Unitary representations of the universal cover of $\mathrm{SU}(1,1)$ and tensor products

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Abstract In this paper we study the discrete spectrum of tensor products of irreducible unitary representations of the universal covering group of $\mathrm{SU}(1,1)$. As a consequence of these results, we show that the set of smooth vectors of the tensor product intersects trivially some of the representations in the discrete spectrum. These results illustrate aspects of the much larger program of branching laws for reductive groups.

1. Introduction

Understanding a unitary representation π of a Lie group G often involves understanding its restriction to suitable subgroups H. In physics, this is referred to as breaking the symmetry, and it often means exhibiting a nice basis of the representation space of π. Similarly, decomposing a tensor product of two representations of G is also an important branching problem, namely, the restriction to the diagonal in $G \times G$. This kind of branching law plays a prominent role in quantum mechanics. The most classical situation is that of the group $\mathrm{SU}(2)$. The set of irreducible unitary representations of $\mathrm{SU}(2)$ (up to isomorphism) is in bijection with the set of nonnegative integers n. The decomposition

$$\pi_n \otimes \pi_m = \pi_{n+m} \oplus \pi_{n+m-2} \oplus \cdots \oplus \pi_{|n-m|+2} \oplus \pi_{|n-m|}$$

is well known (see, e.g., [11]) and leads to the so-called Clebsch–Gordan coefficients.

Generally speaking, the more branching laws we know for a given representation, the more we know about its structure. An important example is given by using the maximal compact subgroup K of a semisimple Lie group G. The K-spectrum of a representation π is an important invariant which serves to describe the structure of π. It is also important to give good models of both π and its explicit K-types. There has been much progress in recent years (and of course a large number of more classical works), both for abstract theory and for concrete examples of branching laws.

The aim of this paper is to study carefully tensor products of irreducible unitary representations of $\mathrm{SU}(1,1)$ and its universal cover. Indeed, the Lie group

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$\mathrm{SU}(1,1)$ has the same complexified Lie algebra as $\mathrm{SU}(2)$, namely, $\mathfrak{sl}(2,\mathbb{C})$. Therefore, the algebraic picture is quite simple and well known. Nevertheless, since irreducible unitary representations of $\mathrm{SU}(1,1)$ are infinite-dimensional, the decomposition of tensor products usually involves both a discrete and a continuous spectrum. There already exist effective methods to deal with the continuous spectrum (see, e.g., [27]). Unfortunately, the analysis of the discrete spectrum seems more tedious. Hence, our specific goal here is to determine when a discrete spectrum does occur in the restriction of a given tensor product, and then to describe explicitly the representations arising in this discrete spectrum.

Here our results should be seen in the light of the larger program of T. Kobayashi, in particular, the papers [12], [13], and [14], which treat in a very general framework branching laws for unitary representations for reductive groups. Note especially the theorem that a continuous spectrum appears only if none of the K'-finite vectors are K-finite, where these are the maximal compact subgroups for the reductive pair of groups in question. In [13] there is a very general criterion for discrete decomposability of restrictions, and in [12] and [14] there are further conditions that ensure admissibility, that is, finite multiplicity in the branching law. In the present paper we consider the very special case of weight modules of degree 1, so that the subalgebra is a Cartan algebra, and the corresponding multiplicities are all at most one.

Let us now review the main results of this paper. Let G denote the universal covering group of $G_0 = \mathrm{SU}(1,1)$. Let $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $\{e^{itH}, 0 \le t < 2\pi\}$ generates a maximal compact subgroup K_0 of G_0. Its covering group is $K = \{\exp(itH), t \in \mathbb{R}\}$. The center of G is generated by $\exp(2i\pi H)$. Let ρ denote an irreducible unitary representation of G. From Schur's lemma we conclude that $\rho(\exp(2i\pi H)) = e^{-2i\pi\tau_0} I$. Therefore, $\tilde{\rho}(\exp(itH)) := e^{i\tau_0 t}\rho(\exp(itH))$ is a unitary representation of $\mathbb{R}$, with period 2π, and hence is completely reducible. As a consequence, H possesses a complete system of eigenelements. In other words, the corresponding representation of the complexified Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is a weight module (see Definition 2.1). We review the basics of weight modules in Section 2.

Section 3 deals with unitarizable weight modules for $\mathfrak{su}(1,1)$. Recall that a unitarizable module is a module defined on a Hilbert space which is the differential of a unitary module for the universal covering group G. Using the explicit action of $\mathfrak{sl}(2,\mathbb{C})$ given in Section 2, we recover the classification due to Pukansky of the unitary dual of G, which falls into three series: the principal series $\pi_{\epsilon,it}$ ($0 < \epsilon \le 1$, $t \in \mathbb{R}$), the complementary series $\pi^c_{\sigma,\tau}$ ($0 < \sigma, \tau < 1$), the (continuation of the) discrete series $\pi^{\pm}_{\lambda}$ ($\lambda > 0$), and the extra trivial representation.

In Sections 4 and 5, we study a tensor product V of the form $\pi_1 \otimes \pi^+_\lambda$ where π_1 is $\pi_{\epsilon,it}$, $\pi^c_{\sigma,\tau}$, or π^-_μ. The main result in Section 4 is Theorem 4.4.

THEOREM 4.4′

Every simple weight $\mathfrak{sl}(2,\mathbb{C})$-module W whose support is included in the support of V appears as a quotient of the algebraic tensor product V.

One should think of this result as an algebraic counterpart for the notion of continuous spectrum.

Then Section 5 is devoted to the study of the discrete part in the Hilbert space V (the completion of the algebraic tensor product V). The main result is Theorem 5.17. Let us state a particular case of this theorem.

THEOREM 5.17′

(1) *If $0 < \mu + \lambda < 1$, then the Hilbert representation $\pi_\mu^- \otimes \pi_\lambda^+$ contains the representation $\pi_{\lambda,\mu}^c$, belonging to the complementary series.*

(2) *If $0 < \sigma + \tau + \lambda < 1$, then the Hilbert representation $\pi_{\sigma,\tau}^c \otimes \pi_\lambda^+$ contains the representation $\pi_{\sigma+\lambda,\tau}^c$, belonging to the complementary series.*

(3) *If $1 < \sigma + \tau - \lambda < 2$, then the Hilbert representation $\pi_{\sigma,\tau}^c \otimes \pi_\lambda^+$ contains the representation $\pi_{\sigma,\tau-\lambda}^c$, belonging to the complementary series.*

The discrete spectrum in item (1) is well known (see [26], [6], [17], [18], [28]), but even in this case our method yields new insight, in particular, giving explicit expressions.

In proving Theorem 5.17′, we provide an explicit generator for all modules in the discrete spectrum of V. This yields results about the smooth vectors.

PROPOSITION 5.19′

If $0 < \lambda + \mu < 1$ (resp., $0 < \sigma + \tau + \lambda < 1$ and $1 < \sigma + \tau - \lambda < 2$), then the Hilbert submodule $\pi_{\lambda,\mu}^c$ (resp., $\pi_{\sigma+\lambda,\tau}^c$ and $\pi_{\sigma,\tau-\lambda}^c$) of the Hilbert representation $\pi_\mu^- \otimes \pi_\lambda^+$ (resp., $\pi_{\sigma,\tau}^c \otimes \pi_\lambda^+$) intersects trivially the set of smooth vectors in $\pi_\mu^- \otimes \pi_\lambda^+$ (resp., $\pi_{\sigma,\tau}^c \otimes \pi_\lambda^+$).

The proofs involve the algebraic structure of weight modules and asymptotic analysis of hypergeometric functions. Also, we give some new results about discrete analogues of the hypergeometric equation, for example, (5.12) and the discussion there. It should be noted that the existence of a continuous spectrum in a branching law seems to give interesting restrictions on the regularity of the discrete spectrum; thus, we may see here a deep connection with Kobayashi's criteria in [12], [13], and [14] involving K-finite vectors.

2. Weight modules

In this section, we recall the definition of a weight module, and the construction of those weight modules which are of degree 1.

Let $\mathfrak{g}$ denote a reductive Lie algebra, and let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra. Let $\mathfrak{h}$ be a fixed Cartan subalgebra, and denote by $\mathcal{R}$ the corresponding set of roots. For $\alpha \in \mathcal{R}$, we denote by $\mathfrak{g}_\alpha$ the root space for the root α.

2.1. The category of weight modules

DEFINITION 2.1

A $\mathfrak{g}$-module M is a *weight module* if it is finitely generated and $\mathfrak{h}$-diagonalizable in the sense that

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda, \quad \text{where } M_\lambda = \{m \in M : H \cdot m = \lambda(H)m, \forall H \in \mathfrak{h}\},$$

with weight spaces M_λ of finite dimension.

REMARK 2.2

Note that we require finite-dimensional weight spaces in our definition, which is not always the case in the literature. This category also appears as a particular case of several other categories (e.g., [20], [21], [5], [8]).

The set of all weight modules forms a full subcategory of the category of all modules, denoted by $\mathcal{M}(\mathfrak{g}, \mathfrak{h})$. Given a weight module M, we call the *support* of M the set

$$\mathrm{Supp}(M) = \{\lambda \in \mathfrak{h}^* : M_\lambda \neq 0\}.$$

The *degree* of a weight module M is the (possibly infinite) number

$$\deg(M) = \sup_{\lambda \in \mathfrak{h}^*} \{\dim(M_\lambda)\}.$$

For instance, a *degree* 1 *module* is a weight module, all of whose nonzero weight spaces are one-dimensional. Such modules have been classified by Benkart, Britten, and Lemire [4]. They are the main object of investigation of this paper.

2.2. The modules of degree 1

Let us review the classification of degree 1 modules for simple Lie algebras. First we have the following theorem.

THEOREM 2.3 (BENKART, BRITTEN, AND LEMIRE [4, PROPOSITION 1.4])

Let $\mathfrak{g}$ be a simple Lie algebra. Let M be a simple infinite-dimensional degree 1 *weight module. Then*

(1) *the Lie algebra $\mathfrak{g}$ is of type A or C;*
(2) *the Gelfand–Kirillov dimension of M is given by the rank of $\mathfrak{g}$.*

2.2.1. Modules over the Weyl algebra

Let n be a positive integer. Recall that the Weyl algebra W_n is the associative algebra generated by the $2n$ generators $\{q_i, p_i, 1 \leq i \leq n\}$ submitted to the following relations:

$$[q_i, q_j] = 0 = [p_i, p_j], \qquad [p_i, q_j] = \delta_{i,j} \cdot 1,$$

where the bracket is the usual commutator for associative algebras.

Define a vector space as follows. Fix some $a \in \mathbb{C}^n$. Let

$$\mathcal{K}(a) = \{\underline{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n : \text{ if } a_i \in \mathbb{Z}, \text{ then } a_i + k_i < 0 \iff a_i < 0\}.$$

Now our vector space $W(a)$ is the $\mathbb{C}$-vector space whose basis is indexed by $\mathcal{K}(a)$. For each $\underline{k} \in \mathcal{K}(a)$, we fix a basis vector $x(\underline{k})$. Let $(\epsilon_i)_{1\leq i\leq n}$ denote the canonical basis of $\mathbb{Z}^n$. Define an action of W_n on $W(a)$ by the following recipe:

$$q_i \cdot x(\underline{k}) = \begin{cases} (a_i + k_i + 1)x(\underline{k} + \epsilon_i) & \text{if } a_i \in \mathbb{Z}_{<0}, \\ x(\underline{k} + \epsilon_i) & \text{otherwise;} \end{cases}$$

$$p_i \cdot x(\underline{k}) = \begin{cases} x(\underline{k} - \epsilon_i) & \text{if } a_i \in \mathbb{Z}_{<0}, \\ (a_i + k_i)x(\underline{k} - \epsilon_i) & \text{otherwise.} \end{cases}$$

This basis shall be referred to as the *standard basis* of $W(a)$.

Then we have the following.

THEOREM 2.4 (BENKART, BRITTEN, AND LEMIRE [4, THEOREM 2.9])
Let $a \in \mathbb{C}^n$. Then $W(a)$ is a simple W_n-module.

2.2.2. Type A case

In this section only, $\mathfrak{g}$ denotes a simple Lie algebra of type A. We construct weight $\mathfrak{g}$-modules of degree 1 by using the previous construction. We realize the Lie algebra $\mathfrak{g}$ inside some W_n. Let $n-1$ be the rank of $\mathfrak{g}$. Then, we can embed $\mathfrak{g}$ into W_n as follows: To an elementary matrix $E_{i,j}$ we associate the element $q_i p_j$ of W_n. This is easily seen to define an embedding of $\mathfrak{g}$ into W_n. Let

$$\mathcal{K}_0(a) = \Big\{\underline{k} = (k_1, \ldots, k_n) \in \mathcal{K}(a) : \sum_{i=1}^{n} k_i = 0\Big\}.$$

Let $N(a)$ be the subspace of $W(a)$ whose basis is indexed by $\mathcal{K}_0(a)$. Then we have the following.

THEOREM 2.5 (BENKART, BRITTEN, AND LEMIRE [4, THEOREM 5.8])

(1) *The vector subspace $N(a)$ of $W(a)$ is a simple weight $\mathfrak{sl}(n,\mathbb{C})$-module of degree 1.*

(2) *Conversely, if M is a simple weight $\mathfrak{sl}(n,\mathbb{C})$-module of degree 1, then there exist $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that the module M is isomorphic to $N(a)$.*

2.2.3. Type C case

In this section only, $\mathfrak{g}$ denotes a simple Lie algebra of type C. We construct weight $\mathfrak{g}$-modules of degree 1 in the same way as above, so we need to realize the Lie algebra $\mathfrak{g}$ inside some W_n. Let n be the rank of $\mathfrak{g}$. Then, $\mathrm{span}_{\mathbb{C}}\{q_i p_j, p_i p_j, q_i q_j, 1 \leq i, j \leq n\}$ is a subalgebra of W_n isomorphic to $\mathfrak{g}$. More specifically, the Cartan subalgebra is given by

$$\mathrm{span}_{\mathbb{C}}\Big(\{q_i p_i - q_{i+1} p_{i+1}, i = 1, \ldots, n-1\} \cup \Big\{q_n p_n + \frac{1}{2}\Big\}\Big),$$

the $n-1$ weight vectors corresponding to the short simple roots are given by $q_i p_{i+1}$ with $i=1,\dots,n-1$, and the weight vector corresponding to the long simple root is given by $\frac{1}{2}q_n^2$. Note that this is not the same kind of embedding as for Lie algebras of type A.

Let

$$\mathcal{K}_{\bar{0}}(a) = \Big\{\underline{k} = (k_1,\dots,k_n) \in \mathcal{K}(a) : \sum_{i=1}^{n} k_i \in 2\mathbb{Z}\Big\}.$$

Let $M(a)$ be the subspace of $W(a)$ whose basis is indexed by $\mathcal{K}_{\bar{0}}(a)$. Then we have the following.

THEOREM 2.6 (BENKART, BRITTEN, AND LEMIRE [4, THEOREM 5.21])

(1) *The vector subspace $M(a)$ of $W(a)$ is a simple weight $\mathfrak{sp}(n,\mathbb{C})$-module of degree 1.*

(2) *Conversely, if M is an infinite-dimensional simple weight $\mathfrak{sp}(n,\mathbb{C})$-module of degree 1, then there exists $a=(a_1,\dots,a_n)\in\mathbb{C}^n$ such that $M\cong M(a)$.*

2.3. The case of $\mathfrak{sl}(2,\mathbb{C})$

In this section, we review the classification of all weight modules for $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$. We consider the standard $\mathfrak{sl}(2,\mathbb{C})$-triple (F,H,E), given by

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We therefore have the following relations:

$$[H,E]=2E, \qquad [H,F]=-2F, \qquad [E,F]=H.$$

PROPOSITION 2.7

Let M be a simple weight $\mathfrak{sl}(2,\mathbb{C})$-module. Then the degree of M is 1.

