Blocks of normal subgroups, automorphisms of groups, and the Alperin–McKay conjecture

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In memory of my father –Tamio Murai

Abstract For a block b of a normal subgroup of a finite group G, E. C. Dade has defined a normal subgroup $G[b]$ of the inertial group of b in G. Let $S^0_G(b)$ be the subgroup of G consisting of all elements of G fixing all irreducible characters of height 0 in b. Under the Alperin–McKay conjecture we show that $S^0_G(b)/G[b]$ has a normal Sylow p-subgroup. Using this theorem, we show that (under the Alperin–McKay conjecture) the class-preserving outer automorphism group $\mathrm{Out}_c(G)$ of a group G has p-length at most one for any prime p. This rectifies C. H. Sah's incorrect proof that this group is solvable (under the Schreier conjecture). We obtain also other results on the structures of $S^0_G(b)/G[b]$ and $\mathrm{Out}_c(G)$ which are derived from the Alperin–McKay conjecture. Main results of the present paper depend on the classification theorem of finite simple groups.

Introduction

Let G be a finite group, and let p be a prime. Let $(\mathcal{K}, R, k)$ be a p-modular system. We assume that $\mathcal{K}$ contains a primitive $|G|$th root of unity and that k is algebraically closed. In this paper a block of G means a block ideal of RG. We recall the Alperin–McKay conjecture. Let B be a block of G with a defect group D, and let $\tilde{B}$ be the Brauer correspondent of B with respect to D in $N_G(D)$. Then the Alperin–McKay conjecture (AM conjecture, for short) states that B and $\tilde{B}$ have the same number of irreducible characters of height zero. As is well known, the AM conjecture is a consequence of a more general conjecture due to Dade [Da9]. Furthermore, Späth [Sp] has recently obtained a reduction theorem for the AM-conjecture. On the other hand, it is natural to seek consequences of the AM-conjecture. The present paper concerns this direction.

For a normal subgroup K of G and a block b of K, Dade [Da5] has defined a normal subgroup $G[b]$ of the inertial group G_b of b in G such that $G[b] \geq K$.

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More precisely, put $\bar{G} = G/K$ and $C = C_{RG}(K)$. Then $C = \bigoplus_{\bar{x}\in\bar{G}} C_{\bar{x}}$, where $C_{\bar{x}} = C \cap RKx$. Let e_b be the block idempotent of b. The subgroup $G[b]$ is defined by

$$G[b] = \{x \in G \mid (e_b C_{\bar{x}})(e_b C_{\bar{x}^{-1}}) = e_b C_{\bar{1}}\}.$$

(Strictly speaking, Dade defines a subgroup $(G/K)[b]$ of G/K. The subgroup $G[b]$ is the preimage of $(G/K)[b]$ in G.) It is known that $G[b]$ consists of all elements of G_b which induce inner automorphisms of b (cf. [Kü2]; see also [Mu5, Lemma 2.1]). We also consider the subgroup $S^0_G(b)$, which consists of all elements of G fixing all irreducible characters of height zero in b. Naturally $G[b] \le S^0_G(b) \le G_b$, and $G[b]$ and $S^0_G(b)$ are normal subgroups of G_b. In Section 4, employing an idea from [Na], we shall show the following (for a precise statement, see Theorem 4.1).

THEOREM A

Assume that the AM conjecture is true. Then $S^0_G(b)/G[b]$ has a normal Sylow p-subgroup.

Let $\mathrm{Aut}_c(G)$ be the class-preserving automorphism group of G; that is, $\mathrm{Aut}_c(G)$ is the group of all automorphisms of G fixing all conjugacy classes of G. Put $\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the inner automorphism group of G. Concerning $\mathrm{Out}_c(G)$, Burnside [Bu, Note B] stated that this group was abelian. But Sah [Sa] found a counterexample. To remedy this situation, two (obvious) ways are possible. One is to consider subgroups of $\mathrm{Out}_c(G)$, and the other is to weaken the conclusion. The first way was considered by Dade in a series of papers [Da1], [Da2], [Da3], [Da4], [Da6], and [Da7]. Let $\mathrm{Aut}(G)$ be the automorphism group of G, and put $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. For a subgroup H of G, let $C_{\mathrm{Out}(G)}(H) = C_{\mathrm{Aut}(G)}(H)\,\mathrm{Inn}(G)/\mathrm{Inn}(G)$ (for the definition of $C_X(Z)$, see the definition at the end of this introduction), and for a family $\mathfrak{K}$ of subgroups of G, let

$$C_{\mathrm{Out}(G)}(\mathfrak{K}) = \bigcap_{H\in\mathfrak{K}} C_{\mathrm{Out}(G)}(H).$$

Let $\mathrm{Cyc}(G), \mathrm{Ab}(G), \mathrm{Nil}(G), \mathrm{Syl}_*(G)$ be the families of all cyclic, abelian, nilpotent Sylow subgroups of G, respectively. The results obtained by Dade are as follows: $C_{\mathrm{Out}(G)}(\mathrm{Syl}_*(G))$ is nilpotent (see [Da7]) and abelian if G is solvable ([Da1]); $C_{\mathrm{Out}(G)}(\mathrm{Nil}(G))$ is abelian (see [Da4]); $C_{\mathrm{Out}(G)}(\mathrm{Ab}(G))$ is nilpotent of class at most two (and abelian if G is solvable) (see [Da6]). Note that $\mathrm{Out}_c(G) = C_{\mathrm{Out}(G)}(\mathrm{Cyc}(G))$. Dade [Da6] says that his method for treating the family $\mathrm{Ab}(G)$ cannot be applied to the family $\mathrm{Cyc}(G)$. The second way was considered by Sah [Sa] himself. He has “proved” that this group is solvable under the Schreier conjecture. His proof is, however, incorrect (see Remark 8.5 below). These investigations are mostly group-theoretical ones. Since $\mathrm{Aut}_c(G)$ consists of all automorphisms of G fixing all irreducible characters of G, it is natural to ask whether representation theory can be applied to the study of $\mathrm{Out}_c(G)$. Indeed, this is the case. We shall show the following (for a precise statement, see Theorem 8.3).

THEOREM B

Assume that the AM conjecture is true. Then $\mathrm{Out}_c(G)$ *has p-length at most one for any group G and any prime p. In particular,* $\mathrm{Out}_c(G)$ *is solvable.*

Theorems A and C below are the main ingredients for the proof of Theorem B.

Let P be a Sylow p-subgroup of G. If $\mathrm{O}_{p'}(G)$ is trivial, then $C_{\mathrm{Aut}(G)}(P)$ is p-nilpotent. This fact follows from Glauberman [Gl, Theorem 1] for $p = 2$ and from Gross [Gr, Theorem A] for p odd. If $\mathrm{O}_{p'}(G)$ is nontrivial, then $C_{\mathrm{Aut}(G)}(P)$ is no longer p-nilpotent in general. In Section 7 we shall show the following (see Theorem 7.7).

THEOREM C

$C_{\mathrm{Aut}(G)}(P) \cap \mathrm{Aut}_c(G)$ *is p-nilpotent for any group G and any prime p.*

Hertweck and Kimmerle [HeKi] has given a positive solution to Dade's conjecture [Da7] that $C_{\mathrm{Out}(G)}(\mathrm{Syl}_*(G))$ is abelian. In their proof, Dade's theorem [Da7] that $C_{\mathrm{Out}(G)}(\mathrm{Syl}_*(G))$ is nilpotent is necessary. In Section 9, we give a generalization of that theorem.

The present paper is organized as follows: Sections 1, 2, and 3 are preliminaries for Theorem A. In Section 1, we give a positive solution of Brauer's height zero conjecture in a special case. In Section 2 we recall some results on $G[b]$ from [Mu5]. In Section 3 we obtain a relative version of the Dade–Okuyama–Wajima theorem on the AM-conjecture for p-solvable groups.

In Section 4 we prove Theorem A and investigate the structure of $S_G^0(b)/G[b]$. In Section 5 we investigate the relationship between the characters of height zero in b and a defect group of a block of G covering b. As an application we obtain a relative version of the Gluck–Wolf theorem on Brauer's height zero conjecture for p-blocks of p-solvable groups. In Section 6 we give an alternative proof of Gross [Gr, Theorem A]. Although we use Gross's Theorem B for the proof (as in [Gr]), our proof is straightforward and as a by-product we can strengthen Glauberman's theorem in certain cases. In Section 7 we prove Theorem C. In Section 8 we prove Theorem B. General theorems on the structure of $S_G^0(b)/G[b]$ and $\mathrm{Out}_c(G)$ will also be given.

In Section 9 we consider $C_{\mathrm{Out}(G)}(\mathrm{Syl}_*(G))$. In Section 10 we consider mainly automorphisms of groups with abelian Sylow p-subgroups. In Section 11 the structure of $S_G^0(b)/G[b]$ and $\mathrm{Out}_c(G)$ will be determined in several cases (under the AM conjecture). In particular, we show that $\mathrm{Out}_c(G)$ is nilpotent if G is either a supersolvable group or an A-group.

In many places in the present paper we need the AM conjecture, apart from some results obtained so far only by using the classification theorem of finite simple groups.

NOTATION

Let ν be the valuation of $\mathcal{K}$ normalized so that $\nu(p) = 1$. For a block B, let $\mathrm{d}(B)$

be the defect of B. A block idempotent of B will be denoted by e_B. Let $\mathrm{Irr}_0(B)$ be the set of irreducible characters of height zero in B. Let $k_0(B) = |\,\mathrm{Irr}_0(B)|$. For a block b of a normal subgroup of G, let $G_b = T_G(b)$ be the inertial group of b in G, and let $\mathrm{BL}(G\,|\,b)$ be the set of blocks of G covering b.

Let $\mathrm{Aut}_c(G)$ be the group of automorphisms of G fixing all conjugacy classes of G. An automorphism σ of G is said to be p-Coleman if for a Sylow p-subgroup P of G, there is $x \in G$ such that $u^\sigma = u^x$ for all u in P. This definition is independent of the choice of P. Let $\mathrm{Aut}_{p\text{-Col}}(G)$ be the subgroup of $\mathrm{Aut}(G)$ consisting of all p-Coleman automorphisms of G. We note that $\mathrm{Aut}_{p\text{-Col}}(G) = C_{\mathrm{Aut}(G)}(P)\,\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ is the inner automorphism group of G. Put $\mathrm{Aut}_{\mathrm{Col}}(G) = \bigcap_p \mathrm{Aut}_{p\text{-Col}}(G)$, where p runs through all primes. An element in $\mathrm{Aut}_{\mathrm{Col}}(G)$ is said to be a Coleman automorphism of G ([HeKi]). Let $\mathrm{Aut}_{c,p}(G)$ be the group of automorphisms of G fixing all conjugacy classes of G consisting of p-elements. $\mathrm{Aut}_c(G)$, $\mathrm{Aut}_{p\text{-Col}}(G), \mathrm{Aut}_{\mathrm{Col}}(G)$, and $\mathrm{Aut}_{c,p}(G)$ are all normal subgroups of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$. Put $\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\,\mathrm{Inn}(G)$. $\mathrm{Out}_{p\text{-Col}}(G), \mathrm{Out}_{\mathrm{Col}}(G)$, and $\mathrm{Out}_{c,p}(G)$ are defined in a similar way. Note that $\mathrm{Out}_{\mathrm{Col}}(G) = C_{\mathrm{Out}(G)}(\mathrm{Syl}_*(G))$. For an automorphism σ of G and a subgroup L of G such that $L^\sigma = L$, let $\sigma|_L$ be the restriction of σ to L. Let $\mathrm{conj} : G \to \mathrm{Inn}(G)$ be the natural map; that is, $\mathrm{conj}(x) : g \mapsto x^{-1}gx$ for $x, g \in G$. For subgroups L, M, and N of a group, let $[L, M]$ be the commutator subgroup of L and M, and let $[L, M, N] = [[L, M], N]$. L' is the commutator subgroup $[L, L]$ of L. Let $F(G)$ be the Fitting subgroup of G, let $F^*(G)$ be the generalized Fitting subgroup of G, and let $E(G)$ be the maximal semisimple normal subgroup of G. Let $\pi(G)$ be the set of distinct prime divisors of the order of G. If a group X acts on a group Y as automorphisms and Z is a subgroup of Y, then put

$$C_X(Z) = \{x \in X \mid z^x = z \text{ for all } z \in Z\}.$$

1. A special case of Brauer's height zero conjecture

We prove a special case of Brauer's height zero conjecture, which will be generalized in Proposition 5.5.

PROPOSITION 1.1

Let N be a normal subgroup of G. Let b be a block of N with a defect group Q. Assume that Q is normal in N and that G/N is a cyclic p-group. Let B be a block of G covering b. Let D be a defect group of B. If any irreducible character in B has height zero, then D is abelian.

Proof

By the Fong–Reynolds theorem, we may assume that b is G-invariant. Then $G = N_G(Q)N$ by the Frattini argument. So $Q \lhd G$. By [Mu2, Lemma 2.2], any irreducible character in b has height zero. So Q is abelian by Reynolds [Re, Theorem 9]. Let β be a block of $C_N(Q)$ covered by b. Let $\tilde{B}$ (resp., $\tilde{b}$) be the

Fong–Reynolds correspondent of B (resp., b) over β in $T_G(\beta)$ (resp., $T_N(\beta)$). Then $\tilde{B}$ covers $\tilde{b}$, since $\tilde{b}$ is a unique block of $T_N(\beta)$ covering β. We see that Q is a defect group of $\tilde{b}$ and a G-conjugate of D is a defect group of $\tilde{B}$. Further, any irreducible character in $\tilde{B}$ has height zero, and $T_G(\beta)/T_N(\beta)$ is a cyclic p-group. Therefore we may assume that β is G-invariant. Then $N/C_N(Q)$ is a p'-group. Further, B is a unique block of G covering β. (Indeed, if B_1 is a block of G covering β, then B_1 covers b. Since G/N is a p-group, we have $B_1 = B$.) Since β is nilpotent and G-invariant, B is isomorphic to the full matrix algebra of some degree n over a twisted group algebra $R^{\alpha}[L]$ by [KP, 1.20.3], where the group L is an extension of $G/C_N(Q)$ by Q (see [KP, 1.8.1]), the 2-cocycle α takes p'th roots of unity as values (see [KP, 2.4]), and D is isomorphic to a Sylow p-subgroup of L (see [KP, Remark 1.9]). We show that $\nu(n) = \nu(|C_N(Q):Q|)$. By the above, $\operatorname{rank}_R B = n^2|L| = n^2|G/C_N(Q)||Q|$. On the other hand, since $B = RGe_B$, we have $\operatorname{rank}_R B = |G/C_N(Q)| \operatorname{rank}_R \beta$. Let m be the dimension of a unique simple $kC_N(Q)$-module in β. Then $\operatorname{rank}_R \beta = |Q|m^2$. Thus we obtain $n = m$. Hence $\nu(n) = \nu(|C_N(Q):Q|)$.

On the other hand, since B is weakly regular with respect to $C_N(Q)$ and β is G-invariant, we have $\nu(|G:D|) = \nu(|C_N(Q):Q|)$. So $\nu(n) = \nu(|G:D|)$. Therefore any irreducible character of $\mathcal{K} \otimes_R R^{\alpha}[L]$ has p'-degree by Morita equivalence. Now there is a p'-central extension H of L and a block B' of H such that $R^{\alpha}[L]$ is isomorphic to B'. Then any irreducible character in B' has p'-degree. Since $G/C_N(Q)$ is p-nilpotent, H is p-solvable. Therefore, by [GW, Theorem B] a defect group of B', that is, a Sylow p-subgroup of H, is abelian. So the same is true for a Sylow p-subgroup of L. Hence D is abelian. This completes the proof. □

2. The subgroup $G[b]$

Throughout this section, let K be a normal subgroup of a group G, let b be a block of K, and let Q be a defect group of b. We will recall some facts on $G[b]$ from [Mu5].

THEOREM 2.1 ([Mu5, THEOREM 3.5])

Let b be G-invariant. Let B be a block of G covering b. We choose a block B' of $G[b]$ so that B covers B' (and B' covers b; note that $G[b] \lhd G$). Let D, S be defect groups of B, B', respectively, such that $Q \le S \le D$. Then

- (i) *(Dade [Da5]) B is a unique block of G covering B';*
- (ii) $S = QC_D(Q)$.

We add a simple fact.

COROLLARY 2.2

Let b be G-invariant. Let N be a normal subgroup of G containing $G[b]$. Let B_1 be a block of N which is covered by a block B of G and covers b. Then B is a unique block of G covering B_1.

Proof
Let B' be a block of $G[b]$ which is covered by B_1 and covers b. Then B covers B'. If B_2 is a block of G covering B_1, then B_2 covers B'. So $B_2 = B$ by Theorem 2.1. $\square$

PROPOSITION 2.3 ([Mu5, PROPOSITION 3.9])
Assume that G/K is a cyclic p'-group. The following are equivalent:

(i) $G = G[b]$,
(ii) $|\mathrm{BL}(G \mid b)| = |G/K|$.

We introduce some notation. Let β be a block of $QC_K(Q)$ such that $\beta^K = b$. Put $L_0 = QC_K(Q)$. Let β_0 be a block of $C_K(Q)$ covered by β. Let θ be the canonical character of β, and let φ be the restriction of θ to $C_K(Q)$. So φ is the canonical character of β_0. Let $T = N_K(Q)_\beta$. So T is the inertial group of β_0 in $N_K(Q)$. Put $L = QC_G(Q)$ and $C = C_G(Q)$.

