-x\sqrt{-\lambda+1/4}}, \\ C_1(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}) + C_2(\frac{1}{2} - \sqrt{-\lambda+\frac{1}{4}}) = 0. \end{cases} \tag{7.6}$$

When u satisfies (7.6),

$$\begin{aligned}
\frac{1}{2}|C_1|^p \int_0^\infty & e^{(\mathrm{Re}\sqrt{-\lambda+1/4})px} e^{(p/2-1)x}\,dx - |C_2|^p \int_0^\infty e^{-(\mathrm{Re}\sqrt{-\lambda+1/4})px} e^{(p/2-1)x}\,dx \\
&\le \int_0^\infty \left|u(x)\right|^p e^{(p/2-1)x}\,dx \\
&\le 2|C_1|^p \int_0^\infty e^{(\mathrm{Re}\sqrt{-\lambda+1/4})px} e^{(p/2-1)x}\,dx \\
&\quad + 2|C_2|^p \int_0^\infty e^{-(\mathrm{Re}\sqrt{-\lambda+1/4})px} e^{(p/2-1)x}\,dx.
\end{aligned}$$

This implies that

$$u \in L^p(\tilde{\nu}_p) \quad \text{if and only if} \quad p\,\mathrm{Re}\sqrt{-\lambda+1/4} + \frac{p}{2} - 1 < 0 \quad \text{or } C_1 = 0. \tag{7.7}$$

By (7.6), if $C_1 = 0$, then $\lambda = 0$ or $C_2 = 0$. Therefore, (7.4) and (7.7) imply that $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\} \cup \{\lambda \in \mathbb{C}; \mathrm{Re}\sqrt{-\lambda+1/4} < \frac{1}{p} - \frac{1}{2}\}$. □

LEMMA 7.2

If $1 \le p < 2$, then

$$\rho(-\tilde{\mathfrak{A}}_p) \supset \left\{x + iy; x, y \in \mathbb{R}, y^2 > \left(\frac{2}{p} - 1\right)^2 \left(x - \frac{p-1}{p^2}\right)\right\} \setminus \{0\}.$$

Proof

It is sufficient to show that $\{z \in \mathbb{C} \setminus \{0\}; \mathrm{Re}\sqrt{-z+1/4} > \frac{1}{p} - \frac{1}{2}\} \subset \rho(-\tilde{\mathfrak{A}}_p)$. For $\lambda \in \{z \in \mathbb{C} \setminus \{0\}; \mathrm{Re}\sqrt{-z+1/4} > \frac{1}{p} - \frac{1}{2}\}$ let

$$\begin{aligned}
\phi_\lambda(x) &:= \Big(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\Big) e^{x\sqrt{-\lambda+1/4}} \\
&\qquad - \Big(\frac{1}{2} + \sqrt{-\lambda+\frac{1}{4}}\Big) e^{-x\sqrt{-\lambda+1/4}}, \quad x \in [0,\infty); \\
\psi_\lambda(x) &:= e^{-x\sqrt{-\lambda+1/4}}, \quad x \in [0,\infty); \\
W_\lambda &:= -2\sqrt{-\lambda+\frac{1}{4}}\Big(\frac{1}{2} - \sqrt{-\lambda+\frac{1}{4}}\Big),
\end{aligned}$$

and define a $\mathbb{C}$-valued function g_λ on $[0,\infty)\times[0,\infty)$ by

$$g_\lambda(x,y) := \begin{cases} \frac{1}{W_\lambda}\phi_\lambda(x)\psi_\lambda(y), & x\le y, \\ \frac{1}{W_\lambda}\phi_\lambda(y)\psi_\lambda(x), & y\le x. \end{cases}$$

Let $G_\lambda f(x) := \int_0^\infty g_\lambda(x,y)f(y)\,dy$ for $f\in C_0([0,\infty);\mathbb{C})$. Then, by explicit calculation, we have for $f\in C_0([0,\infty);\mathbb{C})$

$$\{\lambda-(-\tilde{\mathfrak{A}}_p)\}G_\lambda f = f, \qquad \text{and} \qquad \frac{1}{2}G_\lambda f(0) + (G_\lambda f)'(0) = 0.$$

