Nef cone of flag bundles over a curve

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Abstract Let X be a smooth projective curve defined over an algebraically closed field k, and let E be a vector bundle on X. Let $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ be the tautological line bundle over the Grassmann bundle $\mathrm{Gr}_r(E)$ parameterizing all the r-dimensional quotients of the fibers of E. We give necessary and sufficient conditions for $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ to be ample and nef, respectively. As an application, we compute the nef cone of $\mathrm{Gr}_r(E)$. This yields a description of the nef cone of any flag bundle over X associated to E.

1. Introduction

Let E be a semistable vector bundle over a smooth projective curve defined over an algebraically closed field of characteristic zero. Miyaoka [Mi, p. 456, Theorem 3.1] computed the nef cone of $\mathbb{P}(E)$. Our aim here is to compute the nef cone of the flag bundles associated to vector bundles over curves.

Let X be an irreducible smooth projective curve defined over an algebraically closed field k. (The characteristic is not necessarily zero.) If the characteristic of k is positive, the absolute Frobenius morphism of X will be denoted by F_X. A vector bundle E on X is called *strongly semistable* if all the pullbacks of E by the iterations of F_X are semistable.

Let E be a vector bundle on X. Let

$$E_1 \subset E_2 \subset \cdots \subset E_{m-1} \subset E_m = E \tag{1.1}$$

be the Harder–Narasimhan filtration of E. If the characteristic of k is zero and

$$f : Y \longrightarrow X$$

is a nonconstant morphism, where Y is an irreducible smooth projective curve, then the pulled-back filtration

$$f^*E_1 \subset f^*E_2 \subset \cdots \subset f^*E_{m-1} \subset f^*E_m = f^*E$$

coincides with the Harder–Narasimhan filtration of f^*E. If the characteristic of k is positive, then this is not true in general. However, there is an integer n_E, which depends on E, such that the Harder–Narasimhan filtration of $(F_X^n)^*E$ has this property if $n \geq n_E$, meaning that the Harder–Narasimhan filtration of $f^*(F_X^n)^*E$ is the pullback, by f, of the Harder–Narasimhan filtration of $(F_X^n)^*E$, where f is any nonconstant morphism to X from an irreducible smooth projective curve.

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Fix an integer $r \in [1, \text{rank}(E) - 1]$. Let $\text{Gr}_r(E)$ be the Grassmann bundle on X parameterizing all the r-dimensional quotients of the fibers of E. The tautological line bundle on $\text{Gr}_r(E)$ is denoted by $\mathcal{O}_{\text{Gr}_r(E)}(1)$.

If the characteristic of k is positive, consider the Harder–Narasimhan filtration of $(F_X^{n_E})^*E$

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{d-1} \subset V_d = (F_X^{n_E})^*E,$$

where n_E is as above; if the characteristic of k is zero, then simply take the Harder–Narasimhan filtration of E. So V_i is E_i in (1.1) if the characteristic of k is zero. Using only the numerical data associated to this filtration, we can compute a rational number $\theta_{E,r}$ (see (3.5)). The following theorem shows that $\theta_{E,r}$ controls the positivity of the tautological line bundle $\mathcal{O}_{\text{Gr}_r(E)}(1)$ on $\text{Gr}_r(E)$.

THEOREM 1.1

If $\theta_{E,r} > 0$, then the tautological line bundle $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is ample.

If $\theta_{E,r} = 0$, then $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is nef but not ample.

If $\theta_{E,r} < 0$, then $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is not nef.

(See Theorem 3.4 for a proof of this theorem.)

As an application of Theorem 1.1, we compute the nef cone of $\text{Gr}_r(E)$. (This is done in Section 4.)

In order to know the nef cone of a flag bundle over X associated to E, it is enough to know the nef cones of the corresponding Grassmann bundles associated to E. Therefore, using our description of the nef cone of the Grassmann bundles, we obtain a description of the nef cone of any flag bundle over X associated to E (see Theorem 5.1).

Let $K_\varphi^{-1} := K_{\text{Gr}_r(E)}^{-1} \otimes \varphi^* K_X$ be the relative anticanonical line bundle for the natural projection $\varphi : \text{Gr}_r(E) \longrightarrow X$. It is known that K_φ^{-1} is never ample. If the characteristic of k is zero, then K_φ^{-1} is nef if and only if E is semistable (see [BB]); if the characteristic of k is positive, then K_φ^{-1} is nef if and only if E is strongly semistable (see [BH]). These criteria for semistability and strong semistability follow from the description of the nef cone of $\text{Gr}_r(E)$ given in Propositions 4.1 and 4.4.

2. Preliminaries

Let k be an algebraically closed field. Let X be an irreducible smooth projective curve defined over k. If the characteristic of k is positive, then we have the absolute Frobenius morphism

$$F_X : X \longrightarrow X.$$

For convenience, if the characteristic of k is zero, we denote by F_X the identity morphism of X. For any integer $m \geq 1$, let

$$F_X^m := \overbrace{F_X \circ \cdots \circ F_X}^{m \text{ times}} : X \longrightarrow X$$

be the m-fold iteration of F_X. For notational convenience, we denote by F_X^0 the identity morphism of X.

For a vector bundle E over X of positive rank, define the number

$$\mu(E) := \frac{\text{degree}(E)}{\text{rank}(E)} \in \mathbb{Q}.$$

A vector bundle E over X is called *semistable* if, for every nonzero subbundle $V \subset E$, the inequality

$$\mu(V) \leq \mu(E)$$

holds. The vector bundle E is called *strongly semistable* if the pullback $(F_X^m)^*E$ is semistable for all $m \geq 0$.