Proof

Recall that $\Omega=\frac{1}{4}H^2+\frac{1}{2}H+FE$ is in the center of the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$. Therefore, M being simple, Ω acts as a scalar operator. On the other hand, as M is a weight module, H acts on each weight space by some constant (the weight). Therefore, on each weight space, FE acts by some constant. From this, we conclude that $\mathcal{U}(\mathfrak{g})_0$, the commutant of $\mathbb{C}H$, acts by some constant on each weight space. But, since M is simple, given two nonzero vectors v and w in the same weight space, there should exist some element $u\in\mathcal{U}(\mathfrak{g})$ sending v to w. The fact that v and w have the same weight forces u to be in the commutant of $\mathbb{C}H$. From the above we know that u acts by some constant. This forces v and w to be proportional, and therefore the corresponding weight space is one-dimensional. This completes the proof. □

For a simple weight module M, the action of Ω on M is called the *infinitesimal character*. From Theorem 2.5, the simple weight modules are indexed by $a=$

$(a_1, a_2) \in \mathbb{C}^2$. Recall that we set

$$\mathcal{K}_0(a) = \left\{ \underline{k} = (k_1, k_2) \in \mathbb{Z}^2 : \begin{array}{l} \text{if } a_i \in \mathbb{Z}, \text{ then } a_i + k_i < 0 \iff a_i < 0, \\ k_1 + k_2 = 0 \end{array} \right\}.$$

Setting $k = k_1$ (and thus $-k = k_2$), we have that this reduces to

$$\mathcal{K}_0(a) = \left\{ k \in \mathbb{Z} : \text{ if } a_i \in \mathbb{Z}, \text{ then } a_i + (-1)^{i-1} k < 0 \iff a_i < 0 \right\}.$$

Recall then that the weight module $N(a)$ has a basis $x(k)$ indexed by $\mathcal{K}_0(a)$. The following disjoint cases exhaust all possibilities:

(I) Both a_1 and a_2 are not negative integers.
(II) a_1 is not a negative integer but a_2 is a negative integer.
(III) a_2 is not a negative integer but a_1 is a negative integer.
(IV) Both a_1 and a_2 are negative integers.

Then we have the following action of $\mathfrak{g}$ on $N(a)$:

$$\text{(I)} \quad \begin{cases} H \cdot x(k) = (a_1 - a_2 + 2k) x(k), \\ E \cdot x(k) = (a_2 - k) x(k+1), \\ F \cdot x(k) = (a_1 + k) x(k-1); \end{cases}$$

$$\text{(II)} \quad \begin{cases} H \cdot x(k) = (a_1 - a_2 + 2k) x(k), \\ E \cdot x(k) = x(k+1), \\ F \cdot x(k) = (a_1 + k)(a_2 - k + 1) x(k-1); \end{cases}$$

$$\text{(III)} \quad \begin{cases} H \cdot x(k) = (a_1 - a_2 + 2k) x(k), \\ E \cdot x(k) = (a_1 + k + 1)(a_2 - k) x(k+1), \\ F \cdot x(k) = x(k-1); \end{cases}$$

$$\text{(IV)} \quad \begin{cases} H \cdot x(k) = (a_1 - a_2 + 2k) x(k), \\ E \cdot x(k) = (a_1 + k + 1) x(k+1), \\ F \cdot x(k) = (a_2 - k + 1) x(k-1). \end{cases}$$

3. Unitarizability

The goal of this section is to find which weight modules for $\mathfrak{sl}(2, \mathbb{C})$ are unitarizable. Even though this result is quite classical, we give a self-contained proof (using a criterion of Nelson), in order to stress the Hilbert structure corresponding to each unitarizable weight module.

We keep the previous notations. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{H, E, F\}$. Set $h = -i(E - F)$, set $e = \frac{1}{2}(-iH + E + F)$, and set $f = \frac{1}{2}(iH + E + F)$. Then $\mathrm{span}_{\mathbb{R}}\{h, e, f\}$ is a real Lie algebra isomorphic to $\mathfrak{su}(1,1)$.

Let G denote the simply connected Lie group with Lie algebra $\mathfrak{su}(1,1)$. Recall the following result of Nelson [16, Corollary 9.3].

THEOREM 3.1 (NELSON)
Let ρ be a representation of $\mathfrak{su}(1,1)$ on a Hilbert space by skew-symmetric operators with domain $\mathfrak{D}$. Then there is a unitary representation U of G such that $\mathfrak{D}$ is the space of infinitely differentiable vectors for U and $dU(X) = \rho(X)$, $\forall X \in \mathfrak{g}$, if and only if

$$A = \rho(h)^2 + \rho(e)^2 + \rho(f)^2$$

is essentially self-adjoint and $\mathfrak{D} = \bigcap_{k=1}^{\infty} \mathfrak{D}(\bar{A}^k)$, $\bar{A}$ being the closure of A.

We remark that we have $A = \rho(\Omega) - \frac{1}{2}(\rho(E) - \rho(F))^2$. A $\mathfrak{g}$-module giving rise to a representation ρ of $\mathfrak{su}(1,1)$ satisfying the assumptions of Nelson's theorem will be referred to as a *unitarizable* module.

Thanks to this theorem, to find which $N(a)$ are unitarizable we need to construct on $N(a)$ a Hilbert space structure such that h, e, and f act by skew-symmetric operators. It is then equivalent to construct on $N(a)$ a Hilbert space structure such that $H^* = H$, $E^* = -F$, and $F^* = -E$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $N(a)$. By construction, H acts on $N(a)$ by a semisimple operator. So, for H to be self-adjoint it is necessary that weight vectors for different weights are orthogonal and that all the weights are real numbers. This means that the basis $\{x(k)\}_{k \in \mathcal{K}_0}$ is an orthogonal basis and that $a_1 - a_2 \in \mathbb{R}$.

Besides, we must also have

$$\big\langle F \cdot x(k+1), x(k) \big\rangle = -\big\langle x(k+1), E \cdot x(k) \big\rangle, \quad \forall k \in \mathcal{K}_0.$$

We work with this condition in the different cases (I), (II), (III), and (IV).

3.1. Case (I)

In this case, the condition becomes

$$(a_1 + k + 1)\big\|x(k)\big\|^2 = -(\bar{a}_2 - k)\big\|x(k+1)\big\|^2.$$

Let us distinguish the various situations.

(i) Assume that both a_1 and a_2 are not integers. In this case, $\mathcal{K}_0 = \mathbb{Z}$ and we have $a_1 + k + 1 \neq 0$ and $\bar{a}_2 - k \neq 0$. So, for the condition to hold it is necessary and sufficient that

$$\forall k \in \mathbb{Z}, \quad \frac{k - \bar{a}_2}{k + a_1 + 1} \in \mathbb{R}_{>0}.$$

But we have seen that $a_1 - a_2 \in \mathbb{R}$, so we can set $a_1 = a_2 + r$ for some $r \in \mathbb{R}$. Therefore, we must have either $\mathfrak{Im}(a_2) = 0$ or $2\mathfrak{Re}(a_2) + r + 1 = 0$. In the first situation we must also have

$$\forall k \in \mathbb{Z}, \quad \frac{k - a_2}{k + a_2 + r + 1} > 0.$$

This is true if and only if

$$-2 - [a_2] < a_2 + r < -1 - [a_2],$$

where $[a_2]$ is the integer such that $[a_2] \leq a_2 < [a_2]+1$. Then we can express $\|x(k)\|^2$ uniquely in terms of $\|x(0)\|^2$, via the formula

$$\|x(k)\|^2 = \frac{\prod_{j=1}^{k}(j+a_1)}{\prod_{j=1}^{k}(j-1-a_2)} \|x(0)\|^2 \quad \text{if } k>0, \tag{3.1a}$$

$$\|x(k)\|^2 = \frac{\prod_{j=1}^{-k}(j+a_2)}{\prod_{j=1}^{-k}(j-1-a_1)} \|x(0)\|^2 \quad \text{if } k<0. \tag{3.1b}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.1), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

In the second situation, we have

$$\forall k \in \mathbb{Z}, \quad \frac{k-\bar{a}_2}{k+a_1+1} = 1 \in \mathbb{R}_{>0}.$$

Then we can express $\|x(k)\|^2$ uniquely in terms of $\|x(0)\|^2$, via the formula

$$\|x(k)\|^2 = \|x(0)\|^2, \quad \forall k \in \mathbb{Z}. \tag{3.2}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.2), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

(ii) Assume that a_1 is not an integer but a_2 is a nonnegative integer. In this case, an integer k belongs to $\mathcal{K}_0$ if and only if $k \leq a_2$. Moreover, since $a_1 - a_2 \in \mathbb{R}$, we must have $a_1 \in \mathbb{R}$. Then the condition becomes

$$\forall k < a_2, \quad \frac{k-a_2}{k+a_1+1} \in \mathbb{R}_{>0}.$$

Therefore, we must have $k+1+a_1<0$ for all $k<a_2$. This is true if and only if $a_1 < -a_2$. Then we can express $\|x(k)\|^2$ uniquely in terms of $\|x(a_2)\|^2$, via the formula

$$\|x(a_2-k)\|^2 = \frac{k!}{\prod_{j=1}^{k}(j-1-a_1-a_2)} \|x(a_2)\|^2, \quad \forall k>0. \tag{3.3}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.3), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

(iii) Assume that a_2 is not an integer but a_1 is a nonnegative integer. In this case, an integer k belongs to $\mathcal{K}_0$ if and only if $k \geq -a_1$. Moreover, since $a_1 - a_2 \in \mathbb{R}$, we must have $a_2 \in \mathbb{R}$. Then the condition becomes

$$\forall k \geq -a_1, \quad \frac{k-a_2}{k+a_1+1} \in \mathbb{R}_{>0}.$$

Therefore, we must have $k-a_2>0$ for all $k \geq -a_1$. This is true if and only if $a_2 < -a_1$. Then we can express $\|x(k)\|^2$ uniquely in terms of $\|x(-a_1)\|^2$, via the formula

$$\|x(k-a_1)\|^2 = \frac{k!}{\prod_{j=1}^{k}(j-1-a_1-a_2)} \|x(-a_1)\|^2, \quad \forall k>0. \tag{3.4}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.4), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

(iv) Assume that both a_1 and a_2 are nonnegative integers. In this case, an integer k belongs to $\mathcal{K}_0$ if and only if $-a_1 \le k \le a_2$. Let $-a_1 \le k < a_2$. Then the condition becomes

$$\forall k \in \mathbb{Z}, \quad \frac{k-a_2}{k+a_1+1} \in \mathbb{R}_{>0}.$$

This is not true, unless $a_1 = a_2 = 0$. This choice corresponds to the trivial (one-dimensional) module, which is of course unitarizable. In this case, we recovered the fact that a finite-dimensional representation of a noncompact group cannot be unitary unless it is trivial.

3.2. Case (II)

In this case, the condition becomes

$$(a_1+k+1)(a_2-k)\|x(k)\|^2 = -\|x(k+1)\|^2.$$

Therefore, we must have $(a_1+k+1)(k-a_2) > 0$. For an integer k to belong to $\mathcal{K}_0$ it is necessary that $a_2 - k < 0$. Therefore, we must have $a_1 + k + 1 > 0$. Let us distinguish between the two disjoint situations which may occur.

(i) If $a_1 \notin \mathbb{Z}$, then the condition $a_1 + k + 1 > 0$ for all $k > a_2$ is true if and only if $a_1 + a_2 + 2 > 0$. In this case, we can express $\|x(k)\|^2$ via the formula

$$\|x(k+a_2+1)\|^2 = (k!)\prod_{j=1}^{k}(j+1+a_1+a_2)\|x(a_2+1)\|^2, \quad \forall k > 0. \tag{3.5}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.5), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

(ii) If a_1 is a nonnegative integer, then an integer $k > a_2$ is in $\mathcal{K}_0$ if and only if $k + a_1 \ge 0$. Hence, in this case the condition is fulfilled. Then we can express $\|x(k)\|^2$ via the formula

$$\begin{aligned}&\|x(k-a_1)\|^2\\ &\quad = (k!)\prod_{j=1}^{k}(j-1-a_1-a_2)\|x(-a_1)\|^2, \quad \forall k > 0, \text{if } -a_1 > a_2,\end{aligned} \tag{3.6a}$$

$$\begin{aligned}&\|x(k+a_2+1)\|^2\\ &\quad = (k!)\prod_{j=1}^{k}(j+1+a_1+a_2)\|x(a_2+1)\|^2, \quad \forall k > 0, \text{if } -a_1 \le a_2.\end{aligned} \tag{3.6b}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.6), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

3.3. Case (III)

This case is analogous to the previous one. More specifically, we have exactly two possible situations:

(i) If $a_2 \notin \mathbb{Z}$, then we find the condition $a_1 + a_2 + 2 > 0$. In this case, we can express $\|x(k)\|^2$ via the formula

$$\big\|x(-k-a_1-1)\big\|^2 = (k!)\prod_{j=1}^{k}(j+1+a_1+a_2)\big\|x(-a_1-1)\big\|^2, \quad \forall k>0. \tag{3.7}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.7), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

(ii) If a_2 is a nonnegative integer, then the unitarizability condition is fulfilled, and we can express $\|x(k)\|^2$ via the formula

$$\begin{aligned}&\big\|x(a_2-k)\big\|^2\\ &\quad = (k!)\prod_{j=1}^{k}(j-1-a_1-a_2)\big\|x(a_2)\big\|^2, \quad \forall k>0, \text{ if } -a_1 > a_2,\end{aligned} \tag{3.8a}$$

$$\begin{aligned}&\big\|x(-k-a_1-1)\big\|^2\\ &\quad = (k!)\prod_{j=1}^{k}(j+1+a_1+a_2)\big\|x(-a_1-1)\big\|^2, \quad \forall k>0, \text{ if } -a_1 \le a_2.\end{aligned} \tag{3.8b}$$

Conversely, if we define an inner product on $N(a)$ such that $\{x(k)\}$ is an orthogonal basis satisfying (3.8), then Nelson's theorem applies, and thus the corresponding module is unitarizable.

3.4. Case (IV)

In this case, the condition becomes

$$(a_2-k)\big\|x(k)\big\|^2 = -(a_1+k+1)\big\|x(k+1)\big\|^2.$$

Furthermore, an integer k belongs to $\mathcal{K}_0$ if and only if $k+a_1<0$ and $a_2-k<0$. Therefore, the condition is never fulfilled. Of course, in this case the corresponding module $N(a)$ is finite-dimensional, so we know a priori that it is not unitarizable.

3.5. Statement

We summarize the above results as follows.

THEOREM 3.2

Let $a=(a_1,a_2)\in\mathbb{C}^2$. The module $N(a)$ is unitarizable if and only if a is of one of the following forms:

(1) $a=(-1-x+iy,x+iy)$, *with* $x\in\mathbb{R},y\in\mathbb{R}$.

(2) $a=(a_1,a_2)$, *with* a_1, a_2 *noninteger real numbers and* $-2-[a_2]<a_1<-1-[a_2]$.

(3) $a=(a_1,a_2)$, *with* $a_1\in\mathbb{Z}_{\geq 0}$ *and* $a_2\in\mathbb{R}\setminus\mathbb{Z}$ *such that* $a_1+a_2+2<0$.

(4) $a=(a_1,a_2)$, *with* $a_2\in\mathbb{Z}_{\geq 0}$ *and* $a_1\in\mathbb{R}\setminus\mathbb{Z}$ *such that* $a_1+a_2+2<0$.

(5) $a=(a_1,a_2)$, *with* $a_1\in\mathbb{Z}_{\geq 0}$ *and* $a_2\in\mathbb{Z}_{<0}$.