In view of Lemmas 6.1 and 7.1, to show that $\lambda\in\rho(-\tilde{\mathfrak{A}}_p)$, it is sufficient to prove the boundedness of the operator G_λ on $L^p(\tilde{\nu}_p)$. Let

$$C_\lambda(\varepsilon) := \sup_{y\in[0,\infty)} e^{(1-p/2)y}\Big(\int_0^\infty \big|g_\lambda(x,y)\big|^{(1-\varepsilon)p} e^{(p/2-1)x}\,dx\Big),$$

$$C'_\lambda(\varepsilon) := \sup_{x\in[0,\infty)} \Big(\int_0^\infty \big|g_\lambda(x,y)\big|^{\varepsilon p^*}\,dy\Big)^{p/p^*}$$

for $\varepsilon\in(0,1)$. By explicit calculation, we have $C_\lambda(\varepsilon)<\infty$ when $\operatorname{Re}\sqrt{-\lambda+1/4} > [1/(1-\varepsilon)](\frac{1}{p}-\frac{1}{2})$, and $C'_\lambda(\varepsilon)<\infty$. By Hölder's inequality we have that

$$\begin{aligned}
&\|G_\lambda f\|^p_{L^p(\tilde{\nu}_p)} \\
&= \int_0^\infty \Big|\int_0^\infty g_\lambda(x,y)f(y)\,dy\Big|^p e^{(p/2-1)x}\,dx \\
&\le \int_0^\infty \Big[\int_0^\infty \big|g_\lambda(x,y)\big|^{1-\varepsilon}\big|f(y)\big|\cdot\big|g_\lambda(x,y)\big|^{\varepsilon}\,dy\Big]^p e^{(p/2-1)x}\,dx \\
&\le \int_0^\infty \Big(\int_0^\infty \big|g_\lambda(x,y)\big|^{(1-\varepsilon)p}\big|f(y)\big|^p\,dy\Big)\Big(\int_0^\infty \big|g_\lambda(x,y)\big|^{\varepsilon p^*}\,dy\Big)^{p/p^*} e^{(p/2-1)x}\,dx \\
&\le C'_\lambda(\varepsilon)\int_0^\infty \Big(\int_0^\infty \big|g_\lambda(x,y)\big|^{(1-\varepsilon)p} e^{(p/2-1)x}\,dx\Big)\big|f(y)\big|^p\,dy \\
&\le C'_\lambda(\varepsilon)C_\lambda(\varepsilon)\|f\|^p_{L^p(\tilde{\nu}_p)}.
\end{aligned}$$

Since this estimate holds for all $\varepsilon\in(0,1)$, $\{z\in\mathbb{C}\setminus\{0\};\operatorname{Re}\sqrt{-z+1/4} > \frac{1}{p}-\frac{1}{2}\}\subset\rho(\tilde{\mathfrak{A}}_p)$. □

By the lemmas above, the spectra of $-\tilde{\mathfrak{A}}_p$ are determined exactly.

THEOREM 7.3

The following hold for $1\le p<2$.

(i) $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\}\cup\{x+iy; x,y\in\mathbb{R}, x>(p-1)/p^2,$ *and* $|y| < (\frac{2}{p}-1)\times\sqrt{x-(p-1)/p^2}\}$,

(ii) $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_p) = \{x+iy; x,y\in\mathbb{R}, x\ge(p-1)/p^2,$ *and* $|y| = (\frac{2}{p}-1)\times\sqrt{x-(p-1)/p^2}\}\setminus\{0\}$,

(iii) $\rho(-\tilde{\mathfrak{A}}_p) = \{x+iy; x,y\in\mathbb{R}$ *and* $y^2 > (\frac{2}{p}-1)^2(x-(p-1)/p^2)\}\setminus\{0\}$.

