Quantization dimension for Gibbs-like measures on cookie-cutter sets

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Abstract In this paper using the Banach limit we have determined a Gibbs-like measure μ_h supported by a cookie-cutter set E which is generated by a single cookie-cutter mapping f. For such a measure μ_h and $r \in (0, +\infty)$ we have shown that there exists a unique $\kappa_r \in (0, +\infty)$ such that κ_r is the quantization dimension function of the probability measure μ_h, and we established its functional relationship with the temperature function of the thermodynamic formalism. The temperature function is commonly used to perform the multifractal analysis, in our context of the measure μ_h. In addition, we have proved that the κ_r-dimensional lower quantization coefficient of order r of the probability measure is positive.

1. Introduction

Quantization dimension is one of the most important objects in the quantization problem, which has a deep background in information theory and engineering technology (see [BW], [GG], [GN], [Z]). It characterizes in a natural way the asymptotic property of the error when approximating a given probability measure by a discrete probability measure of finite support in the sense of L_r-metrics. Given a Borel probability measure μ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$, and a natural number $n \in \mathbb{N}$, the nth *quantization error* of order r of μ is defined by

$$V_{n,r}(\mu) := \inf\left\{ \int d(x, \alpha)^r \, d\mu(x) : \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \le n \right\},$$

where $d(x, \alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. Note that if $\int \|x\|^r \, d\mu(x) < \infty$, then there is some set α for which the infimum is achieved (see [GL1]). The set α for which the infimum is achieved is called an *optimal set of n-means* or an *n-optimal set of order r* for $0 < r < +\infty$. The upper and lower quantization dimensions of order r of μ are defined to be

$$\overline{D}_r(\mu) := \limsup_{n \to \infty} \frac{r \log n}{-\log V_{n,r}(\mu)}; \qquad \underline{D}_r(\mu) := \liminf_{n \to \infty} \frac{r \log n}{-\log V_{n,r}(\mu)}.$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call the common value the *quantization dimension of order r* of the probability measure μ, and it is denoted by $D_r := D_r(\mu)$.

For $s > 0$, we define the s-dimensional upper and lower quantization coefficients of order r of μ by $\limsup_n nV_{n,r}^{s/r}(\mu)$ and $\liminf_n nV_{n,r}^{s/r}(\mu)$, respectively. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $V_{n,r}$ goes to zero. If D_r exists, then one can write

$$\log V_{n,r} \sim \log\Big[\Big(\frac{1}{n}\Big)^{r/D_r}\Big].$$

For probabilities with nonvanishing absolutely continuous part the numbers D_r are all equal to the dimension d of the underlying space, but for singular probabilities the family $(D_r)_{r>0}$ gives an interesting description of their geometric (multifractal) structures.

Let $S_1, S_2, \ldots, S_N$ be contractive similitudes from $\mathbb{R}^d$ into itself, where $N \geq 2$ is a positive integer. Let s_i be the contraction ratio of S_i for all $1 \leq i \leq N$. Then for a given probability vector $(p_1, p_2, \ldots, p_N)$ there exists a unique Borel probability measure μ (see [H]) satisfying the condition

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1}.$$

Let the iterated function system $\{S_1, S_2, \ldots, S_N\}$ satisfy the open set condition (OSC): there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{j=1}^{N} S_j(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for $1 \leq i \neq j \leq N$. The iterated function system satisfies the strong OSC if U can be chosen such that $U \cap J \neq \emptyset$, where J is the limit set of the iterated function system. Under the OSC, Graf and Luschgy showed that the quantization dimension function $D_r := D_r(\mu)$ of the probability measure μ exists and satisfies the following relation (see [GL1], [GL3]):

$$\sum_{j=1}^{N} (p_j s_j^r)^{D_r/(r+D_r)} = 1.$$

In fact, they proved a stronger result, namely, that the quantization dimension D_r also satisfies (see [GL2])

(1) $$0 < \liminf_n nV_{n,r}^{D_r/r}(\mu) \leq \limsup_n nV_{n,r}^{D_r/r}(\mu) < +\infty.$$

Under the OSC, Lindsay and Mauldin [LM] determined the quantization dimension for an F-conformal measure m associated with a conformal iterated function system determined by finitely many conformal mappings. They established a relationship between the quantization dimension and the temperature function of the thermodynamic formalism arising in multifractal analysis. Later, in [R1] the author studied the quantization dimension of Moran measures on the Moran sets of which potential functions are defined in terms of the similarity ratios and probability vectors. Then the quantization problem was solved in [R2] for the image measures of Bowen's Gibbs measures supported by the one-sided Bernoulli shifts under the coding maps on the Moran sets. The mixed quantization dimension and its relationship with the temperature function were studied by Wang and Dai [WD]. But from the work in [LM], [R1], [R2], and [WD] it was

not known whether the D_r-dimensional lower quantization coefficient is positive, that is, whether $\liminf_n nV_{n,r}^{D_r/r}(\mu) > 0$, where $D_r := D_r(\mu)$ is the quantization dimension of the probability measure μ.

In this paper, using the Banach limit we have defined a Gibbs-like measure μ_h supported by a cookie-cutter set E, where E is the limit set generated by a cookie-cutter mapping f and $h := \dim_H(E)$ is the Hausdorff dimension of the set E (see [F2]). For this measure μ_h we have shown that for each $r \in (0, +\infty)$ there exists a unique $\kappa_r \in (0, +\infty)$ such that

$$(2) \qquad \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \left(\mu_h(J_\sigma) \|\varphi'_\sigma\|^r \right)^{\kappa_r/(r+\kappa_r)} = 0,$$

and the above κ_r is the quantization dimension $D_r := D_r(\mu_h)$ of order r of the probability measure μ_h. It is known that the singularity exponent $\beta(q)$ (also known as the temperature function) satisfies the usual equation

$$(3) \qquad \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \left(\mu_h(J_\sigma) \right)^q \|\varphi'_\sigma\|^{\beta(q)} = 0$$

and that the spectrum $f(\alpha)$ is the Legendre transform of $\beta(q)$. Comparing (2) and (3), we see that if $q_r = D_r/(r + D_r)$, then $\beta(q_r) = rq_r$; that is, the quantization dimension function of order r of the probability measure μ_h has a relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis. (For thermodynamic formalism, multifractal analysis, and the Legendre transform one could see [F2].) The significant difference between the work in this paper and the work in [LM], [R1], [R2], and [WD] is that, in addition to determining the quantization dimension function κ_r and its relationship with the temperature function of a probability measure, by Proposition 3.10 in this paper, we have proved a stronger result, namely, that the κ_r-dimensional lower quantization coefficient $\liminf_n nV_{n,r}^{\kappa_r/r}(\mu_h)$ of order r of the probability measure μ_h is positive. The quantization problem for a general probability measure is still open.