(6) $a=(a_1,a_2)$, *with* $a_2\in\mathbb{Z}_{\geq 0}$ *and* $a_1\in\mathbb{Z}_{<0}$.

(7) $a=(0,0)$.

In this classification, there are a lot of repetitions. For instance, if a_1 and a_2 are not integers we have $N(a_1,a_2)=N(a_1-k,a_2+k)$, for any integer k. Up to isomorphism, this list reduces to the following:

(i) $N(-1-x+iy,x+iy)$, $-1\leq x<0$, $y\in\mathbb{R}_{>0}$ (principal series);
(ii) $N(a_1,a_2)$, $-1<a_1,a_2<0$ (complementary series);
(iii) $N(a_1,0)$, $a_1<0$ or $N(0,a_2)$, $a_2<0$ (discrete series and continuations);
(iv) $N(0,0)$ (trivial representation).

In the rest of this article we denote in the same way a unitarizable module and the corresponding unitary representation of the universal covering of $\mathrm{SU}(1,1)$.

REMARK 3.3

The first proof of the classification of the unitary dual of the universal covering of $\mathrm{SU}(1,1)$ is due to Pukanszky [22] (see also [24]). Another proof in the same spirit as ours can be found in [10]. There, Jørgensen and Moore proved a stronger result: any simple weight module is the differential of a continuous representation of the universal covering of $\mathrm{SU}(1,1)$ in some Hilbert space.

To conclude this section, we collect the supports and the infinitesimal characters of the unitarizable modules in Table 1.

Table 1

Modules	Support	Infinitesimal character
$\pi_{-x,iy}=N(-1-x+iy,x+iy)$ (principal series)	$-1-2x+2\mathbb{Z}$	$-\frac{1}{4}-y^2$
$\pi^c_{-a_1,-a_2}=N(a_1,a_2)$ (complementary series)	$a_1-a_2+2\mathbb{Z}$	$((a_1+a_2)/2)(1+(a_1+a_2)/2)$
$\pi^+_{-a_1}=N(a_1,0)$ (highest weight)	$a_1-2\mathbb{Z}_{\leq 0}$	$(a_1/2)(1+a_1/2)$
$\pi^-_{-a_2}=N(0,a_2)$ (lowest weight)	$-a_2+2\mathbb{Z}_{\geq 0}$	$(a_2/2)(1+a_2/2)$

4. Tensor products: algebraic approach

In this section we investigate the algebraic structure of tensor products of $\mathfrak{sl}(2,\mathbb{C})$-modules. More precisely, we are interested in tensor products of one of the following forms:

(i) $N(0,b)\otimes N(a,0)$, with $a,b\in\mathbb{R}_{<0}$;
(ii) $N(-1-x+iy,x+iy)\otimes N(a,0)$, with $-1\le x<0$, $y\in\mathbb{R}_{>0}$, and $a\in\mathbb{R}_{<0}$;
(iii) $N(a_1,a_2)\otimes N(a,0)$, with $-1<a_1,a_2<0$, $a\in\mathbb{R}_{<0}$.

In all cases, we denote by V the tensor product. We give a basis of V as follows. Let $x(k)$ be the standard basis of $N(0,b)$ (resp., $N(-1-x+iy,x+iy)$, $N(a_1,a_2)$), where k belongs to $\mathbb{Z}_{\ge 0}$ (resp., $\mathbb{Z}$). Let $y(l)$ be the basis of $N(a,0)$ defined by $y(l)=x(-l)$, where $x(j)$ is the standard basis of $N(a,0)$ and l belongs to $\mathbb{Z}_{\ge 0}$. Set $z(k,l)=x(k)\otimes y(l)$. This is a basis of V. Using formulas (I), (II), (III), and (IV) of Section 2.3, we have

(i) $H\cdot z(k,l)=(-b+a+2(k-l))z(k,l)$,
(ii) $H\cdot z(k,l)=(-1-2x+a+2(k-l))z(k,l)$,
(iii) $H\cdot z(k,l)=(a_1-a_2+a+2(k-l))z(k,l)$.

We deduce then that V is the direct sum of its weight spaces and that all its nonzero weight spaces are infinite-dimensional. Moreover, we have $\mathrm{Supp}(V)=b_1-b_2+a+2\mathbb{Z}$, where $(b_1,b_2)=(0,b)$ (resp., $(-1-x+iy,x+iy)$, (a_1,a_2)). From [7], we know that every submodule (resp., quotient) of V is also the direct sum of its weight spaces. More specifically, if W is a submodule of V, then for any $\lambda\in\mathfrak{h}^*$ we have $W_\lambda=V_\lambda\cap W$ and $(V/W)_\lambda=V_\lambda/(V_\lambda\cap W)$.

Let $\mathcal{U}_0$ denote the commutant of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$. Then, as an algebra, $\mathcal{U}_0$ is generated by H and FE. In other words, a basis of $\mathcal{U}_0$ is given by the vectors $(FE)^tH^s$ for $t,s\in\mathbb{Z}_{\ge 0}$. Now recall the following general result.

THEOREM 4.1 (LEMIRE [15, THEOREM 1])

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Denote by $\mathcal{U}_0$ the commutant of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$.

(1) *Let M be a simple weight $\mathfrak{g}$-module. Then for any $\lambda\in\mathfrak{h}^*$, M_λ is either zero or a simple $\mathcal{U}_0$-module.*

(2) *Let M_0 be a simple $\mathcal{U}_0$-module. Then up to isomorphism, there is a unique simple weight module M such that M_0 is a weight space of M.*

Finally, recall from Proposition 2.7 that a simple weight module for $\mathfrak{sl}(2,\mathbb{C})$ is of degree 1. Thus, Theorem 4.1 implies that to construct all simple submodules (resp., quotients) of V it is sufficient to understand all simple $\mathcal{U}_0$-submodules (resp., quotients) of all the weight spaces of V. We have seen above that weight spaces of V are indexed by integers. Let $n_0\in\mathbb{Z}$. Denote by V_{n_0} the weight space of weight $b_1-b_2+a+2n_0$, where $(b_1,b_2)=(0,b)$ (resp., $(-1-x+iy,x+iy)$, (a_1,a_2)). Then a basis of this weight space is given by all the vectors $z(k,l)$ such that $k-l=n_0$. Using formulas (I), (II), (III), and (IV) of Section 2.3, we see that in general we have

$$(FE)\cdot z(k,l)=a(k,l)z(k-1,l-1)+b(k,l)z(k,l)+c(k,l)z(k+1,l+1)$$

for some complex numbers $a(k,l)$, $b(k,l)$, and $c(k,l)$. Moreover, we always have $c(k,l)\neq 0$.

Let l_0 be the smallest l such that there exists k with $k-l=n_0$. The integer l_0 exists since $l\in\mathbb{Z}_{\geq 0}$. More precisely, in the cases (ii) and (iii) we always have $l_0=0$ since $k\in\mathbb{Z}$. In the case (i), $l_0=0$ when $n_0\geq 0$ and $l_0=-n_0$ when $n_0<0$, since $k\in\mathbb{Z}_{\geq 0}$. The formulas show that we always have $a(l_0+n_0,l_0)=0$. Denote by $\mathfrak{c}$ the one-dimensional Lie algebra $\mathbb{C}(FE)$. We prove the following.

PROPOSITION 4.2

With the notations as above, we have the following:

(1) *As a $\mathfrak{c}$-module, V_{n_0} is cyclic, generated by $z_{n_0}:=z(l_0+n_0,l_0)$.*
(2) *The map $\rho_{n_0}:\mathcal{U}(\mathfrak{c})\to V_{n_0}$ defined by $\rho_{n_0}(u)=u\cdot z_{n_0}$ is a bijection.*

Proof

Denote by $z_{n_0}(j):=z(l_0+n_0+j,l_0+j)$ for $j\in\mathbb{Z}_{\geq 0}$. Then V_{n_0} has a basis given by the vectors $z_{n_0}(j)$ for $j\in\mathbb{Z}_{\geq 0}$.

(1) We have

$$(FE)\cdot z_{n_0}=b(l_0+n_0,l_0)z_N+c(l_0+n_0,l_0)z_{n_0}(1),$$

with $c(l_0+n_0,l_0)\neq 0$. Thus,

$$\frac{(FE)-b(l_0+n_0,l_0)1}{c(l_0+n_0,l_0)}\cdot z_{n_0}=z_{n_0}(1).$$

Therefore, $z_{n_0}(1)\in\mathcal{U}(\mathfrak{c})\cdot z_{n_0}$. We then prove that $z_{n_0}(j)\in\mathcal{U}(\mathfrak{c})\cdot z_{n_0}$ by induction on j, using the relation

$$\begin{aligned}(FE)\cdot z_{n_0}(j)&=a(l_0+n_0+j,l_0+j)z_{n_0}(j-1)\\&\quad+b(l_0+n_0+j,l_0+j)z_{n_0}(j)\\&\quad+c(l_0+n_0+j,l_0+j)z_{n_0}(j+1).\end{aligned}$$

This completes the first part of the proposition.

(2) The map ρ_{n_0} is surjective by the first part. We prove that it is also injective. Let $u=\sum_{m=0}^{M}c_m(FE)^m\in\mathcal{U}(\mathfrak{c})$ such that $c_M\neq 0$. Then using the relation

$$\begin{aligned}(FE)\cdot z_{n_0}(j)&=a(l_0+n_0+j,l_0+j)z_{n_0}(j-1)\\&\quad+b(l_0+n_0+j,l_0+j)z_{n_0}(j)\\&\quad+c(l_0+n_0+j,l_0+j)z_{n_0}(j+1),\end{aligned}$$

we check that

$$u \cdot z_{n_0} = c_M \times c(l_0 + n_0 + M - 1, l_0 + M - 1) z_{n_0}(M) + \sum_{m=0}^{M-1} d_m z_{n_0}(m)$$

for some complex numbers d_m. Since vectors $z_{n_0}(j)$ in V_{n_0} are linearly independent, we conclude that $u \cdot z_{n_0} \neq 0$. Hence, ρ_{n_0} is injective. □

A consequence of Proposition 4.2 is the following.

COROLLARY 4.3
As a $\mathcal{U}(\mathfrak{c})$-module, V_{n_0} is isomorphic to $\mathbb{C}[X]$, the space of polynomials in one indeterminate.

Now remark that H acts on V_{n_0} as a scalar. Therefore, W is a simple $\mathcal{U}_0$-submodule (resp., quotient) of V_{n_0} if and only if it is a simple $\mathcal{U}(\mathfrak{c})$-submodule (resp., quotient) of V_{n_0}. As $V_{n_0} = \mathbb{C}[X]$ is a $\mathcal{U}(\mathfrak{c})$-module, we conclude that it does not have any simple submodule and that simple quotients of V_{n_0} are of the form $V_{n_0}/(FE - \chi)$ for some complex number χ. Such a quotient is one-dimensional (as expected), generated by a vector z satisfying $H \cdot z = (b_1 - b_2 + a + 2n_0)z$ and $FE \cdot z = \chi z$.

Thanks to Theorem 4.1, we conclude that V does not have any simple submodules and that W is a simple quotient of V if and only if W has a one-dimensional weight space generated by a vector z satisfying $H \cdot z = (b_1 - b_2 + a + 2n_0)z$ and $FE \cdot z = \chi z$ for some integer n_0 and some complex number χ. Note that if W is a simple weight $\mathfrak{sl}(2,\mathbb{C})$-module such that $\mathrm{supp}(W) \subset \mathrm{supp}(V)$, then there is an integer n_0 such that $b_1 - b_2 + a + 2n_0$ is a weight of W. The corresponding weight space is one-dimensional as asserted by Proposition 2.7. Let z be any vector of W of weight $b_1 - b_2 + a + 2n_0$. Then the Casimir operator Ω acts on W as a scalar χ', and therefore FE acts on z by a scalar χ. As a conclusion we have proved the following.

THEOREM 4.4
Every simple weight $\mathfrak{sl}(2,\mathbb{C})$-module W whose support is included in the support of V appears as a quotient of the algebraic tensor product V.

5. Tensor products: Hilbertian approach

In this section, we investigate the structure of tensor products of unitarizable $\mathfrak{su}(1,1)$-modules. In what is to follow, we set $V = N(a_1, a_2) \otimes N(a, 0)$, where $a \in \mathbb{R}_{<0}$ and either $a_1 = 0, a_2 \in \mathbb{R}_{<0}$ or $a_1 = -1 - x + iy, a_2 = x + iy$ (with $-1 \leq x < 0$, $y \in \mathbb{R}_{>0}$) or $-1 < a_1, a_2 < 0$. This means that we study the tensor product of either a representation of the principal series or of the complementary series or a lowest weight representation with a highest weight representation. Set $s =$

$a+a_1+a_2$. If $a_1=-1-x+iy, a_2=x+iy$, then $s=-1+a+2iy$, and so $\mathfrak{Re}(s)<-1$. Otherwise, $s\in\mathbb{R}_{<0}$. We consider several disjoint cases which exhaust all possible situations:

(A) $a\notin\mathbb{Z}$ and either $a_1=0, a_2\in\mathbb{R}_{<0}\setminus\mathbb{Z}_{<0}$ or $a_1=-1-x+iy, a_2=x+iy$ (with $-1\le x<0$, $y\in\mathbb{R}_{>0}$) or $-1<a_1,a_2<0$.

(B) $a\in\mathbb{Z}_{<0}$ and either $a_1=0, a_2\in\mathbb{R}_{<0}\setminus\mathbb{Z}_{<0}$ or $a_1=-1-x+iy, a_2=x+iy$ (with $-1\le x<0$, $y\in\mathbb{R}_{>0}$) or $-1<a_1,a_2<0$.

(C) $a\notin\mathbb{Z}_{<0}$ and $a_1=0, a_2\in\mathbb{Z}_{<0}$.

(D) $a\in\mathbb{Z}_{<0}$ and $a_1=0, a_2\in\mathbb{Z}_{<0}$.

Denote by $x(k)$ the standard basis of $N(a_1,a_2)$ given in Section 2.2.1. In particular, $k\in\mathbb{Z}_{\ge 0}$ if $a_1=0$ and $k\in\mathbb{Z}$ otherwise. For $l\in\mathbb{Z}_{\ge 0}$, denote by $y(l)$ the basis of $N(a,0)$ defined by $y(l)=x(-l)$, where $x(j)$ is the standard basis of $N(a,0)$. Now a basis for V is $z(k,l)=x(k)\otimes y(l)$.

Moreover, the modules $N(a_1,a_2)$ and $N(a,0)$ have a Hilbert space structure, given by (3.1), (3.2), (3.3), or (3.4). Therefore, we can construct a Hilbert space structure on V via the formula $\langle x\otimes y, x'\otimes y'\rangle=\langle x,x'\rangle\times\langle y,y'\rangle$. Thus, the completion of V with respect to this Hilbert structure is

$$\hat{V}=\Big\{\sum_{k,l}u_{k,l}z(k,l):\sum_{k,l}|u_{k,l}|^2\big\|z(k,l)\big\|^2<\infty\Big\}.$$

In the rest of this article we shall write V instead of $\hat{V}$. For future use we recall in Table 2 the value of the norms $\|x(k)\|^2$ and $\|y(l)\|^2$ in various situations (which were computed in Section 3).