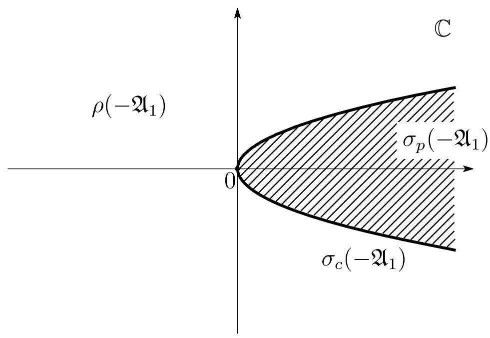

Figure 1. $p = 1$.

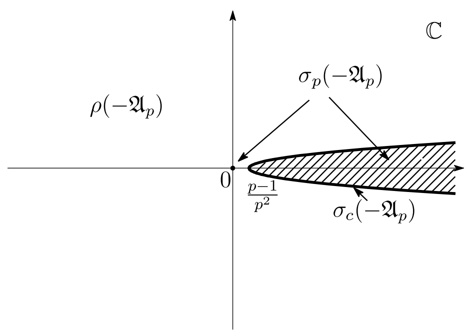

Figure 2. $1 < p < 2$.

Proof
The assertion (i) is obtained in Lemma 7.1. Since any limit point of point spectra is either a point spectrum or a continuous spectrum, by (i) and Lemma 7.2, we have (ii). By (i), (ii), and Lemma 6.1, we obtain (iii). □

By (7.3) we have the following theorem.

THEOREM 7.4
The following hold for $1 \le p < 2$.

(i) $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) = \{0\} \cup \{x + iy; x, y \in \mathbb{R}, x > (p-1)/p^2,$ *and* $|y| < (\frac{2}{p} - 1) \times \sqrt{x - (p-1)/p^2}\}$,
(ii) $\sigma_{\mathrm{c}}(-\mathfrak{A}_p) = \{x + iy; x, y \in \mathbb{R}, x \ge (p-1)/p^2,$ *and* $|y| = (\frac{2}{p} - 1) \times \sqrt{x - (p-1)/p^2}\} \setminus \{0\}$,
(iii) $\rho(-\mathfrak{A}_p) = \{x + iy; x, y \in \mathbb{R}$ *and* $y^2 > (\frac{2}{p} - 1)^2 (x - (p-1)/p^2)\} \setminus \{0\}$.

The pictures of $\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_p)$, and $\rho(-\mathfrak{A}_p)$ for $p = 1$ and for $1 < p < 2$ are described in Figures 1 and 2.

Next we check $\sigma(-\tilde{\mathfrak{A}}_2)$. Note that $\tilde{\nu}_p$ is equal to the Lebesgue measure dx when $p=2$. Since $-\tilde{\mathfrak{A}}_2$ is self-adjoint and nonnegative definite on $L^2(dx)$, we know that $\sigma(-\tilde{\mathfrak{A}}_2)\subset[0,\infty)$ and $\sigma_{\mathrm{r}}(-\tilde{\mathfrak{A}}_2)=\emptyset$ (see Lemma 6.1). The purpose of the argument below is to investigate both $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2)$ and $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2)$ explicitly.

LEMMA 7.5
We have that

$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2)=\{0\}.$$

Proof
The assertion follows in almost the same way as the proof of Lemma 7.5 except the part of checking whether $\frac{1}{4}$ is a point spectrum or not. Let u be the unique solution of the differential equation

$$-\frac{d^2}{dx^2}u+\frac{1}{4}u=\frac{1}{4}u \qquad \text{and} \qquad \frac{1}{2}u(0)+u'(0)=0.$$

Then $u(x)=x-2$. Since $u\notin L^2(dx)$, $\frac{1}{4}\notin\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2)$. The rest of the proof is the same as that of Lemma 7.5. □

We have already obtained $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2)$ and $\sigma_{\mathrm{r}}(-\tilde{\mathfrak{A}}_2)$ explicitly in Lemmas 6.1 and 7.5. Now we investigate $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2)$. Since any limit point of point spectra is either a point spectrum or a continuous spectrum, it was easy to see $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_p)$ for $1\le p<2$. However, it is impossible to discuss continuous spectra in a similar way for the cases in which $p=2$ and $1\le p<2$. Recall that by (7.3) it is sufficient to check the spectra of $\tilde{\mathfrak{A}}_2$ on $L^2(dx)$ defined on (7.2).