Noting that T and L_β are normal subgroups of $N_G(Q)_\beta$, we have $[T, L_\beta] \leq L_\beta \cap T = L_0$. So we can define (after Isaacs [Is1, Section 2]) $\langle\langle t, x\rangle\rangle_\theta \in \mathcal{K}^*$ for $(t, x) \in T \times L_\beta$, where $\mathcal{K}^*$ is the multiplicative group of $\mathcal{K}$. The definition is as follows: let $x \in L_\beta$, and let $\hat{\theta}$ be an extension of θ to $\langle x, L_0\rangle$. Let $t \in T$. Then, since $\hat{\theta}^t$ is also an extension of θ to $\langle x, L_0\rangle$, there exists a unique linear character λ_t of $\langle x, L_0\rangle/L_0$ such that $\hat{\theta}^t = \hat{\theta} \otimes \lambda_t$. Then put $\langle\langle t, x\rangle\rangle_\theta = \lambda_t(x)$. This definition is independent of the choice of $\hat{\theta}$. It is bilinear in the sense that $\langle\langle ts, x\rangle\rangle_\theta = \langle\langle t, x\rangle\rangle_\theta \langle\langle s, x\rangle\rangle_\theta$ for $t, s \in T$ and $x \in L_\beta$ and $\langle\langle t, xy\rangle\rangle_\theta = \langle\langle t, x\rangle\rangle_\theta \langle\langle t, y\rangle\rangle_\theta$ for $t \in T$ and $x, y \in L_\beta$. Similarly we can define $\langle\langle t, x\rangle\rangle_\varphi \in \mathcal{K}^*$ for $(t, x) \in T \times C_{\beta_0}$. It is also bilinear. Define, for $\omega = (\beta, \theta)$,

$$L_\omega = \{x \in L_\beta \mid \langle\langle t, x\rangle\rangle_\theta = 1 \text{ for all } t \in T\},$$

$$C_\omega = \{x \in C_{\beta_0} \mid \langle\langle t, x\rangle\rangle_\varphi = 1 \text{ for all } t \in T\}.$$

By definition, we see that for $x \in C_{\beta_0}$, the condition that $x \in C_\omega$ is equivalent to the condition that any (equivalently, some) extension of φ to $\langle x, C_0\rangle$ is T-invariant.

PROPOSITION 2.4 ([Mu5, LEMMA 3.12])

(i) *L_ω is a normal subgroup of L_β such that L_β/L_ω is a p'-group.*
(ii) *We have $L_\omega K = C_\omega K$.*

We have the following theorem (see also [Mu5, Theorem 3.13, Remark 3.14]).

THEOREM 2.5 (DADE [Da5, COROLLARY 12.6])
We have $G[b] = C_\omega K$.

COROLLARY 2.6 (KÜLSHAMMER [Kü2, PROPOSITION 9]; SEE ALSO [Mu5, COROLLARY 3.15])
We have $G[b] = N_G(Q)[\tilde{b}]K$.

3. A relative version of the Dade–Okuyama–Wajima theorem

For fundamental properties of subpairs, we refer to Alperin and Broué [AB]. We will denote by B^* the central idempotent of kG corresponding to a block B of G.

The following lemma is a slight extension of [CEKL, Lemma 3.4].

LEMMA 3.1

Let N be a normal subgroup of G. Let B be a block of G covering a nilpotent block b of N. Assume that B and b have a common defect group D. If $G = C_G(D)_{b_D} N$ for a maximal b^*-subpair (D, b_D^*), then B is nilpotent.

Proof

The assumption that $G = C_G(D)_{b_D} N$ implies that b is G-invariant. So essentially the same proof as that of [CEKL, Lemma 3.4] yields the result. □

LEMMA 3.2

Let N be a normal subgroup of G. Let b be a block of N covered by a block B of G. Let β be the Fong–Reynolds correspondent of B over b in the inertial group T of b in G. Let D be a defect group of β. Let $\tilde{\beta}$ (resp., $\tilde{B}$) be the Brauer correspondent of β (resp., B) with respect to D in $N_T(D)$ (resp., $N_G(D)$). Then $\tilde{\beta}$ is the Fong–Reynolds correspondent of $\tilde{B}$ over a block of $C_N(D)$.

Proof

Let β_D be a block of $C_T(D)$ covered by $\tilde{\beta}$. Let b_D be a block of $C_N(D)$ covered by β_D. We claim that $\mathrm{Br}_D(b^*)b_D^* = b_D^*$. Since β covers b, $b^*\beta^* = \beta^*$. On the other hand, we have $\mathrm{Br}_D(\beta^*)\beta_D^* = \beta_D^*$. So $\mathrm{Br}_D(b^*)\beta_D^* = \mathrm{Br}_D(b^*)\,\mathrm{Br}_D(\beta^*)\beta_D^* = \mathrm{Br}_D(b^*\beta^*)\beta_D^* = \mathrm{Br}_D(\beta^*)\beta_D^* = \beta_D^*$. So there is a block b'_D of $C_N(D)$ such that $\mathrm{Br}_D(b^*)b_D'^* = b_D'^*$ and that $b_D'^*\beta_D^* \neq 0$. So β_D covers b'_D, and b'_D is $C_T(D)$-conjugate to b_D. This implies that $\mathrm{Br}_D(b^*)b_D^* = b_D^*$. The claim is proved. Then $N_G(D)_{b_D} \leq T$. So $N_G(D)_{b_D} \leq N_T(D)$. Since $\tilde{\beta}$ covers b_D, there is a block B_1 of $N_G(D)$ such that $\tilde{\beta}$ is the Fong–Reynolds correspondent of B_1 over b_D. Then $B_1^G = (\tilde{\beta}^{N_G(D)})^G = \tilde{\beta}^G = (\tilde{\beta}^T)^G = \beta^G = B$. Hence $B_1 = \tilde{B}$ by the first main theorem. The proof is complete. □

The following is a relative version of the Dade–Okuyama–Wajima theorem (see [Da8], [OW]); letting $N = 1$, we recover their theorem.

THEOREM 3.3

Let N be a normal subgroup of G. Let b be a block of N with a defect group Q. Assume that Q is normal in N. Let B be a block of G covering b. If G/N is p-solvable, then the AM conjecture is true for B.

Proof

Let D be a defect group of B. Let $\tilde{B}$ be the Brauer correspondent of B with respect to D in $N_G(D)$. We argue by induction first on $|G/N|$ and second on

$|G|$. Let B_1 be the Fong–Reynolds correspondent of B over b in G_b. We may assume that D is a defect group of B_1. Let $\tilde{B}_1$ be the Brauer correspondent of B_1 in $N_G(D) \cap G_b$. By Lemma 3.2, $\tilde{B}_1$ is the Fong–Reynolds correspondent of $\tilde{B}$ over a block of $C_N(D)$. Thus $k_0(\tilde{B}) = k_0(\tilde{B}_1)$ and $k_0(B_1) = k_0(B)$ by the Fong–Reynolds theorem. If $G \neq G_b$, then by induction, we have $k_0(\tilde{B}_1) = k_0(B_1)$. Thus $k_0(\tilde{B}) = k_0(B)$. So we may assume that b is G-invariant. In particular, $G = N_G(Q)N$ by the Frattini argument. So $Q \lhd G$.

Let β be a block of $QC_N(Q)$ covered by b. We may assume that β is G-invariant. Indeed, let B_1 be the Fong–Reynolds correspondent of B over β in G_β. We may assume that D is a defect group of B_1. Let $\tilde{B}_1$ be the Brauer correspondent of B_1 in $N_G(D) \cap G_\beta$. By Lemma 3.2 $\tilde{B}_1$ is the Fong–Reynolds correspondent of $\tilde{B}$ over a block of $C_N(D) \cap QC_N(Q)$. Thus $k_0(\tilde{B}) = k_0(\tilde{B}_1)$ and $k_0(B_1) = k_0(B)$ by the Fong–Reynolds theorem. Assume $G \neq G_\beta$. Then $|G_\beta/G_\beta \cap N| \leq |G/N|$, $|G_\beta| < |G|$, and B_1 covers a block of $G_\beta \cap N$ with a defect group Q. So, by induction, we have $k_0(\tilde{B}_1) = k_0(B_1)$. Thus $k_0(\tilde{B}) = k_0(B)$. So we may assume tat β is G-invariant. In particular, $N/QC_N(Q)$ is a p'-group, and $G/QC_N(Q)$ is p-solvable.

Put $H = N_G(D)QC_N(Q)$. Put $B_1 = \tilde{B}^H$. Since $G/QC_N(Q)$ is p-solvable, we obtain $k_0(B_1) = k_0(B)$ by [Mu3, Corollary 8]. If $H \neq G$, then by induction as in the second paragraph, $k_0(\tilde{B}) = k_0(B_1)$. Thus $k_0(\tilde{B}) = k_0(B)$. So we may assume $G = N_G(D)QC_N(Q)$. Since $Q \lhd G$, we have $Q \leq D$. So $G = N_G(D)C_N(Q)$ and $DC_N(Q) \lhd G$.

Let $\hat{\beta}$ be a unique block of $DC_N(Q)$ covering β. Since B covers β, B covers $\hat{\beta}$. Since β is G-invariant, $\hat{\beta}$ is G-invariant. In particular, D is a defect group of $\hat{\beta}$. Let β_0 be a root of $\hat{\beta}$. Since $\hat{\beta}$ is G-invariant, $G = N_G(D)_{\beta_0} DC_N(Q)$. Put $K = C_G(D)_{\beta_0} DC_N(Q)$. Then $K \lhd G$. Let B' be a block of K which is covered by B and covers $\hat{\beta}$. Then D is a defect group of B'. Since $DC_N(Q)/QC_N(Q)$ is a p-group and β is nilpotent, $\hat{\beta}$ is nilpotent (see [Ca, Theorem 2]). Then by Lemma 3.1, B' is nilpotent. Now $G[\hat{\beta}] = C_G(D)_\omega DC_N(Q) \leq K$ by Theorem 2.5. So B is a unique block of G covering B' by Corollary 2.2.

Let T be the inertial group of B' in G. Let B_1 be the Fong–Reynolds correspondent of B over B' in T. Then B_1 is a unique block of T covering B' by the Fong–Reynolds theorem. We see that D is a defect group of B_1. Let $\tilde{B}_1$ (resp., $\tilde{B}'$) be the Brauer correspondent of B_1 (resp., B') with respect to D in $N_T(D)$ (resp., $N_K(D)$). By the Harris–Knörr theorem, $\tilde{B}_1$ is a unique block of $N_T(D)$ covering $\tilde{B}'$. Let β_1 be a block of $C_K(D)$ covered by $\tilde{B}'$. Then $\tilde{B}_1$ is a unique block of $N_T(D)$ covering β_1. So, by the Fong–Reynolds theorem, there is a unique block B_2 of $N_T(D)_{\beta_1}$ covering β_1. Let D_δ be the defect pointed group of the pointed group $K_{\{e_{B'}\}}$ on RK which is associated with (D, β_1^*). Then $N_T(D_\delta) = N_T(D)_{\beta_1}$ (cf. [Th, Proposition 40.13(b)]). Therefore, by [KP, 1.20.3], B_1 and B_2 are isomorphic to full matrix algebra over the same R-algebra. This implies $k_0(B_2) = k_0(B_1)$. By the Fong–Reynolds theorem and Lemma 3.2, $k_0(\tilde{B}) = k_0(\tilde{B}_1)$. Also, by the Fong–Reynolds theorem, $k_0(B_2) = k_0(\tilde{B}_1)$ and $k_0(B) = k_0(B_1)$. Hence $k_0(B) = k_0(\tilde{B})$. The proof is complete. $\square$

4. Consequences of the AM conjecture, I: The structure of $S^0_G(b)/G[b]$

In Sections 4, 8, and 11 we use the following notation: K is a normal subgroup of G; b is a (p-)block of K with defect group Q, $H = N_G(Q), \tilde{K} = N_K(Q)$; $\tilde{b}$ is the Brauer correspondent of b with respect to Q in $\tilde{K}$; β is a block of $QC_K(Q)$ covered by $\tilde{b}$; and b_1 is the Fong–Reynolds correspondent of $\tilde{b}$ over β in $N_K(Q)_\beta$.

We say that the AM conjecture is true around (K, b) in G if, whenever L is a subgroup of G containing K such that L/K is cyclic, the AM conjecture is true for any block of L covering b. (The case where $(L, B) = (K, b)$ is included.)

We say, as usual, that a group is p-closed if it has a normal Sylow p-subgroup.

THEOREM 4.1

Assume that the AM conjecture is true around (K, b) in G. Then $S^0_G(b)/G[b]$ is p-closed.

LEMMA 4.2

Assume that the AM conjecture is true around (K, b) in G. We have

(i) $S^0_G(b) = S^0_H(\tilde{b})K$ *and* $S^0_G(b) \cap H = S^0_H(\tilde{b})$,
(ii) $S^0_G(b)/G[b] \simeq S^0_H(\tilde{b})/H[\tilde{b}]$.

Proof

(i) Since $S^0_G(b) = (S^0_G(b) \cap H)K$ by the Frattini argument, it suffices to show that $S^0_G(b) \cap H = S^0_H(\tilde{b})$. We show that both sides contain the same p-elements and p'-elements. The assertion for p'-elements follows from the proof of Navarro [Na, Lemma 3.1]. Let x be any p-element in H. We must show that $x \in S^0_G(b)$ if and only if $x \in S^0_H(\tilde{b})$. We note that in both cases b and $\tilde{b}$ are $\langle x\rangle$-invariant (by the first main theorem). Put $L = \langle x, K\rangle$ and $\tilde{L} = \langle x, \tilde{K}\rangle$. Let B be a unique block of L covering b. Let D be a defect group of B with $D \cap K = Q$. Let B_1 be the Brauer correspondent of B with respect to D in $N_L(D)$. Here $N_L(D) \leq \tilde{L} = N_L(Q)$. Let $\tilde{B} = B_1^{\tilde{L}}$. Then, by the Harris–Knörr theorem, $\tilde{B}$ is a unique block of $\tilde{L}$ covering $\tilde{b}$. Then, since we may assume that b and $\tilde{b}$ are, respectively, L-invariant and $\tilde{L}$-invariant, we obtain $|\operatorname{Irr}_0(B)| = |\operatorname{Irr}^L_0(b)||L/K|$, where $\operatorname{Irr}^L_0(b)$ is the set of L-invariant members of $\operatorname{Irr}_0(b)$. Similarly $|\operatorname{Irr}_0(\tilde{B})| = |\operatorname{Irr}^{\tilde{L}}_0(\tilde{b})||\tilde{L}/\tilde{K}|$. By assumption $|\operatorname{Irr}_0(B)| = |\operatorname{Irr}_0(B_1)|$. By Theorem 3.3, $|\operatorname{Irr}_0(\tilde{B})| = |\operatorname{Irr}_0(B_1)|$. So $|\operatorname{Irr}_0(B)| = |\operatorname{Irr}_0(\tilde{B})|$. Since $|L/K| = |\tilde{L}/\tilde{K}|$, we obtain $|\operatorname{Irr}^L_0(b)| = |\operatorname{Irr}^{\tilde{L}}_0(\tilde{b})|$.

Assume $x \in S^0_G(b)$. Then $|\operatorname{Irr}_0(b)| = |\operatorname{Irr}^L_0(b)|$. Since $|\operatorname{Irr}_0(b)| = |\operatorname{Irr}_0(\tilde{b})|$ by assumption, we obtain $|\operatorname{Irr}^{\tilde{L}}_0(\tilde{b})| = |\operatorname{Irr}_0(\tilde{b})|$. Thus $x \in S^0_H(\tilde{b})$. The converse is proved in a similar way.

(ii) This follows from (i) and Corollary 2.6. Indeed, by (i), $S^0_G(b) = S^0_H(\tilde{b})K = S^0_H(\tilde{b})G[b]$. On the other hand, by Corollary 2.6, $G[b] = H[\tilde{b}]K$, so that $S^0_H(\tilde{b}) \cap G[b] = H[\tilde{b}](S^0_H(\tilde{b}) \cap K) = H[\tilde{b}]$. Therefore

$$S^0_G(b)/G[b] \simeq S^0_H(\tilde{b})/S^0_H(\tilde{b}) \cap G[b] \simeq S^0_H(\tilde{b})/H[\tilde{b}].$$

The proof is complete. □

Proof of Theorem 4.1.
By Lemma 4.2 it suffices to show that $S^0_H(\tilde{b})/H[\tilde{b}]$ is p-closed. We write K, b instead of $\tilde{K}, \tilde{b}$. Put $\bar{H} = H/Q'$, where Q' is the commutator subgroup of Q. We use the bar convention. Let $\bar{b}$ be a unique block of $\bar{K}$ dominated by b (cf. [Kü1, Corollary 4]). Then $\bar{Q}$ is a defect group of $\bar{b}$. So the set of irreducible characters of height zero in b is identified with that of $\bar{b}$. Therefore $\overline{S^0_H(b)} = S^0_{\bar{H}}(\bar{b})$. Thus it suffices to show the following:

(1) $S^0_{\bar{H}}(\bar{b})/\bar{H}[\bar{b}]$ is a p'-group;
(2) $\bar{H}[\bar{b}]/\overline{H[b]}$ is a p-group.

To prove (1), let $\bar{x}$ be a p-element in $S^0_{\bar{H}}(\bar{b})$. Let $\bar{D}$ be a defect group of a unique block $\bar{B}$ of $\langle \bar{x}, \bar{K} \rangle$ covering $\bar{b}$. Since a defect group $\bar{Q}$ of $\bar{b}$ is an abelian normal subgroup of $\bar{K}$, any irreducible character in $\bar{b}$ has height zero by Reynolds [Re, Theorem 9]. Since any irreducible character in $\bar{b}$ is $\langle \bar{x}, \bar{K} \rangle$-invariant and $\langle \bar{x}, \bar{K} \rangle / \bar{K}$ is a cyclic p-group, any irreducible character in $\bar{B}$ has height zero. Hence $\bar{D}$ is abelian by Proposition 1.1. Further, $\langle \bar{x}, \bar{K} \rangle = \bar{D}\bar{K}$. Now a block of $\langle \bar{x}, \bar{K} \rangle[\bar{b}]$ covered by $\bar{B}$ has defect group $\bar{Q}C_{\bar{D}}(\bar{Q}) = \bar{D}$ by Theorem 2.1. Therefore $\langle \bar{x}, \bar{K} \rangle[\bar{b}] \geq \bar{D}\bar{K} = \langle \bar{x}, \bar{K} \rangle$. Thus $\bar{x} \in \bar{H}[\bar{b}]$, and (1) follows.