From now on, we assume that $\|x(0)\|^2=1=\|y(0)\|^2$. We conclude this paragraph by giving the action of H, E, and F in the above four cases. In fact, the action of H is the same in all cases and is given by

$$H\cdot z(k,l)=\big(a_1-a_2+a+2(k-l)\big)z(k,l).$$

We remark, in particular, that $\mathrm{Supp}(V)=a_1-a_2+a+2\mathbb{Z}$ and that all nonzero weight spaces are infinite-dimensional. The actions of E and F are given by

(A)

$$E\cdot z(k,l)=(a_2-k)z(k+1,l)+lz(k,l-1), \tag{5.1a}$$

$$F\cdot z(k,l)=(a_1+k)z(k-1,l)+(a-l)z(k,l+1); \tag{5.1b}$$

Table 2

$a\notin\mathbb{Z}_{<0}$	$\|y(l)\|^2=(l!/[\prod_{j=1}^{l}(j-a-1)])\|y(0)\|^2, l>0$
$a\in\mathbb{Z}_{<0}$	$\|y(l)\|^2=l!\prod_{j=1}^{l}(j-a-1)\|y(0)\|^2, l>0$
$a_2\notin\mathbb{Z}_{<0}$	$\|x(k)\|^2=([\prod_{j=1}^{k}(j+a_1)]/[\prod_{j=1}^{k}(j-\bar{a}_2-1)])\|x(0)\|^2, k>0$
$a_1=0, a_2\in\mathbb{Z}_{<0}$	$\|x(k)\|^2=k!\prod_{j=1}^{k}(j-a_2-1)\|x(0)\|^2, k>0$

(B)

$$E\cdot z(k,l)=(a_2-k)z(k+1,l)+l(a-l+1)z(k,l-1), \tag{5.2a}$$

$$F\cdot z(k,l)=(a_1+k)z(k-1,l)+z(k,l+1); \tag{5.2b}$$

(C)

$$E\cdot z(k,l)=z(k+1,l)+lz(k,l-1), \tag{5.3a}$$

$$F\cdot z(k,l)=k(a_2-k+1)z(k-1,l)+(a-l)z(k,l+1); \tag{5.3b}$$

(D)

$$E\cdot z(k,l)=z(k+1,l)+l(a-l+1)z(k,l-1), \tag{5.4a}$$

$$F\cdot z(k,l)=k(a_2-k+1)z(k-1,l)+z(k,l+1). \tag{5.4b}$$

5.1. Highest and lowest weight modules

In this section, we investigate which highest or lowest weight modules are submodules of V. We remark that $z(k,l)$ and $z(k',l')$ have the same weights if $k-l=k'-l'$. Let $n_0\in\mathbb{Z}$. Assume first that $n_0\geq 0$. Consider a vector of the form

$$v=\sum_{l\geq 0}u_l z(l+n_0,l).$$

We want to determine u_l such that $E\cdot v=0$ and $\sum_{l\geq 0}|u_l|^2\|z(l+n_0,l)\|^2<\infty$. From (5.1), (5.2), (5.3), and (5.4), we see that in general

$$E\cdot z(l+n_0,l)=a(l+n_0)z(l+n_0+1,l)+b(l)z(l+n_0,l-1),$$

where $a(l+n_0)\neq 0$, $b(0)=0$, and $b(l)\neq 0$ for positive l. Now the equation $E\cdot v=0$ gives

$$\sum_{l\geq 0}\big(u_l a(l+n_0)+u_{l+1}b(l+1)\big)z(l+n_0+1,l)=0.$$

Therefore, we must have

$$\forall l\geq 0,\quad u_l a(l+n_0)+u_{l+1}b(l+1)=0.$$

Hence,

$$u_l=(-1)^l\times\frac{\prod_{j=1}^{l}a(j+n_0-1)}{\prod_{j=1}^{l}b(j)}u_0.$$

We assume now that $u_0=1$. To check the convergence condition we give the asymptotic behavior of $|u_l|^2$ in the four cases:

(A) $|u_l|^2\sim l^{n_0-1-a_2}$,
(B) $|u_l|^2\sim l^{n_0+a-a_2}/l!$,
(C) $|u_l|^2\sim 1/l!$,
(D) $|u_l|^2\sim l^{a+1}/(l!)^2$.

Now, recall that $\|z(l+n_0,l)\|^2 = \|x(l+n_0)\|^2\|y(l)\|^2$. Using the asymptotic behavior given in Table 2, we conclude that in all cases we have

$$|u_l|^2\big\|z(l+n_0,l)\big\|^2 \sim l^{a+a_1-a_2+2n_0}.$$

Thus, the convergence condition holds if and only if $2n_0 < -1-a-a_1+a_2$.

Assume now that $n_0 < 0$. Then if $a_1 \neq 0$, the vector $z(l+n_0)$ exists for all nonnegative l. In this case, the above computation still holds. Hence, we find a highest weight vector of weight $a_1-a_2+a+2n_0$ in V if and only if $2n_0 < -1-a-a_1+a_2$. If $a_1 = 0$, the vector $z(l+n_0)$ exists only for $l+n_0 \geq 0$. In this case, the equation $E\cdot\sum_{l\geq -n_0} u_l z(l+n_0,l) = 0$ gives $u_{-n_0} = 0$ and by induction $u_l = 0$ for all l.

We can now summarize our results in the following.

PROPOSITION 5.1

If a simple highest weight module $N(\lambda,0)$, of highest weight λ, is a Hilbert submodule of V, then $\lambda = a_1+a-a_2+2n_0$ for some integer n_0. Conversely:

(1) *Assume that $a_1 = 0$. Then the simple highest module $N(a-a_2+2n_0,0)$, of highest weight $a-a_2+2n_0$, is a Hilbert submodule of V if and only if $0 \leq 2n_0 < -1-a+a_2$.*

(2) *Assume that $a_1 \neq 0$. Then the simple highest module $N(a_1+a-a_2+2n_0,0)$, of highest weight $a_1+a-a_2+2n_0$, is a Hilbert submodule of V if and only if $2n_0 < -1-a-a_1+a_2$.*

Let us now turn to lowest weight modules. First assume that $n_0 \geq 1$. We want to determine u_l such that

$$F\cdot\sum_{l\geq 0} u_l z(l+n_0,l) = 0 \qquad \text{and} \qquad \sum_{l\geq 0} |u_l|^2\big\|z(l+n_0,l)\big\|^2 < \infty.$$

As above, we write in general

$$F\cdot z(l+n_0,l) = a'(l+n_0)z(l+n_0-1,l) + b'(l)z(l+n_0,l+1).$$

We remark that we have $u_0 = 0$ (since $a'(n_0) \neq 0$) and by induction $u_l = 0$. This still holds if $n_0 < 1$ and $a_1 \neq 0$.

Assume then that $n_0 < 1$, and assume that $a_1 = 0$. We want to determine u_l such that $F\cdot\sum_{l\geq -n_0} u_l z(l+n_0,l) = 0$ and $\sum_{l\geq -n_0}|u_l|^2\|z(l+n_0,l)\|^2 < \infty$. As above, we write in general

$$F\cdot z(l+n_0,l) = a'(l+n_0)z(l+n_0-1,l) + b'(l)z(l+n_0,l+1).$$

Now we have $a'(0) = 0$. As above, we write

$$0 = F\cdot\sum_{l\geq -n_0} u_l z(l+n_0,l) = \sum_{l\geq -n_0}\big(u_{l+1}a'(l+n_0+1) + u_l b'(l)\big)z(l+n_0,l+1).$$

We deduce from that the expression of u_l, that is,

$$u_l = (-1)^l \times \frac{\prod_{j=1}^{l} b'(j-1)}{\prod_{j=1}^{l} a'(j+n_0)} u_0.$$

We then find the asymptotic behavior of $|u_l|^2$ from which we conclude that the asymptotic behavior of $|u_l|^2 \|x(l+n_0,l)\|^2$ is $l^{a_2-a-2n_0}$. Thus, we have proved the following.

PROPOSITION 5.2

If a simple lowest weight module $N(0,-\lambda)$, of lowest weight λ, is a Hilbert submodule of V, then $\lambda = a_1 + a - a_2 + 2n_0$ for some integer n_0. Conversely:

(1) *Assume that $a_1 = 0$. Then the simple lowest module $N(0,-a+a_2-2n_0)$, of lowest weight $a-a_2+2n_0$, is a Hilbert submodule of V if and only if $1+a_2-a < 2n_0 \leq 0$.*

(2) *Assume that $a_1 \neq 0$. Then V has no Hilbert submodule isomorphic to the simple lowest module $N(0,-a_1-a+a_2-2n_0)$, of lowest weight $a_1+a-a_2+2n_0$, for any n_0.*

5.2. Principal and complementary series

In this section, we investigate which modules from the principal or the complementary series are submodules of V. Recall that the support of such a module M is of the form $b+2\mathbb{Z}$. As we have $\operatorname{supp}(V) = a_1 - a_2 + a + 2\mathbb{Z}$, we can assume that $b = a_1 - a_2 + a$, that is, $M = N(b_1,b_2)$ with $b_1 - b_2 = a_1 - a_2 + a$ and either $b_1 = -1 - x' + iy', b_2 = x' + iy'$, or $-1 < b_1, b_2 < 0$. Let v denote a weight vector of $N(b_1,b_2)$ having weight $b_1 - b_2$. Then from the action of E and F given in Section 2.3, we find that $FE \cdot v = \xi v$ with

$$\xi \leq -\Big(\frac{1+a+a_1-a_2}{2}\Big)^2, \quad \text{if } b_1 = -1-x'+iy', b_2 = x'+iy', \tag{5.5a}$$

$$-\Big(\frac{1+a+a_1-a_2}{2}\Big)^2 < \xi < -\Big(\frac{a+a_1-a_2}{2}\Big)\Big(\frac{a+a_1-a_2+2}{2}\Big), \tag{5.5b}$$
$$\text{if } -1 < b_1, b_2 < 0.$$

Now remark that the vector $z(k,l)$ has weight $a_1 - a_2 + a$ if and only if $k = l$. Therefore, we are looking for a vector

$$v = \sum_{n \geq 0} u_n z(n,n)$$

such that $FE \cdot v = \xi v$ (with ξ satisfying one of the conditions from (5.5)) and

$$\sum_{n\geq 0} |u_n|^2 \big\|z(n,n)\big\|^2 < \infty.$$

Conversely, if the vector v satisfies both these conditions it is easy to check that v generates a simple submodule of V.

Using (5.1), (5.2), (5.3), and (5.4), we compute $FE \cdot v$ in the four cases and write it in the form

$$FE \cdot v = \sum_{n \geq 0} v_n z(n,n),$$

for some sequence v_n completely determined by the u_k's. Then we can identify the coefficients of $z(n,n)$ in $FE \cdot v$ and in ξv. We obtain

(A)

$$(a_1+1)u_1 + \big(a_2(a_1+1)\big)u_0 = \xi u_0, \tag{5.6a}$$

$$\begin{aligned} &(n+2)(n+2+a_1)u_{n+2} \\ &\quad + \big((a_2-n-1)(a_1+n+2) + (n+1)(a-n)\big)u_{n+1} \\ &\quad + (a-n)(a_2-n)u_n = \xi u_{n+1}, \quad \forall n \geq 0; \end{aligned} \tag{5.6b}$$

(B)

$$a(a_1+1)u_1 + \big(a_2(a_1+1)\big)u_0 = \xi u_0, \tag{5.7a}$$

$$\begin{aligned} &(n+2)(n+2+a_1)(a-n-1)u_{n+2} \\ &\quad + \big((a_2-n-1)(a_1+n+2) + (n+1)(a-n)\big)u_{n+1} \\ &\quad + (a_2-n)u_n = \xi u_{n+1}, \quad \forall n \geq 0; \end{aligned} \tag{5.7b}$$

(C)

$$a_2 u_1 + a_2 u_0 = \xi u_0, \tag{5.8a}$$

$$\begin{aligned} &(n+2)^2(a_2-n-1)u_{n+2} \\ &\quad + \big((a_2-n-1)(n+2) + (n+1)(a-n)\big)u_{n+1} \\ &\quad + (a-n)u_n = \xi u_{n+1}, \quad \forall n \geq 0; \end{aligned} \tag{5.8b}$$

(D)

$$aa_2 u_1 + a_2 u_0 = \xi u_0, \tag{5.9a}$$

$$\begin{aligned} &(n+2)^2(a-n-1)(a_2-n-1)u_{n+2} \\ &\quad + \big((a_2-n-1)(n+2) + (n+1)(a-n)\big)u_{n+1} \\ &\quad + u_n = \xi u_{n+1}, \quad \forall n \geq 0. \end{aligned} \tag{5.9b}$$

In the first case, we see using Table 2 that $\|z(n,n)\|^2 \sim (n+1)^{2+\mathfrak{Re}(s)}$. Therefore, the sequence u_n belongs to the Hilbert space V if and only if $\sum_{n \geq 0} |u_n|^2 (n+1)^{2+\mathfrak{Re}(s)} < \infty$.

Now we consider the following renormalization:

(B) $v_0 = u_0$ and $v_n = \prod_{j=1}^{n}(a+1-j) \times u_n, \forall n > 0$;
(C) $v_0 = u_0$ and $v_n = \prod_{j=1}^{n}(a_2+1-j) \times u_n, \forall n > 0$;
(D) $v_0 = u_0$ and $v_n = \prod_{j=1}^{n}(a+1-j)(a_2+1-j) \times u_n, \forall n > 0$.

Then it is easily checked that the sequence v_n satisfies (5.6). Moreover, the condition $\sum_{n\geq 0}|u_n|^2\|z(n,n)\|^2<\infty$ is then equivalent to the condition $\sum_{n\geq 0}|v_n|^2(n+1)^{2+\mathfrak{Re}(s)}<\infty$, which is the condition satisfied by the sequence u_n in case (A).

Set $\mu=\xi-a_2(1+a+a_1)$, and set $p=aa_2$. Note that in all cases $\mu+[(1+s)/2]^2$ is a real number which satisfies the following:

- If x generates a module from the principal series,
$$\mu+\Big(\frac{1+s}{2}\Big)^2\leq 0.$$
- If x generates a module from the complementary series,
$$0<\mu+\Big(\frac{1+s}{2}\Big)^2<\frac{1}{4}.$$

From the above discussion, we are left with the following two equations:

$$(a_1+1)u_1=(p+\mu)u_0, \tag{5.10a}$$

$$\begin{aligned}&(n+2)(n+2+a_1)u_{n+2}\\&\quad+\big(s+2-\mu-(n+2)(n+2+a_1)-(n-a)(n-a_2)\big)u_{n+1}\\&\quad+(n-a)(n-a_2)u_n=0.\end{aligned}\tag{5.10b}$$

It is clear that this difference equation has a unique solution for a given u_0. In the following, we assume without loss of generality that $u_0=1$. To check whether

$$\sum_{n\geq 0}|u_n|^2n^{2+\mathfrak{Re}(s)}<\infty \tag{5.11}$$

holds, we need to understand the asymptotic behavior of this unique solution. We use two different approaches.

5.2.1. Asymptotics using a discrete derivative

There are two independent fundamental solutions to (5.10b). We denote them by u_n^1 and u_n^2. Then our sequence u_n satisfying (5.10) is an unknown linear combination of these solutions.

Define an operator D (discrete derivative) by the formula $D(u_n)=u_{n+1}-u_n$. Then we can rewrite (5.10b) with D as follows:

$$\begin{aligned}&(n+2)(n+2+a_1)D^2(u_n)+\big(6+s-p+2a_1-\mu+n(4+s)\big)D(u_n)\\&\quad+(s+2-\mu)u_n=0.\end{aligned}\tag{5.12}$$

This is a discrete version of the hypergeometric equation.