Let $\mathscr{E}$ and $\tilde{\mathscr{E}}$ be the bilinear forms associated with $\mathfrak{A}_2$ and $\tilde{\mathfrak{A}}_2$, respectively. Then, for $f,g\in C_b^2([0,\infty))$ such that $f(x)=g(x)=0$ for $x>M$ with some $M>0$, we have

$$\begin{aligned}
\tilde{\mathscr{E}}(f,g)&=\mathscr{E}(I^{-1}f,I^{-1}g)\\
&=\int_0^\infty \bigl(e^{x/2}f(x)\bigr)'\bigl(e^{x/2}g(x)\bigr)'e^{-x}\,dx\\
&=\int_0^\infty \Bigl(f'(x)g'(x)+\frac{1}{2}f'(x)g(x)+\frac{1}{2}f(x)g'(x)+\frac{1}{4}f(x)g(x)\Bigr)\,dx \qquad (7.8)\\
&=\int_0^\infty f'(x)g'(x)\,dx+\frac{1}{4}\int_0^\infty f(x)g(x)\,dx+\frac{1}{2}\int_0^\infty \bigl(f(x)g(x)\bigr)'\,dx\\
&=\int_0^\infty f'(x)g'(x)\,dx+\frac{1}{4}\int_0^\infty f(x)g(x)\,dx-\frac{1}{2}f(0)g(0).
\end{aligned}$$

Denote the Sobolev space on $[0,\infty)$ with measure dx and indices k,p by $W^{k,p}(dx)$, where k is the index for differentiability and p is the index for integrability. Let

$$\mathrm{Dom}(\tilde{\mathfrak{A}}_2^{(0)}):=\bigl\{f\in W^{2,2}(dx);f'(0)=0\bigr\},$$

$$\tilde{\mathfrak{A}}_2^{(0)}:=\frac{d^2}{dx^2}-\frac{1}{4},$$

and let $\tilde{\mathscr{E}}^{(0)}$ be the bilinear form associated with $\tilde{\mathfrak{A}}_2^{(0)}$. Then, by using integration by parts, we have for $f, g \in W^{2,2}(dx) \cap \{f \in C_b^2([0,\infty)); f'(0) = 0$ and $\lim_{x\to\infty} f(x) = 0\}$

$$\begin{aligned} \tilde{\mathscr{E}}^{(0)}(f,g) &= -\int_0^\infty (\tilde{\mathfrak{A}}_2^{(0)} f)(x) g(x)\, dx \\ &= -\int_0^\infty f''(x) g(x)\, dx + \frac{1}{4}\int_0^\infty f(x)g(x)\, dx \\ &= \int_0^\infty f'(x)g'(x)\, dx + \frac{1}{4}\int_0^\infty f(x)g(x)\, dx. \end{aligned} \tag{7.9}$$

Define a norm $\|\cdot\|_{\tilde{\mathscr{E}}_1^{(0)}}$ by

$$\|f\|^2_{\tilde{\mathscr{E}}_1^{(0)}} = \tilde{\mathscr{E}}^{(0)}(f,f) + \int_0^\infty |f(x)|^2\, dx.$$

Then, by standard calculation we have that the closure of $\mathrm{Dom}(\tilde{\mathfrak{A}}_2^{(0)})$ with respect to $\|\cdot\|_{\tilde{\mathscr{E}}_1^{(0)}}$ is equal to $W^{1,2}(dx)$. Hence, $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) = W^{1,2}(dx)$. Now we have the following proposition.

PROPOSITION 7.6
We have that $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) = \mathrm{Dom}(\tilde{\mathscr{E}})$.