For every vector bundle E on X, there is a unique filtration of subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{d_E - 1} \subset E_{d_E} = E$$

such that E_i/E_{i-1} is semistable for each $i \in [1, d_E]$, and $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for all $i \in [1, d_E - 1]$. It is known as the *Harder–Narasimhan filtration* of E. If E is semistable, then $d_E = 1$.

Given any E, there is a nonnegative integer δ satisfying the condition that, for all $i \geq 1$,

$$0 = (F_X^i)^*V_0 \subset (F_X^i)^*V_1 \subset \cdots \subset (F_X^i)^*V_{d-1} \subset (F_X^i)^*V_d = (F_X^{i+\delta})^*E \tag{2.1}$$

is the Harder–Narasimhan filtration of $(F_X^{i+\delta})^*E$, where

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = (F_X^\delta)^*E \tag{2.2}$$

is the Harder–Narasimhan filtration of $(F_X^\delta)^*E$ (see [Lan, p. 259, Theorem 2.7]). (This is vacuously true if the characteristic of k is zero.) It should be emphasized that δ in (2.1) depends on E.

Note that the quotient V_i/V_{i-1} in the filtration in (2.2) is strongly semistable for all $i \in [1, d]$. If δ satisfies the above condition, then clearly $\delta + j$ also satisfies the above condition for all $j \geq 0$.

For a vector bundle E on X, let $\mathbb{P}(E)$ denote the projective bundle over X parameterizing all the hyperplanes in the fibers of E. The vector bundle E is called *ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is ample (see [Ha] for properties of ample bundles).

A line bundle L over an irreducible projective variety Z defined over k is called *numerically effective* (nef for short) if for all pairs of the form (C, f), where C is a smooth projective curve and f is a morphism from C to Z, the inequality

$$\text{degree}(f^*L) \geq 0$$

holds. A vector bundle E is called *nef* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is nef.

The following lemma is well known.

LEMMA 2.1
Let $0 \longrightarrow W \longrightarrow E \longrightarrow Q \longrightarrow 0$ *be a short exact sequence of vector bundles. If both* W *and* Q *are ample (resp., nef), then* E *is ample (resp., nef).*

See [Ha, p. 71, Corollary 3.4] for the case of ample bundles and [DPS, p. 308, Proposition 1.15(ii)] for the case of nef vector bundles.

3. (Semi)Positivity criterion

Let E be a vector bundle over X of rank at least two. Fix an integer $r \in [1, \mathrm{rank}(E) - 1]$. Let

$$\varphi : \mathrm{Gr}_r(E) \longrightarrow X \tag{3.1}$$

be the Grassmann bundle over X parameterizing all the quotients, of dimension r, of the fibers of E. Let

$$\mathcal{O}_{\mathrm{Gr}_r(E)}(1) \longrightarrow \mathrm{Gr}_r(E) \tag{3.2}$$

be the tautological line bundle; the fiber of $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ over any quotient Q of E_x is $\bigwedge^r Q$. So the line bundle $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is relatively ample.

Take any δ satisfying the condition in (2.1). Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = (F_X^\delta)^* E \tag{3.3}$$

be the Harder–Narasimhan filtration of $(F_X^\delta)^* E$. We recall that V_i/V_{i-1} is strongly semistable for all $i \in [1, d]$. Let

$$t \in [1, d]$$

be the unique largest integer such that

$$\sum_{i=t}^{d} \mathrm{rank}(V_i/V_{i-1}) \geq r, \tag{3.4}$$

so either $t = d$, or t is the smallest integer with

$$\sum_{i=t+1}^{d} \mathrm{rank}(V_i/V_{i-1}) = \mathrm{rank}\big(((F_X^\delta)^* E)/V_t\big) < r.$$

Define

$$\theta_{E,r} := \big(r - \mathrm{rank}\big(((F_X^\delta)^* E)/V_t\big)\big) \cdot \mu(V_t/V_{t-1}) + \mathrm{degree}\big(((F_X^\delta)^* E)/V_t\big), \tag{3.5}$$

where t is defined above using (3.4). If E is strongly semistable, then we may take $\delta = 0$; in that case, $\theta_{E,r} = r \cdot \mu(E)$. Note that the condition that $\theta_{E,r}$ is nonzero, or the condition that $\theta_{E,r}$ is positive, does not depend on the choice of the integer δ in (3.3).

LEMMA 3.1

Assume that $\theta_{E,r} > 0$. Then the line bundle $\mathcal{O}_{\mathrm{Gr}_r(E)}(1) \longrightarrow \mathrm{Gr}_r(E)$ in (3.2) is ample.

Proof

Consider the Plücker embedding

$$\rho : \mathrm{Gr}_r(E) \longrightarrow \mathbb{P}\Big(\bigwedge^r E\Big). \tag{3.6}$$

We have that

$$\rho^* \mathcal{O}_{\mathbb{P}(\bigwedge^r E)}(1) = \mathcal{O}_{\mathrm{Gr}_r(E)}(1). \tag{3.7}$$

Therefore, to prove that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is ample, it suffices to show that the vector bundle $\bigwedge^r E$ is ample. Since F_X^δ is a finite flat surjective morphism, it follows that $\bigwedge^r E$ is ample if and only if $(F_X^\delta)^* \bigwedge^r E$ is ample (see [Ha, p. 73, Proposition 4.3]).