2. Basic definitions, lemmas, and propositions

In this paper, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space equipped with a metric d compatible with the Euclidean topology. Let us write

$$V_{n,r}(\mu) := \inf\left\{ \int d(x,\alpha)^r \, d\mu(x) : \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \leq n \right\},$$

$$u_{n,r}(\mu) := \inf\left\{ \int d(x, \alpha \cup U^c)^r \, d\mu(x) : \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \leq n \right\},$$

where U is a set which comes from the OSC and U^c denotes the complement of U. We see that

$$u_{n,r}^{1/r} \leq V_{n,r}^{1/r} := e_{n,r}.$$

We call sets $\alpha_n \subset \mathbb{R}^d$, for which the above infimums are achieved, *n-optimal sets* for $e_{n,r}$, $V_{n,r}$, or $u_{n,r}$, respectively. As stated above, Graf and Luschgy have shown that n-optimal sets exist when $\int \|x\|^r \, d\mu(x) < \infty$.

2.1. Cookie-cutter set

A mapping f is called a cookie cutter if there exists a finite collection of disjoint closed intervals $J_1, J_2, \ldots, J_N \subset J = [0,1]$ such that

(C1) f is defined in a neighborhood of each J_j, $1 \le j \le N$, the restriction of f to each initial interval J_j maps J_j bijectively onto J, and the corresponding branch inverse is denoted by $\varphi_j := (f|_{J_j})^{-1} : J \to J_j$;

(C2) f is differentiable with Hölder continuous derivative f', that is, there exist constants $c > 0$ and $\gamma \in (0,1]$ such that for $x, y \in J_j$, $1 \le j \le N$,

$$\left| f'(x) - f'(y) \right| \le c|x - y|^\gamma;$$

(C3) f is boundedly expanding in the sense that there exist constants b and B such that

$$1 < b := \inf_x \left\{ \left| f'(x) \right| \right\} \le \sup_x \left\{ \left| f'(x) \right| \right\} := B < +\infty.$$

We call $[\bigcup_{j=1}^N J_j; c, \gamma, b, B]$ the *defining data* of the cookie-cutter mapping f. Let Ω_0 be the empty set. For $n \ge 1$, define

$$\Omega_n = \{1, 2, \ldots, N\}^n, \qquad \Omega_\infty = \{1, 2, \ldots, N\}^\mathbb{N}, \qquad \text{and} \qquad \Omega = \bigcup_{k=0}^\infty \Omega_k.$$

Elements of Ω are called *words*. For any $\sigma \in \Omega$ if $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Omega_n$, we write $\sigma^- = (\sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ to denote the word obtained by deleting the last letter of σ, $|\sigma| = n$ to denote the length of σ, and $\sigma|_k := (\sigma_1, \sigma_2, \ldots, \sigma_k)$, $k \le n$, to denote the truncation of σ to the length k. For any two words $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$, we write $\sigma\tau = \sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m)$ to denote the juxtaposition of $\sigma, \tau \in \Omega$. A word of length zero is called the *empty word* and is denoted by $\emptyset$. For $\sigma \in \Omega$ and $\tau \in \Omega \cup \Omega_\infty$ we say that τ is an *extension* of σ, written as $\sigma \prec \tau$, if $\tau|_{|\sigma|} = \sigma$. For $\sigma \in \Omega_k$, the cylinder set $C(\sigma)$ is defined as $C(\sigma) = \{\tau \in \Omega_\infty : \tau|_k = \sigma\}$. For $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Omega_n$, let us write $\varphi_\sigma = \varphi_{\sigma_1} \circ \cdots \circ \varphi_{\sigma_n}$, and we define the rank-n basic interval corresponding to σ by

$$J_\sigma = J_{(\sigma_1, \sigma_2, \ldots, \sigma_n)} = \varphi_\sigma(J),$$

where $1 \le \sigma_k \le N$, $1 \le k \le n$. If $\sigma = \emptyset$, then we identify $\varphi_\emptyset$ to be the identity mapping on J and write $J_\emptyset = J$. By $|J_\sigma|$ we mean the diameter of the set J_σ for all $\sigma \in \Omega$. It is easy to see that the set of basic intervals $\{J_\sigma : \sigma \in \Omega\}$ has the following net properties:

(i) $J_{\sigma * j} \subset J_\sigma$ for each $\sigma \in \Omega_n$ and $1 \le j \le N$ for all $n \ge 1$;
(ii) $J_\sigma \cap J_\tau = \emptyset$ if $\sigma, \tau \in \Omega_n$ for all $n \ge 1$ and $\sigma \ne \tau$.

Since φ_j is a branch inverse of f, where $1 \le j \le N$, for all $x \in J$, we have $f(\varphi_j(x)) = x$, and so $|f'(\varphi_j(x))| \cdot |\varphi_j'(x)| = 1$, which yields

$$B^{-1} \le \left|\varphi_j'(x)\right| \le b^{-1}. \tag{4}$$

Choose x, y to be the endpoints of J. Then $\varphi_\sigma(x), \varphi_\sigma(y)$ are the endpoints of J_σ for each $\sigma \in \Omega$, and so by the mean value theorem, we have that

$$|J_\sigma| = \left|\varphi_\sigma(x) - \varphi_\sigma(y)\right| = \left|\varphi_\sigma'(w)\right||x - y| = \left|\varphi_\sigma'(w)\right|$$

for some $w \in J_\sigma$. Thus, $B^{-n} \le |J_\sigma| \le b^{-n}$ for any $\sigma \in \Omega_n$, and thus the diameter $|J_\sigma| \to 0$ as $|\sigma| \to \infty$. Since given $\sigma = (\sigma_i)_{i=1}^\infty \in \Omega_\infty$ the diameters of the compact sets $J_{\sigma|_k}, k \ge 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{k=0}^\infty J_{\sigma|_k}$$

is a singleton, and therefore, if we denote its element by $\pi(\sigma)$, this defines the coding map $\pi : \Omega_\infty \to J$. The main object of our interest is the limit set

$$E := \pi(\Omega_\infty) = \bigcup_{\sigma \in \Omega_\infty} \bigcap_{k=0}^\infty J_{\sigma|_k}.$$

Moreover, $\pi(C(\sigma)) = E \cap J_\sigma$ for $\sigma \in \Omega$. With the net properties it follows that E is a perfect, nowhere dense, and totally disconnected subset of J. The set E is called the *cookie-cutter set*.

Let ℓ^∞ be the set of all bounded sequences $x = (x_n)_{n \in \mathbb{N}}$ of real or complex numbers which form a vector space with respect to pointwise addition and multiplication by a scalar. It is equipped with the norm $\|x\| = \sup_n |x_n|$. The normed space ℓ^∞ is complete with respect to the metric $\|x - y\|$, and so it forms a Banach space. By the Hahn–Banach theorem (see [Y, pp. 102–104]), there exists a linear functional $L : \ell^\infty \to \mathbb{R}$ for which

(i) L is linear;
(ii) $L((x_n)_{n \in \mathbb{N}}) = L((x_{n+1})_{n \in \mathbb{N}})$;
(iii) $\liminf_{n \to \infty}(x_n) \le L((x_n)_{n \in \mathbb{N}}) \le \limsup_{n \to \infty}(x_n)$.