Proof
Since $\tilde{\mathscr{E}}(f,f) \le \mathscr{E}^{(0)}(f,f)$ for $f \in C_0^\infty([0,\infty))$, $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) \subset \mathrm{Dom}(\tilde{\mathscr{E}})$. To show that $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) \supset \mathrm{Dom}(\tilde{\mathscr{E}})$, it is sufficient to show that $f \mapsto f(0)$ is a continuous linear functional on $W^{1,2}(dx)$. Let $f \in C_0^\infty([0,\infty))$. Since $f(x) = f(0) + \int_0^x f'(y)\, dy$, we have that

$$\begin{aligned} |f(0)|^2 &= \int_0^1 \Big| f(x) - \int_0^x f'(y)\, dy \Big|^2 dx \\ &\le 2\int_0^1 |f(x)|^2\, dx + 2\int_0^1 \Big| \int_0^x f'(y)\, dy\Big|^2 dx \\ &\le 2\int_0^\infty |f(x)|^2\, dx + 2\int_0^1 \sqrt{x}\Big(\int_0^x |f'(y)|^2\, dy\Big)\, dx \\ &\le 2\int_0^\infty |f(x)|^2\, dx + 2\int_0^\infty |f'(y)|^2\, dy. \end{aligned}$$

Hence, $f \mapsto f(0)$ is a continuous linear functional $W^{1,2}(dx)$. □

Now we extend the operators $\tilde{\mathfrak{A}}_2$ and $\tilde{\mathfrak{A}}_2^{(0)}$ in the same way as in the argument written in [11, Section 2.2]. Let $H := L^2(dx)$, let $V := \mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) = \mathrm{Dom}(\tilde{\mathscr{E}})$, and let V^* be the dual space of V. By the Riesz theorem, the dual of H can be identified with H^*. By this identification, we can regard $V \subset H = H^* \subset V^*$. Noting that V and H are dense subsets of H and V^*, respectively, the operator $\tilde{\mathfrak{A}}_2$ can be extended to an operator from V to V^*. Denote the extension of

$\tilde{\mathfrak{A}}_2$ by $\mathfrak{B}$. For $\lambda \in (0,\infty)$, $\lambda - \mathfrak{B}$ is a bijection from V to V^*, and the inverse $(\lambda - \mathfrak{B})^{-1} : V^* \to V$ is an extension of the resolvent $(\lambda - \tilde{\mathfrak{A}}_2)^{-1} : H \to \mathrm{Dom}(\tilde{\mathfrak{A}}_2)$. We also define $\mathfrak{B}^{(0)}$ from $\tilde{\mathfrak{A}}_2^{(0)}$ similarly. Note that $\mathfrak{B}^{(0)}$ has the same properties as $\mathfrak{B}$.

Denote the essential spectra of a linear operator A by $\sigma_{\mathrm{ess}}(A)$. The definition of essential spectra is in [6, Chapter XII, Section 2]. Then, we have the following proposition.

PROPOSITION 7.7
We have that $\sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2) = \sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2^{(0)}) = [\frac{1}{4}, \infty)$.

Proof
It is well known that $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2^{(0)}) = \emptyset$ and $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2^{(0)}) = [\frac{1}{4}, \infty)$. Since $-\tilde{\mathfrak{A}}_2$ and $-\tilde{\mathfrak{A}}_2^{(0)}$ are nonnegative definite, $-1 \in \rho(-\tilde{\mathfrak{A}}_2) \cap \rho(-\tilde{\mathfrak{A}}_2^{(0)})$. Once we have the compactness of the bounded linear operator $(1 - \tilde{\mathfrak{A}}_2)^{-1} - (1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}$ on H, we obtain the conclusion by Weyl's theorem (see [6, Theorem XIII.14]):

$$\begin{aligned}
&(1 - \tilde{\mathfrak{A}}_2)^{-1} - (1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} \\
&\quad = (1 - \tilde{\mathfrak{A}}_2)^{-1}(1 - \tilde{\mathfrak{A}}_2^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} - (1 - \tilde{\mathfrak{A}}_2)^{-1}(1 - \tilde{\mathfrak{A}}_2)(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} \\
&\quad = (1 - \mathfrak{B})^{-1}(1 - \mathfrak{B}^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} - (1 - \mathfrak{B})^{-1}(1 - \mathfrak{B})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} \\
&\quad = (1 - \mathfrak{B})^{-1}(\mathfrak{B} - \mathfrak{B}^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}.
\end{aligned}$$