By the filtration in (3.3) it follows that the vector bundle $(F_X^\delta)^* \bigwedge^r E$ admits a filtration of subbundles such that each successive quotient is of the form

$$V_{\underline{a}} := \bigotimes_{i=1}^{d} \bigwedge^{a_i} (V_i/V_{i-1}) \tag{3.8}$$

with $\sum_{i=1}^d a_i = r$; we use the standard convention that $\bigwedge^0 F$ is the trivial line bundle for every vector bundle F. Since each V_i/V_{i-1} is strongly semistable, the above vector bundle $V_{\underline{a}}$ is also strongly semistable (see [RR, p. 285, Theorem 3.18] for $\mathrm{Char}(k) = 0$ and [RR, p. 288, Theorem 3.23] for $\mathrm{Char}(k) > 0$). From the assumption that $\theta_{E,r} > 0$, it follows immediately that

$$\mathrm{degree}(V_{\underline{a}}) > 0. \tag{3.9}$$

Since $V_{\underline{a}}$ is strongly semistable of positive degree, it can be shown that $V_{\underline{a}}$ is ample (see [BP]). We include the details for completeness.

To prove that $V_{\underline{a}}$ is ample, we need to show that, for any coherent sheaf $\mathcal{E}$ on X, there is a positive integer $b_{\mathcal{E}}$ such that

$$H^1\big(X, \mathrm{Sym}^j(V_{\underline{a}}) \otimes \mathcal{E}\big) = 0 \tag{3.10}$$

for all $j \geq b_{\mathcal{E}}$ (see [Ha, p. 70, Proposition 3.3]). Since $H^1(X, \mathrm{Sym}^j(V_{\underline{a}}) \otimes \mathcal{E}) = 0$, if $\mathcal{E}$ is a torsion sheaf, and any vector bundle on X admits a filtration of subbundles such that each successive quotient is a line bundle, it is enough to prove (3.10) for all line bundles $\mathcal{E}$. Take a line bundle $\mathcal{E}$. Since $V_{\underline{a}}$ is strongly semistable, it follows that $\mathrm{Sym}^j(V_{\underline{a}})$ is semistable for all $j \geq 1$ (see [RR, p. 285, Theorem 3.18] for $\mathrm{Char}(k) = 0$ and [RR, p. 288, Theorem 3.23] for $\mathrm{Char}(k) > 0$). Therefore, the vector bundle $\mathrm{Sym}^j(V_{\underline{a}})^* \otimes \mathcal{E}^* \otimes K_X$ is semistable. Now, from (3.9), we conclude that

$$\mu\big(\mathrm{Sym}^j(V_{\underline{a}})^* \otimes \mathcal{E}^* \otimes K_X\big) = -j \cdot \mu(V_{\underline{a}}) - \mathrm{degree}(\mathcal{E}) + 2\big(\mathrm{genus}(X) - 1\big) < 0$$

for all j sufficiently large and positive. Consequently,

$$H^0\big(X, \mathrm{Sym}^j(V_{\underline{a}})^* \otimes \mathcal{E}^* \otimes K_X\big) = 0$$

for all j sufficiently large and positive. Therefore, from Serre duality,

$$H^1\big(X, \mathrm{Sym}^j(V_{\underline{a}}) \otimes \mathcal{E}\big) = 0$$

for all j sufficiently large and positive. Hence, $V_{\underline{a}}$ is ample.

We note that if the characteristic of k is zero, then the nef cone of the projective bundle $\mathbb{P}(V_{\underline{a}})$ is explicitly described in [Mi, p. 456, Theorem 3.1(4)]. It is straightforward to check that the tautological line bundle $\mathcal{O}_{\mathbb{P}(V_{\underline{a}})}(1)$ lies in the interior of the nef cone of $\mathbb{P}(V_{\underline{a}})$. This also proves that $V_{\underline{a}}$ is ample under the assumption that the characteristic of k is zero.

Since $V_{\underline{a}}$ is ample and $(F_X^\delta)^* \bigwedge^r E$ admits a filtration of subbundles such that each successive quotient is of the form $V_{\underline{a}}$, using Lemma 2.1 we conclude that the vector bundle $(F_X^\delta)^* \bigwedge^r E$ is ample. We noted earlier that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is ample if $(F_X^\delta)^* \bigwedge^r E$ is ample. □

LEMMA 3.2

Assume that $\theta_{E,r}$ defined in (3.5) satisfies the inequality $\theta_{E,r} < 0$. Then $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is not nef.

Proof

Consider the strongly semistable vector bundle V_t/V_{t-1} (see (3.5)). Given any real number $\epsilon > 0$ and any $s \in [1, \mathrm{rank}(V_t/V_{t-1})]$, there exist an irreducible smooth projective curve Y, a nonconstant morphism

$$f : Y \longrightarrow X,$$

and a subbundle

$$W \subset f^*(V_t/V_{t-1}) \tag{3.11}$$

of rank s such that (see [PS, p. 525, Theorem 4.1])

$$\mu(V_t/V_{t-1}) - \frac{\mu(W)}{\mathrm{degree}(f)} = \frac{\mu(f^*(V_t/V_{t-1})) - \mu(W)}{\mathrm{degree}(f)} < \epsilon. \tag{3.12}$$