The functional L, defined above, is called a *Banach limit*. The use of the Banach limit is a rather standard tool in producing an invariant measure from a given measure.

Let us now prove the following lemma.

LEMMA 2.2 (BOUNDED-VARIATION PRINCIPLE)
There exists a constant $1 < \xi < +\infty$ such that, for each $\sigma \in \Omega_n$ and $x, y \in J_\sigma$, we have

$$\xi^{-1} \le \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \le \xi,$$

where $f^n = f \circ f \circ \cdots \circ f$ represents the n-fold composition of f with itself.

Proof

Note that, for each $k \leq n$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Omega_n$, f^{k-1} maps J_σ diffeomorphically to the set $\varphi_{\sigma_k} \circ \varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(J)$, and so

$$\left| f^{k-1}(x) - f^{k-1}(y) \right| \leq \operatorname{diam}\left(\varphi_{\sigma_k} \circ \varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(J) \right)$$

$$= \left| \varphi_{\sigma_k} \circ \varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(J) \right|.$$

By the mean value theorem,

$$\left| \varphi_{\sigma_k} \circ \varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(J) \right|$$

$$= \sup_{x,y \in J} \left| \varphi_{\sigma_k}\left(\varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(x) \right) - \varphi_{\sigma_k}\left(\varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(y) \right) \right|$$

$$\leq b^{-1} \left| \varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(J) \right|.$$

Thus, proceeding inductively, we have that

$$\left| \varphi_{\sigma_k} \circ \varphi_{\sigma_{k+1}} \circ \cdots \circ \varphi_{\sigma_n}(J) \right| \leq b^{-(n-k+1)}.$$

Then, the Hölder continuity of f' gives

$$\left| f'\left(f^{k-1}(x) \right) - f'\left(f^{k-1}(y) \right) \right| \leq c \left| f^{k-1}(x) - f^{k-1}(y) \right|^\gamma \leq c b^{-(n-k+1)\gamma},$$

and so by the mean value theorem and the assumption that $|f'| > 1$, we have

$$\left| \log\left| f'\left(f^{k-1}(x) \right) \right| - \log\left| f'\left(f^{k-1}(y) \right) \right| \right|$$

$$\leq \left| \left| f'\left(f^{k-1}(x) \right) \right| - \left| f'\left(f^{k-1}(y) \right) \right| \right|$$

$$\leq c b^{-(n-k+1)\gamma}.$$

Therefore, by the above inequality and the chain rule,

$$\left| \log\left| (f^n)'(x) \right| - \log\left| (f^n)'(y) \right| \right|$$

$$= \left| \sum_{k=1}^{n} \log\left| f'\left(f^{k-1}(x) \right) \right| - \sum_{k=1}^{n} \log\left| f'\left(f^{k-1}(y) \right) \right| \right|$$

$$\leq \sum_{k=1}^{n} \left| \log\left| f'\left(f^{k-1}(x) \right) \right| - \log\left| f'\left(f^{k-1}(y) \right) \right| \right|$$

$$\leq \sum_{k=1}^{n} c b^{-(n-k+1)\gamma} \leq \frac{c b^{-\gamma}}{1 - b^{-\gamma}}.$$

Take $\xi = \exp\{c/(b^\gamma - 1)\}$. Since $c/(b^\gamma - 1) > 0$, we have $1 < \xi < +\infty$, and thus the lemma follows. $\qquad \square$

Let us now prove the following proposition.

PROPOSITION 2.3 (BOUNDED DISTORTION PRINCIPLE)

For any $n \geq 1$, $\sigma \in \Omega_n$, $x \in J_\sigma$, we have

$$\xi^{-1} \leq \left| (f^n)'(x) \right| \cdot |J_\sigma| \leq \xi.$$

Moreover, for each $1 \leq j \leq N$, we get $|J_{\sigma*j}| \geq \xi^{-2}B^{-1}|J_\sigma|$, where ξ is the constant of Lemma 2.2.

Proof

Note that, for $\sigma \in \Omega_n$, $f^n : J_\sigma \to J$ is a differentiable bijection. So by the mean value theorem, if $y, z \in J_\sigma$, there exists $w \in J_\sigma$ such that

$$f^n(y) - f^n(z) = (f^n)'(w)(y - z).$$

Choose y, z to be the endpoints of J_σ. Then $f^n(y), f^n(z)$ are the endpoints of J, and so

$$|J| = |(f^n)'(w)| \cdot |J_\sigma|; \qquad \text{that is,} \qquad |(f^n)'(w)| \cdot |J_\sigma| = 1.$$

Hence, using the bounded-variation principle, we have

$$\xi^{-1} \leq |(f^n)'(x)| \cdot |J_\sigma| \leq \xi \tag{5}$$

for all $x \in J_\sigma$. Now let $1 \leq j \leq N$ and $x \in J_{\sigma*j}$. Then using (5), we have

$$\xi^{-1} \leq |(f^{n+1})'(x)| \cdot |J_{\sigma*j}|$$
$$= |(f'(f^n(x))| \cdot |(f^n)'(x)| \cdot |J_{\sigma*j}| \leq B|(f^n)'(x)| \cdot |J_{\sigma*j}|.$$

Since $J_{\sigma*j} \subseteq J_\sigma$, we have $x \in J_\sigma$. Hence, using (5) again, we have

$$|J_{\sigma*j}| \geq \xi^{-2}B^{-1}|J_\sigma|.$$

Thus, the proof of the proposition is yielded. □

PROPOSITION 2.4

For any $n \geq 1$, let $\sigma \in \Omega_n$, and let $x, y \in J$. Let ξ be the constant of Lemma 2.2. Then,

$$\xi^{-1}|\varphi'_\sigma(y)| \leq |\varphi'_\sigma(x)| \leq \xi|\varphi'_\sigma(y)|.$$

Proof

For $\sigma \in \Omega_n$ and $x \in J$, we know that $f^n(\varphi_\sigma(x)) = x$. Thus,

$$|(f^n)'(\varphi_\sigma(x))| \cdot |\varphi'_\sigma(x)| = 1.$$

Again, for all $x \in J$, $\varphi_\sigma(x) \in J_\sigma$. Hence, Lemma 2.2 yields

$$\xi^{-1}|\varphi'_\sigma(y)| \leq |\varphi'_\sigma(x)| \leq \xi|\varphi'_\sigma(y)|,$$

and thus the proposition is obtained. □

Let us now prove the following two lemmas.

LEMMA 2.5

Let $\sigma, \tau \in \Omega$. Then

$$\xi^{-1}\|\varphi'_\sigma\|\|\varphi'_\tau\| \leq \|\varphi'_{\sigma\tau}\| \leq \|\varphi'_\sigma\|\|\varphi'_\tau\|,$$

where ξ is the constant of Lemma 2.2.