Now we use a discrete version of the local analysis of differential equations (see [3]). For (5.12), the point ∞ is regular-singular (see [3, Section 5.2]). Therefore, we know from the discrete version of the Fuchs theorem (see [3, Section 5.2]) that the fundamental solutions of the difference equation have asymptotics of the form $n^\alpha\sum_{n\geq 0}A_kn^{-k}$ and $n^\beta\sum_{n\geq 0}B_kn^{-k}$ or $n^\alpha\ln(n)\sum_{n\geq 0}A_kn^{-k}$, for some complex numbers α and β, where A_0 and B_0 are not zero. To find α (and β), we write $u_n^1 = A_0 n^\alpha + A_1 n^{\alpha-1} + A_2 n^{\alpha-2} + o(n^{\alpha-2})$ for large n. Evaluating (5.10b) gives $(\alpha^2 + (3+s)\alpha + s + 2 - \mu)A_0 + o(1/n^2) = 0$, from which we conclude that $\alpha^2 + (3+s)\alpha + s + 2 - \mu = 0$. Thus, we find that

$$\alpha = \frac{-s-3}{2} \pm \sqrt{\mu + \Big(\frac{1+s}{2}\Big)^2}.$$

Now, if v generates a module from the principal series, we have seen that $\mu + [(1+s)/2]^2 \leq 0$. First, if $\mu + [(1+s)/2]^2 < 0$, then the square-norm of both fundamental solutions is equivalent to $n^{2\mathfrak{Re}(\alpha)} = n^{-\mathfrak{Re}(s)-3}$. But any solution is a linear combination of these fundamental solutions. Hence, the square-norm of every solution is equivalent to $n^{-\mathfrak{Re}(s)-3}$. If $\mu + [(1+s)/2]^2 = 0$, then the square-norm of the fundamental solutions are equivalent to $n^{-\mathfrak{Re}(s)-3}$ or to $n^{-\mathfrak{Re}(s)-3}\ln^2(n)$. Hence, the square-norm of every solution is equivalent to $n^{-\mathfrak{Re}(s)-3}$ or to $n^{-\mathfrak{Re}(s)-3}\ln^2(n)$. Thus, if v generates a module from the principal series, we have $|u_n|^2 n^{2+\mathfrak{Re}(s)} \sim n^{-1}$ or $|u_n|^2 n^{2+\mathfrak{Re}(s)} \sim n^{-1}\ln^2(n)$. Hence, the sequence u_n is not in the Hilbert space V. Thus, we have proved the following.

PROPOSITION 5.3
Let $b_1 = -1 - x' + iy'$, and let $b_2 = x' + iy'$, with $y' \neq 0$. Then the simple weight module $N(b_1, b_2)$ is never a Hilbert submodule of V.

In other words, the tensor product V never discretely contains a module from the principal series.

Note, however, that the principal series whose support is $a_1 - a_2 + a + 2\mathbb{Z}$ is almost contained in the Hilbert space, in the sense that it is contained in

$$V_\epsilon := \Big\{\sum_{k,l} u_{k,l} z(k,l) : \sum_{k,l} |u_{k,l}|^2 \big\|z(k,l)\big\|^2 (k^2+l^2)^{-\epsilon} < \infty\Big\},$$

for any positive ϵ. This might be seen as an analogous condition for being in the continuous spectrum.

On the other hand, if v generates a module from the complementary series, we have seen that $0 < \mu + [(1+s)/2]^2 < 1/4$. Hence, we find that $|u_n^1|^2 n^{2+\mathfrak{Re}(s)} \sim n^{-1-\sqrt{\mu+[(1+s)/2]^2}}$ and $|u_n^2|^2 n^{2+\mathfrak{Re}(s)} \sim n^{-1+\sqrt{\mu+[(1+s)/2]^2}}$. As the solution u_n is an unknown linear combination of u_n^1 and u_n^2, we cannot conclude from these asymptotics whether $u_n \in V$.

5.2.2. Asymptotics using a differential equation
Therefore, we need another approach to deal with complementary series. From now on, we assume that $0 < \mu + [(1+s)/2]^2 < 1/4$. For $-1 < t < 1$, set $S(t) = \sum_{n\geq 0} u_n t^n$. Then, the sequence u_n satisfies (5.10) if and only if $S(t)$ is a solution of the following differential equation:

$$t(1-t)S''(t) + \big(1 + a_1 - (1+a_1-s)t\big)S'(t) - \Big(p + \frac{\mu}{1-t}\Big)S(t) = 0. \tag{5.13}$$

Moreover, we must have $S(0)=1$ and $S'(0)=(p+\mu)/(1+a_1)$, since we assumed that $u_0=1$. The unique solution to this Cauchy problem is the function

$$S(t)=(1-t)^r{}_2F_1(r-a,r-a_2;1+a_1;t),$$

where $r=(1+s)/2-\sqrt{\mu+[(1+s)/2]^2}$ and ${}_2F_1(r-a,r-a_2;1+a_1;t)$ is the corresponding hypergeometric function

$${}_2F_1(r-a,r-a_2;1+a_1;t)=\sum_{n\geq 0}\frac{(r-a)_n(r-a_2)_n}{(1+a_1)_n}\frac{t^n}{n!}.$$

Here, $(b)_n$ is the Pochhammer symbol, that is,

$$(b)_0=1, \qquad (b)_n=\prod_{j=1}^{n}(b-1+j), \quad \forall n>0.$$

As $1+a_1\notin\mathbb{Z}_{\leq 0}$, the function $S(z)$ is well defined on the (open) unit disc D and is holomorphic. We remark that a and a_2 play a symmetric role in the definition of S. In the rest of this article, we shall write $a_{(2)}$ to refer to either a or a_2.

Before going further, we need to collect several facts about the hypergeometric function. We refer the reader to [2] or [1]. On first reading, the reader might want to skip these technicalities and go straight to page 337.

LEMMA 5.4 (GAUSS THEOREM)
Let $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\alpha\notin\mathbb{Z}_{\leq 0}$, $\beta\notin\mathbb{Z}_{\leq 0}$, and $\gamma\notin\mathbb{Z}_{\leq 0}$. Then:

(1) *The function $z\mapsto {}_2F_1(\alpha,\beta;\gamma;z)$ is holomorphic on D.*

(2) *If $\mathfrak{Re}(\gamma-\alpha-\beta)>0$, then the function $z\mapsto {}_2F_1(\alpha,\beta;\gamma;z)$ is continuous in $\bar{D}$.*

(3) *If $\mathfrak{Re}(\gamma-\alpha-\beta)<0$, then there is a nonzero constant C such that*

$${}_2F_1(\alpha,\beta;\gamma;z)\sim C(1-z)^{\gamma-\alpha-\beta}, \quad \text{when } z\to 1.$$

(4) *If $\gamma-\alpha-\beta=0$, then there is a nonzero constant C such that*

$${}_2F_1(\alpha,\beta;\gamma;z)\sim C\log(1-z), \quad \text{when } z\to 1.$$

Proof
The first and the second assertions are [1, Theorem 2.1.2]. The last two assertions are [1, Theorem 2.1.3]. □

LEMMA 5.5
Assume that $r-a$ or $r-a_2$ is a nonpositive integer.

(1) *Then ${}_2F_1(r-a,r-a_2;1+a_1;z)$ is polynomial and is therefore holomorphic on $\mathbb{C}$.*

(2) *We cannot have $r-a\in\mathbb{Z}_{\leq 0}$ and $r-a_2\in\mathbb{Z}_{\leq 0}$ unless $r=a=a_2$.*

(3) *We have ${}_2F_1(r-a,r-a_2;1+a_1;1)\neq 0$.*

(4) *If $\mathfrak{Re}(s)\geq -2$, then $r-a\in\mathbb{Z}_{\leq 0}$ (resp., $r-a_2\in\mathbb{Z}_{\leq 0}$) implies that $r=a$ (resp., $r=a_2$), and therefore ${}_2F_1(r-a,r-a_2;1+a_1;x)=1$.*

Proof
The first assertion follows from the definition of the Pochhammer symbol.

Assume that $r-a=-n$, and assume that $r-a_2=-m$. Then $2r-a-a_2+n+m=0$; that is, $a_1+1+n+m-2\sqrt{\mu+[(s+1)/2]^2}=0$. Therefore, we should have $0<1+a_1+m+n<1$. But we have $1+a_1=-x+iy$ or $1+a_1>0$. In the first case the equality $a_1+1+n+m-2\sqrt{\mu+[(s+1)/2]^2}=0$ can hold only when $y=0$. Then in both cases $0<1+a_1\leq 1$. Therefore, the equality can hold only for $n+m=0$, that is, $n=m=0$. This proves the second part of the lemma.

The third assertion is a consequence of the Chu–Vandermonde theorem [1, Corollary 2.2.3]. Indeed, assume, for instance, that $r-a=-n$. Then $1+a_1-r+a_2=1+s-a-r=1+s-n-2r=2\sqrt{\mu+[(s+1)/2]^2}-n$. This last quantity is never an integer. But the Chu–Vandermonde theorem implies that ${}_2F_1(r-a,r-a_2;1+a_1;1)=(1+a_1-r+a_2)_n/(1+a_1)_n$. Therefore, this is not zero.

Finally, assume that $r-a_{(2)}=-n$ for $n\in\mathbb{Z}_{>0}$. Then we must have $0<n+[(\mathfrak{Re}(s)+1)/2]-\mathfrak{Re}(a_{(2)})<\frac{1}{2}$. By our hypothesis $(\mathfrak{Re}s+1)/2\geq-\frac{1}{2}$. But we have $\mathfrak{Re}(a_{(2)})\leq 0$. Thus, such a condition never holds. □

LEMMA 5.6

Assume that $r-a$ and $r-a_2$ are not nonpositive integers.

(i) *The hypergeometric function is well defined and continuous on the closed unit disc $\bar{D}$.*

(ii) *We have that*

$$ {}_2F_1(r-a,r-a_2;1+a_1;1)=\frac{\Gamma(1+a_1)\Gamma(1+s-2r)}{\Gamma(1+a_1+a-r)\Gamma(1+a_1+a_2-r)}. $$

(iii) *For $z\in D$, we have that*

$$ {}_2F_1'(r-a,r-a_2;1+a_1;z)=\frac{(r-a)(r-a_2)}{1+a_1}{}_2F_1(r-a+1,r-a_2+1;2+a_1;z). $$

(iv) *The derivative of ${}_2F_1(r-a,r-a_2;1+a_1;z)$ is well defined and continuous on the domain $\bar{D}\setminus\{1\}$.*

(v) *When $z\to 1$, there is a nonzero constant C such that*

$$ {}_2F_1'(r-a,r-a_2;1+a_1;z)\sim C(1-z)^{s-2r}. $$

Proof
We remark that $1+a_1-(r-a)-(r-a_2)=1+s-2r=2\sqrt{\mu+[(1+s)/2]^2}$ and that $0<\mu+[(1+s)/2]^2<\frac{1}{4}$. Now (i) is a consequence of [1, Theorem 2.1.2], (ii) is the Gauss theorem [1, Theorem 2.2.2], (iii) is [1, (2.5.1)], (iv) is a consequence of (iii) and [1, Theorem 2.1.2], and (v) is a consequence of (iii) and [1, Theorem 2.1.3]. □

COROLLARY 5.7

Assume that $r-a$ and $r-a_2$ are not nonpositive integers. If $1+a_1+a-r\notin\mathbb{Z}_{\leq 0}$ and $1+a_1+a_2-r\notin\mathbb{Z}_{\leq 0}$, then ${}_2F_1(r-a,r-a_2;1+a_1;1)\neq 0$.

LEMMA 5.8

Assume that $r-a$ and $r-a_2$ are not nonpositive integers. Assume also that $1+a_1+a-r\in\mathbb{Z}_{\le 0}$ or $1+a_1+a_2-r\in\mathbb{Z}_{\le 0}$.

(1) *We cannot have $1+a_1+a-r\in\mathbb{Z}_{\le 0}$ and $1+a_1+a_2-r\in\mathbb{Z}_{\le 0}$.*

(2) *If $1+a_1+a-r=-n$, we have ${}_2F_1(1+a_1+n, r-a_2; 1+a_1; z)=(1-z)^{a_2-r-n}P_n(z)$, where $P_n(z)$ is a polynomial of degree n. If $1+a_1+a_2-r=-n$, we have ${}_2F_1(1+a_1+n, r-a; 1+a_1; z)=(1-z)^{a-r-n}Q_n(z)$, where $Q_n(z)$ is a polynomial of degree n.*

(3) *With the notations as above, we have $P_0=1=Q_0$ and, for $n>0$,*

$$P_n(1)=\frac{(r-a_2)(r-a_2+1)\cdots(r-a_2+n-1)}{(1+a_1)(2+a_1)\cdots(n+a_1)}\neq 0$$

and

$$Q_n(1)=\frac{(r-a)(r-a+1)\cdots(r-a+n-1)}{(1+a_1)(2+a_1)\cdots(n+a_1)}\neq 0.$$

(4) *If $\mathfrak{Re}(s)\ge -2$, then $1+a_1+a-r=-n$ (resp., $1+a_1+a_2-r=-n$) implies that $\mathfrak{Re}(s)<-1$ and $n=0$, and therefore ${}_2F_1(1+a_1, r-a_2; 1+a_1; z)=(1-z)^{a_2-r}$ (resp., ${}_2F_1(1+a_1, r-a; 1+a_1; z)=(1-z)^{a-r}$).*

Proof

Assume that $1+a_1+a-r=-n$, and assume that $1+a_1+a_2-r=-m$. Then we have $1+s+1+a_1-2r+n+m=0$; that is, $1+a_1+n+m+2\sqrt{\mu+[(1+s)/2]^2}=0$. So we must have $1+a_1\in\mathbb{R}$ and $-1<1+a_1+n+m<0$. As $1+a_1>0$, this is not possible.

The second assertion of the lemma is Euler's theorem [1, Theorem 2.2.5]. The value of P_n at $z=1$ is Chu–Vandermonde's theorem [1, Corollary 2.2.3].

To prove the fourth part of the lemma, assume, for instance, that $1+a_1-r+a+n=0$. Therefore we have $(1+\mathfrak{Re}(s))/2+n-\mathfrak{Re}(a_2)+\sqrt{\mu+[(s+1)/2]^2}=0$. So we must have $0<-(1+\mathfrak{Re}(s))/2-n+\mathfrak{Re}(a_2)<\frac{1}{2}$. As we have $\mathfrak{Re}(a_2)<0$ and $\mathfrak{Re}(s)\ge -2$ by hypothesis, such an equality can hold only if $n=0$. Moreover, we must have $\mathfrak{Re}(s)>-1$. The proof when $1+a_1+a_2-r=-n$ is analogous. $\square$

From now on, we set $F(z)={}_2F_1(r-a, r-a_2; 1+a_1; z)$. For future use, we compute some asymptotics.