To prove (2), we first claim that $\overline{H[b]} \leq \bar{H}[\bar{b}]$. Let $x \in H[b]$. Then x induces an inner automorphism of b, so that $\bar{x}$ induces an inner automorphism of $\bar{b}$. Thus $\bar{x} \in \bar{H}[\bar{b}]$, and the claim is proved. Let $\bar{x}$ be a p'-element in $\bar{H}[\bar{b}]$. Then domination gives a bijection of $\mathrm{BL}(\langle x, K \rangle \mid b)$ onto $\mathrm{BL}(\langle \bar{x}, \bar{K} \rangle \mid \bar{b})$ by [Kü1, Corollary 4]. Since $\bar{x} \in \bar{H}[\bar{b}]$, the last set consists exactly of $|\langle \bar{x}, \bar{K} \rangle / \bar{K}| = |\langle x, K \rangle / K|$ blocks by Proposition 2.3. Hence $x \in H[b]$ by Proposition 2.3. Thus (2) follows. The proof is complete. □

COROLLARY 4.3
Assume that the AM conjecture is true around (K, b) in G. If Q is abelian, then $S^0_G(b)/G[b]$ is a p'-group.

Proof
This is clear from the proof of Theorem 4.1. □

Since the AM-conjecture is true for p-blocks of p-solvable groups (see [Da8], [OW]), we obtain the following.

COROLLARY 4.4
If K is p-solvable, then $S^0_G(b)/G[b]$ is p-closed.

LEMMA 4.5
It holds that $S^0_H(\tilde{b})/H[\tilde{b}] \simeq S^0_{H_\beta}(b_1)/H_\beta[b_1]$.

Proof
For simplicity we write K, b instead of $\tilde{K}, \tilde{b}$. We claim $H_b = H_\beta K$. Indeed, since b

covers K-conjugates of β, we have $H_b \leq H_\beta K$. Since $b = \beta^K$, we have $H_\beta K \leq H_b$. Thus the claim follows. Then, since $K \leq S^0_H(b) \leq H_b$, we obtain $S^0_H(b) = (S^0_H(b) \cap H_\beta)K$. Here we show that $S^0_H(b) \cap H_\beta = S^0_{H_\beta}(b_1)$. Indeed, b_1 is a unique block of K_β such that b_1 covers β and $(b_1)^K = b$. This implies that $S^0_H(b) \cap H_\beta$ fixes b_1. Since induction is a bijection of $\mathrm{Irr}_0(b_1)$ onto $\mathrm{Irr}_0(b)$, the equality follows. Thus $S^0_H(b) = S^0_{H_\beta}(b_1)K$. Then $S^0_H(b) = S^0_{H_\beta}(b_1)H[b]$.

We claim that $H[b] \cap H_\beta = H_\beta[b_1]$. By Theorem 2.5, $H[b] = C_H(Q)_\omega K$ and $H_\beta[b_1] = C_{H_\beta}(Q)_\omega K_\beta$. We have $C_H(Q)_\beta = C_{H_\beta}(Q)$, so that $C_H(Q)_\omega = C_{H_\beta}(Q)_\omega$ by definition. Thus $H[b] \cap H_\beta = C_{H_\beta}(Q)_\omega (K \cap H_\beta) = C_{H_\beta}(Q)_\omega K_\beta = H_\beta[b_1]$. The claim is proved. Then $S^0_{H_\beta}(b_1) \cap H[b] = S^0_{H_\beta}(b_1) \cap H_\beta \cap H[b] = S^0_{H_\beta}(b_1) \cap H_\beta[b_1] = H_\beta[b_1]$.

Hence

$$S^0_H(b)/H[b] = S^0_{H_\beta}(b_1)H[b]/H[b] \simeq S^0_{H_\beta}(b_1)/S^0_{H_\beta}(b_1) \cap H[b] \simeq S^0_{H_\beta}(b_1)/H_\beta[b_1].$$

The proof is complete. □

CONVENTION 4.6

We will always assume that the AM conjecture is true around (K, b) in G when necessary. In that case, by Lemmas 4.2 and 4.5, so far as the structure of $S^0_G(b)/G[b]$ is concerned we may assume $G = H_\beta$; more precisely, it suffices to consider $(H_\beta, N_K(Q)_\beta, b_1)$ in place of (G, K, b). When we do so, we will always use the notation (H, K, b) instead of $(H_\beta, N_K(Q)_\beta, b_1)$ for simplicity.

As usual, the group $N_K(Q)_\beta/QC_K(Q)$ is called the inertial quotient group of b, and its order is the inertial index of b and denoted by $e(b)$.

LEMMA 4.7

Suppose $G = H_\beta$, and use Convention 4.6. Let $\bar{H} = H/Q'$, and use the bar convention. Let $\bar{b}$ be a unique block of $\bar{K}$ dominated by b. Let $\bar{\beta}$ be a block of $C_{\bar{K}}(\bar{Q})$ covered by $\bar{b}$. Then

(i) *$\bar{\beta}$ is $\bar{H}$-invariant and has a defect group $\bar{Q}$;*

(ii) *the inertial quotient group of $\bar{b}$ is a factor group of that of b. In particular, $e(\bar{b})$ is a divisor of $e(b)$;*

(iii) *a p-complement of $S^0_H(b)/H[b]$ is isomorphic to $S^0_{\bar{H}}(\bar{b})/\bar{H}[\bar{b}]$;*

(iv) *if the canonical character of β extends to K, then the canonical character of $\bar{\beta}$ extends to $\bar{K}$.*

Proof

First we note that b is uniquely determined (cf. the proof of Theorem 4.1); (iii) is proved in the proof of Theorem 4.1.

Let β' be a unique block of $\overline{QC_K(Q)}$ dominated by β. Since β is H-invariant, β' is $\bar{H}$-invariant. Since b covers β, $\bar{b}$ covers β'. So $\bar{\beta}$ covers β'. Since $C_{\bar{K}}(\bar{Q})/\overline{QC_K(Q)}$ is a p-group, $\bar{\beta}$ is uniquely determined. Thus $\bar{\beta}$ also is $\bar{H}$-invariant.

Since $\bar{Q}$ is a defect group of $\bar{b}$ and $\bar{b}$ covers $\bar{\beta}$, a defect group of $\bar{\beta}$ is contained in $\bar{Q}$. Since $\bar{Q} \lhd C_{\bar{K}}(\bar{Q})$, we see that $\bar{Q}$ is a defect group of $\bar{\beta}$. Since $\bar{\beta}$ is $\bar{K}$-invariant, the inertial quotient group of $\bar{b}$ is $\bar{K}/C_{\bar{K}}(\bar{Q})$ and it is a factor group of $K/QC_K(Q)$, the inertial quotient group of b. Let θ be the canonical character of β, and let $\hat{\theta}$ be an extension of θ to K. Then θ and $\hat{\theta}$ are regarded as characters of $\overline{QC_K(Q)}$ and $\bar{K}$, respectively. Put $\varphi = \hat{\theta}_{C_{\bar{K}}(\bar{Q})}$. Since $\hat{\theta}$ belongs to $\bar{b}$, φ belongs to $\bar{\beta}$. Since φ is irreducible and $\operatorname{Ker}\varphi \geq \bar{Q}$, φ is the canonical character of $\bar{\beta}$. The proof is complete. □

For a positive integer n, let $\pi(n)$ be the set of distinct prime divisors of n. The structure of $S_G^0(b)/G[b]$ is closely related with the inertial quotient group of b as the following theorem shows. See also Proposition 8.7 below.

THEOREM 4.8

Assume that the AM conjecture is true around (K,b) *in* G. *Then* $\pi(S_G^0(b)/G[b]) \subseteq \{p\} \cup \pi(e(b))$.

Proof

We may assume $G = H_\beta$. Use Convention 4.6. Let the notation be as in Lemma 4.7. By Lemma 4.7(ii) and (iii), instead of (H,K,b,Q) it suffices to consider $(\bar{H},\bar{K},\bar{b},\bar{Q},\bar{\beta})$, which we denote from here by (H,K,b,Q,β). So $H = H_\beta$, Q is abelian, and $S_H^0(b)/H[b]$ is a p'-group. We must show that $S_H^0(b)/H[b]$ is a $\pi(e(b))$-group. Assume that this is false. Then there is a prime divisor q of $|S_H^0(b)/H[b]|$ such that $q \notin \pi(e(b))$. Then $q \neq p$. We can choose a q-element $x \in S_H^0(b) - H[b]$. We show $x \in C_H(Q)$ in the next paragraph by modifying slightly the proof of [Na, Theorem 3.2].

Put $A = \langle x \rangle$, and consider A to be an operator group acting on K. We must show $[Q,A] = 1$. Let E be the semidirect product of K/Q and Q with respect to the natural action of K/Q on Q. Put $N = C_K(Q)/Q$. Let θ be the canonical character of β, let $\bar{\theta}$ be the corresponding character of N, and put $\eta = \bar{\theta}$. Then there is a bijection of $\operatorname{Irr}(b)$ onto $\operatorname{Irr}(E \mid \eta)$ which commutes the action of A. Since A acts trivially on $\operatorname{Irr}(b)$ (as all irreducible characters in b have height zero), A acts trivially on $\operatorname{Irr}(E \mid \eta)$. Let $A \ltimes E$ be the semidirect product with respect to the natural action of A on E. As in [Na], it suffices to show that for any irreducible character γ of $A \ltimes E$ lying over η, it holds that $\gamma(1)/\eta(1)$ is a $\{p,q\}'$-number. Let τ, μ be as in [Na]. Then $\gamma(1)/\eta(1) = \tau(1)/(\mu \times \eta)(1)$ divides $|E/(N \times Q)| = e(b)$. Since $e(b)$ is a $\{p,q\}'$-number, the result follows.

Now put $H_1 = \langle x, K \rangle$. Then $H_1 = C_{H_1}(Q)K$ by the preceding paragraph. We see that $C_{H_1}(Q)/C_K(Q)$ and $K/C_K(Q)$ have relatively prime orders and $C_{H_1}(Q)/C_K(Q) \lhd H_1/C_K(Q)$. Let Ω be the set of all extensions of the canonical character θ to $C_{H_1}(Q)$. Note that Ω is not empty since θ is $C_{H_1}(Q)$-invariant and $C_{H_1}(Q)/C_K(Q)$ is cyclic. Applying Glauberman's lemma (see [Is2, Lemma 13.8]) to the action of $K/C_K(Q)$ on Ω and the transitive action (multiplication) of the character group of $C_{H_1}(Q)/C_K(Q)$ on Ω, we see that there is an extension of θ

to $C_{H_1}(Q)$ which is $K/C_K(Q)$-invariant. This implies $C_{H_1}(Q) = C_{H_1}(Q)_\omega$. Thus $H_1 = H_1[b]$ by Theorem 2.5. Hence $x \in H[b]$, which is a contradiction. Thus $S^0_H(b)/H[b]$ is a $\pi(e(b))$-group. The proof is complete. □

The following generalizes [Na, Theorems A, B]; for an analogous result, see Proposition 5.4 below.

COROLLARY 4.9

Suppose that $G = A \ltimes K$ for a subgroup A with $(|A|, |N_K(Q)/C_K(Q)|) = 1$. Let b be an A-invariant block of K with an A-invariant defect group Q. Assume that the AM conjecture is true around (K, b) in G. Then the following are equivalent.

(i) *A centralizes Q.*
(ii) *A fixes all irreducible characters in b.*
(iii) *A fixes all irreducible characters of height zero in b.*

Proof

(i) ⇒ (ii). Since $(|A|, |N_K(Q)/QC_K(Q)|) = 1$, this follows from [Wa, Proposition 1] (see also [Mu4, Theorem B']).

(ii) ⇒ (iii). This is trivial.

(iii) ⇒ (i). We claim that $G = G[b]$. We first consider the case where $|N_K(Q)/C_K(Q)|$ is a multiple of p. Then by assumption $|A|$ is prime to p. Since $G = S^0_G(b)$, we have $\pi(G/G[b]) \subseteq (\{p\} \cup \pi(e(b))) \cap \pi(A)$ by Theorem 4.8. So $\pi(G/G[b])$ is empty by assumption, and $G = G[b]$. Next, assume that $|N_K(Q)/C_K(Q)|$ is prime to p. Then Q is abelian. Since $G = S^0_G(b)$, by Corollary 4.3 and Theorem 4.8 we have $\pi(G/G[b]) \subseteq \pi(e(b)) \cap \pi(A) = \emptyset$ by assumption. So $G = G[b]$. The claim is proved. Therefore $G = C_G(Q)K$ by Theorem 2.5. Then $N_G(Q) = C_G(Q)N_K(Q)$. Hence $|N_G(Q)/C_G(Q)| = |N_K(Q)/C_K(Q)|$ is prime to $|A|$ by assumption. Since $A \le N_G(Q)$, we obtain $A \le C_G(Q)$. The proof is complete. □

Next we determine a defect group of a block of $S^0_G(b)$, which may be compared to Theorem 2.1(ii). We begin with the following, which follows from the proof of the Harris–Knörr theorem [HaKn, Theorem].

LEMMA 4.10

Let N be a normal subgroup of G. Let β be a block of N with a defect group S. Let B be a block of G covering β. Let D be a defect group of B with $D \cap N = S$. Let L be a subgroup of G with $L \ge N_G(S)$. Let $\tilde{B}$ be the Brauer correspondent of B with respect to D in L. Let $\tilde{\beta}$ be the Brauer correspondent of β with respect to S in $L \cap N$. Then $\tilde{B}$ covers $\tilde{\beta}$.

THEOREM 4.11

Assume that the AM conjecture is true around (K, b) in G. Assume that b is G-invariant. Let B and B' be blocks of G and $S^0_G(b)$, respectively, such that B

covers B' and B' covers b. (Note that $S_G^0(b) \lhd G$.) We choose defect groups S and D of B' and B, respectively, so that $Q \le S \le D$. Then $S = C_D(Q/Q')$, where Q' is the commutator subgroup of Q.

Proof
Since $D \cap S_G^0(b) = S$ and $S \cap K = Q$, we have $N_G(D) \le N_G(S) \le N_G(Q) = H$. Let $\tilde{B}$ be the Brauer correspondent of B with respect to D in H. Noting that $S_G^0(b) \cap H = S_H^0(\tilde{b})$ by Lemma 4.2, let $\tilde{B}'$ be the Brauer correspondent of B' with respect to S in $S_H^0(\tilde{b})$. By Lemma 4.10, $\tilde{B}$ covers $\tilde{B}'$. By the proof of the Harris–Knörr theorem $\tilde{B}'$ covers $\tilde{b}$.

Let $\bar{H} = H/Q'$. We use the bar convention. As in the proof of Theorem 4.1, let $\bar{B}$ (resp., $\bar{B}', \bar{b}$) be a unique block of $\bar{H}$ (resp., $S_{\bar{H}}^0(\bar{b}), \bar{\tilde{K}}$) dominated by $\tilde{B}$ (resp., $\tilde{B}', \tilde{b}$). (Note that $S_{\bar{H}}^0(\bar{b}) = \overline{S_H^0(\tilde{b})}$; see the proof of Theorem 4.1.) Also, $\bar{D}$ (resp., $\bar{S}, \bar{Q}$) is a defect group of $\bar{B}$ (resp., $\bar{B}', \bar{b}$). Then $\bar{B}$ covers $\bar{B}'$, and $\bar{B}'$ covers $\bar{b}$. Since b is G-invariant, $\tilde{b}$ is H-invariant. Therefore $\bar{b}$ is $\bar{H}$-invariant. Now, since $S_{\bar{H}}^0(\bar{b})/\bar{H}[\bar{b}]$ is a p'-group by Corollary 4.3, we have $\bar{S} \le \bar{H}[\bar{b}]$. Since $\bar{b}$ is $\bar{H}$-invariant, we see that there is a block β of $\bar{H}[\bar{b}]$ such that $\bar{B}'$ covers β, β covers $\bar{b}$, and $\bar{S}$ is a defect group of β. Since $\bar{B}$ covers $\bar{B}'$, $\bar{B}$ covers β. (Note that $\bar{H}[\bar{b}] \lhd \bar{H}$.) Thus, by Theorem 2.1, $\bar{S} = \bar{Q}C_{\bar{D}}(\bar{Q}) = C_{\bar{D}}(\bar{Q})$. Hence $S = C_D(Q/Q')$. The proof is complete. □

Put $\mathrm{Aut}_c^0(Q) = \{\sigma \in \mathrm{Aut}(Q) \mid [Q/Q', \sigma] = 1\}$ and $\mathrm{Out}_c^0(Q) = \mathrm{Aut}_c^0(Q)/\mathrm{Inn}(Q)$. As is well known, $\mathrm{Out}_c^0(Q)$ is a p-group. The following extends Corollary 4.3.

PROPOSITION 4.12
Assume that the AM conjecture is true around (K, b) in G.

(i) *A Sylow p-subgroup of $S_G^0(b)/G[b]$ is isomorphic to $S_{H_\beta}^0(\beta)/H_\beta[\beta]$.*
(ii) *A Sylow p-subgroup of $S_G^0(b)/G[b]$ is isomorphic to a subgroup of $\mathrm{Out}_c^0(Q)$.*

Proof
We may assume $G = H_\beta$. Use Convention 4.6. Then, since $b = \beta^K$, b is H-invariant. Let B be a weakly regular block of H covering b with a defect group D. Then $D \rhd Q$ and $|H : DK|$ is prime to p. We choose a block B' of $S_H^0(b)$ such that B covers B' and $D \cap S_H^0(b) := D_1$ is a defect group of B'. Next we choose a block B'' of $H[b]$ such that B' covers B'' and $D_1 \cap H[b] := D_2$ is a defect group of B''. Let $M/H[b]$ be a normal Sylow p-subgroup of $S_H^0(b)/H[b]$. Then $|M : M \cap H[b]D| = |MD : H[b]D|$ is prime to p. (Note that M and $H[b]$ are normal subgroups of H.) Hence $M = M \cap H[b]D = H[b](M \cap D)$. Now $M \cap D = M \cap S_H^0(b) \cap D = M \cap D_1 = D_1$. So

$$
\begin{aligned}
M/H[b] &= H[b](M \cap D)/H[b] \simeq M \cap D/M \cap D \cap H[b] \\
&= D_1/D_2 \\
&= C_D(Q/Q')/QC_D(Q)
\end{aligned}
$$

by Theorems 2.1 and 4.11. Since b is a unique block of K covering β, B is also a weakly regular block of H covering β. Therefore we obtain similarly that a Sylow p-subgroup of $S^0_H(\beta)/H[\beta]$ is isomorphic to $C_D(Q/Q')/QC_D(Q)$.