Proof
For any $x \in J$, we have $|\varphi'_{\sigma\tau}(x)| = |\varphi'_\sigma(\varphi_\tau(x))| \cdot |\varphi'_\tau(x)|$. Thus, by Proposition 2.4, for any $y \in J$,

$$\xi^{-1}|\varphi'_\sigma(y)| \cdot |\varphi'_\tau(x)| \leq |\varphi'_\sigma(\varphi_\tau(x))| \cdot |\varphi'_\tau(x)|$$

$$= |\varphi'_{\sigma\tau}(x)| \leq \|\varphi'_\sigma\| \cdot \|\varphi'_\tau\|,$$

and thus the lemma follows. $\qquad\qquad\Box$

LEMMA 2.6
Let $\sigma \in \Omega$, and let $x \in J$. Then

$$\xi^{-1}|J_\sigma| \leq |\varphi'_\sigma(x)| \leq \xi|J_\sigma|,$$

where ξ is the constant of Lemma 2.2.

Proof
Let $x \in J$, and then $\varphi_\sigma(x) \in J_\sigma$ for $\sigma \in \Omega_n$, $n \geq 1$. We know that $f^n(\varphi_\sigma(x)) = x$, and so $|(f^n)'(\varphi_\sigma(x))| \cdot |\varphi'_\sigma(x)| = 1$. Now use Proposition 2.3 to obtain the lemma. $\qquad\Box$

Let us now prove the following lemma.

LEMMA 2.7
Let $\sigma, \tau \in \Omega$. Then

$$\xi^{-3}|J_\sigma||J_\tau| \leq |J_{\sigma\tau}| \leq \xi^3|J_\sigma||J_\tau|,$$

where ξ is the constant of Lemma 2.2.

Proof
For $\sigma, \tau \in \Omega$, we have $|\varphi'_{\sigma\tau}(x)| = |\varphi'_\sigma(y)||\varphi'_\tau(x)|$, where $y = \varphi_\tau(x)$ and $x \in J$. Again by Proposition 2.4, for any $x, y \in J$, we have

$$\xi^{-1}|\varphi'_\sigma(y)| \leq |\varphi'_\sigma(x)| \leq \xi|\varphi'_\sigma(y)|.$$

Hence, Lemma 2.6 implies that

$$\xi^{-3}|J_\sigma||J_\tau| \leq \xi^{-1}|\varphi'_\sigma(y)||\varphi'_\tau(x)| = \xi^{-1}|\varphi'_{\sigma\tau}(x)| \leq |J_{\sigma\tau}| \leq \xi|\varphi'_{\sigma\tau}(x)| \leq \xi^3|J_\sigma||J_\tau|,$$

and thus the lemma is obtained. $\qquad\qquad\Box$

By Lemma 2.5 and the standard theory of subadditive sequences, the function $Q(t)$ given by

$$Q(t) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \|\varphi'_\sigma\|^t,$$

for any real t, exists. It is easy to see that the function $Q(t)$ is strictly decreasing, convex and hence continuous in t.

LEMMA 2.8

There exists a unique $h \in (0, +\infty)$ such that $Q(h) = 0$.

Proof

Since the function $Q(t)$ is strictly decreasing and continuous on $\mathbb{R}$, there exists a unique $h \in \mathbb{R}$ such that $Q(h) = 0$. Note that

$$Q(0) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} 1 \geq \lim_{k \to \infty} \frac{1}{k} \log N^k = \log N \geq \log 2 > 0.$$

In order to conclude the proof, it therefore suffices to show that $\lim_{t \to +\infty} Q(t) = -\infty$. For $t > 0$,

$$Q(t) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \|\varphi'_\sigma\|^t \leq \lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} b^{-kt}$$

$$= \lim_{k \to \infty} \frac{1}{k} \log N^k - t \log b = \log N - t \log b.$$

Since $b > 1$, it follows that $\lim_{t \to +\infty} Q(t) = -\infty$, and hence the lemma follows.
□

NOTE 2.9

Lemma 2.6 implies that $\xi^{-1}|J_\sigma| \leq \sup_{x \in J} |\varphi'_\sigma(x)| = \|\varphi'_\sigma\| \leq \xi|J_\sigma|$, and so the topological pressure $Q(t)$ can be written as follows:

$$Q(t) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} |J_\sigma|^t.$$

The unique $h \in (0, +\infty)$ given by Lemma 2.8 is the Hausdorff dimension $\dim_{\mathrm{H}}(E)$ of the cookie-cutter set E (see [F2]).

Let us now prove the following proposition, which plays a vital role in the paper.

PROPOSITION 2.10

Let $h \in (0, +\infty)$ be unique such that $Q(h) = 0$, and let s_* and s^* be any two arbitrary real numbers with $0 < s_* < h < s^*$. Then for all $n \geq 1$,

$$\xi^{-3s_*} < \sum_{\sigma \in \Omega_n} |J_\sigma|^{s_*} \qquad and \qquad \sum_{\sigma \in \Omega_n} |J_\sigma|^{s^*} < \xi^{3s^*},$$

where ξ is the constant of Lemma 2.2.

Proof

Let $s_* < h$. As the pressure function $Q(t)$ is strictly decreasing, $Q(s_*) > Q(h) = 0$. Then for any positive integer n, by Lemma 2.7, we have

$$0 < Q(s_*) = \lim_{p \to \infty} \frac{1}{np} \log \sum_{\omega \in \Omega_{np}} |J_\omega|^{s_*} \leq \lim_{p \to \infty} \frac{1}{np} \log \xi^{3(p-1)s_*} \left(\sum_{\sigma \in \Omega_n} |J_\sigma|^{s_*} \right)^p,$$

which implies that

$$0 < \frac{1}{n}\log\Big(\xi^{3s_*}\sum_{\sigma\in\Omega_n}|J_\sigma|^{s_*}\Big), \qquad \text{and so} \qquad \sum_{\sigma\in\Omega_n}|J_\sigma|^{s_*} > \xi^{-3s_*}.$$

Now if $h < s^*$, then $Q(s^*) < 0$ as $Q(t)$ is strictly decreasing. Then for any positive integer n, by Lemma 2.7, we have

$$0 > Q(s^*) = \lim_{p\to\infty}\frac{1}{np}\log\sum_{\omega\in\Omega_{np}}|J_\omega|^{s^*} \geq \lim_{p\to\infty}\frac{1}{np}\log\xi^{-3(p-1)s^*}\Big(\sum_{\sigma\in\Omega_n}|J_\sigma|^{s^*}\Big)^p,$$

which implies that

$$0 > \frac{1}{n}\log\Big(\xi^{-3s^*}\sum_{\sigma\in\Omega_n}|J_\sigma|^{s^*}\Big), \qquad \text{and so} \qquad \sum_{\sigma\in\Omega_n}|J_\sigma|^{s^*} < \xi^{3s^*}.$$

Thus, the proposition is obtained. $\square$

COROLLARY 2.11

Since s_* and s^* are any two arbitrary real numbers with $0 < s_* < h < s^*$, from the above proposition it follows that, for all $n \geq 1$,

$$\xi^{-3h} \leq \sum_{\sigma\in\Omega_n}|J_\sigma|^h \leq \xi^{3h}.$$

Let us now prove the following proposition.