LEMMA 5.9

(1) *Let $d\in\mathbb{R}$. Then, when $te^{i\theta}\to 1$, there is a nonzero constant C such that*

$$(1-te^{i\theta})^d\sim C|1-te^{i\theta}|^d.$$

(2) *Assume that $r-a$ and $r-a_2$ are not nonpositive integers. Assume also that $F(1)\neq 0$. Then, when $te^{i\theta}\to 1$, there is a nonzero constant C such that*

$$F'(te^{i\theta})\sim C|1-te^{i\theta}|^{s-2r}.$$

(3) *Let $d \in \mathbb{C}$. Then there is a nonzero constant such that*

$$\left|(1-z)^d\right|^2 \sim C|1-z|^{2\mathfrak{Re}(d)}, \quad \text{when } z \to 1.$$

Proof
The first part of the lemma follows from the equality

$$(1-te^{i\theta})^d = |1-te^{i\theta}|^d \times \exp\Big(2id\arctan\Big(\frac{-t\sin\theta}{1-t\cos\theta+\sqrt{1+t^2-2t\cos\theta}}\Big)\Big).$$

The second assertion follows from the first part together with Lemma 5.6. (Note that $s-2r \in \mathbb{R}$.) The last part of the lemma follows from the equality

$$(1-z)^d = |1-z|^{\mathfrak{Re}(d)} \times \exp\Big(i\mathfrak{Im}(d)\ln|1-z| + 2id\arctan\Big(\frac{-\mathfrak{Im}(z)}{1-\mathfrak{Re}(z)+|1-z|}\Big)\Big). \qquad \square$$

LEMMA 5.10
Let $a \in D$. Then we have

$$\frac{1}{2\pi}\int_0^{2\pi}\big(1+a^2-2a\cos(\theta)\big)^\nu\,d\theta = {}_2F_1(-\nu,-\nu;1;a^2).$$

Proof
This is [9, p. 427, (3.665(2))]. $\square$

COROLLARY 5.11
Let ν be such that $\nu \notin \mathbb{Z}_{\le 0}$ and $\mathfrak{Re}(1+2\nu) < 0$. When $t \to 1$, there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\nu}\,d\theta \sim C(1-t^2)^{1+2\nu}.$$

Proof
We remark that $|1-te^{i\theta}|^{2\nu} = (1+a^2-2a\cos(\theta))^\nu$. Then apply Lemmas 5.10 and 5.4. $\square$

LEMMA 5.12
Assume that $r-a \notin \mathbb{Z}_{\le 0}$, $r-a_2 \notin \mathbb{Z}_{\le 0}$, $1+a_1+a-r \notin \mathbb{Z}_{\le 0}$, and $1+a_1+a_2-r \notin \mathbb{Z}_{\le 0}$. Then:

(1) *We have*

$$S'(z) = (1-z)^r F'(z) + (-r)(1-z)^{r-1}F(z).$$

(2) *We have*

$$S(z)\overline{S'(z)z} = \left|(1-z)^r\right|^2 F(z)\overline{F'(z)z} + (-\bar r)\left|(1-z)^{r-1}\right|^2\left|F(z)\right|^2\left(\bar z - |z|^2\right).$$

When $z \to 1$, there are nonzero constants C and C' such that

$$\left|(1-z)^r\right|^2 F(z)\overline{F'(z)z} \sim C|1-z|^{\mathfrak{Re}(s)}$$

and

$$\left|(1-z)^{r-1}\right|^2 \left|F(z)\right|^2 \bar{z}(1-z) \sim C'|1-z|^{2(\mathfrak{Re}(r)-1/2)}.$$

Furthermore, if $r \neq 0$, then there is a nonzero constant C'' such that

$$S(z)\overline{S'(z)z} \sim C''|1-z|^{2(\mathfrak{Re}(r)-1/2)}.$$

(3) *We have*

$$\begin{aligned}\left|S'(z)\right|^2 &= \left|(1-z)^r\right|^2 \left|F'(z)\right|^2 + |r|^2 \left|(1-z)^{r-1}\right|^2 \left|F(z)\right|^2 \\ &\quad + 2\left|(1-z)^{r-1}\right|^2 \left((-r)(1-z)F(z)\overline{F'(z)} + (-\bar{r})(1-\bar{z})F'(z)\overline{F(z)}\right).\end{aligned}$$

When $z \to 1$, there are nonzero constants C, C', and C'' such that

$$\left|(1-z)^r\right|^2 \left|F'(z)\right|^2 \sim C|1-z|^{2(\mathfrak{Re}(s)-\mathfrak{Re}(r))},$$

$$\left|(1-z)^{r-1}\right|^2 \left|F(z)\right|^2 \sim C'|1-z|^{2(\mathfrak{Re}(r)-1)},$$

$$\left|(1-z)^{r-1}\right|^2 (1-z)F(z)\overline{F'(z)} \sim C''|1-z|^{\mathfrak{Re}(s)-1}.$$

Furthermore, if $r \neq 0$, then there is a nonzero constant C''' such that

$$\left|S'(z)\right|^2 \sim C'''|1-z|^{2(\mathfrak{Re}(r)-1)}.$$

Proof
The equalities are clear. The equivalents are consequences of the fact that $F(1) \neq 0$ by Corollary 5.7 and of those equivalents in Lemma 5.9. $\square$

Now, we need to transform the infinitesimal condition $\sum_{n\geq 0} |u_n|^2 (n+1)^{2+\mathfrak{Re}(s)} < \infty$ from (5.11) into an equivalent condition satisfied by the function $S(z) = \sum_{n\geq 0} u_n t^n = (1-z)^r F(z)$, where $F(z) = {}_2F_1(r-a, r-a_2; 1+a_1; z)$. Thus, we need to define a Hilbert space $\mathcal{H}_s$ such that $\sum_{n\geq 0} |u_n|^2 (n+1)^{2+\mathfrak{Re}(s)} < \infty$ if and only if $S \in \mathcal{H}_s$.

We denote by $\mathcal{O}(D)$ the set of holomorphic functions of the unit disc. Let $d\,\mathrm{vol}(z)$ denote the measure on D such that $\int_D d\,\mathrm{vol}(z) = 1$. First, we remark that $\mathfrak{Re}(s) < 0$. We will distinguish several cases.

Assume that $\mathfrak{Re}(s) < -2$, and consider the following space:

$$\mathcal{H}_s := \left\{ f \in \mathcal{O}(D) : \int_D \left|f(z)\right|^2 \left(1-|z|^2\right)^{-3-\mathfrak{Re}(s)} d\,\mathrm{vol}(z) < \infty \right\}.$$

Then it is clear that for all nonnegative n the function $f_n(z) = z^n$ belongs to the Hilbert space $\mathcal{H}_s$. Moreover, we have $\langle f_n, f_m \rangle = 0$ if $n \neq m$, and $\langle f_n, f_n \rangle \sim (n+1)^{2+\mathfrak{Re}(s)}$ for large n. Therefore, $\langle S, S \rangle = \sum_{n\geq 0} |u_n|^2 \langle f_n, f_n \rangle$. Hence, $u_n \in V$ if and only if $S \in \mathcal{H}_s$.

Now assume that $\mathfrak{Re}(s) = -2$. Then the infinitesimal condition is $\sum_{n\geq 0} |u_n|^2 < \infty$. Therefore, we need to consider the Hilbert space

$$\mathcal{H}_s := \Big\{ f \in \mathcal{O}(D) : \lim_{\rho\to 1} \frac{1}{2\pi} \int_0^{2\pi} \big| f(\rho e^{i\theta}) \big|^2 \, d\theta < \infty \Big\}.$$

It is well known that the functions $f_n(z) = z^n$ give an orthonormal basis of $\mathcal{H}_s$. Hence, $u_n \in V$ if and only if $S \in \mathcal{H}_s$.

Assume now that $-2 < \mathfrak{Re}(s) < -1$, and consider the Hilbert space

$$\mathcal{H}_s := \Big\{ f \in \mathcal{O}(D) : \int_D \big(f(z)\overline{f'(z)z} \big) \big(1 - |z|^2\big)^{-2-\mathfrak{Re}(s)} \, d\operatorname{vol}(z) < \infty \Big\}.$$

It is easy to check that the functions $f_n(z) = z^n$ belong to $\mathcal{H}_s$, are mutually orthogonal, and satisfy $\|f_n\|_s^2 \sim (n+1)^{2+\mathfrak{Re}(s)}$ for large n. Thus, the sequence u_n belongs to V if and only if $S \in \mathcal{H}_s$.

Assume now that $\mathfrak{Re}(s) = -1$, in which case $s \in \mathbb{R}$, and consider the Hilbert space

$$\mathcal{H}_s := \Big\{ f \in \mathcal{O}(D) : \lim_{\rho\to 1} \int_0^{2\pi} \frac{1}{2\pi} \big(f(\rho e^{i\theta})\overline{f'(\rho e^{i\theta})e^{i\theta}} \big) \, d\theta + \big| f(0) \big|^2 < \infty \Big\}.$$

It is easy to check that the functions $f_n(z) = z^n$ belong to $\mathcal{H}_s$, are mutually orthogonal, and satisfy $\|f_n\|_s^2 = n$ for large n. Thus, the sequence u_n belongs to V if and only if $S \in \mathcal{H}_s$.

Assume now that $-1 < \mathfrak{Re}(s) < 0$, in which case $s \in \mathbb{R}$, and consider the Hilbert space

$$\mathcal{H}_s := \Big\{ f \in \mathcal{O}(D) : \int_D \big| f'(z) \big|^2 \big(1 - |z|^2\big)^{-1-\mathfrak{Re}(s)} \, d\operatorname{vol}(z) + \big| f(0) \big|^2 < \infty \Big\}.$$

It is easy to check that the functions $f_n(z) = z^n$ belong to $\mathcal{H}_s$, are mutually orthogonal, and satisfy $\|f_n\|_s^2 \sim (n+1)^{2+\mathfrak{Re}(s)}$ for large n. Thus, the sequence u_n belongs to V if and only if $S \in \mathcal{H}_s$.

Now we are in position to prove the following three propositions, examining whether S belongs to $\mathcal{H}_s$.

PROPOSITION 5.13

Assume that $r - a \in \mathbb{Z}_{\leq 0}$ or $r - a_2 \in \mathbb{Z}_{\leq 0}$. Then the function $S(z)$ does not belong to $\mathcal{H}_s$.

Proof

Assume first that $r = a$ or $r = a_2$. (In particular, $r \neq 0$.) Then $F(z) = 1$ and therefore $S(z) = (1 - z)^r$.

(1) If $\mathfrak{Re}(s) < -2$, then $S(z) \in \mathcal{H}_s$ if and only if

$$\int_D \big| (1 - z)^r \big|^2 \big(1 - |z|^2\big)^{-3-\mathfrak{Re}(s)} \, d\operatorname{vol}(z) < \infty.$$

The function $|(1-z)^r|^2(1-|z|^2)^{-3-\mathfrak{Re}(s)}$ is integrable if and only if it is integrable near $z=1$. By Lemma 5.9, this is equivalent to the following condition:

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\mathfrak{Re}(r)}\,d\theta\Big)(1-t^2)^{-3-\mathfrak{Re}(s)}t\,dt<\infty.$$

Now, we remark that $\mathfrak{Re}(-r)>0$ and $\mathfrak{Re}(1+2r)<0$. So, from Corollary 5.11, we know that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\mathfrak{Re}(r)}\,d\theta\sim C(1-t^2)^{1+2\mathfrak{Re}(r)}.$$

So the condition of integrability becomes

$$\int_0^1(1-t^2)^{2\mathfrak{Re}(r)-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

But $-2+2\mathfrak{Re}(r)-\mathfrak{Re}(s)=-1-2\sqrt{\mu+[(s+1)/2]^2}<-1$; therefore, S is not integrable.

(2) If $\mathfrak{Re}(s)=-2$, then $S(z)\in\mathcal{H}_s$ if and only if

$$\lim_{\rho\to1}\frac{1}{2\pi}\int_0^{2\pi}\big|(1-\rho e^{i\theta})^r\big|^2\,d\theta<\infty,$$

or also via Lemma 5.9 if and only if

$$\lim_{\rho\to1}\frac{1}{2\pi}\int_0^{2\pi}|1-\rho e^{i\theta}|^{2\mathfrak{Re}(r)}\,d\theta<\infty.$$

Now, we remark that $\mathfrak{Re}(-r)>0$ and $\mathfrak{Re}(1+2r)<0$. So, from Corollary 5.11, we know that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-\rho e^{i\theta}|^{2\mathfrak{Re}(r)}\,d\theta\sim C(1-\rho^2)^{1+2\mathfrak{Re}(r)}.$$

Therefore, the above limit is infinite.

(3) If $-2<\mathfrak{Re}(s)<-1$, then $S(z)\in\mathcal{H}_s$ if and only if

$$\int_D-\bar{r}\big|(1-z)^{r-1}\big|^2(1-z)\bar{z}\big(1-|z|^2\big)^{-2-\mathfrak{Re}(s)}\,d\,\mathrm{vol}(z)<\infty.$$

The function $-\bar{r}|(1-z)^{r-1}|^2(1-z)\bar{z}(1-|z|^2)^{-2-\mathfrak{Re}(s)}$ is integrable if and only if it is integrable near $z=1$. Thanks to Lemma 5.9, there is some nonzero constant C such that

$$-\bar{r}\big|(1-z)^{r-1}\big|^2(1-z)\bar{z}\big(1-|z|^2\big)^{-2-\mathfrak{Re}(s)}\sim C|1-z|^{2(\mathfrak{Re}(r)-1/2)}\times\big(1-|z|^2\big)^{-2-\mathfrak{Re}(s)}.$$

The integral $\int_D-\bar{r}|(1-z)^{r-1}|^2(1-z)\bar{z}(1-|z|^2)^{-2-\mathfrak{Re}(s)}\,d\,\mathrm{vol}(z)$ becomes

$$C\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(r)-1/2)}\,d\theta\Big)(1-t^2)^{-2-\mathfrak{Re}(s)}t\,dt.$$

Now we remark that $\mathfrak{Re}(\frac{1}{2}-r)>0$ and $\mathfrak{Re}(2r)<0$. Then from Corollary 5.11, we know that there is a nonzero constant C' such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(r)-1/2)}\,d\theta \sim C'(1-t^2)^{2\mathfrak{Re}(r)}.$$

So the condition of integrability becomes

$$\int_0^1 (1-t^2)^{2\mathfrak{Re}(r)-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

But $-2+2\mathfrak{Re}(r)-\mathfrak{Re}s=-1-2\sqrt{\mu+[(s+1)/2]^2}<-1$, and therefore S is not integrable.

(4) If $\mathfrak{Re}(s)=-1$, then $S(z)\in\mathcal{H}_s$ if and only if

$$\lim_{\rho\to 1}\frac{1}{2\pi}\int_0^{2\pi}-\bar{r}\big|(1-\rho e^{i\theta})^{r-1}\big|^2(1-\rho e^{i\theta})\rho e^{-i\theta}\,d\theta<\infty.$$

Using Lemma 5.9, we see that this limit is finite if and only if

$$\lim_{\rho\to 1}\frac{1}{2\pi}\int_0^{2\pi}|1-\rho e^{i\theta}|^{2(\mathfrak{Re}(r)-1/2)}\,d\theta<\infty.$$

Now we remark that $\mathfrak{Re}(\frac{1}{2}-r)>0$ and $\mathfrak{Re}(2r)<0$. Thus, from Corollary 5.11, we know that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-\rho e^{i\theta}|^{2(\mathfrak{Re}(r)-1/2)}\,d\theta \sim C(1-\rho^2)^{2\mathfrak{Re}(r)}.$$

So the condition becomes

$$\lim_{\rho\to 1}(1-\rho^2)^{2\mathfrak{Re}(r)}<\infty.$$

Therefore, the above limit is not finite.

(5) If $-1<\mathfrak{Re}(s)<0$, then $S(z)\in\mathcal{H}_s$ if and only if

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|r|^2\big|(1-te^{i\theta})^{r-1}\big|^2\,d\theta\Big)(1-t^2)^{-1-\mathfrak{Re}(s)}t\,dt<\infty.$$

By Lemma 5.9, this is equivalent to the condition

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|r|^2|1-te^{i\theta}|^{2(\mathfrak{Re}(r)-1)}\,d\theta\Big)(1-t^2)^{-1-\mathfrak{Re}(s)}t\,dt<\infty.$$

Now, we remark that $\mathfrak{Re}(1-r)>0$ and $\mathfrak{Re}(2r-1)<0$. Hence, from Corollary 5.11, we know that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(r)-1)}\,d\theta \sim C(1-t^2)^{2\mathfrak{Re}(r)-1}.$$

So the condition of integrability becomes

$$\int_0^1 (1-t^2)^{2\mathfrak{Re}(r)-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

But $-2+2\mathfrak{Re}(r)-\mathfrak{Re}s=-1-2\sqrt{\mu+[(s+1)/2]^2}<-1$, and therefore S is not integrable.

