A theta expression of the Hilbert modular functions for $\sqrt{5}$ via the periods of $K3$ surfaces

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Abstract In this paper, we give an extension of the classical story of the elliptic modular function to the Hilbert modular case for $\mathbb{Q}(\sqrt{5})$. We construct the period mapping for a family $\mathcal{F}=\{S(X,Y)\}$ of $K3$ surfaces with 2 complex parameters X and Y. The inverse correspondence of the period mapping gives a system of generators of Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. Moreover, we show an explicit expression of this inverse correspondence by theta constants.

Introduction

The symmetric Hilbert modular surface $(\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}_K),\tau\rangle$, where $\mathcal{O}_K$ is the ring of integers in a real quadratic field K and τ exchanges the factors of $\mathbb{H}\times\mathbb{H}$, gives the moduli space for the family $\mathcal{F}_K=\{A\}$ of the principally polarized Abelian surfaces with an extra endomorphism structure $K(\subset \mathrm{End}^0(A))$.

In classical theory, the elliptic modular function $\lambda(z)$ on the moduli space $\mathbb{H}/\Gamma(2)$ is given by the inverse of the multivalued period mapping for a family of elliptic curves. This period mapping gives the Schwarz mapping of the Gauss hypergeometric differential equation $E(\frac{1}{2},\frac{1}{2},1)$. It is important that the modular function $\lambda(z)$ have an explicit expression given by the Jacobi theta constants.

For the Hilbert modular cases, although there are various studies on the structure of the field of modular functions and the ring of modular forms (e.g., Gundlach [Gu], Hirzebruch [H], Müller [M]), still now, to the best of the author's knowledge, there has not appeared an explicit expression of Hilbert modular functions as an inverse correspondence of the period mapping for a family of algebraic varieties. In this paper, we give an extension of the above classical story to the Hilbert modular functions for $K=\mathbb{Q}(\sqrt{5})$ by using a family of $K3$ surfaces that gives the same variation of Hodge structures of weight 2 with the family $\mathcal{F}_K$ of Abelian surfaces. Namely, we show that the inverse of the period mapping for our family of $K3$ surfaces gives Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$.

Kyoto Journal of Mathematics, Vol. 53, No. 4 (2013), 815–843
DOI 10.1215/21562261-2366102,
Received August 28, 2012. Revised October 23, 2012. Accepted October 24, 2012.
2010 Mathematics Subject Classification: Primary 14J28; Secondary 11F46, 33C05.
Author's work supported by Grant-in-Aid IIJ04746 from the Japan Society for the Promotion of Science fellows.

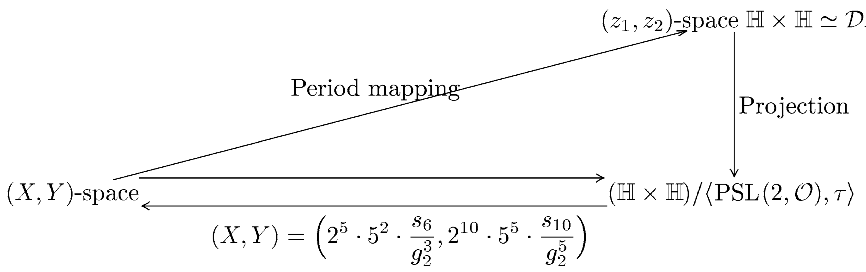

Figure 1

Moreover, we obtain an explicit theta expression of this inverse correspondence. As our method, we use the fact that our period integrals of $K3$ surfaces satisfy a system of partial differential equations determined in [N1].

Our result is obtained as a combined work with [N1] and based on the results of Hirzebruch [H] and Müller [M] also.

In this paper, we consider the family $\mathcal{F} = \{S(X, Y)\}$ of $K3$ surfaces with 2 complex parameters given by an affine equation in (x, y, z)-space:

$$S(X, Y) : z^2 = x^3 - 4y^2(4y - 5)x^2 + 20Xy^3x + Yy^4.$$

We show that a system of generators of the field of the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$ is given by the inverse of the period mapping for $\mathcal{F}$ and obtain an explicit expression of these Hilbert modular functions given by theta constants.

We use the following results of the Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. Hirzebruch [H] studied the Hilbert modular orbifold $\overline{(\mathbb{H} \times \mathbb{H})/\langle \mathrm{PSL}(2, \mathcal{O}), \tau \rangle}$, where $\mathcal{O} = \mathbb{Z} + \frac{1+\sqrt{5}}{2}\mathbb{Z}$ and τ is an involution of $\mathbb{H} \times \mathbb{H}$, by an algebrogeometric method. He determined the structure of the ring of the symmetric Hilbert modular forms. This ring is isomorphic to the Klein icosahedral ring $\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}]/(R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) = 0)$. Müller [M] obtained a system of generators $\{g_2, s_6, s_{10}, s_{15}\}$ of the ring of the symmetric Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$ and found the relation $M(g_2, s_6, s_{10}, s_{15}) = 0$. These generators are given by the theta constants. Then, they are holomorphic functions on $\mathbb{H} \times \mathbb{H}$.