Set

$$s = r - \mathrm{rank}\big(((F_X^\delta)^*E)/V_t\big), \quad \text{and set } \epsilon = -\frac{\theta_{E,r}}{2s}.$$

Let Q be the quotient of $f^*(F_X^\delta)^*E$ defined by the composition

$$f^*(F_X^\delta)^*E \longrightarrow f^*\big(((F_X^\delta)^*E)/V_{t-1}\big) \longrightarrow f^*\big(((F_X^\delta)^*E)/V_{t-1}\big)/W,$$

where f and W are as in (3.11) for the above choices of s and ϵ. Note that

$$\mathrm{degree}(Q) = \mathrm{degree}(f) \cdot \mathrm{degree}\big(((F_X^\delta)^*E)/V_t\big) + \big(\mathrm{degree}(f) \cdot \mathrm{degree}(V_t/V_{t-1}) - \mathrm{degree}(W)\big).$$

Hence from (3.5),

$$\operatorname{degree}(Q)=\operatorname{degree}(f)\Big(\theta_{E,r}+\Big(\mu(V_t/V_{t-1})-\frac{\mu(W)}{\operatorname{degree}(f)}\Big)\cdot s\Big).$$

But from (3.12), we have $\mu(V_t/V_{t-1})-\mu(W)/\operatorname{degree}(f)<\epsilon$. Consequently,

$$\operatorname{degree}(Q)<0. \tag{3.13}$$

The quotient bundle $f^*(F_X^\delta)^*E\longrightarrow Q$ of rank r defines a morphism

$$\phi:Y\longrightarrow \operatorname{Gr}_r\big((F_X^\delta)^*E\big)=(F_X^\delta)^*\operatorname{Gr}_r(E),$$

where $\operatorname{Gr}_r((F_X^\delta)^*E)$ is the Grassmann bundle parameterizing all r-dimensional quotients of the fibers of $(F_X^\delta)^*E$, and $(F_X^\delta)^*\operatorname{Gr}_r(E)$ is the pullback of the fiber bundle $\operatorname{Gr}_r(E)\longrightarrow X$ using the morphism F_X^δ. Consider the commutative diagram

$$\begin{array}{ccc}(F_X^\delta)^*\operatorname{Gr}_r(E) & \stackrel{\beta}{\longrightarrow} & \operatorname{Gr}_r(E)\\ \downarrow & & \downarrow\\ X & \stackrel{F_X^\delta}{\longrightarrow} & X\end{array} \tag{3.14}$$

of morphisms. We have $\beta^*\mathcal{O}_{\operatorname{Gr}_r(E)}(1)=\mathcal{O}_{\operatorname{Gr}_r((F_X^\delta)^*E)}(1)$, where $\mathcal{O}_{\operatorname{Gr}_r((F_X^\delta)^*E)}(1)$ is the tautological line bundle, and β is the morphism in (3.14). Hence, from the definition of ϕ it follows immediately that

$$(\beta\circ\phi)^*\mathcal{O}_{\operatorname{Gr}_r(E)}(1)=\bigwedge^r Q.$$

Now from (3.13) we conclude that $\mathcal{O}_{\operatorname{Gr}_r(E)}(1)$ is not nef. □

LEMMA 3.3

Assume that $\theta_{E,r}=0$ (defined in (3.5)). Then $\mathcal{O}_{\operatorname{Gr}_r(E)}(1)$ is nef but not ample.

Proof

The proof that $\mathcal{O}_{\operatorname{Gr}_r(E)}(1)$ is nef is very similar to the proof of Lemma 3.1.

We know that $\bigwedge^r E$ is nef if and only if $(F_X^\delta)^*\bigwedge^r E$ is nef (see [Fu, p. 360, Propositions 2.2 and 2.3]). Consider the vector bundles $V_{\underline{a}}$ in (3.8). We noted earlier that $V_{\underline{a}}$ is strongly semistable. The condition that $\theta_{E,r}=0$ implies that

$$\operatorname{degree}(V_{\underline{a}})\geq 0.$$

A strongly semistable vector bundle W over X of nonnegative degree is nef. To prove this, take any morphism

$$\psi:Y\longrightarrow \mathbb{P}(W),$$

where Y is an irreducible smooth projective curve. Let $h:\mathbb{P}(W)\longrightarrow X$ be the natural projection. The pullback ψ^*h^*W is semistable because W is strongly semistable. Since $\psi^*\mathcal{O}_{\mathbb{P}(W)}(1)$ is a quotient of ψ^*h^*W and $\operatorname{degree}(\psi^*h^*W)\geq 0$, we conclude that $\operatorname{degree}(\psi^*\mathcal{O}_{\mathbb{P}(W)}(1))\geq 0$. Hence, $\mathcal{O}_{\mathbb{P}(W)}(1)$ is nef, meaning that W is nef.

The above observation implies that the vector bundle $V_{\underline{a}}$ is nef.

Since each successive quotient of the filtration of $(F_X^\delta)^* \bigwedge^r E$ is nef (as they are of the form $V_{\underline{a}}$), from Lemma 2.1 we know that $(F_X^\delta)^* \bigwedge^r E$ is nef. We noted earlier that $\bigwedge^r E$ is nef if $(F_X^\delta)^* \bigwedge^r E$ is so. Now using (3.6) and (3.7) we conclude that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is nef.

To complete the proof of the lemma we need to show that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is not ample.