We claim $S^0_H(\beta)/H[\beta]$ is isomorphic to a subgroup of $\mathrm{Out}^0_c(Q)$. First we show that $H[\beta] = QC_H(Q)$. Put $L = QC_H(Q)$. By Theorem 2.5, $H[\beta] \le L$. On the other hand, by Theorem 2.5 and Proposition 2.4, $L[\beta] = L_\beta$. Since $L = L_\beta$, we obtain $L = L[\beta] \le H[\beta]$. Thus $H[\beta] = QC_H(Q)$.

Let $x \in S^0_H(\beta)$. From the structure of the irreducible characters in β (see [NT, Theorem 5.8.14]), we see that x fixes all linear characters of Q. Thus $[Q/Q', x] = 1$. Further, x induces an inner automorphism of Q if and only if x belongs to $QC_{H_\beta}(Q)$. Since $QC_{H_\beta}(Q) = H_\beta[\beta]$, the claim follows. Since $\mathrm{Out}^0_c(Q)$ is a p-group, so is $S^0_H(\beta)/H[\beta]$. Thus (i) and (ii) follow.

The proof is complete. □

REMARK 4.13

For any p-group Q and for any subgroup $\bar{A}$ of $\mathrm{Out}^0_c(Q)$, we consider the case where $G = A \ltimes Q$, which is the semidirect product with respect to the natural action of A on Q, where A is the preimage of $\bar{A}$ in $\mathrm{Aut}^0_c(Q)$. Let $K = Q$, and let b be the principal block of K. Then $G = S^0_G(b)$, and by Theorem 2.5 $G[b] = C_G(Q)Q = \mathrm{Inn}(Q) \ltimes Q$, so that $S^0_G(b)/G[b] \simeq \bar{A}$. Therefore we cannot improve Proposition 4.12(ii) in general.

5. Consequences of the AM conjecture, II: Height zero irreducible characters in b and the defect group

In this section we assume b is a G-invariant block of K. Let B be a block of G covering b, and let D be a defect group of B containing the defect group Q of b.

We have considered the following conditions in [Mu1].

(I) Every irreducible character in b of height zero is D-invariant.

(II) Every D-invariant irreducible character in b of height zero extends to DK.

We have proved the following (see [Mu1, Proposition 4.11]), which clarifies the meaning of the condition (II) completely.

LEMMA 5.1

The following conditions are equivalent.

(i) *Every D-invariant irreducible character in b of height zero extends to DK;*

(ii) *every D-invariant linear character of Q extends to D;*

(iii) $[D, Q] = D' \cap Q$.

The following clarifies the condition (I) completely (under the AM conjecture).

PROPOSITION 5.2

Assume that the AM conjecture is true around (K,b) *in* DK*. Then the following conditions are equivalent.*

(i) *Every irreducible character in* b *of height zero is* D*-invariant;*
(ii) *Every linear character of* Q *is* D*-invariant;*
(iii) $W[D,Q] \leq Q'$.

Proof

Since D is a defect group of a unique block of DK covering b ([Mu1, Lemma 2.2]), we may assume $G = DK$. We can choose a block B' of $S_G^0(b)$ such that B covers B' and B' covers b and that for a defect group S of B' it holds that $S = D \cap S_G^0(b)$. Then by Theorem 4.11, $S = C_D(Q/Q')$.

(i) $\Rightarrow$ (iii): Condition (i) yields $D = C_D(Q/Q')$, and (iii) holds.

(iii) $\Rightarrow$ (i): Condition (iii) yields $S = D$. Hence $D \leq S_G^0(b)$, and (i) holds.

(ii) $\Leftrightarrow$ (iii): This is easy to see. □

The following clarifies the meaning of a certain condition which is stronger than (II) (under the AM conjecture).

PROPOSITION 5.3

Assume that the AM conjecture is true around (K,b) *in* DK*. Then the following conditions are equivalent.*

(i) *Every irreducible character in* b *of height* 0 *extends to* DK*;*
(ii) *Every linear character of* Q *extends to* D*;*
(iii) $D' \cap Q = Q'$.

Proof

This follows from Lemma 5.1 and Proposition 5.2. □

We give two applications of Proposition 5.2. The following deals with a situation analogous to Corollary 4.9.

PROPOSITION 5.4

Suppose that $G = A \ltimes K$ *for a subgroup* A *with* $(|A|, |N_K(Q)/QC_K(Q)|) = 1$*. Let* b *be an* A*-invariant block of* K *with an* A*-invariant defect group* Q*. Assume that the AM conjecture is true around* (K,b) *in* G*. Then the following are equivalent.*

(i) A *centralizes* Q/Q'*.*
(ii) A *fixes all irreducible characters of height zero in* b*.*

Proof

We may assume that A is a cyclic q-group for a prime q. If $q \neq p$, then condition (i) is equivalent to the condition that A centralizes Q. So the equivalence follows from Corollary 4.9. Assume $q = p$. We may assume $A \neq 1$. So $N_K(Q)/QC_K(Q)$ is a p'-group by assumption. Let D be a defect group of a unique block of G covering b such that $D \geq Q$. So $D \triangleright Q$. Since b is G-invariant and G/K is a p-group, we have $G = DK$. Since $G = AK = DK$, A fixes all irreducible characters of height zero in b if and only if D fixes all irreducible characters of height zero in b. So, by Proposition 5.2, it suffices to show that A centralizes Q/Q' if and only if D centralizes Q/Q'.

Put $C = C_{N_G(Q)}(Q/Q')$. Clearly $C \triangleleft N_G(Q)$. Assume that A centralizes Q/Q'. Then, since $G = AK$ and $A \leq N_G(Q)$, we have $N_G(Q) = AN_K(Q) = CN_K(Q)$. So $N_G(Q)/C \simeq N_K(Q)/N_K(Q) \cap C$. Since $QC_K(Q) \leq N_K(Q) \cap C$, we see that $N_G(Q)/C$ is a p'-group. Thus $D \leq C$. So D centralizes Q/Q'. The converse is proved in a similar way. The proof is complete. □

The following is a relative version of the Gluck–Wolf theorem [GW, Theorem B]; letting $N = 1$, we recover their theorem.

PROPOSITION 5.5

Let N be a normal subgroup of G. Let b_0 be a block of N with a defect group Q_0. Assume that Q_0 is normal in N and that G/N is p-solvable. Let B be a block of G covering b_0. Let D be a defect group of B. If any irreducible character in B has height zero, then D is abelian.

Proof

We argue by induction on $|G/N|$. If $G = N$, the result follows by Reynolds [Re, Theorem 9]. Assume $G \neq N$, and let K/N be a maximal normal subgroup of G/N. If G/K is a p'-group, then the result follows by induction. Assume that G/K is of order p. Let b be a block of K which is covered by B and covers b_0. By the Fong–Reynolds theorem, we may assume b is G-invariant. Then, since $\mathrm{d}(B) - \mathrm{d}(b) = 1$, any irreducible character in b has height zero and D-invariant. Also, $Q = D \cap K$ is a defect group of b. So Q is abelian by induction. By Theorem 3.3, the AM conjecture is true for B and b. Thus, by Proposition 5.2, we have $[D, Q] \leq Q' = 1$. Hence $Q \leq Z(D)$. Since D/Q is cyclic (of order p), we obtain that D is abelian. The proof is complete. □

6. Automorphisms centralizing a Sylow p-subgroup

Glauberman [Gl] proves the following.

THEOREM 6.1 (GLAUBERMAN [GL, THEOREM 1])

Let G be a group with $\mathrm{O}_{2'}(G) = 1$. Let P be a Sylow 2-subgroup of G. Then $C_{\mathrm{Aut}(G)}(P) = C_1 \rtimes C_2$, where C_1 is a group of odd order and C_2 is an abelian 2-group.

For odd primes, Gross [Gr] proves the following two theorems.

THEOREM 6.2 (GROSS [Gr, THEOREM A])

Let p be an odd prime. Let G be a group with $O_{p'}(G) = 1$. Let P be a Sylow p-subgroup of G. Then

(i) $C_{\mathrm{Aut}(G)}(P) = C_1 \times C_2$, *where C_1 is a p'-group and C_2 is an abelian p-group.*

(ii) *If $O_p(G) = 1$, then $C_2 \le \mathrm{Inn}(G)$; that is, $C_2 = \mathrm{conj}(Z(P))$.*

THEOREM 6.3 (GROSS [Gr, THEOREM B])

Let G be a simple group, and let p be an odd prime dividing the order of G. We identify G with $\mathrm{Inn}(G)$. Let S be a Sylow p-subgroup of $\mathrm{Aut}(G)$ and $P = S \cap G$. Then $C_S(P) = Z(P)$.

Gross proves Theorem 6.2 by using Theorem 6.3. In this section we give an alternative proof, which is straightforward.

LEMMA 6.4

Let P be a Sylow p-subgroup of G. Put $C = C_{\mathrm{Aut}(G)}(P)$. Then

(i) $[\mathrm{conj}(P), C] = 1$;

(ii) $\mathrm{conj}(Z(P))$ *is a central subgroup of C.*

Proof

(i) Let $u \in P$ and $\sigma \in C$. Then $\{\mathrm{conj}(u)\}^{\sigma} = \mathrm{conj}(u^{\sigma}) = \mathrm{conj}(u)$.

(ii) Clearly $\mathrm{conj}(Z(P)) \le C$. (i) yields $\mathrm{conj}(Z(P)) \le Z(C)$. □

LEMMA 6.5

Let p be any prime. Let P be a Sylow p-subgroup of G. The following are equivalent.

(i) $\mathrm{Out}_{p\text{-Col}}(G)$ *is a p'-group;*

(ii) *any p-element of $C_{\mathrm{Aut}(G)}(P)$ is inner;*

(iii.a) $C_{\mathrm{Aut}(G)}(P) = C_1 \times C_2$, *where C_1 is a p'-group and C_2 is an abelian p-group; and*

(iii.b) $C_2 = \mathrm{conj}(Z(P))$.

Proof

(i) $\Leftrightarrow$ (ii). Since $\mathrm{Aut}_{p\text{-Col}}(G) = C_{\mathrm{Aut}(G)}(P)\,\mathrm{Inn}(G)$, this is clear.

(ii) $\Rightarrow$ (iii). Put $C = C_{\mathrm{Aut}(G)}(P)$. Let $\sigma \in C$ be a p-element. Then $\sigma = \mathrm{conj}(x)$ for some $x \in G$ by assumption. Then $x \in C_G(P)$. Since $C_G(P)$ is a direct product of $Z(P)$ and a p'-group, we obtain $\sigma \in \mathrm{conj}(Z(P))$. On the other hand, $\mathrm{conj}(Z(P)) \le Z(C)$ by Lemma 6.4. Thus $\mathrm{conj}(Z(P))$ is a central Sylow p-subgroup of C, and the result follows.

(iii) $\Rightarrow$ (ii). This is obvious. The proof is complete. □

PROPOSITION 6.6

If p is an odd prime, $\mathrm{O}_{p'}(G)=1$*, and* $G=F^*(G)$*, then* $\mathrm{Out}_{p\text{-Col}}(G)$ *is a* p'*-group.*

Proof

We argue by induction on $|G|$. Since $\mathrm{O}_{p'}(G)=1$, $G=\mathrm{O}_p(G)E(G)$. Let P be a Sylow p-subgroup of G. Let $\sigma\in C_{\mathrm{Aut}(G)}(P)$ be a p-element. It suffices to show that σ is inner by Lemma 6.5. We may assume that p divides the order of G, since otherwise $G=1$ and the conclusion is trivial. Assume $G\neq E(G)$. By induction, there is $x\in E(G)$ such that $e^\sigma=e^x$ for all $e\in E(G)$. Since $[\mathrm{O}_p(G),\sigma]=1$ and $[\mathrm{O}_p(G),E(G)]=1$, we have $(ue)^\sigma=ue^x=(ue)^x$ for all $u\in\mathrm{O}_p(G)$ and $e\in E(G)$. So $\sigma=\mathrm{conj}(x)$, and σ is inner. Assume $G=E(G)$. Put $G=Q_1\cdots Q_r$ $(r\geq 1)$, where Q_i is a component of G for each i. Put $P_i=P\cap Q_i$ for each i. Fix i. Since $\mathrm{O}_{p'}(G)=1$, Q_i is not a p'-group. So $P_i\neq 1$. If $P_i\leq Z(Q_i)$, then Q_i is a direct product of P_i and a p'-group. Then $Q_i'<Q_i$, a contradiction. So $P_i\not\leq Z(Q_i)$. Put $Q_i^\sigma=Q_j$. Then $P_i=P_i^\sigma\leq Q_i\cap Q_j$. Since $P_i\not\leq Z(Q_i)$, we obtain $i=j$. So each Q_i is σ-invariant. Assume $r>1$. Then by induction, $\sigma|_{Q_i}$ is inner for each i. Thus σ is inner. Assume that G is quasi-simple but not simple. If $G/Z(G)$ is a p'-group, then $G'<G$ (as above), a contradiction. So p divides $|G/Z(G)|$. By induction, there is $x\in G$ such that $\bar{g}^\sigma=\bar{g}^{\bar{x}}$ for all $\bar{g}\in G/Z(G)$. Put $\rho=\sigma\,\mathrm{conj}(x^{-1})$. Then $[G,\rho]\leq Z(G)$. By the three subgroup lemma, $[G,G,\rho]=1$. So $[G,\rho]=1$. Thus $\rho=1$ and σ is inner. Finally assume that G is simple. We identify G with $\mathrm{Inn}(G)$. Let R be a Sylow p-subgroup of $C_{\mathrm{Aut}(G)}(P)$. Let S be a Sylow p-subgroup of $\mathrm{Aut}(G)$ containing RP. Then $R\leq C_S(P)=Z(P)$ by Theorem 6.3. So $R\leq G$. Thus $\mathrm{Out}_{p\text{-Col}}(G)$ is a p'-group by Lemma 6.5. The proof is complete. □

We say a group G has the p-Gross property if for a Sylow p-subgroup P of G, $C_{\mathrm{Aut}(G)}(P)$ is a direct product of an abelian p-group and a p'-group; equivalently, $C_{\mathrm{Aut}(G)}(P)$ has a central Sylow p-subgroup.

THEOREM 6.7

Let p be any prime. Let G be a group with $\mathrm{O}_{p'}(G)=1$*. Assume that for any component Q of G (if any),* $Q/Z(Q)$ *has the p-Gross property. Then G has the p-Gross property.*

Proof

Put $F=F^*(G)$. Let P be a Sylow p-subgroup of G, and put $C=C_{\mathrm{Aut}(G)}(P)$.

We first consider the case where $G=F$. Then $G=\mathrm{O}_p(G)Q_1\cdots Q_r$, where Q_i is a component of G for each i. Let $\alpha,\beta\in C$. Assume that α is a p-element. As in the proof of Proposition 6.6, $Q_i^\alpha=Q_i=Q_i^\beta$ for each i. Assume that the conclusion holds for quasisimple groups. Then $\alpha^\beta|_{Q_i}=(\alpha|_{Q_i})^{\beta|_{Q_i}}=\alpha|_{Q_i}$ for each i. Since $\alpha^\beta|_{\mathrm{O}_p(G)}=\alpha|_{\mathrm{O}_p(G)}$, we obtain $\alpha^\beta=\alpha$. Thus we may assume that G is quasisimple. By assumption $\alpha^{-1}\alpha^\beta$ acts trivially on $G/Z(G)$. Put $\varphi(g)=g^{-1}g^{[\alpha,\beta]}$. Then $\varphi:G\to Z(G)$ is a group homomorphism. Since $G=G'$, it is a trivial homomorphism. Thus $[\alpha,\beta]=1$. So G has the p-Gross property.

For a general G, we first show that C is a direct product of a p-group and a p'-group. Let $\alpha, \beta \in C$. Assume that α is a p-element. By the preceding paragraph, $C_{\mathrm{Aut}(F)}(P \cap F)$ is a direct product of an abelian p-group and a p'-group. Hence $\alpha^{\beta}|_F = (\alpha|_F)^{\beta|_F} = \alpha|_F$. Put $\sigma = \alpha^{-1}\alpha^{\beta}$. Then $[F, \sigma] = 1$ and $[P, \sigma] = 1$. Put $\varphi(g) = g^{-1}g^{\sigma}$ for $g \in G$. By the three subgroup lemma, $\varphi(g) \in Z(F)$, and $\varphi: G \to Z(F)$ is a 1-cocycle. Since $Z(F)$ is an abelian p-group and $\varphi(u) = 1$ for all $u \in P$, φ is a 1-coboundary (see [Gr, Lemma 2.3], [Hu, I.16.18]). It follows that $\sigma = \mathrm{conj}(x)$ for some $x \in Z(F)$. Then $x \in Z(F) \cap C_G(P) \leq P \cap C_G(P) = Z(P)$. Now by Lemma 6.4, $\mathrm{conj}(Z(P))$ is a central p-subgroup of C and the above shows $\alpha^{-1}\alpha^{\beta} \equiv 1 \mod \mathrm{conj}(Z(P))$. It follows that $C/\mathrm{conj}(Z(P))$ is a direct product of an abelian p-group and a p'-group. By elementary group theory, then C is a direct product of a p-group, say, C_p, and a p'-group.