Assume now that $r-a=-n$ or $r-a_2=-n$ for some positive integer n. Then from Lemma 5.5, we know that necessarily $\mathfrak{Re}(s)<-2$. We also know that $F(x)$ is polynomial (of degree n) and that $F(1)\neq 0$. Now, $S(z)\in\mathcal{H}_s$ if and only if

$$\int_D \left|(1-z)^r\right|^2 \left|F(z)\right|^2 \left(1-|z|^2\right)^{-3-\mathfrak{Re}(s)} d\,\mathrm{vol}(z)<\infty.$$

The function $|(1-z)^r|^2|F(z)|^2(1-|z|^2)^{-3-\mathfrak{Re}(s)}$ is integrable if and only if it is integrable near $z=1$. But then as $F(1)\neq 0$ and by Lemma 5.9, there is some nonzero constant C such that $|(1-z)^r|^2|F(z)|^2(1-|z|^2)^{-3-\mathfrak{Re}(s)}\sim C|1-z|^{2\mathfrak{Re}(r)}(1-|z|^2)^{-3-\mathfrak{Re}(s)}$ when $z\to 1$. Hence, we are left with the previous situation. □

PROPOSITION 5.14

Assume that $r-a\notin\mathbb{Z}_{\leq 0}$, and assume that $r-a_2\notin\mathbb{Z}_{\leq 0}$. Assume also that $1+a_1+a-r\in\mathbb{Z}_{\leq 0}$ or $1+a_1+a_2-r\in\mathbb{Z}_{\leq 0}$. Then the function $S(z)$ belongs to $\mathcal{H}_s$.

Proof

By Lemma 5.8, we have $F(1)=0$. More precisely, there is a polynomial $P(z)$ of degree n such that $P(1)\neq 0$ and $F(z)=(1-z)^{a_{(2)}-r-n}P(z)$. (Recall that $a_{(2)}$ denotes either a or a_2.) Thus, $S(z)=(1-z)^{a_{(2)}-n}P(z)$.

Assume first that $n=0$, that is, $1+s-a_{(2)}-r=0$. Then Lemma 5.8 implies that $P=1$ and that $\mathfrak{Re}(s)>-1$.

(1) If $-2<\mathfrak{Re}(s)<-1$, then $S\in\mathcal{H}_s$ if and only if

$$\int_D -\overline{a_{(2)}}\left|(1-z)^{a_{(2)}-1}\right|^2(1-z)\bar{z}\left(1-|z|^2\right)^{-2-\mathfrak{Re}(s)} d\,\mathrm{vol}(z)<\infty.$$

The function $-\overline{a_{(2)}}|(1-z)^{a_{(2)}-1}|^2(1-z)\bar{z}(1-|z|^2)^{-2-\mathfrak{Re}(s)}$ is integrable if and only if it is integrable near $z=1$. Thanks to Lemma 5.9, there is a nonzero constant C such that

$$\begin{aligned}\left|(1-z)^{a_{(2)}-1}\right|^2(1-z)\bar{z}\left(1-|z|^2\right)^{-2-\mathfrak{Re}(s)} &\sim C|1-z|^{2(\mathfrak{Re}(a_{(2)})-1/2)}\\ &\quad\times\left(1-|z|^2\right)^{-2-\mathfrak{Re}(s)}.\end{aligned}$$

So the condition of integrability becomes

$$\int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(a_{(2)})-1/2)}\,d\theta\right)(1-t^2)^{-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

Now we remark that $\mathfrak{Re}(2a_{(2)})<0$. Then from Corollary 5.11, we know that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(a_{(2)})-1/2)}\,d\theta\sim C(1-t^2)^{2\mathfrak{Re}(a_{(2)})}.$$

Thus, the condition is now

$$\int_0^1(1-t^2)^{2\mathfrak{Re}(a_{(2)})-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

As

$$\begin{aligned} 2\mathfrak{Re}(a_{(2)}) - 2 - \mathfrak{Re}(s) &= 2 + 2\mathfrak{Re}(s) - 2\mathfrak{Re}(r) - 2 - \mathfrak{Re}(s) \\ &= -1 + 2\sqrt{\mu + [(s+1)/2]^2} > -1, \end{aligned}$$

we conclude that S is integrable.

(2) If $\mathfrak{Re}(s) = -2$, then $S \in \mathcal{H}_s$ if and only if

$$\lim_{\rho\to 1} \frac{1}{2\pi}\int_0^{2\pi} \left|(1-\rho e^{i\theta})^{a_{(2)}}\right|^2 d\theta < \infty.$$

By Lemma 5.9, this is equivalent to

$$\lim_{\rho\to 1} \frac{1}{2\pi}\int_0^{2\pi} |1-\rho e^{i\theta}|^{2\mathfrak{Re}(a_{(2)})}\, d\theta < \infty.$$

Now we remark that $-\mathfrak{Re}(a_{(2)}) > 0$ and

$$\mathfrak{Re}(1+2a_{(2)}) = 2\sqrt{\mu + \Big(\frac{s+1}{2}\Big)^2} > 0.$$

From Lemma 5.10, we know that

$$\frac{1}{2\pi}\int_0^{2\pi} |1-\rho e^{i\theta}|^{2\mathfrak{Re}(a_{(2)})}\, d\theta = {}_2F_1\big(-\mathfrak{Re}(a_{(2)}), -\mathfrak{Re}(a_{(2)}); 1; \rho^2\big).$$

So the condition becomes

$$\lim_{\rho\to 1} {}_2F_1\big(-\mathfrak{Re}(a_{(2)}), -\mathfrak{Re}(a_{(2)}); 1; \rho^2\big) < \infty.$$

From the Gauss theorem (Lemma 5.4), we know that

$$\rho \mapsto {}_2F_1\big(-\mathfrak{Re}(a_{(2)}), -\mathfrak{Re}(a_{(2)}); 1; \rho^2\big)$$

is continuous on $[0,1]$ and hence has a limit when $\rho \to 1$.

(3) If $\mathfrak{Re}(s) < -2$, then $S \in \mathcal{H}_s$ if and only if

$$\int_0^1 \Big(\frac{1}{2\pi}\int_0^{2\pi} \left|(1-te^{i\theta})^{\mathfrak{Re}(a_{(2)})}\right|^2 d\theta\Big)(1-t^2)^{-3-\mathfrak{Re}(s)}\, t\, dt < \infty.$$

By Lemma 5.9, this is equivalent to

$$\int_0^1 \Big(\frac{1}{2\pi}\int_0^{2\pi} |1-te^{i\theta}|^{2\mathfrak{Re}(a_{(2)})}\, d\theta\Big)(1-t^2)^{-3-\mathfrak{Re}(s)}\, t\, dt < \infty.$$

Note that we always have $\mathfrak{Re}(-a_{(2)}) > 0$.

If $\mathfrak{Re}(1+2a_{(2)}) < 0$, then from Corollary 5.11, we know that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi} |1-te^{i\theta}|^{2\mathfrak{Re}(a_{(2)})}\, d\theta \sim C(1-t^2)^{1+2\mathfrak{Re}(a_{(2)})}.$$

So the condition of integrability becomes

$$\int_0^1 (1-t^2)^{2\mathfrak{Re}(a_{(2)})-2-\mathfrak{Re}(s)}\, t\, dt < \infty.$$

But $-2+2\mathfrak{Re}(a_{(2)})-\mathfrak{Re}s=-1+2\sqrt{\mu+[(s+1)/2]^2}>-1$, and therefore S is integrable.

If $\mathfrak{Re}(1+2a_{(2)})=0$, then Lemma 5.10 together with Lemma 5.4 implies that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\mathfrak{Re}(a_{(2)})}\,d\theta\sim C\log(1-t^2).$$

So the condition of integrability becomes

$$\int_0^1\log(1-t^2)(1-t^2)^{-3-\mathfrak{Re}(s)}t\,dt<\infty.$$

Since $\mathfrak{Re}(s)<-2$, the function $\log(1-t^2)(1-t^2)^{-3-\mathfrak{Re}(s)}t$ is integrable. If $\mathfrak{Re}(1+2a_{(2)})>0$, then Lemma 5.10 together with Lemma 5.4 implies that

$$t\mapsto\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\mathfrak{Re}(a_{(2)})}\,d\theta$$

is continuous on $[0,1]$, and therefore the function

$$\Big(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\mathfrak{Re}(a_{(2)})}\,d\theta\Big)(1-t^2)^{-3-\mathfrak{Re}(s)}t$$

is integrable on $[0,1[$.

Assume now that $n>0$. So Lemma 5.8 implies that $\mathfrak{Re}(s)>-2$. Then $S\in\mathcal{H}_s$ if and only if

$$\int_D\big|(1-z)^{a_{(2)}-n}\big|^2\big|P(z)\big|^2\big(1-|z|^2\big)^{-3-\mathfrak{Re}(s)}\,d\,\mathrm{vol}(z)<\infty.$$

The function $|(1-z)^{a_{(2)}-n}|^2|P(z)|^2(1-|z|^2)^{-3-\mathfrak{Re}(s)}$ is integrable if and only if it is integrable near $z=1$. As $P(1)\neq 0$ and by Lemma 5.9, this function is integrable near 1 if and only if

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(a_{(2)})-n)}\,d\theta\Big)(1-t^2)^{-3-\mathfrak{Re}(s)}t\,dt<\infty.$$

We remark that $\mathfrak{Re}(n-a_{(2)})>0$ and $\mathfrak{Re}(1+2a_{(2)}-2n)<0$. Thus, from Corollary 5.11, we know that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(a_{(2)})-n)}\,d\theta\sim C(1-t^2)^{1+2\mathfrak{Re}(a_{(2)})-2n}.$$

So the condition of integrability becomes

$$\int_0^1(1-t^2)^{2\mathfrak{Re}(a_{(2)})-2n-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

But $-2+2\mathfrak{Re}(a_{(2)})-2n-\mathfrak{Re}(s)=-1+2\sqrt{\mu+[(s+1)/2]^2}>-1$, and therefore S is integrable. □

PROPOSITION 5.15

Assume that $r-a\notin\mathbb{Z}_{\leq 0}$, assume that $r-a_2\notin\mathbb{Z}_{\leq 0}$, assume that $1+a_1+a-r\notin$

$\mathbb{Z}_{\le 0}$, and assume that $1+a_1+a_2-r \notin \mathbb{Z}_{\le 0}$. Then the function $S(z)$ belongs to $\mathcal{H}_s$ if and only if s is real, $-1<s<0$, and $r=0$.

Proof

Thanks to Corollary 5.7, we know that $F(1)\neq 0$.

(1) If $\mathfrak{Re}(s)<-2$, then $S\in\mathcal{H}_s$ if and only if

$$\int_D \big|(1-z)^r\big|^2\big|F(z)\big|^2\big(1-|z|^2\big)^{-3-\mathfrak{Re}(s)}\,d\operatorname{vol}(z)<\infty.$$

It is clear that the function $|(1-z)^r|^2|F(z)|^2(1-|z|^2)^{-3-\mathfrak{Re}(s)}$ is integrable if and only if it is integrable near $z=1$. But near $z=1$ we have $|F(z)|^2\sim|F(1)|^2$ since F is continuous by Lemma 5.6. So by Lemma 5.9, the function is integrable near $z=1$ if and only if

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\mathfrak{Re}(r)}\,d\theta\Big)(1-t^2)^{-3-\mathfrak{Re}(s)}t\,dt<\infty.$$

We remark that $\mathfrak{Re}(-r)>0$ and $\mathfrak{Re}(2r+1)<0$. So by Corollary 5.11 there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2\mathfrak{Re}(r)}\,d\theta\sim C\times(1-t^2)^{1+2\mathfrak{Re}(r)}.$$

So $S\in\mathcal{H}_s$ if and only if

$$\int_0^1(1-t^2)^{2\mathfrak{Re}(r)-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

But $-2+2\mathfrak{Re}(r)-\mathfrak{Re}(s)=-1-2\sqrt{\mu+[(s+1)/2]^2}<-1$. Therefore, S is not integrable.

(2) If $\mathfrak{Re}(s)=-2$, then $S\in\mathcal{H}_s$ if and only if

$$\lim_{\rho\to1}\frac{1}{2\pi}\int_0^{2\pi}\big|(1-\rho e^{i\theta})^{\mathfrak{Re}(r)}\big|^2\big|F(\rho e^{i\theta})\big|^2\,d\theta<\infty.$$

By Lemma 5.9, this is equivalent to

$$\lim_{\rho\to1}\frac{1}{2\pi}\int_0^{2\pi}|1-\rho e^{i\theta}|^{2\mathfrak{Re}(r)}\big|F(\rho e^{i\theta})\big|^2\,d\theta<\infty.$$

As above this holds if and only if

$$\lim_{\rho\to1}\frac{1}{2\pi}\int_0^{2\pi}|1-\rho e^{i\theta}|^{2\mathfrak{Re}(r)}\,d\theta<\infty.$$

We have $\mathfrak{Re}(1+2r)<0$. So by Corollary 5.11, the limit is not finite. Hence, $S\notin\mathcal{H}_s$.

(3) If $-2<\mathfrak{Re}(s)<-1$, then $S\in\mathcal{H}_s$ if and only if

$$\int_D S(z)S'(z)\bar z\big(1-|z|^2\big)^{-2-\mathfrak{Re}(s)}\,d\operatorname{vol}(z)<\infty.$$

The function $S(z)S'(z)\bar z(1-|z|^2)^{-2-\mathfrak{Re}(s)}$ is integrable if and only if it is integrable near $z=1$. Now we remark that $2\mathfrak{Re}(r)<1+\mathfrak{Re}(s)<0$. Thus, according to Lemma 5.12, this function is integrable near $z=1$ if and only if

$$\int_D |1-z|^{2(\mathfrak{Re}(r)-1/2)}\left(1-|z|^2\right)^{-2-\mathfrak{Re}(s)} d\operatorname{vol}(z)<\infty$$

if and only if

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(r)-1/2)}\,d\theta\Big)(1-t^2)^{-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

As $\frac{1}{2}-\mathfrak{Re}(r)>0$ and $\mathfrak{Re}(r)<0$, Corollary 5.11 implies that there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(\mathfrak{Re}(r)-1/2)}\,d\theta\sim C(1-t^2)^{2\mathfrak{Re}(r)}.$$

So the condition of integrability becomes

$$\int_0^1(1-t^2)^{2\mathfrak{Re}(r)-2-\mathfrak{Re}(s)}t\,dt<\infty.$$

But we have $2\mathfrak{Re}(r)-2-\mathfrak{Re}(s)=-1-2\sqrt{\mu+[(s+1)/2]^2}<-1$. Hence, $S\notin\mathcal{H}_s$.

(4) If $\mathfrak{Re}(s)=-1$, then $S\in\mathcal{H}_s$ if and only if

$$\lim_{\rho\to 1}\frac{1}{2\pi}\int_0^{2\pi}S(\rho e^{i\theta})S'(\rho e^{-i\theta})e^{-i\theta}\,d\theta<\infty.$$

This case is analogous to the previous one and is left to the reader.

(5) Finally, assume that $-1<\mathfrak{Re}(s)<0$. Then s and r are real. We have $S\in\mathcal{H}_s$ if and only if

$$\int_D\left|S'(z)\right|^2\left(1-|z|^2\right)^{-1-s}d\operatorname{vol}(z)<\infty.$$

This holds if and only if the function $|S'(z)|^2(1-|z|^2)^{-1-s}$ is integrable near $z=1$.