Consider V_t/V_{t-1} in (3.5). Let

$$f : \mathrm{Gr}_s(V_t/V_{t-1}) \longrightarrow X \tag{3.15}$$

be the Grassmann bundle parameterizing quotients of the fibers of V_t/V_{t-1} of dimension

$$s := r - \mathrm{rank}\big(((F_X^\delta)^*E)/V_t\big). \tag{3.16}$$

Let

$$\gamma : \mathrm{Gr}_s(V_t/V_{t-1}) \longrightarrow \mathrm{Gr}_r\big((F_X^\delta)^*E\big) \tag{3.17}$$

be the morphism of fiber bundles over X that sends any quotient $q : (V_t/V_{t-1})_x \longrightarrow Q$ to the quotient defined by the composition

$$\big((F_X^\delta)^*E\big)_x \longrightarrow \big(((F_X^\delta)^*E)/V_{t-1}\big)_x \longrightarrow \big(\big(((F_X^\delta)^*E)/V_{t-1}\big)_x\big)/\,\mathrm{kernel}(q).$$

To define γ using the universal property of a Grassmannian, let

$$f^*(V_t/V_{t-1}) \xrightarrow{\widetilde{q}} \mathcal{Q} \longrightarrow 0$$

be the universal quotient bundle of rank s over $\mathrm{Gr}_s(V_t/V_{t-1})$. Now consider the diagram of homomorphisms

$$\begin{array}{cccccc} & & \mathrm{kernel}(\widetilde{q}) & \hookrightarrow & V_t/V_{t-1} & \xrightarrow{\widetilde{q}} \ \mathcal{Q} \\ & & \bigcap & & \bigcap & \\ f^*(F_X^\delta)^*E & \xrightarrow{\widehat{q}} & \big((F_X^\delta)^*E\big)/V_{t-1} & = & \big((F_X^\delta)^*E\big)/V_{t-1} & \\ & & \downarrow h & & & \\ & & \big(((F_X^\delta)^*E)/V_{t-1}\big)/\,\mathrm{kernel}(\widetilde{q}) & & & \end{array}$$

Note that $\mathrm{rank}((((F_X^\delta)^*E)/V_{t-1})/\,\mathrm{kernel}(\widetilde{q})) = r$ by (3.16). Let

$$\widetilde{\gamma} : \mathrm{Gr}_s(V_t/V_{t-1}) \longrightarrow \mathrm{Gr}_r\big(f^*(F_X^\delta)^*E\big) = \mathrm{Gr}_s(V_t/V_{t-1}) \times_X \mathrm{Gr}_r\big((F_X^\delta)^*E\big)$$

be the morphism representing the surjective homomorphism $h \circ \widehat{q}$ in the above diagram. The morphism γ in (3.17) is the composition of $\widetilde{\gamma}$ with the natural projection $\mathrm{Gr}_s(V_t/V_{t-1}) \times_X \mathrm{Gr}_r((F_X^\delta)^*E) \longrightarrow \mathrm{Gr}_r((F_X^\delta)^*E)$.

The morphism γ in (3.17) is clearly an embedding. Define the line bundle

$$\mathcal{L} := \det\big(((F_X^\delta)^*E)/V_t\big) = \bigotimes_{i=t+1}^{d} \bigwedge^{\mathrm{rank}(V_i/V_{i-1})}(V_i/V_{i-1})$$

on X. We note that

$$\gamma^*\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(1) = \mathcal{O}_{\mathrm{Gr}_s(V_t/V_{t-1})}(1) \otimes f^*\mathcal{L}, \tag{3.18}$$

where $\mathcal{O}_{\mathrm{Gr}_s(V_t/V_{t-1})}(1) \longrightarrow \mathrm{Gr}_s(V_t/V_{t-1})$ is the tautological line bundle.

For any integer n, the line bundles $\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(1)^{\otimes n}$ and $\mathcal{O}_{\mathrm{Gr}_s(V_t/V_{t-1})}(1)^{\otimes n}$ are denoted by $\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(n)$ and $\mathcal{O}_{\mathrm{Gr}_s(V_t/V_{t-1})}(n)$, respectively.

Assume that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is ample. Since F_X^δ is a finite morphism, this implies that $\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(1)$ is ample. Therefore, the pullback $\gamma^*\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(1)$ is ample because γ is an embedding. Hence, for sufficiently large positive n, we have

$$\dim H^0\big(\mathrm{Gr}_s(V_t/V_{t-1}), \gamma^*\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(n)\big) = cn^{d_0} + \sum_{j=0}^{d_0-1} a_j n^j \tag{3.19}$$

with $c > 0$, where $d_0 = \dim \mathrm{Gr}_s(V_t/V_{t-1})$.

For convenience, the integer $\mathrm{rank}(V_t/V_{t-1})$ is denoted by r_t.