The linear operator $(1 - \mathfrak{B})^{-1}(\mathfrak{B} - \mathfrak{B}^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}$ is the following mapping:

$$H \xrightarrow{(1-\tilde{\mathfrak{A}}_2^{(0)})^{-1}} \mathrm{Dom}(\tilde{\mathfrak{A}}_2^{(0)}) \hookrightarrow V \xrightarrow{\mathfrak{B}-\mathfrak{B}^{(0)}} V^* \xrightarrow{(1-\mathfrak{B})^{-1}} V \hookrightarrow H.$$

Since $(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}$ and $(1 - \mathfrak{B})^{-1}$ are continuous, it is sufficient to show the compactness of the operator $\mathfrak{B} - \mathfrak{B}^{(0)}$ from V to V^*. By (7.8) and (7.9) we have for $f, g \in V$ that

$${}_{V^*}\big\langle (\mathfrak{B} - \mathfrak{B}^{(0)}) f, g \big\rangle_V = \frac{1}{2} f(0) g(0).$$

This implies that $\mathfrak{B} - \mathfrak{B}^{(0)}$ is a mapping $f \mapsto f(0)\delta$, where $\delta \in V^*$ is a bounded linear operator on V defined by $\delta(g) = g(0)$ for V. Hence, the range of $\mathfrak{B} - \mathfrak{B}^{(0)}$ is one-dimensional. This concludes the compactness of $\mathfrak{B} - \mathfrak{B}^{(0)}$. □

By Lemma 7.5 and Proposition 7.7 we obtain explicit information on the spectra of $\tilde{\mathfrak{A}}_2$ as follows.

THEOREM 7.8
It holds that

$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2) = \{0\}, \qquad \sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2) = \Big[\frac{1}{4}, \infty\Big).$$

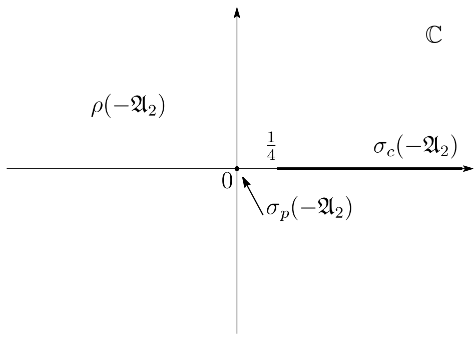

Figure 3. $p = 2$.

Proof
We have already obtained that $\sigma_{\rm p}(-\tilde{\mathfrak{A}}_2) = \{0\}$ in Lemma 7.5. Noting that $\sigma_{\rm p}(-\tilde{\mathfrak{A}}_2) \cap \sigma_{\rm ess}(-\tilde{\mathfrak{A}}_2) = \emptyset$, by the definition of essential spectra we have that $\sigma_{\rm c}(-\tilde{\mathfrak{A}}_2) = \sigma_{\rm ess}(-\tilde{\mathfrak{A}}_2) = [\frac{1}{4}, \infty)$. □

By (7.3) we have the following theorem.

THEOREM 7.9
It holds that

$$\sigma_{\rm p}(-\mathfrak{A}_2) = \{0\}, \qquad \sigma_{\rm c}(-\mathfrak{A}_2) = \Big[\frac{1}{4}, \infty\Big).$$

The picture of $\sigma_{\rm p}(-\mathfrak{A}_2)$, $\sigma_{\rm c}(-\mathfrak{A}_2)$, and $\rho(-\mathfrak{A}_2)$ is described in Figure 3.