It remains to show that C_p is abelian. If $p = 2$, then C_p is abelian by Theorem 6.1. Assume that p is odd. Let α, β be p-elements of C. By Proposition 6.6, $\mathrm{Out}_{p\text{-Col}}(F)$ is a p'-group. So by Lemma 6.5 there are $f, f_1 \in F$ such that $\alpha|_F = \mathrm{conj}(f)|_F$ and $\beta|_F = \mathrm{conj}(f_1)|_F$ with $f, f_1 \in Z(P \cap F)$. Put $\sigma = \alpha\,\mathrm{conj}(f^{-1})$ and $\rho = \beta\,\mathrm{conj}(f_1^{-1})$. Then $\sigma\rho = \rho\sigma$. Indeed, since $[F, \sigma] = 1$, we have $[G, \sigma, F] = 1$ by the three subgroup lemma. Thus $[G, \sigma] \leq C_G(F) = Z(F)$. So for any $g \in G$, $g^{\sigma} = g f_{\sigma}(g)$ for $f_{\sigma}(g) \in Z(F)$. Likewise, $g^{\rho} = g f_{\rho}(g)$ for $f_{\rho}(g) \in Z(F)$. Then, since $Z(F) \leq P \cap F \leq P$, $f_{\sigma}(g)^{\rho} = f_{\sigma}(g)^{f_1^{-1}} = f_{\sigma}(g)$. So $g^{\sigma\rho} = g f_{\rho}(g) f_{\sigma}(g)$. Therefore $g^{\sigma\rho} = g^{\rho\sigma}$, and $\sigma\rho = \rho\sigma$. Then

$$\alpha\,\mathrm{conj}(f^{-1})\beta\,\mathrm{conj}(f_1^{-1}) = \beta\,\mathrm{conj}(f_1^{-1})\alpha\,\mathrm{conj}(f^{-1}).$$

Here $\mathrm{conj}(f^{-1})\beta = \beta\,\mathrm{conj}(f^{-1})$ by Lemma 6.4. Likewise, $\mathrm{conj}(f_1^{-1})\alpha = \alpha\,\mathrm{conj}(f_1^{-1})$. Therefore $\alpha\beta = \beta\alpha$. Thus C_p is abelian.

The proof is complete. □

Theorem 6.7 shows that Glauberman's theorem (Theorem 6.1) can be strengthened in certain cases.

COROLLARY 6.8

Let G be a group with $\mathrm{O}_{2'}(G) = 1$. Assume that the simple factor group of any component of G (if any) is either a group of Lie type of characteristic 2 or a sporadic simple group. Then $C_{\mathrm{Aut}(G)}(P)$ is a direct product of an abelian 2-group and a group of odd order, where P is a Sylow 2-subgroup of G.

Proof

Let Q be a component of G. Any automorphism of $Q/Z(Q)$ centralizing a Sylow 2-subgroup is inner by [HeKi, Theorem 13, proof of Theorem 14]. So we can apply Theorem 6.7. □

Proof of Theorem 6.2

(i) For any component Q of G, $\mathrm{O}_{p'}(Q/Z(Q)) = 1$, compare the proof of Proposition 6.6. So $\mathrm{Out}_{p\text{-Col}}(Q/Z(Q))$ is a p'-group by Proposition 6.6. Then by

Lemma 6.5, $Q/Z(Q)$ has the p-Gross property. Thus G has the p-Gross property by Theorem 6.7. The proof is complete. □

By Lemma 6.5, Theorem 6.2(ii) is equivalent to the following.

PROPOSITION 6.9

If p is an odd prime and $\mathrm{O}_{p'}(G) = \mathrm{O}_p(G) = 1$, then $\mathrm{Out}_{p\text{-Col}}(G)$ is a p'-group.

Proof

Let $\sigma \in C_{\mathrm{Aut}(G)}(P)$ be a p-element, where P is a Sylow p-subgroup of G. We must show that σ is inner. Put $F = F^*(G)$. By Proposition 6.6, there is an $f \in F$ such that $\sigma|_F = \mathrm{conj}(f)|_F$. Put $\rho = \sigma\,\mathrm{conj}(f^{-1})$. Then $[F, \rho] = 1$. So $[G, \rho, F] = 1$ by the three subgroup lemma. So $[G, \rho] \leq C_G(F) = Z(F) = 1$. Thus $\rho = 1$, and σ is inner. The proof is complete. □

Gross [Gr, p. 203] conjectures that the following is true whenever p is odd.

(G_p) If $\mathrm{O}_{p'}(G) = 1$, then any p-element of $C_{\mathrm{Aut}(G)}(P)$ is inner, where P is a Sylow p-subgroup of G; that is, $\mathrm{Out}_{p\text{-Col}}(G)$ is a p'-group.

By Theorem 6.2(ii) and Proposition 6.6, (G_p) is true if p is odd and either $\mathrm{O}_p(G) = 1$ or $G = F^*(G)$. Concerning this conjecture, we have the following.

PROPOSITION 6.10

If p is an odd prime and $\mathrm{O}_{p'}(G) = 1$, then $\mathrm{Out}_{p\text{-Col}}(G)/\mathrm{Out}_{\mathrm{Col}}(G)$ is a p'-group.

Proof

Let P be a Sylow p-subgroup of G. Let α be a p-element of $C_{\mathrm{Aut}(G)}(P)$. We must show $\alpha \in \mathrm{Aut}_{\mathrm{Col}}(G)$. Put $F = F^*(G)$. By Proposition 6.6 and Lemma 6.5, there is $f \in F$ such that $\alpha|_F = \mathrm{conj}(f)|_F$ with $f \in Z(P \cap F)$. Put $\sigma = \alpha\,\mathrm{conj}(f^{-1})$. Then $[F, \sigma] = 1$. So $[G, \sigma] \leq Z$ by the three subgroup lemma, where $Z = Z(F)$. Of course $\sigma \in \mathrm{Aut}_{p\text{-Col}}(G)$. Let q be a prime distinct from p. Let Q be a Sylow q-subgroup of G. Then $QZ = Q \ltimes Z$ is σ-invariant. So there is a $z \in Z$ such that $Q^\sigma = Q^z$. So for any $a \in Q$, there is $b \in Q$ such that $a^\sigma = b^z$. Further, there is $w \in Z$ such that $a^\sigma = aw$. Then $aw = b(z^{-1})^b z$. So $a = b$. Thus $a^\sigma = a^z$. So $\sigma \in \mathrm{Aut}_{q\text{-Col}}(G)$. Thus $\alpha \in \mathrm{Aut}_{\mathrm{Col}}(G)$. The proof is complete. □

REMARK 6.11

Hertweck and Kimmerle [HeKi, Question 2, p.213] ask whether $\mathrm{Out}_{\mathrm{Col}}(G/\mathrm{O}_{p'}(G)) = 1$ for any group G and any prime p. If the answer to this question is affirmative, then Proposition 6.10 yields that the Gross conjecture is true.

7. The structure of $\mathrm{Aut}_c(G) \cap C_{\mathrm{Aut}(G)}(P)$

In this section we prove Theorem C.

THEOREM 7.1

Let S be a nonabelian simple group. If p is a prime not dividing the order of S, then $\mathrm{Out}(S)$ is p-nilpotent and has a cyclic Sylow p-subgroup.

Proof

This is a consequence of the classification theorem of finite simple groups (see [GLS, Theorems 7.1.1, 7.1.2]). □

The following lemma will be used several times.

LEMMA 7.2

Let Y be a normal subgroup of a group X such that $[Y, X]$ is a p'-group.

(i) *If X/Y is p-nilpotent, then so is X.*
(ii) *If X/Y has p-length at most one, then so does X.*

Proof

Let $Z = \mathrm{O}_{p'}(Y)$, and let $U/Y = \mathrm{O}_{p'}(X/Y)$. Since $Y/[Y, X]$ is abelian, Y is p-nilpotent. So $Z \geq [Y, X]$, and Y/Z is a central Sylow p-subgroup of U/Z. Hence U/Z is a direct product of Y/Z and a subgroup V/Z. Then V is a normal p'-subgroup of X.

(i) If X/Y is p-nilpotent, then X/V is a p-group. So X is p-nilpotent.

(ii) If X/Y has p-length at most one, then X/V is p-closed. So X has p-length at most one.

The proof is complete. □

We need modular representation theory for the proof of the following proposition.

PROPOSITION 7.3

Let M be a minimal normal subgroup of G. Assume that M is an elementary abelian q-group for a prime q with $q \neq p$. Let $\mathrm{Inn}(G) \leq H \leq \mathrm{Aut}_c(G)$. Let $\varphi : \mathrm{Aut}_c(G) \to \mathrm{Aut}_c(G/M)$ and $\psi : \mathrm{Aut}_c(G/M) \to \mathrm{Out}_c(G/M)$ be the natural maps. Then we have the following.

(i) *If $\psi\varphi(H)$ is p-nilpotent, then so is $H/\mathrm{Inn}(G)$.*
(ii) *If $\psi\varphi(H)$ has p-length at most one, then so does $H/\mathrm{Inn}(G)$.*

Proof

First we note that φ is well defined (since M is $\mathrm{Aut}_c(G)$-invariant). We regard M as an irreducible $GF(q)G$-module. Let $H \ltimes G$ be the semidirect product with respect to the natural action of H on G.

Then M is naturally extended to an irreducible $GF(q)[H \ltimes G]$-module. Indeed, it suffices to define $m \cdot (\sigma g) := (m^\sigma)^g$ for $m \in M, \sigma \in H, g \in G$. Let F

be the algebraic closure of $GF(q)$. Then $F \otimes_{GF(q)} M$ is an $F[H \ltimes G]$-module. We have $(F \otimes_{GF(q)} M)_G = \bigoplus_i V_i$, where V_i are nonisomorphic absolutely irreducible FG-modules (see [NT, Theorem 3.1.32(iii)]). Let $\sigma \in H$. If β_i is the Brauer character of V_i, then $\beta_i^\sigma = \beta_i$, since $\sigma \in \mathrm{Aut}_c(G)$. Here β_i^σ is the Brauer character of $V_i \cdot \sigma$. Hence $V_i \cdot \sigma \simeq V_i$. Since $V_i \cdot \sigma = V_j$ for some j (see [NT, Theorem 1.7.3(ii)]), we obtain $V_i \cdot \sigma = V_i$. Let $\alpha : H \cap \mathrm{Ker}\,\varphi \to \mathrm{Aut}(M)$ be the action of $H \cap \mathrm{Ker}\,\varphi$ on M. Let $\tau \in H \cap \mathrm{Ker}\,\varphi$. Then for any $g \in G$, $[g, \tau] \in M$. Therefore $m \cdot (\tau g) = m \cdot (g\tau)$ for all $m \in M$. So $v \cdot (\tau g) = v \cdot (g\tau)$ for any $v \in V_i$ for any i. So the restriction of τ to V_i is a scalar for any i by Schur's lemma. Thus $\alpha([\sigma, \tau]) = 1$. Hence H acts trivially on $(H \cap \mathrm{Ker}\,\varphi)\,\mathrm{Inn}(G)/\,\mathrm{Ker}\,\alpha\,\mathrm{Inn}(G)$. It is easy to see that $\mathrm{Ker}\,\alpha$ is a q-group. (Indeed, let $\rho \in \mathrm{Ker}\,\alpha$. For any $g \in G$, $g^\rho = gm$ for some $m \in M$. Since $m^\rho = m$, we obtain $g^{\rho^q} = gm^q = g$. So $\rho^q = 1$.) Put $X = H/\,\mathrm{Inn}(G)$ and $Y = (H \cap \mathrm{Ker}\,\varphi)\,\mathrm{Inn}(G)/\,\mathrm{Inn}(G)$. Then $X/Y \simeq H/(H \cap \mathrm{Ker}\,\varphi)\,\mathrm{Inn}(G) \simeq \psi\varphi(H)$ and $[Y, X] \le \mathrm{Ker}\,\alpha\,\mathrm{Inn}(G)/\,\mathrm{Inn}(G)$ is a p'-group. Thus applying Lemma 7.2, we obtain the assertion. The proof is complete. □

COROLLARY 7.4

Let $G = L \ltimes A$ be a semidirect product such that A is a solvable p'-group.

(i) *If $\mathrm{Out}_c(L)$ is p-nilpotent, then so is $\mathrm{Out}_c(G)$.*
(ii) *If $\mathrm{Out}_c(L)$ has p-length at most one, then so does $\mathrm{Out}_c(G)$.*

Proof

We argue by induction on $|A|$. If $A = 1$, the assertion is trivial. Assume $A > 1$, and let $M \le A$ be a minimal normal subgroup of G.

(i) By induction $\mathrm{Out}_c(G/M) \simeq \mathrm{Out}_c(L \ltimes (A/M))$ is p-nilpotent. By applying Proposition 7.3 with $H = \mathrm{Aut}_c(G)$ we see that $\mathrm{Out}_c(G)$ is p-nilpotent.

(ii) The proof is similar to that of (i). □

LEMMA 7.5

Let P be a Sylow p-subgroup of G. The following are equivalent:

(i) $\mathrm{Aut}_c(G) \cap C_{\mathrm{Aut}(G)}(P)$ *is p-nilpotent;*
(ii) $\mathrm{Out}_c(G) \cap \mathrm{Out}_{p\text{-Col}}(G)$ *is p-nilpotent.*

Proof

Since $\mathrm{Aut}_{p\text{-Col}}(G) = C_{\mathrm{Aut}(G)}(P)\,\mathrm{Inn}(G)$, we have

$$\mathrm{Out}_c(G) \cap \mathrm{Out}_{p\text{-Col}}(G) \simeq \mathrm{Aut}_c(G) \cap C_{\mathrm{Aut}(G)}(P)/\,\mathrm{conj}\big(C_G(P)\big).$$

(i) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (i). We see that $\mathrm{conj}(C_G(P))$ is a direct product of $\mathrm{conj}(Z(P))$ and a p'-group. Since $\mathrm{conj}(Z(P))$ is central in $\mathrm{Aut}_c(G) \cap C_{\mathrm{Aut}(G)}(P)$ by Lemma 6.4, the result follows by Lemma 7.2. □

We need the following.

THEOREM 7.6 (GLAUBERMAN AND GROSS)
If $\mathrm{O}_{p'}(G) = 1$*, then* $\mathrm{Out}_{p\text{-Col}}(G)$ *is* p*-nilpotent and has an abelian Sylow* p*-subgroup.*

Proof
For a Sylow p-subgroup P of G, we have $\mathrm{Out}_{p\text{-Col}}(G) = C_{\mathrm{Aut}(G)}(P)\,\mathrm{Inn}(G)/\mathrm{Inn}(G)$. Therefore the result follows from Theorem 6.1 when $p=2$ and Theorem 6.2 when p is odd. □

THEOREM 7.7
Let P *be a Sylow* p*-subgroup of* G*. Then* $\mathrm{Aut}_c(G) \cap C_{\mathrm{Aut}(G)}(P)$ *is* p*-nilpotent.*

Proof
We argue by induction on the order of G. If $G=1$, the statement is trivial. Assume $G \neq 1$. Put $B_p(G) = \mathrm{Aut}_c(G) \cap \mathrm{Aut}_{p\text{-Col}}(G)$. Let $\bar{B}_p(G) = B_p(G)/\mathrm{Inn}(G) = \mathrm{Out}_c(G) \cap \mathrm{Out}_{p\text{-Col}}(G)$. By Lemma 7.5 it suffices to show that $\bar{B}_p(G)$ is p-nilpotent. We divide the proof into several steps.

Step 1. We may assume $\mathrm{O}_{p'}(G) \neq 1$.

Proof
If $\mathrm{O}_{p'}(G) = 1$, then $\mathrm{Out}_{p\text{-Col}}(G)$ is p-nilpotent by Theorem 7.6. So the result follows. □

Step 2. We may assume that G has a unique minimal normal subgroup.

Proof
Let N_1 and N_2 be two distinct minimal normal subgroups of G. The natural maps $\varphi_i : G \to G/N_i$ $(i=1,2)$ induce $\bar{\varphi}_i : B_p(G) \to \bar{B}_p(G/N_i)$. Consider $\bar{\varphi}_1 \times \bar{\varphi}_2 : B_p(G) \to \bar{B}_p(G/N_1) \times \bar{B}_p(G/N_2)$. We claim that $\mathrm{Ker}(\bar{\varphi}_1 \times \bar{\varphi}_2)/\mathrm{Inn}(G) \leq Z(\bar{B}_p(G))$. Let $\sigma \in \mathrm{Ker}(\bar{\varphi}_1 \times \bar{\varphi}_2)$. Let $\rho \in B_p(G)$. For $i=1,2$, there is $y_i \in G$ such that $x^\sigma \equiv x^{y_i} \mod N_i$ for all $x \in G$. Then $x^{y_1 y_2^{-1}} \equiv x \mod N_1N_2$ for all $x \in G$. Thus $y_1y_2^{-1}N_1N_2 \in Z(G/N_1N_2)$. Therefore $(y_1y_2^{-1})^\rho \equiv y_1y_2^{-1} \mod N_1N_2$. Hence we can write $(y_1y_2^{-1})^\rho = y_1n_1n_2y_2^{-1}$ for $n_i \in N_i$ $(i=1,2)$. Put $n_1^{-1}y_1^{-1}y_1^\rho = n_2y_2^{-1}y_2^\rho =: w$. For any $x \in G$, $x^{\rho^{-1}\sigma\rho} \equiv ((x^{\rho^{-1}})^{y_1})^\rho \equiv x^{y_1^\rho} \mod N_1$. So $x^{[\sigma,\rho]} \equiv (x^{\sigma^{-1}})^{y_1^\rho} \equiv x^{y_1^{-1}y_1^\rho} \equiv x^{n_1w} \equiv x^w \mod N_1$. Similarly $x^{[\sigma,\rho]} \equiv x^w \mod N_2$. Since $N_1 \cap N_2 = 1$, we obtain $x^{[\sigma,\rho]} = x^w$. Thus $[\sigma,\rho] \in \mathrm{Inn}(G)$. The claim is proved. By induction, we obtain that $B_p(G)/\mathrm{Ker}(\bar{\varphi}_1 \times \bar{\varphi}_2)$ is p-nilpotent. Hence $\bar{B}_p(G)$ is p-nilpotent by the claim and Lemma 7.2. □

Step 3. We may assume $F(G) = 1$.