Suppose first that $r\neq 0$. Then according to Lemma 5.12, this function is integrable near $z=1$ if and only if

$$\int_D|1-z|^{2(r-1)}\left(1-|z|^2\right)^{-1-s}d\operatorname{vol}(z)<\infty$$

if and only if

$$\int_0^1\Big(\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(r-1)}\,d\theta\Big)(1-t^2)^{-1-s}t\,dt<\infty.$$

Now remark that $1-r>0$ and $2r-1<0$. Thus, from Corollary 5.11, there is a nonzero constant C such that

$$\frac{1}{2\pi}\int_0^{2\pi}|1-te^{i\theta}|^{2(r-1)}\,d\theta\sim C(1-t^2)^{2r-1}.$$

Therefore, the condition of integrability becomes

$$\int_0^1(1-t^2)^{2r-2-s}t\,dt<\infty.$$

But we have $2r-2-s=-1-2\sqrt{\mu+[(s+1)/2]^2}<-1$. Hence, $S\notin\mathcal{H}_s$.

Suppose now that $r=0$. Then according to Lemma 5.12, the function $|S'(z)|^2(1-|z|^2)^{-1-s}$ is integrable near $z=1$ if and only if

$$\int_D |1-z|^{2s}\big(1-|z|^2\big)^{-1-s}\,d\operatorname{vol}(z)<\infty$$

if and only if

$$\int_0^1 \Big(\frac{1}{2\pi}\int_0^{2\pi} |1-te^{i\theta}|^{2s}\,d\theta\Big)(1-t^2)^{-1-s}t\,dt<\infty.$$

From Lemma 5.10, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |1-te^{i\theta}|^{2s}\,d\theta = {}_2F_1(-s,-s;1;t^2).$$

Thus, the condition of integrability is now

$$\int_0^1 {}_2F_1(-s,-s;1;t^2)(1-t^2)^{-1-s}t\,dt<\infty.$$

Note that we always have $-s>0$. If $2s+1<0$, then Lemma 5.4 implies that

$${}_2F_1(-s,-s;1;t^2)(1-t^2)^{-1-s}t\sim(1-t^2)^s,$$

which is integrable since $s>-1$. If $2s+1=0$, then Lemma 5.4 implies that

$${}_2F_1(-s,-s;1;t^2)(1-t^2)^{-1-s}t\sim\log(1-t^2)(1-t^2)^{-1-s},$$

which is integrable since $-1-s>-1$. If $2s+1>0$, then Lemma 5.4 implies that the function $t\mapsto {}_2F_1(-s,-s;1;t^2)$ is continuous on $[0,1]$, and therefore

$$\int_0^1 {}_2F_1(-s,-s;1;t^2)(1-t^2)^{-1-s}t\,dt<\infty,$$

since $-1-s>-1$. Consequently, when $r=0$, we always have $S\in\mathcal{H}_s$. □

Let us now state a consequence of these three propositions.

PROPOSITION 5.16

A simple weight module $N(b_1,b_2)$ in the complementary series is a Hilbert submodule of (the Hilbert completion of) $N(a_1,a_2)\otimes N(a,0)$ if and only if

- *either $a_1,a_2\in\mathbb{R}$, $-1<a_1+a_2+a<0$, and $N(b_1,b_2)\cong N(a+a_1,a_2)$;*
- *or $-1<a_1,a_2<0$, $-2<a_1+a_2-a<-1$, and $N(b_1,b_2)\cong N(a_1,a_2-a)$.*

Moreover, in the first case the submodule $N(a+a_1,a_2)$ is generated by the vector

$$w(0)=\sum_{n\geq 0}\frac{(-a)_n(-a_2)_n}{(1+a_1)_n}\frac{z(n,n)}{n!}.$$

In the second case, the submodule $N(a_1,a_2-a)$ is generated by the vector

$$w(0)=\sum_{n\geq 0}\frac{(-a)_n}{n!}z(n,n).$$

Proof
Recall we set $s = a + a_1 + a_2$, and denote by ξ the infinitesimal character of $V = N(a_1, a_2) \otimes N(a, 0)$. Recall also that we set $\mu = \xi - a_2(1 + a + a_1)$ and

$$r = \frac{1+s}{2} - \sqrt{\mu + \Big(\frac{1+s}{2}\Big)^2}.$$

Let v be the standard basis vector of $N(b_1, b_2) \subset V$ of weight $b_1 - b_2$. Then it is straightforward to check that $FE \cdot v = b_2(b_1 + 1)v$. Moreover, as we already mentioned, we can assume that $b_1 - b_2 = a + a_1 - a_2$. In the above notations, we have $\xi = b_2(b_1 + 1)$. From the above three propositions, we conclude that a vector w of weight $a + a_1 - a_2$ such that $FE \cdot w = \xi w$ generates a submodule of V if and only either s is real, $-1 < s < 0$, and $r = 0$ or there is a nonnegative integer n such that $r = 1 + a_1 + a + n$ or $r = 1 + a_1 + a_2 + n$.

In the first case, s being real implies that a_1, a_2 are real; $-1 < s < 0$ and $r = 0$ imply that $\mu = 0$. Therefore, we have $b_1 - b_2 = a + a_1 - a_2$ and $b_2(b_1 + 1) = a_2(1 + a + a_1)$. Hence, up to isomorphism, $N(b_1, b_2) = N(a + a_1, a_2)$.

In the second case, if $r = 1 + a_1 + a + n = 1 + s - a_2 + n$, then we have $2\sqrt{\mu + [(s+1)/2]^2} = 2a_2 - 2n - 1 - s$. Therefore, we must have $2a_2 - 2n - 1 - s \in \mathbb{R}$ and $0 < 2a_2 - 2n - 1 - s < 1$. The first condition is always fulfilled. The second condition reads $1 + 2n < a_2 - a_1 - a < 2 + 2n$. But then we also have $\mu = (n - a_2)(n + 1 + a + a_1)$, which implies that $\xi = n(1 + n + a + a_1 - a_2)$. Therefore, we have $b_1 - b_2 = a + a_1 - a_2$ and $b_2(b_1 + 1) = n(1 + n + a + a_1 - a_2)$. The solutions of this system are $b_1 = n + a + a_1 - a_2, b_2 = n$ or $b_1 = -1 - n, b_2 = a_2 - a_1 - a - n - 1$. In both cases, the corresponding module is a highest weight module and therefore does not belong to the complementary series.

If $r = 1 + a_1 + a_2 + n = 1 + s - a + n$, then we have $2\sqrt{\mu + [(s+1)/2]^2} = 2a - 2n - 1 - s$. Therefore, we must have $2a - 2n - 1 - s \in \mathbb{R}$ and $0 < 2a - 2n - 1 - s < 1$. But $2a - 2n - 1 - s \in \mathbb{R}$ implies that $a_1, a_2 \in \mathbb{R}$. Now the second condition reads $-2 - 2n < a_1 + a_2 - a < -1 - 2n$. But then we also have $\mu = (n - a)(n - a + 1 + s)$, which implies that $\xi = (n + a_2 - a)(1 + n + a_1)$. Therefore, we have $b_1 - b_2 = a + a_1 - a_2$ and $b_2(b_1 + 1) = (n + a_2 - a)(1 + n + a_1)$. Consequently, we have $b_1 = n + a_1$ and $b_2 = n + a_2 - a$ or $b_1 = a - n - a_2 - 1$ and $b_2 = -1 - n - a_1$. If $a_1 = 0$, then the corresponding module is a lowest weight module and therefore does not belong to the complementary series. If $a_1 \neq 0$, then $-1 < a_1, a_2 < 0$. However, in this case, $a_2 - a > a_2 > -1$. Therefore, the condition $-2 - 2n < a_1 + a_2 - a < -1 - 2n$ can hold only if $n = 0$, which gives the asserted condition. Then the submodule is isomorphic to $N(a_1, a_2 - a)$ or to $N(a - a_2 - 1, -1 - a_1)$, which turn out to be isomorphic. □

Let us now state a final result about the discrete spectrum of the tensor products.

THEOREM 5.17

(1) *Let $a, b < 0$. Then the discrete spectrum of the Hilbert tensor product $N(0, a) \otimes N(b, 0)$ is*

$$N(a,b), \quad \textit{if } -1 < a+b < 0,$$

$$\bigoplus_{0 \leq 2n < a-b-1} N(b-a+2n,0), \quad \textit{if } 0 < a-b-1,$$

$$\bigoplus_{a-b+1 < 2n \leq 0} N(0,a-b-2n), \quad \textit{if } a-b+1 < 0.$$

(2) *Let* $-1 \leq x < 0$*, let* $y \in \mathbb{R} \setminus \{0\}$*, and let* $a < 0$*. Then the discrete spectrum of the Hilbert tensor product* $N(-1-x+iy, x+iy) \otimes N(a,0)$ *is*

$$\bigoplus_{2n < 2x-a} N(-1-2x+a+2n,0).$$

(3) *Let* $-1 < a_1, a_2 < 0$*, and let* $a < 0$*. Then the discrete spectrum of the Hilbert tensor product* $N(a_1,a_2) \otimes N(a,0)$ *is*

$$\Big(\bigoplus_{2n < a_2-a_1-a-1} N(a+a_1-a_2+2n,0)\Big) \oplus N(a+a_1,a_2),$$

$$\textit{if } -1 < a+a_1+a_2 < 0,$$

$$\Big(\bigoplus_{2n < a_2-a_1-a-1} N(a+a_1-a_2+2n,0)\Big) \oplus N(a_1,a_2-a),$$

$$\textit{if } -2 < a_1+a_2-a < -1,$$

$$\Big(\bigoplus_{2n < a_2-a_1-a-1} N(a+a_1-a_2+2n,0)\Big), \quad \textit{otherwise.}$$

We collect in Table 3 the results of Theorem 5.17 in a more classical setting (see page 322 for the correspondence between the setting $N(a,b)$ and the classical one).

Proof
This is a consequence of Propositions 5.1, 5.2, 5.3, and 5.16. Note also that the different conditions in (1) (and in (3)) are mutually exclusive because of the restriction on the parameters found on page 322. □

REMARK 5.18
In [23], Repka gives the decomposition of tensor products of unitary representations of $\mathrm{SU}(1,1)$. Theorem 5.17 recovers in particular (some of) these results. Note also that the particular case $N(0,a) \otimes N(a,0)$ was obtained in [19].

5.3. Application to smooth vectors

Let a_1, a_2, and a be real numbers such that $-1 < a_1 \leq 0$, $-1 < a, a_2 < 0$, and $-1 < a+a_1+a_2 < 0$. From Proposition 5.16, we know that the (completed) tensor product $V = N(a_1,a_2) \otimes N(a,0)$ contains a (Hilbert) submodule W isomorphic to $N(a_1+a,a_2)$. We want to determine the possible relation between the spaces of

Table 3

$\pi \otimes \pi'$	Conditions on the parameters	Discrete spectrum
$\pi_a^- \otimes \pi_b^+$	$a > 0,\ b > 0,$ $0 < a + b < 1$	$\pi_{a,b}^c$
$\pi_a^- \otimes \pi_b^+$	$a > 0,\ b > 0,$ $1 < b - a$	$\bigoplus_{0 \le 2n < b-a-1} \pi_{b-a-2n}^+$
$\pi_a^- \otimes \pi_b^+$	$a > 0,\ b > 0,$ $1 < a - b$	$\bigoplus_{0 \le 2n < a-b-1} \pi_{a-b+2n}^-$
$\pi_{x,iy} \otimes \pi_a^+$	$0 < x \le 1,\ y > 0,$ $a > 0$	$\bigoplus_{2n < a-2x} \pi_{1+a-2x-2n}^+$
$\pi_{a_1,a_2}^c \otimes \pi_a^+$	$0 < a_1,\ a_2 < 1,\ a > 0,$ $0 < a + a_1 + a_2 < 1$	$\bigoplus_{2n < a+a_1-a_2-1} \pi_{a+a_1-a_2-2n}^+ \oplus \pi_{a+a_1,a_2}^c$
$\pi_{a_1,a_2}^c \otimes \pi_a^+$	$0 < a_1,\ a_2 < 1,\ a > 0,$ $1 < a_1 + a_2 - a < 2$	$\bigoplus_{2n < a+a_1-a_2-1} \pi_{a+a_1-a_2-2n}^+ \oplus \pi_{a_1,a_2-a}^c$
$\pi_{a_1,a_2}^c \otimes \pi_a^+$	$0 < a_1,\ a_2 < 1,\ a > 0$	$\bigoplus_{2n < a+a_1-a_2-1} \pi_{a+a_1-a_2-2n}^+$

smooth vectors in W and the smooth vectors in V. We denote them, respectively, by W_∞ and V_∞.

PROPOSITION 5.19
With notations as above, we have $W \cap V_\infty = \{0\}$. In particular, $W_\infty \cap V_\infty = \{0\}$.

Proof
According to Nelson's Theorem 3.1, the set of smooth vectors of a unitarizable module is the common domain of the definition of the various operators $\rho(u)$ for $u \in \mathcal{U}(\mathfrak{g})$. Denote by $\{w(k), k \in \mathbb{Z}\}$ the standard basis of $W = N(a_1 + a, a_2)$. Recall that the action of the triple (H, E, F) on W is given by

$$\begin{cases} H \cdot w(k) = (a + a_1 - a_2 + 2k)w(k), \\ E \cdot w(k) = (a_2 - k)w(k+1), \\ F \cdot w(k) = (a + a_1 + k)w(k-1). \end{cases}$$

On the other hand, (3.1) gives $\|w(k)\|^2 \sim |k|^{1+a+a_1+a_2}\|w(0)\|^2$, and so

$$W = \Big\{\sum_{k\in\mathbb{Z}} \alpha_k w(k) : \sum_{k\in\mathbb{Z}} |\alpha_k|^2 |k|^{1+a+a_1+a_2} < \infty\Big\}.$$

Hence, Nelson's theorem implies that

$$W_\infty = \Big\{\sum_{k\in\mathbb{Z}} \alpha_k w(k) : \forall N \in \mathbb{Z}_{\ge 0}, \sum_{k\in\mathbb{Z}} |\alpha_k|^2 |k|^{1+a+a_1+a_2} k^{2N} < \infty\Big\}.$$

Recall then that

$$V=\Big\{\sum_{k,l}u_{k,l}z(k,l):\sum_{k,l}|u_{k,l}|^2|k|^{1+a_1+a_2}|l|^{1+a}<\infty\Big\}.$$

Using (5.1), giving the action of E and F on V, we check that

$$V_\infty=\Big\{\sum_{k,l}u_{k,l}z(k,l):\forall N\in\mathbb{Z}_{\geq 0},\sum_{k,l}|u_{k,l}|^2|k|^{1+a_1+a_2+2N}|l|^{1+a+2N}<\infty\Big\}.$$

Now, we know that

$$w(0)=\sum_{n\geq 0}\frac{(-a)_n(-a_2)_n}{(1+a_1)_n}\frac{z(n,n)}{n!}.$$

Then the standard basis is given for $k>0$ by the following formulas:

$$\begin{cases} w(k)=\prod_{j=1}^k\frac{a+a_1+j}{a_1+j}\times\sum_{n\geq 0}\frac{(-a)_n(k-a_2)_n}{(k+1+a_1)_n}\frac{z(k+n,n)}{n!},\\ w(-k)=\prod_{j=1}^k\frac{a_1+1-j}{a+a_1+1-j}\times\sum_{n\geq 0}\frac{(-a)_n(-k-a_2)_n}{(-k+1+a_1)_n}\frac{z(-k+n,n)}{n!}.\end{cases}$$

It is now easy to check that $w(k)\notin V_\infty$. We remark then that the weight vectors that occur in the decomposition of $w(k)$ and $w(l)$ are all distinct if $k\neq l$. Therefore, we conclude that $W\cap V_\infty=\{0\}$, as asserted. $\square$

REMARK 5.20

In [25], Speh and Venkataramana proved an analogous result about the K-finite vectors for the restriction of some complementary series of $\mathrm{SO}(n,1)$ to $\mathrm{SO}(n-1,1)$.

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