Let $K_f^{-1} := K^{-1}_{\mathrm{Gr}_s(V_t/V_{t-1})} \otimes f^*K_X$ be the relative anticanonical line bundle for the projection f in (3.15). We have that

$$K_f^{-1} = \mathcal{O}_{\mathrm{Gr}_s(V_t/V_{t-1})}(r_t) \otimes \Big(\Big(\bigwedge^{r_t}(V_t/V_{t-1})\Big)^{\otimes s}\Big)^*, \tag{3.20}$$

where s is defined in (3.16). The given condition that $\theta_{E,r} = 0$ implies that

$$-s \cdot \mathrm{degree}(V_t/V_{t-1}) = r_t \cdot \mathrm{degree}\big(((F_X^\delta)^*E)/V_t\big).$$

Hence, the two line bundles $((\bigwedge^{r_t}(V_t/V_{t-1}))^{\otimes s})^*$ and $\mathcal{L}^{\otimes r_t}$ differ by tensoring with a line bundle of degree zero. Therefore, from (3.20) we conclude that

$$\big(\mathcal{O}_{\mathrm{Gr}_s(V_t/V_{t-1})}(1) \otimes \mathcal{L}\big)^{\otimes r_t} = K_f^{-1} \otimes f^*\mathcal{L}_0,$$

where $\mathcal{L}_0$ is a line bundle on X of degree zero. Now, from (3.18),

$$\gamma^*\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(r_t) = K_f^{-1} \otimes f^*\mathcal{L}_0. \tag{3.21}$$

From the projection formula and (3.21),

$$H^0\big(\mathrm{Gr}_s(V_t/V_{t-1}), \gamma^*\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(n \cdot r_t)\big) = H^0\big(X, \big(f_*(K_f^{-1})^{\otimes n}\big) \otimes \mathcal{L}_0^{\otimes n}\big). \tag{3.22}$$

We show that the line bundle $\det(f_*(K_f^{-1})^{\otimes n}) \longrightarrow X$ is trivial. For that, let $F_{\mathrm{GL}_{r_t}}$ be the principal $\mathrm{GL}_{r_t}(k)$-bundle on X defined by the vector bundle V_t/V_{t-1}; the fiber of $F_{\mathrm{GL}_{r_t}}$ over any point $x \in X$ is the space of all linear isomorphisms from $k^{\oplus r_t}$ to the fiber $(V_t/V_{t-1})_x$. Let $F_{\mathrm{PGL}_{r_t}} := F_{\mathrm{GL}_{r_t}}/\mathbb{G}_m$ be the corresponding principal $\mathrm{PGL}_{r_t}(k)$-bundle. The vector bundle $f_*(K_f^{-1})^{\otimes n}$ is the one associated to the principal $\mathrm{PGL}_{r_t}(k)$-bundle $F_{\mathrm{PGL}_{r_t}}$ for the $\mathrm{PGL}_{r_t}(k)$-module $H^0(\mathrm{Gr}_s(k^{\oplus r_t}), (K^{-1}_{\mathrm{Gr}_s(k^{\oplus r_t})})^{\otimes n})$. (The action of $\mathrm{PGL}_{r_t}(k)$ on the space of sections is given by the standard action of $\mathrm{PGL}_{r_t}(k)$ on $\mathrm{Gr}_s(k^{\oplus r_t})$.) Since $\mathrm{PGL}_{r_t}(k)$ does not have any nontrivial character, the line bundle $\det(f_*(K_f^{-1})^{\otimes n})$ associated to $F_{\mathrm{PGL}_{r_t}}$ for the $\mathrm{PGL}_{r_t}(k)$-module $\bigwedge^{\mathrm{top}} H^0(\mathrm{Gr}_s(k^{\oplus r_t}), (K^{-1}_{\mathrm{Gr}_s(k^{\oplus r_t})})^{\otimes n})$ is trivial.

As $\det(f_*(K_f^{-1})^{\otimes n})$ is trivial and $\mathrm{degree}(\mathcal{L}) = 0$,

$$\mathrm{degree}\big(\big(f_*(K_f^{-1})^{\otimes n}\big) \otimes \mathcal{L}_0^{\otimes n}\big) = 0.$$

Since V_t/V_{t-1} is strongly semistable, the corresponding principal $\mathrm{GL}_{r_t}(k)$-bundle $F_{\mathrm{GL}_{r_t}}$ is strongly semistable. Therefore, the associated vector bundle $f_*(K_f^{-1})^{\otimes n}$ is also semistable (see [RR, p. 285, Theorem 3.18] and [RR, p. 288, Theorem 3.23]). This implies that $(f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n}$ is semistable.

For a semistable vector bundle $\mathcal{V}$ on X of degree zero, any nonzero section $\sigma : \mathcal{O}_X \longrightarrow \mathcal{V}$ is nowhere vanishing. Indeed, this follows immediately from the semistability condition that the line bundle of $\mathcal{V}$ generated by the image of σ is of nonpositive degree. Consequently,

$$\dim H^0(X,\mathcal{V}) \le \operatorname{rank}(\mathcal{V}).$$

Since $(f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n}$ is semistable of degree zero, we have

$$\begin{aligned}\dim H^0\big(X, \big(f_*(K_f^{-1})^{\otimes n}\big) \otimes \mathcal{L}_0^{\otimes n}\big) &\le \operatorname{rank}\big(\big(f_*(K_f^{-1})^{\otimes n}\big) \otimes \mathcal{L}_0^{\otimes n}\big)\\ &= \operatorname{rank}\big(\big(f_*(K_f^{-1})^{\otimes n}\big)\big)\end{aligned} \tag{3.23}$$

for all $n > 0$.

We have $R^j f_*((K_f^{-1})^{\otimes n}) = 0$ for $j,n \ge 1$. Hence, from the Riemann–Roch theorem for the restriction $(K_f^{-1})^{\otimes n}|_{f^{-1}(x)}$, $x \in X$, we conclude that $\operatorname{rank}((f_*(K_f^{-1})^{\otimes n}))$ is a polynomial of degree at most $d_0 - 1$ (which is the dimension of the fibers of f). Therefore, using (3.22) and (3.23) we conclude that

$$\dim H^0\big(\mathrm{Gr}_s(V_t/V_{t-1}), \gamma^*\mathcal{O}_{\mathrm{Gr}_r((F_X^{\delta})^*E)}\big(n \cdot \operatorname{rank}(V_t/V_{t-1})\big)\big)$$

is a polynomial of degree at most $d_0 - 1$. But this contradicts (3.19).