Proof
By Steps 1 and 2, we obtain $\mathrm{O}_p(G) = 1$. Assume that $F(G)_{p'}$, the Hall

p'-subgroup of $F(G)$, is nontrivial. Choose a minimal normal subgroup M contained in $F(G)_{p'}$. Then M is an elementary abelian q-group for a prime $q \neq p$. Let $\varphi : \mathrm{Aut}_c(G) \to \mathrm{Aut}_c(G/M)$ and $\psi : \mathrm{Aut}_c(G/M) \to \mathrm{Out}_c(G/M)$ be the natural maps. Then $\psi\varphi(B_p(G)) \leq \bar{B}_p(G/M)$ is p-nilpotent by induction. Hence $\bar{B}_p(G)$ is p-nilpotent by Proposition 7.3. □

Step 4. We have $F^*(G) = S_1 \times \cdots \times S_r$ $(r \geq 1)$, where S_i $(1 \leq i \leq r)$ are nonabelian simple groups. S_i $(1 \leq i \leq r)$ are G-conjugate, and $F^*(G)$ is a p'-group.

Proof
Since $G \neq 1$, by Step 3, $F^*(G) = E(G) \neq 1$ and $F^*(G)$ is a direct product of nonabelian simple groups S_i $(1 \leq i \leq r)$, $r \geq 1$. By Step 2, S_i $(1 \leq i \leq r)$ are G-conjugate, and $F^*(G)$ is a (unique) minimal normal subgroup of G. Then, since $\mathrm{O}_{p'}(G) \neq 1$ by Step 1, we have $\mathrm{O}_{p'}(G) \geq F^*(G)$, so that $F^*(G)$ is a p'-group.

Put $F^*(G) = F^*$. By Step 4, we may identify F^* with $S \times \cdots \times S$ (r factors) with S_i with ith factor $(1 \leq i \leq r)$. Since F^* is a characteristic subgroup of G, the restriction defines a homomorphism $f : B_p(G) \to \mathrm{Aut}(F^*)$. We have $\mathrm{Aut}(F^*) = \Sigma_r \ltimes A$, where Σ_r is the symmetric group on $\{1, 2, \ldots, r\}$ and $A = \mathrm{Aut}(S) \times \cdots \times \mathrm{Aut}(S)$ (r-factors). For $x \in F^*$, we write $x = (x_i)$ to mean that the ith component of x is $x_i \in S$ for $1 \leq i \leq r$; $\pi \in \Sigma_r$ acts on F^* as $x^\pi = (x_{\pi(i)})$.

Likewise $a \in A$ is denoted by $a = (a_i), a_i \in \mathrm{Aut}(S)$, and $x^a = (x_i^{a_i})$. For $\sigma \in B_p(G)$, put $f(\sigma) = \pi(\sigma)a(\sigma)$ with $\pi(\sigma) \in \Sigma_r$ and $a(\sigma) \in A$.

For $\sigma, \rho \in B_p(G)$,

(1) $\pi(\sigma\rho)\pi(\sigma)\pi(\rho)$; that is, $\pi : B_p(G) \to \Sigma_r$ is a homomorphism;
(2) $a(\sigma\rho) = a(\sigma)^{\pi(\rho)}a(\rho)$;
(3) $S_i^\sigma = S_{\pi(\sigma)^{-1}(i)}$ for any i.

Put $\bar{G} = G/F^*$. The natural map $\varphi_0 : G \to \bar{G}$ induces $\varphi : B_p(G) \to B_p(\bar{G})$. Recall that $\mathrm{conj} : G \to \mathrm{Inn}(G)(\leq B_p(G))$ is the natural map induced by conjugation. □

Step 5. $B_p(G)$ acts trivially on $\mathrm{Ker}\,\varphi\,\mathrm{Inn}(G)/(\mathrm{Ker}\,\varphi \cap f^{-1}(A))\,\mathrm{Inn}(G)$.

Proof
It suffices to show that $B_p(G)$ acts trivially on $\mathrm{Ker}\,\varphi/\mathrm{Ker}\,\varphi \cap f^{-1}(A)$. We first claim that $\pi(\mathrm{Ker}\,\varphi)$ acts semiregularly on $\{1, 2, \ldots, r\}$. Indeed, assume $\pi(\tau)(i) = i$ for some i and some $\tau \in \mathrm{Ker}\,\varphi$. Then $S_i^\tau = S_i$ by (3). For any j, there is $g \in G$ such that $S_j = S_i^g$ by Step 4. Then, since $g^\tau g^{-1} \in F^*$, we have $S_j^\tau = (S_i^g)^\tau = (S_i^\tau)^{g^\tau} = ((S_i)^{g^\tau g^{-1}})^g = S_i^g = S_j$. Hence $\pi(\tau) = 1$ by (3), and the claim is proved.

Let $\tau \in \mathrm{Ker}\,\varphi$ and $\sigma \in B_p(G)$. We show that $\pi([\tau, \sigma]) = 1$. This holds when $\pi(\tau) = 1$. Thus we may assume $\pi(\tau) \neq 1$. Then $\pi(\tau)(1) \neq 1$ by the claim. Choose $x \in F^*$ so that $x_{\pi(\sigma)(1)}$ and $x_{\pi(\sigma)\pi(\tau)(1)}$ have distinct orders $\neq 1$ and all other x_i equal 1. (This is possible since S is not of prime power order.) We have

$x^\sigma = (x_{\pi(\sigma)(i)}^{a(\sigma)_i})$. Since $\sigma \in \mathrm{Aut}_c(G)$, there is $g \in G$ such that $x^\sigma = x^g$. We have $x^g = (x_{\pi(\mathrm{conj}(g))(i)}^{a(\mathrm{conj}(g))_i})$. Hence, by the choice of x, $\pi(\sigma)(1) = \pi(\mathrm{conj}(g))(1)$ and $\pi(\sigma)\pi(\tau)(1) = \pi(\mathrm{conj}(g))\pi(\tau)(1)$. Further, since $g^{-1}g^\tau \in F^*$, $[\pi(\mathrm{conj}(g)), \pi(\tau)] = \pi([\mathrm{conj}(g), \tau]) = \pi(\mathrm{conj}(g^{-1}g^\tau)) = 1$. Then $\pi(\tau)\pi(\sigma)(1) = \pi(\tau)\pi(\mathrm{conj}(g))(1) = \pi(\mathrm{conj}(g))\pi(\tau)(1) = \pi(\sigma)\pi(\tau)(1)$. So $\pi([\tau,\sigma])(1) = 1$. Since $[\tau,\sigma] \in \mathrm{Ker}\,\varphi$, $\pi([\tau,\sigma]) = 1$ by the claim. Therefore $f([\tau,\sigma]) \in A$ and $[\tau,\sigma] \in \mathrm{Ker}\,\varphi \cap f^{-1}(A)$, as required.

Since S is a p'-group by Step 4, by Theorem 7.1, $\mathrm{Aut}(S)/\mathrm{Inn}(S)$ has a normal p-complement $H_0/\mathrm{Inn}(S)$ such that $\mathrm{Aut}(S)/H_0$ is a cyclic p-group. Put $H = H_0 \times \cdots \times H_0$ (r-factors) $\leq A$. Since $H/\mathrm{Inn}(F^*)$ is a normal p-complement of $A/\mathrm{Inn}(F^*)$, H is normal in $\mathrm{Aut}(F^*)$, so that $f^{-1}(H)$ is normal in $B_p(G)$. □

Step 6. $B_p(G)$ acts trivially on

$$\big(\mathrm{Ker}\,\varphi \cap f^{-1}(A)\big)\,\mathrm{Inn}(G)/\big(\mathrm{Ker}\,\varphi \cap f^{-1}(H)\big)\,\mathrm{Inn}(G).$$

Proof

It suffices to show that $B_p(G)$ acts trivially on $\mathrm{Ker}\,\varphi \cap f^{-1}(A)/\mathrm{Ker}\,\varphi \cap f^{-1}(H)$. Let $\tau \in \mathrm{Ker}\,\varphi \cap f^{-1}(A)$. For any $g \in G$, we obtain by (2),

$$a\big(\mathrm{conj}(g^{-1}g^\tau)\big) = a\big(\mathrm{conj}(g)\big)^{-1}\big(a(\tau)^{-1}\big)^{\pi(\mathrm{conj}(g))}a\big(\mathrm{conj}(g)\big)a(\tau)$$

since $\pi(\tau) = 1$. Therefore,

$$a\big(\mathrm{conj}(g^{-1}g^\tau)\big)_i = \big(a\big(\mathrm{conj}(g)\big)^{-1}\big)_i\big(a(\tau)^{-1}_{\pi(\mathrm{conj}(g))(i)}\big)a\big(\mathrm{conj}(g)\big)_i a(\tau)_i$$

for any i. Since $g^{-1}g^\tau \in F^*$, $a(\mathrm{conj}(g^{-1}g^\tau))_i \in \mathrm{Inn}(S)$. Since $\mathrm{Aut}(S)/H_0$ is abelian, we obtain $1 \equiv a(\tau)^{-1}_{\pi(\mathrm{conj}(g))(i)}a(\tau)_i \mod H_0$. Thus $a(\tau)_{\pi(\mathrm{conj}(g))(i)} \equiv a(\tau)_i \mod H_0$. Since $\pi(\mathrm{Inn}(G))$ acts transitively on $\{1, 2, \ldots, r\}$ by Step 4 and (3), we have $a(\tau)_i \equiv a(\tau)_1 \mod H_0$ for any i.

Let $\sigma \in B_p(G)$. We have $f(\tau) = a(\tau), f(\sigma^{-1}\tau\sigma) = a(\sigma^{-1}\tau\sigma)$ since $\tau, \sigma^{-1}\tau\sigma \in f^{-1}(A)$. Since $\pi(\tau) = 1, a(\sigma^{-1}\tau\sigma) = a(\sigma)^{-1}a(\tau)^{\pi(\sigma)}a(\sigma)$ by (2). Hence for any i, $a(\sigma^{-1}\tau\sigma)_i = a(\sigma)_i^{-1}a(\tau)_{\pi(\sigma)(i)}a(\sigma)_i$. Thus $a(\sigma^{-1}\tau\sigma)_i \equiv a(\tau)_{\pi(\sigma)(i)} \mod H_0$. Hence $a(\sigma^{-1}\tau\sigma)_i \equiv a(\tau)_1 \mod H_0$. Therefore, for any i, $a(\sigma^{-1}\tau\sigma)_i \equiv a(\tau)_i \mod H_0$. This implies $f(\sigma^{-1}\tau\sigma) \equiv f(\tau) \mod H$. Thus $[\tau,\sigma] \in \mathrm{Ker}\,\varphi \cap f^{-1}(H)$, as required. □

Step 7. $\mathrm{Ker}\,\varphi \cap f^{-1}(H)$ is a p'-group.

Proof

We show that f is a monomorphism. Let $\sigma \in \mathrm{Ker}\,f$. Then $[F^*,\sigma] = 1$. So $[F^*,\sigma,G] = [G,F^*,\sigma] = 1$. Therefore $[G,\sigma,F^*] = 1$ by the three subgroup lemma. So $[G,\sigma] \leq C_G(F^*) = Z(F^*) = 1$. Hence $\sigma = 1$. Since f induces an isomorphism of $\mathrm{conj}(F^*)$ onto $\mathrm{Inn}(F^*)$, we obtain $f^{-1}(\mathrm{Inn}(F^*)) = \mathrm{conj}(F^*)$. Thus $\mathrm{Ker}\,\varphi \cap$

$f^{-1}(H)/\operatorname{conj}(F^*)$ is isomorphic to a subgroup of $H/\operatorname{Inn}(F^*)$. So $\operatorname{Ker}\varphi \cap f^{-1}(H)/\operatorname{conj}(F^*)$ is a p'-group. Since $\operatorname{conj}(F^*)$ is a p'-group, the result follows. □

Step 8. Conclusion.

Proof
We see that $B_p(G)/\operatorname{Ker}\varphi\operatorname{Inn}(G) = B_p(G)/\varphi^{-1}(\operatorname{Inn}(\bar{G}))$ is isomorphic to a subgroup of $B_p(\bar{G})/\operatorname{Inn}(\bar{G})$. So $B_p(G)/\operatorname{Ker}\varphi\operatorname{Inn}(G)$ is p-nilpotent by induction. Then by Step 5 and Lemma 7.2, we see that $B_p(G)/(\operatorname{Ker}\varphi \cap f^{-1}(A))\operatorname{Inn}(G)$ is p-nilpotent. Then by Step 6 and Lemma 7.2, $B_p(G)/(\operatorname{Ker}\varphi \cap f^{-1}(H))\operatorname{Inn}(G)$ is p-nilpotent. By Step 7, $(\operatorname{Ker}\varphi \cap f^{-1}(H))\operatorname{Inn}(G)/\operatorname{Inn}(G)$ is a p'-group. Therefore $\bar{B}_p(G)$ is p-nilpotent. The proof is complete. □

□

REMARK 7.8
When $p = 2$, Theorem 7.7 follows from the Glauberman theorem (Theorem 6.1 above) and the Feit–Thompson theorem, and we do not need the classification theorem of finite simple groups.

8. Consequences of the AM conjecture, III: The structures of $S_G^0(b)/G[b]$ and $\operatorname{Out}_c(K)$ (general cases)

In this section we use the notation in Section 4.

LEMMA 8.1
Let X be a (solvable) group with p-length at most one for all primes p. Then the nilpotent length (Fitting length) of X is at most $|\pi(X)|$.

Proof
By Alperin's theorem (see [Hu, Satz 6.14, p. 695]), there is a group Y with the following properties.

X is isomorphic to a subgroup of Y, $\pi(X) = \pi(Y)$, Y is a product of normal subgroups $\{Y_i\}$, and each Y_i has a Sylow tower.

Let $l(X)$ be the nilpotent length of X (see [Su, Exercise 11, p. 118] for the definition of nilpotent length). Then $l(X) \le l(Y) \le \max_i l(Y_i) \le \max_i |\pi(Y_i)| \le |\pi(Y)| = |\pi(X)|$. The proof is complete. □

The following lemma gives a relationship between $S_G^0(b)/G[b]$ and class-preserving outer automorphism groups in one direction; the other direction is given in Lemma 8.6 below.

For any group X and any prime q, let $b_q(X)$ be the principal q-block of X. Let $\operatorname{Aut}(X) \ltimes X$ be the semidirect product with respect to the natural action of $\operatorname{Aut}(X)$ on X.

LEMMA 8.2

Let K be any group. Let $G = \mathrm{Aut}(K) \ltimes K$. Let $b_p = b_p(K)$. Then

(i) $\mathrm{Aut}_c(K) \cap G[b_p] = \mathrm{Aut}_c(K) \cap \mathrm{Aut}_{p\text{-Col}}(K)$;

(ii) $\mathrm{Out}_c(K)/\,\mathrm{Out}_c(K) \cap \mathrm{Out}_{p\text{-Col}}(K)$ *is isomorphic to a subgroup of* $S_G^0(b_p)/G[b_p]$.

Proof

By Theorem 2.5, $G[b_p] = C_G(Q)K$, where Q is a Sylow p-subgroup of K. We claim that $C_G(Q)K = \mathrm{Aut}_{p\text{-Col}}(K)K$. Let $\sigma k \in C_G(Q)$, where $\sigma \in \mathrm{Aut}(K)$ and $k \in K$. Then for all $u \in Q, u^\sigma = u^{k^{-1}}$. So $\sigma \in \mathrm{Aut}_{p\text{-Col}}(K)$, and $C_G(Q) \leq \mathrm{Aut}_{p\text{-Col}}(K)K$. Likewise we see that $\mathrm{Aut}_{p\text{-Col}}(K) \leq C_G(Q)K$. Thus the claim follows. Then $\mathrm{Aut}(K) \cap G[b_p] = \mathrm{Aut}_{p\text{-Col}}(K)$, and $\mathrm{Aut}_c(K) \cap G[b_p] = \mathrm{Aut}_c(K) \cap \mathrm{Aut}_{p\text{-Col}}(K)$. So (i) is proved. Clearly $\mathrm{Aut}_c(K) \leq S_G^0(b_p)$. Thus $\mathrm{Aut}_c(K)/\,\mathrm{Aut}_c(K) \cap \mathrm{Aut}_{p\text{-Col}}(K)$ is isomorphic to a subgroup of $S_G^0(b_p)/G[b_p]$. Since $\mathrm{Aut}_c(K)/\,\mathrm{Aut}_c(K) \cap \mathrm{Aut}_{p\text{-Col}}(K) \simeq \mathrm{Out}_c(K)/\,\mathrm{Out}_c(K) \cap \mathrm{Out}_{p\text{-Col}}(K)$, (ii) follows. □

We say that the AM conjecture is true around (K, b) if, whenever L is a group containing K as a normal subgroup such that L/K is cyclic, the AM conjecture is true for any block of L covering b. (The case where $(L, B) = (K, b)$ is included.)

THEOREM 8.3

Let K be any group. We assume that the AM conjecture is true around (K, b_p) for all primes p, where $b_p = b_p(K)$. Then $\mathrm{Out}_c(K)$ has p-length at most one for all primes p. In particular, $\mathrm{Out}_c(K)$ is solvable and the nilpotent length of $\mathrm{Out}_c(K)$ is at most $|\pi(K)|$.

Proof

Let $G = \mathrm{Aut}(K) \ltimes K$ be the semidirect product. By Lemma 8.2, $\mathrm{Out}_c(K)/\,\mathrm{Out}_c(K) \cap \mathrm{Out}_{p\text{-Col}}(K)$ is isomorphic to a subgroup of $S_G^0(b_p)/G[b_p]$, which is p-closed by Theorem 4.1. On the other hand, $\mathrm{Out}_c(K) \cap \mathrm{Out}_{p\text{-Col}}(K)$ is p-nilpotent by Theorem 7.7. Therefore $\mathrm{Out}_c(K)$ has p-length at most one. By Lemma 8.1 the nilpotent length of $\mathrm{Out}_c(K)$ is at most $|\pi(\mathrm{Out}_c(K))|$. Since $\pi(\mathrm{Out}_c(K)) \subseteq \pi(K)$ by [HeKi, Proposition 1], the result follows. □

The following strengthens [Sa, Theorem 2.9].