By Theorems 7.4 and 7.9 we obtain the spectra of $-\mathfrak{A}_p$ exactly for $p \in [1, 2]$ as described in Figures 1, 2, and 3.

We have considered only the case in which $1 \le p \le 2$. We also obtain $\sigma_{\rm p}(-\mathfrak{A}_p)$, $\sigma_{\rm c}(-\mathfrak{A}_p)$, and $\sigma_{\rm r}(-\mathfrak{A}_p)$ explicitly for $p \in (2, \infty)$ by using Proposition 6.2, Corollary 6.4, and Theorems 7.4 and 7.9.

THEOREM 7.10
For $p \in (2, \infty)$, we have the following:

(i) $\sigma_{\rm p}(-\mathfrak{A}_p) = \{0\}$,
(ii) $\sigma_{\rm c}(-\mathfrak{A}_p) = \{x + iy; x, y \in \mathbb{R}, x \ge (p^* - 1)/{p^*}^2,$ *and* $|y| = (\frac{2}{p^*} - 1) \times \sqrt{x - (p^* - 1)/{p^*}^2}\} \setminus \{0\}$,
(iii) $\sigma_{\rm r}(-\mathfrak{A}_p) = \{x + iy; x, y \in \mathbb{R}, x > (p^* - 1)/{p^*}^2,$ *and* $|y| < (\frac{2}{p^*} - 1) \times \sqrt{x - (p^* - 1)/{p^*}^2}\}$,
(iv) $\rho(-\mathfrak{A}_p) = \{x + iy; x, y \in \mathbb{R}, y^2 > (\frac{2}{p^*} - 1)^2(x - (p^* - 1)/{p^*}^2)\} \setminus \{0\}$.

Proof
Let $p \in (2, \infty)$. Since $\sigma(-\mathfrak{A}_p) = \sigma(-\mathfrak{A}_{p^*})$, we have (iv). By Theorem 7.4 and

Corollary 6.4 we obtain (ii). By Corollary 6.4 again, we have that

$$\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \cup \sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \sigma_{\mathrm{p}}(-\mathfrak{A}_{p^*}). \tag{7.10}$$

On the other hand, applying Proposition 6.2 for $q = 2$, we have that $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \subset \sigma_{\mathrm{p}}(-\mathfrak{A}_2)$. Hence, Theorem 7.9 implies that $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \subset \{0\}$. Since $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \supset \{0\}$, we obtain (i). By (7.10), (i), and Theorem 7.4, we have (iii). □

This operator $-\mathfrak{A}_p$ is an example in which the spectra depend on p, the spectra are not included by $\mathbb{R}$ for $p \neq 2$, and $\sigma_{\mathrm{c}}(-\mathfrak{A}_q) \subset \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$ for some $p < q \leq 2$ even if $-\mathfrak{A}_p$ is a diffusion operator, consistent on $L^p(\nu)$ for $p \in [1,\infty)$, self-adjoint when $p = 2$, and ergodic.

In view of the argument in Section 2, the exact information on the spectra of $-\mathfrak{A}_p$ gives the explicit value of $\gamma_{p\to p}$ as follows.

COROLLARY 7.11
We have that

$$\gamma_{p\to p} = \frac{p-1}{p^2}, \quad p \in [1,\infty].$$

Proof
Since $-\mathfrak{A}_2$ is self-adjoint on $L^2(\nu)$, the argument in Section 2 is available and (2.5) holds. By (2.5) we have that $\gamma_{p\to p} = (p-1)/p^2$ for $p \in (1,2]$. By Theorem 2.4 we have $0 \leq \gamma_{1\to 1} \leq \inf\{\gamma_{p\to p}; p \in [1,2]\} = 0$. Hence, $\gamma_{1\to 1} = 0$. By Theorem 2.4 again $\gamma_{p\to p} = \gamma_{p^*\to p^*}$ for $p \in [1,\infty]$. Therefore, the assertion holds. □

Thus, we obtain an example for which the exponential rate of convergence $\{\gamma_{p\to p}; p \in [1,\infty]\}$ depends on p.

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