We assumed that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is ample, and we are led to the above contradiction. Therefore, we conclude that $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is not ample. This completes the proof of the lemma. □

Lemmas 3.1, 3.2, and 3.3 together give the following.

THEOREM 3.4

If $\theta_{E,r} > 0$, then the line bundle $\mathcal{O}_{\mathrm{Gr}_r(E)}(1) \longrightarrow \mathrm{Gr}_r(E)$ in (3.2) is ample.

If $\theta_{E,r} = 0$, then $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is nef but not ample.

If $\theta_{E,r} < 0$, then $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ is not nef.

4. The nef cone of $\mathrm{Gr}_r(E)$

In this section we compute the nef cone of $\mathrm{Gr}_r(E)$ using Theorem 3.4. Being a closed cone, it is generated by its boundary. For notational reasons, it is convenient to treat the cases of characteristic zero and positive characteristic separately.

For a smooth projective variety Z, the real Néron–Severi group $\mathrm{NS}(Z)_{\mathbb{R}}$ is defined to be

$$\mathrm{NS}(Z)_{\mathbb{R}} := \big(\mathrm{Pic}(Z)/\mathrm{Pic}^0(Z)\big) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{4.1}$$

where $\mathrm{Pic}^0(Z)$ is the connected component, containing the identity element, of the Picard group $\mathrm{Pic}(Z)$ of Z.

4.1. Characteristic is zero

In this case, the number δ in (3.5) is zero.

As in (3.1), φ is the projection of $\mathrm{Gr}_r(E)$ to X. Fix a line bundle L_1 over X of degree one. The line bundle $\varphi^* L_1$ is denoted by $\mathcal{L}$. The real Néron–Severi group $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ is freely generated by $\mathcal{L}$ and $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$.

Although $\theta_{E,r}$ in (3.5) need not be an integer, we note that $\mathcal{L}^{\otimes -\theta_{E,r}}$ is well defined as an element of $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ because $\theta_{E,r} \in \mathbb{Q}$.

PROPOSITION 4.1

The boundary of the nef cone in $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ *is given by* $\mathcal{L}$ *and* $\mathcal{O}_{\mathrm{Gr}_r(E)}(1) \otimes \mathcal{L}^{\otimes -\theta_{E,r}}$.

Proof

We first show that it is enough to treat the case where $\theta_{E,r}$ is a multiple of r. In fact, this argument is standard (see [Laz, p. 23, Lemma 6.2.8]). However, we describe the details for completeness.

Write

$$\theta_{E,r} = \frac{p_1 r}{q_1},$$

where p_1 and q_1 are integers with $q_1 > 0$. Take a pair (Y, f), where Y is an irreducible smooth projective curve and f is a morphism from Y to X such that degree(f) is a multiple of q_1. The natural map

$$\gamma : \mathrm{Gr}_r(f^* E) \longrightarrow \mathrm{Gr}_r(E)$$

produces an isomorphism between $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ and $\mathrm{NS}(\mathrm{Gr}_r(f^*E))_{\mathbb{R}}$. This isomorphism preserves the nef cones. Therefore, it is enough to prove the proposition for (Y, f^*E). Note that $\theta_{f^*E,r} = (\mathrm{degree}(f) p_1 r)/q_1$ is a multiple of r.

Hence, we can assume that $\theta_{E,r}/r$ is an integer.

Consider the vector bundle

$$F := E \otimes L_1^{\otimes -\theta_{E,r}/r}.$$

Note that $\mathrm{Gr}_r(E) = \mathrm{Gr}_r(F)$. From (3.5) and the definition of F it follows immediately that

$$\theta_{F,r} = 0.$$

Since $\theta_{F,r} = 0$, from the second part of Theorem 3.4 we know that the nef cone in $\mathrm{NS}(\mathrm{Gr}_r(F))_{\mathbb{R}}$ is generated by $\mathcal{O}_{\mathrm{Gr}_r(F)}(1)$ and $\mathcal{L}$. (It is considered as a line bundle on $\mathrm{Gr}_r(F)$ using the identification of $\mathrm{Gr}_r(F)$ with $\mathrm{Gr}_r(E)$.) The proposition follows immediately from this description of the nef cone in $\mathrm{NS}(\mathrm{Gr}_r(F))_{\mathbb{R}}$ using the identification of $\mathrm{Gr}_r(F)$ with $\mathrm{Gr}_r(E)$. □

REMARK 4.2

We note that the two generators of the nef cone given in Proposition 4.1 lie in the rational Néron–Severi group $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{Q}} := (\mathrm{Pic}(Z)/\mathrm{Pic}^0(Z)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

4.2. Characteristic is positive

Let $p > 0$ be the characteristic of k. Consider δ in (3.5). Let $\varphi_1 : \mathrm{Gr}_r((F_X^\delta)^*E) \longrightarrow X$ be the natural projection. Define the line bundle

$$\mathcal{L}_1 := \varphi_1^* L_1 \longrightarrow X,$$

where L_1 is a fixed line bundle on X of degree one.