COROLLARY 8.4

(i) *If K is p-solvable, then $\mathrm{Out}_c(K)$ has p-length at most one.*

(ii) *If K is solvable, then the conclusion of Theorem 8.3 holds.*

Proof

(i) This follows from Corollary 4.4 and the proof of Theorem 8.3.

(ii) This follows from (i). □

REMARK 8.5

In [Bu, Note B], Burnside stated that $\mathrm{Out}_c(K)$ was abelian. But in [Sa], Sah found a counterexample to this statement. As a substitute for this statement he "proved" that $\mathrm{Out}_c(K)$ was solvable under the Schreier conjecture (see [Sa, Theorem 2.10] and the remark after that theorem). However, his proof of Theorem 2.10 is incorrect. Indeed, in the last part of the proof of Theorem 2.10 (in the notation there) he asserts that $I/I \cap A_i(G)$ is isomorphic to a section of $A(M_1)/A_i(M_1)$. But the homomorphism theorem yields only that $I/(I \cap A_i(G))\operatorname{Ker}\varphi$ is isomorphic to a section of $A(M_1)/A_i(M_1)$, where $\varphi : I \to A(M_1)$ is the natural homomorphism. Since it is not obvious that $\operatorname{Ker}\varphi$ is contained in $I \cap A_i(G)$, we cannot conclude that $A_c(G)/A_i(G)$ is solvable under the Schreier conjecture. To show that his assertion is illegitimate, let us consider an example. We use our notation for automorphisms. Let $G = H \times L$ for nonabelian simple groups H and L such that $\mathrm{Out}(L) = 1$ (e.g., $L = M_{11}$, the Mathieu group of degree 11). Take L as $M = M_1$ in the proof of Theorem 2.10. If the above assertion were true, we could conclude that $\mathrm{Out}_c(H) = \mathrm{Out}_c(G) = 1$. This conclusion is in fact true, but it is a theorem of Feit and Seitz [FS, Theorem C] and its proof needs detailed analysis based on the classification theorem of finite simple groups. We conclude that Sah misused the homomorphism theorem. Thus Theorem 2.10 remains unproved, and by Theorem 8.3 the solvability of $\mathrm{Out}_c(G)$ is now a consequence of the AM conjecture.

Additional remark. A better counterexample to Sah's assertion would be obtained if we could take H such that $\mathrm{Out}_c(H) \neq 1$ and (since we are in case 2 of the proof of Theorem 2.10) that the Fitting subgroup $F(H)$ is trivial. (L is the same as above.) Is there such a group H?

LEMMA 8.6

Assume that $H = H_\beta$, Q is abelian, and assume that the canonical character of β extends to K. Then $S^0_H(b)/H[b]$ is isomorphic to a section of $\mathrm{Out}_c(K/C_K(Q) \ltimes Q)$.

Proof

We use the argument in the proof of Theorem 4.8. Let $E = K/Q \ltimes Q$ be the natural semidirect product. For the canonical character θ of β, let $\bar\theta$ be the corresponding character of $N = C_K(Q)/Q$. We consider $S^0_H(b)$ to be an operator group acting on K. Then by [Na, Theorem 2.2] (see also the proof of [Na, Theorem 3.2]), there is a bijection of $\mathrm{Irr}(b)$ onto $\mathrm{Irr}(E \mid \bar\theta)$, which commutes with the action of $S^0_H(b)$. Since $S^0_H(b)$ acts trivially on $\mathrm{Irr}(b)$ (as all irreducible characters in b have height zero), $S^0_H(b)$ acts trivially on $\mathrm{Irr}(E \mid \bar\theta)$. By assumption $\bar\theta$ has an extension $\hat\theta$ to $K/Q \ltimes Q$. Regard $\hat\theta$ as a character of K/Q, and then inflate it to E. We write this character $\hat\theta$ again. Then $\hat\theta$ is an extension of $\bar\theta$ to E. Hence any irreducible character χ in $\mathrm{Irr}(E \mid \bar\theta)$ is written as $\chi = \hat\theta \otimes \zeta$ with a unique ζ in $\mathrm{Irr}(E/N) = \mathrm{Irr}(K/C_K(Q) \ltimes Q)$. Then the map sending χ to ζ gives a bijection of $\mathrm{Irr}(E \mid \bar{\theta})$ onto $\mathrm{Irr}(K/C_K(Q) \ltimes Q)$. Since $S_H^0(b)$ acts trivially on $\mathrm{Irr}(E \mid \bar{\theta})$, $S_H^0(b)$ acts trivially on $\mathrm{Irr}(K/C_K(Q) \ltimes Q)$. Thus we obtain a homomorphism $\varphi : S_H^0(b) \to \mathrm{Aut}_c(K/C_K(Q) \ltimes Q)$. Let $\alpha : \mathrm{Aut}_c(K/C_K(Q) \ltimes Q) \to \mathrm{Out}_c(K/C_K(Q) \ltimes Q)$ be the natural map.

We claim that $\mathrm{Ker}\,\alpha\varphi \leq H[b]$. Let $x \in \mathrm{Ker}\,\alpha\varphi$. There are $\bar{y} \in K/C_K(Q)$ and $u \in Q$ such that for any $v \in Q$, it holds that $v^x = v^{\bar{y}u} = v^y$. This implies $x \in C_H(Q)K$. Thus $\mathrm{Ker}\,\alpha\varphi \leq S_H^0(b) \cap C_H(Q)K$. We show that $S_H^0(b) \cap C_H(Q)K \leq H[b]$. Since $S_H^0(b) \cap C_H(Q)K = (S_H^0(b) \cap C_H(Q))K$, it suffices to show that $S_H^0(b) \cap C_H(Q) \leq H[b]$. Let $x \in S_H^0(b) \cap C_H(Q)$. As above let $\hat{\theta}$ be an extension of θ to K. Since $\hat{\theta}$ is $\langle x, K\rangle$-invariant, there is an extension $\tilde{\theta}$ of $\hat{\theta}$ to $\langle x, K\rangle$. Then $\tilde{\theta}_{\langle x, C_K(Q)\rangle}$ is a K-invariant extension of θ to $\langle x, C_K(Q)\rangle$. Therefore $x \in C_H(Q)_\omega \leq H[b]$, as asserted. Thus the claim follows. Then the result follows. The proof is complete. □

PROPOSITION 8.7

Assume that the AM conjecture is true around (K,b) in G. Assume that for any factor group S of the inertial quotient group of b and for any prime q, the AM conjecture is true around $(S, b_q(S))$. If the canonical character of β is extendible to the inertial group of β in $N_K(Q)$, then the following hold:

(i) *$S_G^0(b)/G[b]$ has q-length at most one for any prime q;*

(ii) *$S_G^0(b)/G[b]$ is solvable, and the nilpotent length of $S_G^0(b)/G[b]$ is at most $|\pi(e(b))| + 1$.*

Proof

(i) We may assume $G = H_\beta$. Use Convention 4.6. When $q = p$, $S_H^0(b)/H[b]$ is p-closed. Assume $q \neq p$. It suffices to show that a p-complement of $S_H^0(b)/H[b]$ has q-length at most one. Let the notation be as in Lemma 4.7. By Lemma 4.7(iii) and (iv), instead of (H, K, b, Q, β) it suffices to consider $(\bar{H}, \bar{K}, \bar{b}, \bar{Q}, \bar{\beta})$, which we denote from here by (H, K, b, Q, β). So our new (H, K, b, Q, β) satisfies the assumption of Lemma 8.6. By Lemma 8.6, $S_H^0(b)/H[b]$ is isomorphic to a section of $\mathrm{Out}_c(K/C_K(Q) \ltimes Q)$. Here $K/C_K(Q)$ is a factor group of the inertial quotient group of the original block b. So, by assumption and Theorem 8.3, $\mathrm{Out}_c(K/C_K(Q))$ has q-length at most one. Then $\mathrm{Out}_c(K/C_K(Q) \ltimes Q)$ has q-length at most one by Corollary 7.4. So $S_H^0(b)/H[b]$ has q-length at most one.

(ii) By (i) and Lemma 8.1, the nilpotent length of $S_G^0(b)/G[b]$ is not greater than $|\pi(S_G^0(b)/G[b])|$. By Theorem 4.8, $|\pi(S_G^0(b)/G[b])| \leq |\pi(e(b))| + 1$. The proof is complete. □

Since the AM conjecture is true for solvable groups we obtain the following.

COROLLARY 8.8

Assume that the canonical character of β is extendible to the inertial group of β in $N_K(Q)$. If K is solvable, then the conclusion of Proposition 8.7 holds.

We are not able to answer the following question.

QUESTION 8.9
Is $S_G^0(b)/G[b]$ solvable?

The answer is affirmative at least in the following cases under the AM conjecture.

(i) The inertial index of b is odd (by Theorems 4.1, 4.8, and the Feit–Thompson theorem); for example, K has odd order or $p=2$.

(ii) The canonical character of β is extendible to the inertial group of β in $N_K(Q)$ (by Proposition 8.7); for example, b is the principal block of K, or the inertial quotient group of b is cyclic (in particular, Q is cyclic); if Q is cyclic, the group is in fact trivial (see Proposition 11.2).

9. Coleman automorphisms of finite groups

Hertweck and Kimmerle [HeKi] have proved the following.

THEOREM 9.1
$\mathrm{Out}_{\mathrm{Col}}(G)$ *is abelian for any group* G.

This is a positive solution to Dade's conjecture in [Da7]. They have shown this theorem by using

(1) Glauberman's theorem (Theorem 6.1 above);
(2) Gross' theorem (Theorem 6.2 above); and
(3) Dade's theorem stating that $\mathrm{Out}_{\mathrm{Col}}(G)$ is nilpotent [Da7].

In this section we show that Dade's theorem (3) can be proved by using (1), (2), and some modifications of results in [HeKi]. The point is that we can use the classification theorem of finite simple groups (this theorem is needed for the proof of (2)), while Dade did not. We shall prove a more general result (for a possible generalization of Theorems 9.1, 9.2; see Remark 10.4 below).

THEOREM 9.2
$[\mathrm{Out}_{\mathrm{Col}}(G), \mathrm{Out}_{p\text{-}\mathrm{Col}}(G), \mathrm{Out}_{c,p}(G)]$ *is a* p'*-group for any prime* p *and any group* G.

For the proof we need the following.

PROPOSITION 9.3
Let L *be a normal subgroup of* G. *Let* σ *be an element of* p*-power order in* $\mathrm{Aut}(G)$. *Let* ρ *be an element of* $\mathrm{Aut}_{c,p}(G)$. *Assume that* $L^\sigma = L^\rho = L$ *and that* σ *acts trivially on* G/L.

(i) *If* σ *belongs to* $\mathrm{Aut}_{q\text{-}\mathrm{Col}}(G)$ *for a prime* q *(possibly* $q=p$*), then* $[\sigma,\rho]|_L$ *belongs to* $\mathrm{Aut}_{q\text{-}\mathrm{Col}}(L)$.

(ii) *If* σ *belongs to* $\mathrm{Aut}_{\mathrm{Col}}(G)$, *then* $[\sigma,\rho]|_L$ *belongs to* $\mathrm{Aut}_{\mathrm{Col}}(L)$.

Proof
Clearly (ii) follows from (i). So we prove (i). Let Q be a Sylow q-subgroup of L. There is $x \in G$ such that $u^\sigma = u^x$ for all $u \in Q$. Since $G = N_G(Q)L$ by the Frattini argument, we can write $x = nl_1$ with $n \in N_G(Q)$ and $l_1 \in L$. Put $\alpha = \sigma\,\mathrm{conj}(l_1^{-1})$. Then $u^\alpha = u^n$ for all $u \in Q$. There is $l_2 \in L$ such that $Q^\rho = Q^{l_2}$ by Sylow's theorem. Put $\beta = \rho\,\mathrm{conj}(l_2^{-1})$. Then $Q^\beta = Q$. Noting that $C_G(Q)L$ is normal in G, put $\bar{G} = G/C_G(Q)L$. By [HeKi, Lemma 6], $g^\sigma \equiv g^x \mod C_G(Q)L$ for all $g \in G$. On the other hand, $g^\sigma \equiv g \mod C_G(Q)L$ by assumption. Thus $\bar{x} \in Z(\bar{G})$. By [HeKi, Lemma 2], there is $l \in L$ such that for all $u \in Q, u^x = u^\sigma = u^{lx_p}$. Then $x_{p'} \in C_G(Q)L$. It follows that $\bar{x}$ is a central p-element of $\bar{G}$. Since $Q^\beta = Q$ and $L^\beta = L$, $C_G(Q)L$ is β-invariant. So β induces an automorphism $\bar{\beta}$ of $\bar{G}$. Then $\bar{\beta} \in \mathrm{Aut}_{c,p}(\bar{G})$. Therefore $\bar{x}^{\bar{\beta}} = \bar{x}$. Noting $\bar{x} = \bar{n}$, we have $\bar{n}^{\bar{\beta}} = \bar{n}$. Therefore we have $n^\beta n^{-1} = cl_3$ for $c \in C_G(Q)$ and $l_3 \in L$. We have, for all $u \in Q$, that $u^{\alpha\beta} = (u^n)^\beta = ((u^\beta)^{n^\beta n^{-1}})^n = ((u^\beta)^{l_3})^n$. Since $u^{\alpha\beta} \in Q$ and $n \in N_G(Q)$, we have $(u^\beta)^{l_3} \in Q$. Thus $u^{\alpha\beta\alpha^{-1}} = (u^\beta)^{l_3}$. Therefore $u^{\alpha\beta\alpha^{-1}\beta^{-1}} = u^{l_3^{\beta^{-1}}}$. If we put $A = \langle \alpha, \beta, \mathrm{conj}(L)\rangle = \langle \sigma, \rho, \mathrm{conj}(L)\rangle$, then $\alpha\beta\alpha^{-1}\beta^{-1} \in C_A(Q)\,\mathrm{conj}(L)$. We have that $C_A(Q)\,\mathrm{conj}(L)$ is normal in A, because Q and L are $\langle \alpha, \beta\rangle$-invariant. It follows that $[\alpha, \beta] \equiv 1 \mod C_A(Q)\,\mathrm{conj}(L)$. Hence $[\sigma, \rho] \equiv 1 \mod C_A(Q)\,\mathrm{conj}(L)$. Therefore there exists $m \in L$ such that $u^{[\sigma,\rho]} = u^m$ for all $u \in Q$. The proof is complete. □

Proof of Theorem 9.2.
Put $Y = \mathrm{Aut}_{\mathrm{Col}}(G)$ and $Y_p = \mathrm{Aut}_{p\text{-}\mathrm{Col}}(G)$. We have $Y \le Y_p \le \mathrm{Aut}_{c,p}(G)$. Since Y and Y_p are normal subgroups of $\mathrm{Aut}(G)$, Y and Y_p are normal subgroups of $\mathrm{Aut}_{c,p}(G)$. Put $\overline{\mathrm{Aut}_{c,p}(G)} = \mathrm{Aut}_{c,p}(G)/\mathrm{Inn}(G)$, and use the bar convention. Put $H = [Y, Y_p]\,\mathrm{Inn}(G)$. Then $\bar{H} \lhd \overline{\mathrm{Aut}_{c,p}(G)}$.

We claim that for any p-element $\bar{\sigma}$ of $\bar{H}$ and any element $\bar{\rho}$ of $\overline{\mathrm{Aut}_{c,p}(G)}$, $[\bar{\sigma}, \bar{\rho}]$ is a p'-element. Let $L = \mathrm{O}_{p'}(G)$ and $\bar{G} = G/L$. Let $\varphi : \mathrm{Aut}_{c,p}(G) \to \mathrm{Aut}_{c,p}(\bar{G})$ be the natural map. We have $\varphi(Y) \le \varphi(Y_p) \le \mathrm{Aut}_{p\text{-}\mathrm{Col}}(\bar{G})$. Since $\mathrm{O}_{p'}(\bar{G})$ is trivial, by Theorem 7.6, $\mathrm{Out}_{p\text{-}\mathrm{Col}}(\bar{G})$ is p-nilpotent with an abelian Sylow p-subgroup. Therefore we obtain that $\alpha\varphi([Y, Y_p]\,\mathrm{Inn}(G))$ is a p'-group, where $\alpha : \mathrm{Aut}(\bar{G}) \to \mathrm{Out}(\bar{G})$ is the natural map. Since $\mathrm{Ker}\,\alpha\varphi = \mathrm{Ker}\,\varphi\,\mathrm{Inn}(G)$, we see that

$$\alpha\varphi\big([Y, Y_p]\,\mathrm{Inn}(G)\big) \simeq [Y, Y_p]\,\mathrm{Inn}(G)/\big([Y, Y_p]\,\mathrm{Inn}(G) \cap \mathrm{Ker}\,\varphi\big)\,\mathrm{Inn}(G)$$

is a p'-group. Thus $\bar{\sigma} \in \overline{[Y, Y_p]\,\mathrm{Inn}(G) \cap \mathrm{Ker}\,\varphi}$, and we may take a preimage σ of $\bar{\sigma}$ as a p-element in $[Y, Y_p]\,\mathrm{Inn}(G) \cap \mathrm{Ker}\,\varphi$. Note that $[Y, Y_p]\,\mathrm{Inn}(G) \cap \mathrm{Ker}\,\varphi \le Y \cap \mathrm{Ker}\,\varphi$. Hence by Proposition 9.3, $[\sigma, \rho]|_L$ belongs to $\mathrm{Aut}_{\mathrm{Col}}(L)$. Thus $[\sigma, \rho]|_L$ is a p'-element by [HeKi, Proposition 1]. Since $[\sigma, \rho]$ acts trivially on G/L, it follows that $[\sigma, \rho]$ is a p'-element. (This is well known. We supply a proof for the convenience of the reader. Put $\tau = [\sigma, \rho]$. For a p'-integer n, $\tau^n|_L$ is the identity. Put $\alpha = \tau^n$. Then for any $g \in G$, $g^\alpha = gl$ for some $l \in L$. Since $l^\alpha = l$, we obtain $g^{\alpha^m} = gl^m = g$, where $m = |L|$. Thus $\tau^{mn} = \alpha^m$ is the identity. Since mn is a p'-integer, the result follows.) Thus the claim is proved.