PROPOSITION 2.12 (GIBBS-LIKE MEASURE)

Let $h \in (0, +\infty)$ be such that $Q(h) = 0$. Then there exist a constant $\eta > 1$ and a probability measure μ_h supported by E such that, for any $\sigma \in \Omega$,

$$\eta^{-1}|J_\sigma|^h \leq \mu_h(J_\sigma) \leq \eta|J_\sigma|^h.$$

Proof
For $\sigma \in \Omega$, $n \geq 1$, define

$$\nu_n\big(C(\sigma)\big) = \frac{\sum_{\tau\in\Omega_n}(\operatorname{diam} J_{\sigma\tau})^h}{\sum_{\tau\in\Omega_{|\sigma|+n}}(\operatorname{diam} J_\tau)^h}.$$

Then using Lemma 2.7 and Corollary 2.11, we have

$$\nu_n\big(C(\sigma)\big) \leq \frac{\xi^{3h}(\operatorname{diam} J_\sigma)^h\sum_{\tau\in\Omega_n}(\operatorname{diam} J_\tau)^h}{\sum_{\tau\in\Omega_{|\sigma|+n}}(\operatorname{diam} J_\tau)^h} \leq \xi^{9h}(\operatorname{diam} J_\sigma)^h,$$

and similarly, $\nu_n(C(\sigma)) \geq \xi^{-9h}(\operatorname{diam} J_\sigma)^h$. Thus, for a given $\sigma \in \Omega$, $\{\nu_n(C(\sigma))\}_{n=1}^\infty$ is a bounded sequence of real numbers, and so the Banach limit, denoted by Lim, is defined. For $\sigma \in \Omega$, let

$$\nu\big(C(\sigma)\big) = \operatorname*{Lim}_{n\to\infty}\nu_n\big(C(\sigma)\big).$$

Then

$$\sum_{j=1}^{N} \nu\big(C(\sigma j)\big) = \operatorname*{Lim}_{n \to \infty} \sum_{j=1}^{N} \frac{\sum_{\tau \in \Omega_n} (\operatorname{diam} J_{\sigma j \tau})^h}{\sum_{\tau \in \Omega_{|\sigma|+1+n}} (\operatorname{diam} J_\tau)^h}$$

$$= \operatorname*{Lim}_{n \to \infty} \frac{\sum_{\tau \in \Omega_{n+1}} (\operatorname{diam} J_{\sigma \tau})^h}{\sum_{\tau \in \Omega_{|\sigma|+n+1}} (\operatorname{diam} J_\tau)^h},$$

and so

$$\sum_{j=1}^{N} \nu\big(C(\sigma j)\big) = \operatorname*{Lim}_{n \to \infty} \nu_{n+1}\big(C(\sigma)\big) = \operatorname*{Lim}_{n \to \infty} \nu_n\big(C(\sigma)\big) = \nu\big(C(\sigma)\big).$$

Thus, by Kolmogorov's extension theorem, ν can be extended to a unique Borel probability measure γ on Ω_∞. Let μ_h be the image measure of γ under the coding map π; that is, $\mu_h = \gamma \circ \pi^{-1}$. Then μ_h is a unique Borel probability measure supported by E. Moreover, for any $\sigma \in \Omega$,

$$\mu_h(J_\sigma) = \gamma\big(C(\sigma)\big) = \operatorname*{Lim}_{n \to \infty} \nu_n\big(C(\sigma)\big)$$

$$\le \operatorname*{Lim}_{n \to \infty} \xi^{9h} (\operatorname{diam} J_\sigma)^h = \xi^{9h} (\operatorname{diam} J_\sigma)^h,$$

and similarly,

$$\mu_h(J_\sigma) \ge \xi^{-9h} (\operatorname{diam} J_\sigma)^h.$$

Write $\eta = \xi^{9h}$, and then $\eta > 1$, and thus the proof of the proposition is complete. $\square$

For the above measure μ_h, known as a *Gibbs-like measure*, we will determine the quantization dimension function and its functional relationship with the temperature function of the thermodynamic formalism.

Let us now prove the following lemma.

LEMMA 2.13

Let μ_h be the Gibbs-like measure as defined in Proposition 2.12. Then there exists a constant $L > 1$ such that, for $\sigma, \tau \in \Omega$,

$$L^{-1} \mu_h(J_\sigma) \mu_h(J_\tau) \le \mu_h(J_{\sigma \tau}) \le L \mu_h(J_\sigma) \mu_h(J_\tau).$$

Proof
Let $\sigma, \tau \in \Omega$. Then by Lemma 2.7 and Proposition 2.12, we have

$$\mu_h(J_{\sigma \tau}) \le \eta |J_{\sigma \tau}|^h \le \eta \xi^{3h} |J_\sigma|^h |J_\tau|^h \le \eta^3 \xi^{3h} \mu_h(J_\sigma) \mu_h(J_\eta),$$

and similarly, $\mu_h(J_{\sigma \tau}) \ge \eta^{-3} \xi^{-3h} \mu_h(J_\sigma) \mu_h(J_\eta)$. Take $L = \eta^3 \xi^{3h}$. As $h > 0$, $\xi > 1$, and $\eta > 1$, it follows that $L > 1$, and thus

$$L^{-1} \mu_h(J_\sigma) \mu_h(J_\eta) \le \mu_h(J_{\sigma \tau}) \le L \mu_h(J_\sigma) \mu_h(J_\eta),$$

which is the lemma. $\square$

2.14. Topological pressure

For $q, t \in \mathbb{R}$ and $n \geq 1$, let us write

$$Z_n(q,t) = \sum_{\sigma \in \Omega_n} \big(\mu_h(J_\sigma)\big)^q \|\varphi'_\sigma\|^t.$$

Then for $n, p \geq 1$,

$$Z_{n+p}(q,t) = \sum_{\sigma \in \Omega_n} \sum_{\tau \in \Omega_p} \big(\mu_h(J_{\sigma\tau})\big)^q \|\varphi'_{\sigma\tau}\|^t.$$

Let us first assume that $q \geq 0$. Then by Lemmas 2.5 and 2.13, it follows that if $t \geq 0$, then

$$Z_{n+p}(q,t) \leq L^q Z_n(q,t) Z_p(q,t),$$

and if $t < 0$, then

$$Z_{n+p}(q,t) \leq L^q \xi^{-t} Z_n(q,t) Z_p(q,t).$$

Let us now assume that $q < 0$. Using the same argument, we have that if $t \geq 0$, then

$$Z_{n+p}(q,t) \leq L^{-q} Z_n(q,t) Z_p(q,t),$$

and if $t < 0$, then

$$Z_{n+p}(q,t) \leq L^{-q} \xi^{-t} Z_n(q,t) Z_p(q,t).$$

Hence, by the standard theory of subadditive sequences, $\lim_{k\to\infty}(1/k)\log Z_k(q,t)$ exists (see [F2, Corollary 1.2]). Let us denote it by $P(q,t)$, that is,

$$(6) \qquad P(q,t) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \big(\mu_h(J_\sigma)\big)^q \|\varphi'_\sigma\|^t.$$

The following proposition states the well-known properties of the function $P(q,t)$ (see [F1], [P]).