LEMMA 4.3

The nef cone in $\mathrm{NS}(\mathrm{Gr}_r((F_X^\delta)^*E))_{\mathbb{R}}$ *(defined in (4.1)) is generated by* $\mathcal{L}_1$ *and* $\mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(1) \otimes \mathcal{L}_1^{\otimes -\theta_{(F_X^\delta)^*E,r}}$.

Proof

The proof is exactly identical to the proof of Proposition 4.1. We refrain from repeating it. □

As in (3.1), the projection of $\mathrm{Gr}_r(E)$ to X is denoted by φ. Define

$$\mathcal{L} := \varphi^* L_1.$$

PROPOSITION 4.4

The boundary of the nef cone in $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ *is given by* $\mathcal{L}$ *and* $\mathcal{O}_{\mathrm{Gr}_r(E)}(p^\delta) \otimes \mathcal{L}^{\otimes -\theta_{(F_X^\delta)^*E,r}}$.

Proof

Consider the commutative diagram of morphisms in (3.14). The morphism β in this diagram produces an isomorphism between $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ and $\mathrm{NS}(\mathrm{Gr}_r((F_X^\delta)^*E))_{\mathbb{R}}$. This isomorphism preserves the nef cones.

We have $\beta^*\mathcal{O}_{\mathrm{Gr}_r(E)}(1) = \mathcal{O}_{\mathrm{Gr}_r((F_X^\delta)^*E)}(1)$ and $(F_X^\delta)^*L_1 = L_1^{\otimes p^\delta}$. Hence, the proposition follows from Lemma 4.3. □

REMARK 4.5

The two generators of the nef cone given in Proposition 4.4 lie in $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{Q}}$.

5. The nef cone of flag bundles

Fix integers

$$0 < r_1 < r_2 < \cdots < r_{\nu-1} < r_\nu < \mathrm{rank}(E).$$

Let

$$\Phi : \mathrm{Fl}(E) \longrightarrow X$$

be the corresponding flag bundle, so for any $x \in X$, the fiber $\Phi^{-1}(x)$ parameterizes all filtrations of linear subspaces

$$E_x \supset S_1 \supset S_2 \supset \cdots \supset S_{\nu-1} \supset S_\nu \tag{5.1}$$

such that $\dim E_x - \dim S_i = r_i$ for all $i \in [1, \nu]$.

For each $i \in [1,\nu]$, let $\mathrm{Gr}_{r_i}(E)$ be the Grassmann bundle over X parameterizing all the r_i-dimensional quotients of the fibers of E. Let

$$\phi_i : \mathrm{Fl}(E) \longrightarrow \mathrm{Gr}_{r_i}(E) \tag{5.2}$$

be the natural projection that sends any filtration as in (5.1) to E_x/S_i. Let

$$\omega_i \in \mathrm{NS}\big(\mathrm{Gr}_{r_i}(E)\big)_{\mathbb{R}}$$

be the element $\mathcal{O}_{\mathrm{Gr}_{r_i}(E)}(1) \otimes \mathcal{L}^{\otimes -\theta_{E,r_i}}$ (resp., $\mathcal{O}_{\mathrm{Gr}_{r_i}(E)}(p^\delta) \otimes \mathcal{L}^{\otimes -\theta_{(F_X^\delta)^*E,r_i}}$) in Proposition 4.1 (resp., Proposition 4.4) if the characteristic of k is zero (resp., positive). Define

$$\widetilde{\omega}_i := \phi_i^* \omega_i \in \mathrm{NS}\big(\mathrm{Fl}(E)\big)_{\mathbb{R}},$$

where ϕ_i is the projection in (5.2).

THEOREM 5.1
The nef cone in $\mathrm{NS}(\mathrm{Fl}(E))_{\mathbb{R}}$ *is generated by* $\{\widetilde{\omega}_i\}_{i=1}^{\nu} \cup \Phi^*\mathcal{L}'$*, where* $\mathcal{L}'$ *is a line bundle over* X *of degree one.*

Proof
The dimension of the $\mathbb{R}$-vector space $\mathrm{NS}(\mathrm{Fl}(E))_{\mathbb{R}}$ is $\nu+1$, and the vector space is generated by $\{\widetilde{\omega}_i\}_{i=1}^{\nu} \cup \Phi^*\mathcal{L}'$. We note that $\mathcal{L}'$ and all the $\widetilde{\omega}_i$'s are nef.

Fix any point $x \in X$. For each $i \in [1,\nu]$, define

$$\widetilde{\omega}_{x,i} := \widetilde{\omega}_i|_{\Phi^{-1}(x)} \in \mathrm{NS}\big(\Phi^{-1}(x)\big)_{\mathbb{R}}.$$

The dimension of the $\mathbb{R}$-vector space $\mathrm{NS}(\Phi^{-1}(x))_{\mathbb{R}}$ is ν. It is known that the nef cone of $\mathrm{NS}(\Phi^{-1}(x))_{\mathbb{R}}$ is generated by $\{\widetilde{\omega}_{x,i}\}_{i=1}^{\nu}$ (see [Br, p. 187, Theorem 1] for a general result). In view of this, the theorem follows from Proposition 4.1 (resp., Proposition 4.4) when the characteristic of k is zero (resp., positive). □

REMARK 5.2
All the elements of the generating set of the nef cone in $\mathrm{NS}(\mathrm{Fl}(E))_{\mathbb{R}}$ given in Theorem 5.1 lie in $\mathrm{NS}(\mathrm{Fl}(E))_{\mathbb{Q}}$.

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