Then $\bar{H}$ is p-nilpotent (with an abelian Sylow p-subgroup) by [Su, Theorem 5.2.8]. Now the claim shows that $\overline{[H, \mathrm{Aut}_{c,p}(G)]} \leq \mathrm{O}_{p'}(\bar{H})$. Thus the conclusion follows. The proof is complete. □

From Theorem 9.2 we have the following.

COROLLARY 9.4 (DADE [Da7, COROLLARY])

$\mathrm{Out}_{\mathrm{Col}}(G)$ is nilpotent.

Proof

Since $\mathrm{Out}_{\mathrm{Col}}(G) \leq \mathrm{Out}_{p\text{-}\mathrm{Col}}(G) \leq \mathrm{Out}_{c,p}(G)$, Theorem 9.2 yields that $[\mathrm{Out}_{\mathrm{Col}}(G), \mathrm{Out}_{\mathrm{Col}}(G), \mathrm{Out}_{\mathrm{Col}}(G)]$ is a p'-group for any prime p, so that it is a trivial group. Thus $\mathrm{Out}_{\mathrm{Col}}(G)$ is nilpotent (of class at most two). □

10. Automorphisms of groups with abelian Sylow p-subgroups

In this section we consider mainly automorphisms of groups with abelian Sylow p-subgroups.

For groups with abelian Sylow p-subgroups, (G_p) (in Section 6) is true. To show this we begin with a special case.

LEMMA 10.1

If G has an abelian Sylow p-subgroup, $G = F^*(G)$ and $\mathrm{O}_{p'}(G) = 1$, then $\mathrm{Out}_{p\text{-}\mathrm{Col}}(G)$ is a p'-group.

Proof

When p is odd, this follows from Proposition 6.6 (the assumption that G has an abelian Sylow p-subgroup is unnecessary). Let $p = 2$. By the same proof as that of Proposition 6.6, we may assume that G is simple. By the classification result, G is isomorphic to one of the following groups:

$$\mathrm{SL}(2, 2^n) \quad (n \geq 2), \qquad \mathrm{PSL}(2, q) \quad (q \equiv \pm 3 \mod 8),$$

$$J_1, \qquad \mathrm{Re}(3^{2n+1}) \quad (n \geq 1).$$

For the last two types, $\mathrm{Out}(G)$ has odd order, so $\mathrm{Out}_{2\text{-}\mathrm{Col}}(G)$ has odd order.

If $G = \mathrm{SL}(2, 2^n)$, then $\mathrm{Out}_{2\text{-}\mathrm{Col}}(G) = 1$ by [HeKi, Theorem 13].

If $G = \mathrm{PSL}(2, q)(q \equiv \pm 3 \mod 8)$, then we have $\mathrm{Aut}(G) \rhd \mathrm{PGL}(2, q) \rhd G$, where $\mathrm{Aut}(G)/\mathrm{PGL}(2, q)$ has odd order (see [Su, Chapter 6, (8.8)]). Let P be a Sylow 2-subgroup of G. If $\mathrm{Out}_{2\text{-}\mathrm{Col}}(G)$ had even order, then there would be a 2-group $D \geq P, |D| \geq 8, D \leq \mathrm{PGL}(2, q)$, and D centralizes P. But a Sylow 2-subgroup of $\mathrm{PGL}(2, q)$ is dihedral of order 8, a contradiction. The proof is complete. □

PROPOSITION 10.2

If G has an abelian Sylow p-subgroup and $\mathrm{O}_{p'}(G) = 1$, then $\mathrm{Out}_{p\text{-}\mathrm{Col}}(G)$ is a p'-group.

Proof
Let P be a Sylow p-subgroup of G. Let $u \in P$. Put $F = F^*(G)$. Then $\mathrm{conj}(u)|_F$ has p-power order and centralizes $P \cap F$, so that by Lemma 10.1, $\mathrm{conj}(u)|_F = \mathrm{conj}(f)|_F$ for some $f \in F$. Then $uf^{-1} \in C_G(F) = Z(F)$. Thus $u \in F$. So $P \leq F$. (If p is odd, this follows also from Gross [Gr, Theorem C].) Let $\sigma \in C_{\mathrm{Aut}(G)}(P)$ be a p-element. Then by Lemma 10.1, $\sigma|_F = \mathrm{conj}(f)|_F$ for some $f \in F$. Put $\rho = \sigma\,\mathrm{conj}(f^{-1})$. Since $\rho|_F$ is the identity, the three subgroup lemma yields $[G, \rho] \leq Z(F)$. Put $\varphi(g) = g^{-1}g^{\rho}, g \in G$. Then $\varphi : G \to Z(F)$ is a 1-cocycle. Since $Z(F)$ is an abelian p-group and $\varphi(u) = 1$ for all $u \in P$ (as $P \leq F$), φ is a 1-coboundary (see [Gr, Lemma 2.3], [Hu, I 16.18]). Hence ρ is inner, and σ is inner. The proof is complete. □

The following proposition shows that when the conclusion of (G_p) is true for $G/\mathrm{O}_{p'}(G)$, a conclusion stronger than Theorem 9.2 can be obtained.

PROPOSITION 10.3
Assume either of the following.

(i) $\mathrm{Out}_{p\text{-Col}}(G/\mathrm{O}_{p'}(G))$ *is a* p'*-group.*
(ii) $\mathrm{Out}_{\mathrm{Col}}(G/\mathrm{O}_{p'}(G))$ *is a* p'*-group.*

Then $[\mathrm{Out}_{\mathrm{Col}}(G), \mathrm{Out}_{c,p}(G)]$ *is a* p'*-group.*

Proof
Since $\mathrm{Out}_{\mathrm{Col}}(G/\mathrm{O}_{p'}(G)) \leq \mathrm{Out}_{p\text{-Col}}(G/\mathrm{O}_{p'}(G))$, we may assume that (ii) holds. The proof is similar to that of Theorem 9.2. Put $Y = \mathrm{Aut}_{\mathrm{Col}}(G)$. Then Y is a normal subgroup of $\mathrm{Aut}_{c,p}(G)$. Put $\overline{\mathrm{Aut}_{c,p}(G)} = \mathrm{Aut}_{c,p}(G)/\mathrm{Inn}(G)$, and use the bar convention. We claim that for any p-element $\bar{\sigma}$ of $\bar{Y}$ and any element $\bar{\rho}$ of $\overline{\mathrm{Aut}_{c,p}(G)}$, $[\bar{\sigma}, \bar{\rho}]$ is a p'-element. Put $\bar{G} = G/\mathrm{O}_{p'}(G)$. Let $\varphi : \mathrm{Aut}_{c,p}(G) \to \mathrm{Aut}_{c,p}(\bar{G})$ be the natural map, and let $\alpha : \mathrm{Aut}(\bar{G}) \to \mathrm{Out}(\bar{G})$ be the natural map. Since $\varphi(Y) \leq \mathrm{Aut}_{\mathrm{Col}}(\bar{G}), \alpha\varphi(Y)$ is a p'-group. Now $\alpha\varphi(Y) \simeq Y/(Y \cap \mathrm{Ker}\,\varphi)\,\mathrm{Inn}(G)$, so that we can choose a preimage σ of $\bar{\sigma}$ as a p-element of $Y \cap \mathrm{Ker}\,\varphi$. Then as in the proof of Theorem 9.2, the claim follows. Then $\overline{[Y, \mathrm{Aut}_{c,p}(G)]} \leq \mathrm{O}_{p'}(\bar{Y})$. The proof is complete. □

REMARK 10.4
Proposition 10.3 shows that if the answer to [HeKi, Question 2] (see Remark 6.11 above) is affirmative, then $[\mathrm{Out}_{\mathrm{Col}}(G), \mathrm{Out}_{c,p}(G)]$ is a p'-group for any group G and any prime p.

Note that this implies immediately that $\mathrm{Out}_{\mathrm{Col}}(G)$ is abelian (Theorem 9.1), cf. the proof of Corollary 9.4.

COROLLARY 10.5
If G has an abelian Sylow p-subgroup, then the conclusion of Proposition 10.3 holds.

Proof
Use Proposition 10.2. □

The following strengthens Theorem 9.1 for solvable groups.

COROLLARY 10.6
Let G be a solvable group. Then $\mathrm{Out}_{\mathrm{Col}}(G)$ is a central subgroup of $\bigcap_p \mathrm{Out}_{c,p}(G)$, where p runs through all primes. In particular, $[\mathrm{Out}_{\mathrm{Col}}(G), \mathrm{Out}_c(G)] = 1$, and $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G)$ is a central subgroup of $\mathrm{Out}_c(G)$.

Proof
If G is solvable, then $\mathrm{Out}_{p\text{-Col}}(G/\mathrm{O}_{p'}(G)) = 1$ for any prime p by Gross [Gr, Corollary 2.4]. Hence by Proposition 10.3, $[\mathrm{Out}_{\mathrm{Col}}(G), \bigcap_p \mathrm{Out}_{c,p}(G)]$ is a p'-group for any prime p, so that it is a trivial group. □

11. Consequences of the AM conjecture, IV: The structure of $\mathrm{Out}_c(K)$ and $S_G^0(b)/G[b]$ (special cases)

In this section we use the notation from Section 4. Recall that a group X is said to be an A-group if X is solvable and all Sylow subgroups of X are abelian.

LEMMA 11.1 (HERTWECK [He, COROLLARY 3])
For any meta-abelian A-group X, $\mathrm{Out}_c(X) = 1$.

PROPOSITION 11.2
Assume that the AM conjecture is true around (K, b) in G.

(i) *If the inertial quotient group of b is abelian and the canonical character of β is extendible to $N_K(Q)_\beta$, then $S_G^0(b)/G[b]$ is a p-group.*
(ii) *If Q is cyclic, then $S_G^0(b)/G[b] = 1$.*

Proof
(i) We may assume $G = H_\beta$. Use Convention 4.6. By Theorem 4.1 it suffices to show that a p-complement of $S_H^0(b)/H[b]$ is trivial. Let the notation be as in Lemma 4.7. By Lemma 4.7(ii) and (iii), instead of (H, K, b, Q) it suffices to consider $(\bar{H}, \bar{K}, \bar{b}, \bar{Q})$, which we denote from here by (H, K, b, Q). By Lemma 8.6, $S_H^0(b)/H[b]$ is isomorphic to a section of $\mathrm{Out}_c(K/C_K(Q) \ltimes Q)$. Since $K/C_K(Q)$ is an abelian p'-group and Q is an abelian p-group, $K/C_K(Q) \ltimes Q$ is a meta-abelian A-group. So $\mathrm{Out}_c(K/C_K(Q) \ltimes Q) = 1$ by Lemma 11.1. Thus $S_H^0(b)/H[b] = 1$, and the result follows.

(ii) Since the assumption of (i) holds, $S_G^0(b)/G[b]$ is a p-group. On the other hand, $S_G^0(b)/G[b]$ is a p'-group by Corollary 4.3. Thus the result follows. □

PROPOSITION 11.3

Assume that the AM conjecture is true around $(K, b_p(K))$. If $N_K(Q)/QC_K(Q)$ is abelian, then $\mathrm{Out}_c(K)$ *is p-nilpotent.*

Proof

Let $G = \mathrm{Aut}(K) \ltimes K$ be the semidirect product. By Lemma 8.2, $\mathrm{Out}_c(K)/\mathrm{Out}_c(K) \cap \mathrm{Out}_{p\text{-Col}}(K)$ is isomorphic to a subgroup of $S_G^0(b)/G[b]$, where $b = b_p(K)$. By Proposition 11.2, $S_G^0(b)/G[b]$ is a p-group, and by Theorem 7.7, $\mathrm{Out}_c(K) \cap \mathrm{Out}_{p\text{-Col}}(K)$ is p-nilpotent, so that $\mathrm{Out}_c(K)$ is p-nilpotent. The proof is complete. □

COROLLARY 11.4

Assume that for any prime p, the AM conjecture is true around $(K, b_p(K))$. If, for any prime p, $N_K(Q)/QC_K(Q)$ is abelian for a Sylow p-subgroup Q of K, then $\mathrm{Out}_c(K)$ *is nilpotent.*

The following generalizes part of [Sa, Corollary, p.53].

COROLLARY 11.5

If K is p-supersolvable, then $\mathrm{Out}_c(K)$ *is p-nilpotent. Therefore, if K is supersolvable, then* $\mathrm{Out}_c(K)$ *is nilpotent.*

Proof

If K is p-supersolvable, then K' is p-nilpotent (see [Hu, VI 9.1]). Let $M/K' = \mathrm{O}_p(K/K')$. Then M is a p-nilpotent normal subgroup such that K/M is an abelian p'-group. For a Sylow p-subgroup Q of K, $Q \le M$. Then $K = N_K(Q)M$, and $N_K(Q)/N_M(Q) \simeq K/M$ is abelian. Since $N_M(Q) = QC_M(Q) \le QC_K(Q)$, we see that $N_K(Q)/QC_K(Q)$ is abelian. Since the AM conjecture is true for p-solvable groups, $\mathrm{Out}_c(K)$ is p-nilpotent by Proposition 11.3. The proof is complete. □

PROPOSITION 11.6

Let K be a group with an abelian Sylow q-subgroup for a prime q. Assume that for any section S of K and for any p, the AM conjecture is true around $(S, b_p(S))$. Then the following holds.

(i) *For any group G with $G \rhd K$ and for any prime p, $S_G^0(b)/G[b]$ is q-nilpotent, where $b = b_p(K)$.*

(ii) $\mathrm{Out}_c(K)$ *is q-nilpotent.*

Proof

We prove the proposition by induction on the order of K. Both (i) and (ii) are trivial if $K = 1$. Assume $K > 1$.

(i) If K is a p'-group, then $S_G^0(b)/G[b] = 1$. So we may assume $Q \ne 1$, where Q is a Sylow p-subgroup of K. We may assume $G = H_\beta$. Use Convention 4.6. When

$p = q$, $S^0_H(b)/H[b]$ is a q'-group by Corollary 4.3. Assume $p \neq q$. By Theorem 4.1, it suffices to show that a p-complement of $S^0_H(b)/H[b]$ is q-nilpotent. Let the notation be as in Lemma 4.7. By Lemma 4.7(iii), instead of (H, K, b, Q) it suffices to consider $(\bar{H}, \bar{K}, \bar{b}, \bar{Q})$, which we denote from here on by (H, K, b, Q). Then by Lemma 8.6, $S^0_H(b)/H[b]$ is isomorphic to a section of $\mathrm{Out}_c(K/C_K(Q) \ltimes Q)$. We have $|K/C_K(Q)| \leq |K/Q| < |K|$, since Q is a nontrivial abelian group. By induction, $\mathrm{Out}_c(K/C_K(Q))$ is q-nilpotent by (ii). Then $\mathrm{Out}_c(K/C_K(Q) \ltimes Q)$ is q-nilpotent by Corollary 7.4. Therefore $S^0_H(b)/H[b]$ is q-nilpotent, and the proof is complete.

(ii) Since $\mathrm{Out}_c(K) \cap \mathrm{Out}_{\mathrm{Col}}(K) \leq \mathrm{Out}_{\mathrm{Col}}(K)$ and $\mathrm{Out}_c(K) \leq \mathrm{Out}_{c,q}(K)$, $[\mathrm{Out}_c(K) \cap \mathrm{Out}_{\mathrm{Col}}(K), \mathrm{Out}_c(K)]$ is a q'-group by Corollary 10.5. We claim that $\mathrm{Out}_c(K)/\mathrm{Out}_c(K) \cap \mathrm{Out}_{\mathrm{Col}}(K)$ is q-nilpotent.

To prove the claim, let $G = \mathrm{Aut}(K) \ltimes K$ be the semidirect product. Let $b_p = b_p(K)$. By Lemma 8.2 $\mathrm{Aut}_c(K) \cap \mathrm{Aut}_{p\text{-}\mathrm{Col}}(K) = \mathrm{Aut}_c(K) \cap G[b_p]$ for any p. So we have

$$\mathrm{Aut}_c(K) \cap \mathrm{Aut}_{\mathrm{Col}}(K) = \mathrm{Aut}_c(K) \cap \Big(\bigcap_p G[b_p]\Big),$$

where p runs through all primes. Since $\mathrm{Aut}_c(K) \leq S^0_G(b_p)$ for any p, we see that

$$\mathrm{Out}_c(K)/\mathrm{Out}_c(K) \cap \mathrm{Out}_{\mathrm{Col}}(K) = \mathrm{Aut}_c(K)/\mathrm{Aut}_c(K) \cap \mathrm{Aut}_{\mathrm{Col}}(K)$$

is isomorphic to a subgroup of $\prod_p S^0_G(b_p)/G[b_p]$ (direct product), which is q-nilpotent by (i), and the claim is proved.

Then $\mathrm{Out}_c(K)$ is q-nilpotent by Lemma 7.2. The proof is complete. □

COROLLARY 11.7

Let K be a group all of whose Sylow subgroups are abelian. Assume that for any section S of K and for any p, the AM conjecture is true around $(S, b_p(S))$. Then $\mathrm{Out}_c(K)$ *is nilpotent.*

Since the AM conjecture is true for solvable groups, we obtain the following, which may be compared to Lemma 11.1.

COROLLARY 11.8

Let K be an A-group. Then $\mathrm{Out}_c(K)$ *is nilpotent.*

In view of the above results and Theorem 8.3, we pose the following.

QUESTION 11.9

Is $\mathrm{Out}_c(K)$ always nilpotent?

Added in proof (by T. H.)

While the author of the present paper, Masafumi Murai, was awaiting the referees' report, he died in an accidental fire at his house, and the precious secrets in his personal computer were also lost to eternity. I heartfully regret the tragic loss of this unique mathematician and also the loss of his mathematical treasures. After his graduation from Kyoto University, he educated himself in the theory of finite groups, not following any graduate course, and continued his studies in this field, not affiliated to any institutions. I continue to admire him as a genuine mathematician who reached the top level and was going still higher by himself, without any official support. For the proofreading of the present paper, I have replaced him by the generous permission of his younger sister and of the editors-in-chief of the *Kyoto Journal of Mathematics*, since it is easy to fulfill the referees' requests. □

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