We show that the period mapping for $\mathcal{F}$ gives a biholomorphic correspondence between the monodromy covering of (X, Y)-space and $\mathbb{H} \times \mathbb{H}$, and the projective monodromy group coincides with the extended Hilbert modular group $\langle \mathrm{PSL}(2, \mathcal{O}), \tau \rangle$. Then, the quotient space $(\mathbb{H} \times \mathbb{H})/\langle \mathrm{PSL}(2, \mathcal{O}), \tau \rangle$ becomes the classifying space of the family $\mathcal{F}$. Consequently, we may regard X and Y as Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. This framework enables us to obtain explicit relations between the results of [H] and [M]. Namely, we obtain an expression of the parameters X and Y as quotients of theta constants by use of the period mapping for our family $\mathcal{F}$ of $K3$ surfaces (Figure 1).

In Section 1, we give a survey of the results of [N1] and the properties of the Hilbert modular orbifold for $\mathbb{Q}(\sqrt{5})$. Especially, we recall the family $\mathcal{F}_0$ of $K3$ surfaces and the period differential equation (1.14) for $\mathcal{F}_0$. A generic member of $\mathcal{F}_0$ is transformed to $S(X, Y) \in \mathcal{F}$. The system (1.14) turns out to be the period differential equation for $\mathcal{F}$, which gives an analogy of the Gauss hypergeometric equation ${}_2E_1(\frac{1}{2},\frac{1}{2},1)$.

In Section 2, we study the $K3$ surface $S(X,Y)$. First, we obtain the weighted projective space $\mathbb{P}(1,3,5)$ as a compactification of the (X,Y)-space $\mathbb{C}^2$. This remains a parameter space for $K3$ surfaces except one point (Theorem 2.1). We note that, due to [H] together with Klein [Kl], the orbifold $\overline{(\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle}$ is isomorphic to $\mathbb{P}(1,3,5)$ as algebraic varieties. Secondly, we define the multivalued period mapping $\mathbb{P}(1,3,5)-\{\text{one point}\}\to\mathcal{D}$ for $\mathcal{F}$, where $\mathcal{D}$ is a symmetric Hermitian space of type IV. We have a modular isomorphism between $\mathbb{H}\times\mathbb{H}$ and a connected component $\mathcal{D}_+$ of $\mathcal{D}$. Our period mapping gives an explicit isomorphism between $\mathbb{P}(1,3,5)$ and $\overline{(\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle}$. Then, we obtain the coordinates of $\mathbb{H}\times\mathbb{H}$ given by the quotients of period integrals of $S(X,Y)$:

$$(0.1)\qquad \big(z_1(X,Y),z_2(X,Y)\big)=\Big(-\frac{\int_{\Gamma_3}\omega+\frac{1-\sqrt{5}}{2}\int_{\Gamma_4}\omega}{\int_{\Gamma_2}\omega},-\frac{\int_{\Gamma_3}\omega+\frac{1+\sqrt{5}}{2}\int_{\Gamma_4}\omega}{\int_{\Gamma_2}\omega}\Big),$$

where $\Gamma_1,\dots,\Gamma_4$ are 2-cycles on $S(X,Y)\in\mathcal{F}$ given in Section 2.2.

Then, the inverse correspondence $(z_1,z_2)\mapsto(X(z_1,z_2),Y(z_1,z_2))$ defines a pair of Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. We obtain an expression of X and Y in the following way.

In Section 3, we consider the subfamily $\mathcal{F}_X=\{S(X,0)\}$ of $K3$ surfaces. The period mapping for $\mathcal{F}_X$ gives a correspondence between the X-space and the diagonal $\Delta=\{(z_1,z_2)\in\mathbb{H}\times\mathbb{H}\mid z_1=z_2\}$. We obtain the period differential equation for $\mathcal{F}_X$. The solutions of this period differential equation are described in terms of the solutions of the Gauss hypergeometric equation ${}_2E_1(\frac{1}{12},\frac{5}{12},1)$. Then, we obtain an expression of the parameter X in terms of the elliptic J function (see Theorem 3.2).

In Section 4, we obtain an explicit expression of the inverse of the period mapping (0.1) by theta constants:

$$(X,Y)=\Big(2^5\cdot 5^2\cdot\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)},2^{10}\cdot 5^5\cdot\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big),$$

where g_2,s_6, and s_{10} are Hilbert modular forms given by Müller (see Theorem 4.1).

Our results in this paper are used in the forthcoming paper [N2], in which we shall show simple and new defining equations of the family of Kummer surfaces for the Humbert surface of invariant 5 and a geometric and intuitive interpretation of period mappings for this family.