PROPOSITION 2.15

(i) $P(q,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.
(ii) $P(q,t)$ is strictly decreasing in each variable separately.
(iii) For fixed q we have $\lim_{t\to+\infty} P(q,t) = -\infty$ and $\lim_{t\to-\infty} P(q,t) = +\infty$.
(iv) $P(q,t)$ is convex: if $q_1, q_2, t_1, t_2 \in \mathbb{R}$, $a_1, a_2 \geq 0$, $a_1 + a_2 = 1$, then

$$P(a_1 q_1 + a_2 q_2, a_1 t_1 + a_2 t_2) \leq a_1 P(q_1, t_1) + a_2 P(q_2, t_2).$$

Now for fixed q, $P(q,t)$ is a continuous function of t. Its values range from $-\infty$ (when $t \to +\infty$) to $+\infty$ (when $t \to -\infty$). Therefore, by the intermediate value theorem there is a real number β such that $P(q,\beta) = 0$. The solution β is unique, since $P(q,\cdot)$ is strictly decreasing. This defines β implicitly as a function of q: for each q there is a unique $\beta = \beta(q)$ such that $P(q,\beta(q)) = 0$.

The following proposition gives the well-known properties of the function $\beta(q)$ (see [F1], [P]).

PROPOSITION 2.16

Let $\beta = \beta(q)$ be defined by $P(q, \beta(q)) = 0$. Then

(i) β is a continuous function of the real variable q;
(ii) β is strictly decreasing: if $q_1 < q_2$, then $\beta(q_1) > \beta(q_2)$;
(iii) $\lim_{q \to -\infty} \beta(q) = +\infty$ and $\lim_{q \to +\infty} \beta(q) = -\infty$;
(iv) β is convex: if $q_1, q_2, a_1, a_2 \in \mathbb{R}$ with $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$, then

$$\beta(a_1 q_1 + a_2 q_2) \leq a_1 \beta(q_1) + a_2 \beta(q_2).$$

The function $\beta(q)$ is sometimes denoted by $T(q)$ and called the *temperature function*. A more general discussion of this function can be found in [HJK+], where our $\beta(q)$ function corresponds to $-\tau(q)$ in their notation.

REMARK 2.17

If $q = 0$, then $P(0, \beta(0)) = 0$, which implies that

$$\lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \|\varphi'_\sigma\|^{\beta(0)} = 0;$$

that is, $\beta(0)$ gives the Hausdorff dimension $\dim_H(E)$ of the cookie-cutter set E (see [F2]). Again,

$$P(1, 0) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \mu_h(J_\sigma) = \lim_{k \to \infty} \frac{1}{k} \log 1 = 0,$$
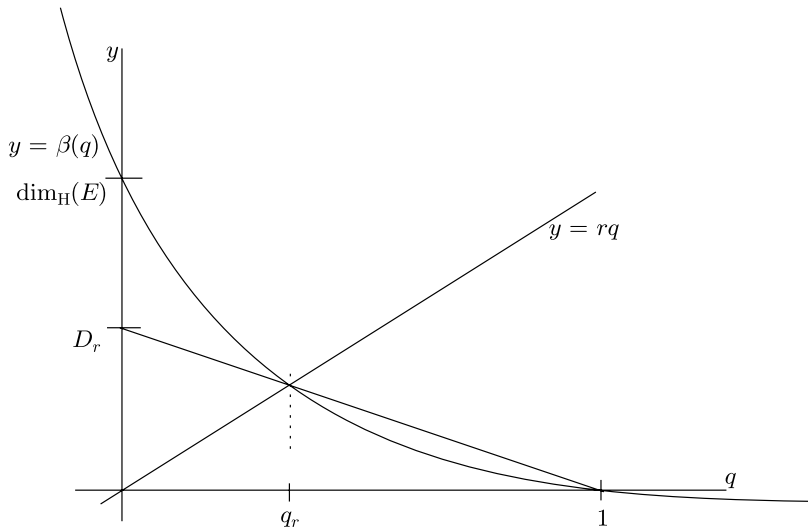
and hence $\beta(1) = 0$ (see Figure 1).

Figure 1. To determine D_r first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then D_r is the y-intercept of the line through this point and the point $(1, 0)$.

3. Main result

The relationship between the quantization dimension function and the temperature function $\beta(q)$ for the Gibbs-like measure μ_h, where the temperature function is the Legendre transform of the $f(\alpha)$ curve (for the definitions of $f(\alpha)$ and the Legendre transform see [F2]), is given by the following theorem, which constitutes the main result of the paper. For its graphical description, see Figure 1.

THEOREM 3.1

Let μ_h be the Gibbs-like measure supported by the cookie-cutter set E. Then, for each $r \in (0, +\infty)$, there exists a unique $\kappa_r \in (0, +\infty)$ such that

$$\kappa_r = \frac{\beta(q_r)}{1 - q_r},$$

where we recall β is the temperature function, that is, $\beta(q_r) = rq_r$, and κ_r is the quantization dimension of order r of the probability measure μ_h. Moreover, the κ_r-dimensional lower quantization coefficient is positive, that is, $\liminf_n nV_{n,r}^{\kappa_r/r}(\mu_h) > 0$.

To prove the theorem we need several lemmas and propositions. Let us first state the following lemma, which has a proof similar to that of [R2, Lemma 3.2].

LEMMA 3.2

Let $0 < r < +\infty$ be fixed. Then there exists exactly one number $\kappa_r \in (0, +\infty)$ such that

$$\lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in \Omega_k} \left(\mu_h(J_\sigma) \|\varphi'_\sigma\|^r \right)^{\kappa_r/(r+\kappa_r)} = 0.$$

Let us now prove the following lemma.

LEMMA 3.3

Let $0 < r < +\infty$ and κ_r be as in Lemma 3.2. Then for $n \geq 1$, we have

$$(L\xi^r)^{-\kappa_r/(r+\kappa_r)} \leq \sum_{\sigma \in \Omega_n} \left(\mu_h(J_\sigma) \|\varphi'_\sigma\|^r \right)^{\kappa_r/(r+\kappa_r)} \leq (L\xi^r)^{\kappa_r/(r+\kappa_r)}.$$

Proof

For $\sigma \in \Omega$ let us write $s_\sigma = \mu_h(J_\sigma)\|\varphi'_\sigma\|^r$. Then for $\sigma \in \Omega_n$ and $\tau \in \Omega_p$ with $n, p \geq 1$, by Lemmas 2.5 and 2.13, we have $L^{-1}\xi^{-r}s_\sigma s_\tau \leq s_{\sigma\tau} \leq Ls_\sigma s_\tau \leq L\xi^r s_\sigma s_\tau$. Since $r > 0$ and $L, \xi > 1$, it is true that $L^{-2}\xi^{-2r}s_\sigma s_\tau \leq s_{\sigma\tau} \leq L^2\xi^{2r}s_\sigma s_\tau$. Then by the standard theory of subadditive sequences, $\lim_{n\to\infty} n^{-1}\log \sum_{\sigma\in\Omega_n} s_\sigma^t$ exists for any $t \in \mathbb{R}$. Then, proceeding as in [R2, Lemma 3.3], we obtain the lemma. $\square$

We call $\Gamma \subset \Omega$ a *finite maximal antichain* if Γ is a finite set of words in Ω such that every sequence in Ω_∞ is an extension of some word in Γ, but no word of Γ is an extension of another word in Γ. By $|\Gamma|$ we denote the cardinality of Γ. Note

that from the definition of Γ it follows that a finite maximal antichain does not contain the empty word $\emptyset$ as all words are extensions of $\emptyset$.

Let us now state and prove the following lemma.

LEMMA 3.4

Let $0 < r < +\infty$ and κ_r be as in Lemma 3.2. Then, for any finite maximal antichain Γ, we have

(a) $L^{-1} \sum_{\sigma \in \Gamma} \mu_h(J_\sigma) \mu_h \circ \varphi_\sigma^{-1} \leq \mu_h \leq L \sum_{\sigma \in \Gamma} \mu_h(J_\sigma) \mu_h \circ \varphi_\sigma^{-1}$, *and*

(b) $(L\xi^r)^{-3\kappa_r/(r+\kappa_r)} \leq \sum_{\sigma \in \Gamma} (\mu_h(J_\sigma)\|\varphi_\sigma'\|^r)^{\kappa_r/(r+\kappa_r)} \leq (L\xi^r)^{3\kappa_r/(r+\kappa_r)}$.

Proof

(a) Let $M = \max\{|\sigma| : \sigma \in \Gamma\}$. Note that the Borel σ-algebra on E is generated by the set $\{J_\sigma : \sigma \in \Omega\}$ of all basic intervals. For any two basic intervals, either they are disjoint or one is contained in the other. Hence, it is enough to prove that, for any $\omega \in \Omega_n$ with $n \geq M$,

$$L^{-1} \sum_{\sigma \in \Gamma} \mu_h(J_\sigma) \mu_h \circ \varphi_\sigma^{-1}(J_\omega) \leq \mu_h(J_\omega) \leq L \sum_{\sigma \in \Gamma} \mu_h(J_\sigma) \mu_h \circ \varphi_\sigma^{-1}(J_\omega),$$

which follows along similar lines as the proof of [R2, Lemma 3.4(a)].

(b) As Γ is a finite maximal antichain, there exists a finite sequence of positive integers $n_1 < n_2 < \cdots < n_K$ such that

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_K,$$

where $\Gamma_j = \{\sigma \in \Gamma : |\sigma| = n_j\}$ for all $1 \leq j \leq K$. Let M be a positive integer such that $M \geq n_K$. Then by Lemma 3.3, we have

$$\sum_{\sigma \in \Gamma} \big(\mu_h(J_\sigma)\|\varphi_\sigma'\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$\geq \sum_{j=1}^{K} \sum_{\tau \in \Gamma_j} \big(\mu_h(J_\tau)\|\varphi_\tau'\|^r\big)^{\kappa_r/(r+\kappa_r)} (L\xi^r)^{-\kappa_r/(r+\kappa_r)}$$

$$\times \sum_{\sigma \in \Omega_{M-n_j}} \big(\mu_h(J_\sigma)\|\varphi_\sigma'\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$\geq (L\xi^r)^{-\kappa_r/(r+\kappa_r)} \sum_{j=1}^{K} \sum_{\tau \in \Gamma_j} \big(\mu_h(J_\tau)\|\varphi_\tau'\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$\times \sum_{\substack{\sigma \in \Omega_{M-n_j} \\ \tau \prec \sigma}} \big(\mu_h(J_\sigma)\|\varphi_\sigma'\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$= (L\xi^r)^{-\kappa_r/(r+\kappa_r)} \sum_{j=1}^{K} \sum_{\tau \in \Gamma_j} \sum_{\substack{\sigma \in \Omega_{M-n_j} \\ \tau \prec \sigma}} \big(\mu_h(J_\tau)\|\varphi_\tau'\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$\times \big(\mu_h(J_\sigma)\|\varphi'_\sigma\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$\geq (L\xi^r)^{-2\kappa_r/(r+\kappa_r)} \sum_{j=1}^{K} \sum_{\tau\in\Gamma_j} \sum_{\sigma\in\Omega_{M-n_j}} \big(\mu_h(J_{\tau\sigma})\|\varphi'_{\tau\sigma}\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$= (L\xi^r)^{-2\kappa_r/(r+\kappa_r)} \sum_{\sigma\in\Omega_M} \big(\mu_h(J_\sigma)\|\varphi'_\sigma\|^r\big)^{\kappa_r/(r+\kappa_r)}$$

$$\geq (L\xi^r)^{-3\kappa_r/(r+\kappa_r)}.$$

Similarly, we have $\sum_{\sigma\in\Gamma}(\mu_h(J_\sigma)\|\varphi'_\sigma\|^r)^{\kappa_r/(r+\kappa_r)} \leq (L\xi^r)^{3\kappa_r/(r+\kappa_r)}$. Hence, the lemma is obtained. $\qquad\square$

Let us now give the following lemma.

LEMMA 3.5
Let $x,y \in J$, and let $\sigma \in \Omega$. Then

$$\xi^{-1}\|\varphi'_\sigma\|d(x,y) \leq d\big(\varphi_\sigma(x),\varphi_\sigma(y)\big) \leq \|\varphi'_\sigma\|d(x,y),$$

where ξ is the constant of Lemma 2.2.

Proof
By the mean value theorem, for any $x,y \in J$ there exists some $w \in (x,y)$ such that

$$d\big(\varphi_\sigma(x),\varphi_\sigma(y)\big) = \big|\varphi'_\sigma(w)\big|d(x,y),$$

and so, by Proposition 2.4, for any $z \in J$,

$$\xi^{-1}\big|\varphi'_\sigma(z)\big|d(x,y) \leq d\big(\varphi_\sigma(x),\varphi_\sigma(y)\big) = \big|\varphi'_\sigma(w)\big|d(x,y) \leq \|\varphi'_\sigma\|d(x,y).$$

Now take the supremum over all $z \in J$, and then

$$\xi^{-1}\|\varphi'_\sigma\|d(x,y) \leq d\big(\varphi_\sigma(x),\varphi_\sigma(y)\big) \leq \|\varphi'_\sigma\|d(x,y),$$

to obtain the assertion of the lemma. $\qquad\square$

Using the above lemma and Lemma 3.4(a), and the parallel lines as [R2, Lemma 3.5], the following lemma can be proved.

LEMMA 3.6
Let $\Gamma \subset \Omega$ be a finite maximal antichain, let $n \in \mathbb{N}$ with $n \geq |\Gamma|$, and let $0 < r < +\infty$. Then $V_{n,r}(\mu_h) \leq \inf\{L\sum_{\sigma\in\Gamma}\mu_h(J_\sigma)\|\varphi'_\sigma\|^r V_{n_\sigma,r}(\mu_h) : 1 \leq n_\sigma, \sum_{\sigma\in\Gamma} n_\sigma \leq n\}$.

By using Lemma 3.6 and proceeding along lines similar to [R2, Proposition 3.6], the following proposition can be proved.

PROPOSITION 3.7

Let $0 < r < +\infty$ and κ_r be as in Lemma 3.2. Then $\limsup_n nV_{n,r}^{\kappa_r/r}(\mu_h) < +\infty$.

NOTE 3.8

We say that the cookie-cutter mapping f *satisfies the OSC* if the corresponding set of branch inverses $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ satisfies the OSC: there exists a bounded nonempty open set $U \subset J$ (in the topology of J) such that $\varphi_j(U) \subset U$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for $1 \leq i \neq j \leq N$. Furthermore, f satisfies the *strong OSC* (SOSC) if U can be chosen such that $U \cap E \neq \emptyset$. Note that we can choose $U = (0, 1)$, and so, because of the net properties of the basic intervals of the cookie-cutter set, it follows that the cookie-cutter mapping f satisfies the SOSC.

As a parallel to [R2, Lemma 3.7] the following lemma can be proved.

LEMMA 3.9

Let $\Gamma \subset \Omega$ be a finite maximal antichain. Then there exists $n_0 = n_0(\Gamma)$ such that for every $n \geq n_0$ there exists a set of positive integers $\{n_\sigma := n_\sigma(n)\}_{\sigma \in \Gamma}$ such that $\sum_{\sigma \in \Gamma} n_\sigma \leq n$ and

$$u_{n,r}(\mu_h) \geq (L\xi^r)^{-1} \sum_{\sigma \in \Gamma} \mu_h(J_\sigma)\|\varphi_\sigma'\|^r u_{n_\sigma,r}(\mu_h).$$

Let us now prove the following proposition, which shows that the κ_r-dimensional lower quantization coefficient of order r of the probability measure μ_h is positive.

PROPOSITION 3.10

Let μ_h be the Gibbs-like measure, and let $0 < r < +\infty$. Moreover, let κ_r be as in Lemma 3.2. Then $\liminf_n nV_{n,r}^{\kappa_r/r}(\mu_h) > 0$.

Proof

Let Γ be a finite maximal antichain. By Lemma 3.9, we have n_0 and for $n \geq n_0$ the numbers $\{n_\sigma := n_\sigma(n)\}_{\sigma \in \Gamma}$ which satisfy the conclusion of the lemma. Set $c = \min\{n^{r/\kappa_r} u_{n,r}(\mu_h) : n \leq n_0\}$. Clearly each $u_{n,r}(\mu_h) > 0$, and hence $c > 0$. Suppose that $n \geq n_0$, and suppose that $k^{r/\kappa_r} u_{k,r}(\mu_h) \geq c$ for all $k < n$. Hence, using Lemma 3.9, we have

$$n^{r/\kappa_r} u_{n,r}(\mu_h)$$

$$\geq n^{r/\kappa_r}(L\xi^r)^{-1} \sum_{\sigma \in \Gamma} \mu_h(J_\sigma)\|\varphi_\sigma'\|^r u_{n_\sigma,r}(\mu_h)$$

$$= n^{r/\kappa_r}(L\xi^r)^{-1} \sum_{\sigma \in \Gamma} \mu_h(J_\sigma)\|\varphi_\sigma'\|^r \big(n_\sigma(n)\big)^{-r/\kappa_r} \big(n_\sigma(n)\big)^{r/\kappa_r} u_{n_\sigma,r}(\mu_h)$$

$$\geq c(L\xi^r)^{-1} \sum_{\sigma \in \Gamma} \mu_h(J_\sigma)\|\varphi_\sigma'\|^r \Big(\frac{n_\sigma(n)}{n}\Big)^{-r/\kappa_r}.$$

Using Hölder's inequality (with exponents less than 1), we have

$$n^{r/\kappa_r} u_{n,r}(\mu_h) \geq c(L\xi^r)^{-1} \Big(\sum_{\sigma \in \Gamma} \big(\mu_h(J_\sigma) \|\varphi'_\sigma\|^r \big)^{\kappa_r/(r+\kappa_r)} \Big)^{(1+r/\kappa_r)}$$

$$\times \Big(\sum_{\sigma \in \Gamma} \Big(\frac{n_\sigma(n)}{n} \Big)^{(-r/\kappa_r)(-\kappa_r/r)} \Big)^{-r/\kappa_r}.$$

By Lemma 3.4(b) and the fact that $\sum_{\sigma \in \Gamma} n_\sigma(n) \leq n$, we have

$$n^{r/\kappa_r} u_{n,r}(\mu_h) \geq c(L\xi^r)^{-1}(L\xi^r)^{-3}.$$

Therefore, by induction,

$$\liminf_n n u_{n,r}^{\kappa_r/r}(\mu_h) \geq \big[c(L\xi^r)^{-4} \big]^{\kappa_r/r} > 0; \text{ that is, } \liminf_n n V_{n,r}^{\kappa_r/r}(\mu_h) > 0,$$

and thus the proposition is obtained. □

Proof of Theorem 3.1
Note that $e_{n,r} = V_{n,r}^{1/r}$, and by [GL1, Proposition 11.3], we know the following:

(a) If $0 \leq t < \underline{D}_r < s$, then

$$\lim_{n \to \infty} n e_{n,r}^t = +\infty \qquad \text{and} \qquad \liminf_{n \to \infty} n e_{n,r}^s = 0.$$

(b) If $0 \leq t < \overline{D}_r < s$, then

$$\limsup_{n \to \infty} n e_{n,r}^t = +\infty \qquad \text{and} \qquad \lim_{n \to \infty} n e_{n,r}^s = 0.$$

From (a) and Proposition 3.10, we have $\kappa_r \leq \underline{D}_r$. From (b) and Proposition 3.7, we have $\overline{D}_r \leq \kappa_r$. Hence, $\kappa_r \leq \underline{D}_r \leq \overline{D}_r \leq \kappa_r$; that is, the quantization dimension D_r exists and $D_r = \kappa_r$. Note that if $q_r = \kappa_r/(r+\kappa_r)$, by Lemma 3.2, we have $\beta(q_r) = r q_r$, and then $D_r = \beta(q_r)/(1-q_r)$. Moreover, by Proposition 3.10, we have $\liminf_n n V_{n,r}^{\kappa_r/r}(\mu_h) > 0$. Thus, the proof of the theorem is complete. □

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