1. Preliminaries

1.1. The family $\mathcal{F}_0$

In [N1], we studied the family $\mathcal{F}_0=\{S_0(\lambda,\mu)\}$ of $K3$ surfaces defined by the equation

$$(1.1)\qquad S_0(\lambda,\mu): x_0y_0z_0^2(x_0+y_0+z_0+1)+\lambda x_0y_0z_0+\mu=0,$$

where $(\lambda,\mu) \in \Lambda = \{(\lambda,\mu) \mid \lambda\mu(\lambda^2(4\lambda-1)^3 - 2(2+25\lambda(20\lambda-1))\mu - 3125\mu^2) \neq 0)\}$. First, we recall the results of this family.

Set

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}. \tag{1.2}$$

Put

$$\mathcal{D} = \{\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4) \in \mathbb{P}^3(\mathbb{C}) \mid \xi A^t\xi = 0, \xi A^t\overline{\xi} > 0)\}.$$

This is a 2-dimensional symmetric Hermitian space of type IV. Note that $\mathcal{D}$ is composed of two connected components: $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$. We let $(1:1:-\sqrt{-1}:0) \in \mathcal{D}_+$. Set $\mathrm{PO}(A,\mathbb{Z}) = \{g \in \mathrm{PGL}(4,\mathbb{Z}) \mid {}^t gAg = A\}$. It acts on $\mathcal{D}$ by ${}^t\xi \mapsto g{}^t\xi$. Let $\mathrm{PO}^+(A,\mathbb{Z}) = \{g \in \mathrm{PO}(A,\mathbb{Z}) \mid g(\mathcal{D}_+) = \mathcal{D}_+\}$.

In [N1, Section 2], we had the multivalued period mapping $\Phi_0 : \Lambda \to \mathcal{D}_+$ for $\mathcal{F}_0$ given by

$$\Phi_0(\lambda,\mu) = \Big(\int_{\Gamma_1} \omega : \cdots : \int_{\Gamma_4} \omega\Big), \tag{1.3}$$

where ω is the unique holomorphic 2-form on $S_0(\lambda,\mu)$ up to a constant factor and 2-cycles $\Gamma_1,\dots,\Gamma_4 \in H_2(S_0(\lambda,\mu),\mathbb{Z})$ are given by this construction.

Let $\mathrm{NS}(S)$ be the Néron–Severi lattice of a $K3$ surface S. The orthogonal complement $\mathrm{Tr}(S) = \mathrm{NS}(S)^\perp$ in $H_2(S,\mathbb{Z})$ is called the *transcendental lattice* of S. We proved the following.

THEOREM 1.1

(1) *For a generic point* $(\lambda,\mu) \in \Lambda$*, the intersection matrix of* $\mathrm{NS}(S_0(\lambda,\mu))$ *is given by*

$$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{1.4}$$

and the intersection matrix of $\mathrm{Tr}(S_0(\lambda,\mu))$ *is given by*

$$U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = A \tag{1.5}$$

(see [N1, Theorems 2.2, 3.1]).

(2) *The projective monodromy group of the period mapping* $\Phi_0 : \Lambda \to \mathcal{D}_+$ *is isomorphic to* $\mathrm{PO}^+(A,\mathbb{Z})$ *(see [N1, Theorem 5.2]).*

Moreover, we determined the partial differential equation in 2 variables λ and μ of rank 4 that is satisfied by the periods for the family $\mathcal{F}_0$. We call this equation the *period differential equation* for $\mathcal{F}_0$. This equation has the singular locus Λ (see [N1, Theorem 4.1]).

1.2. The Hilbert modular orbifold $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$

Here, we recall the action of the Hilbert modular group on $\mathbb{H}\times\mathbb{H}$. Let $\mathcal{O}$ be the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{5})$. Set $\mathbb{H}_\pm = \{z\in\mathbb{C} \mid \pm \mathrm{Im}(z) > 0\}$. The Hilbert modular group $PSL(2,\mathcal{O})$ acts on $(\mathbb{H}_+\times\mathbb{H}_+)\cup(\mathbb{H}_-\times\mathbb{H}_-)$ by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} : (z_1,z_2)\mapsto\Big(\frac{\alpha z_1+\beta}{\gamma z_1+\delta},\frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\Big),$$

for $g=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in PSL(2,\mathcal{O})$, where $'$ means the conjugate in $\mathbb{Q}(\sqrt{5})$. We also consider the involution

$$\tau : (z_1,z_2)\mapsto(z_2,z_1).$$

DEFINITION 1.1

If a holomorphic function g on $\mathbb{H}\times\mathbb{H}$ satisfies the transformation law

$$g\Big(\frac{az_1+b}{cz_1+d},\frac{a'z_2+b'}{c'z_2+d'}\Big)=(cz_1+d)^k(c'z_2+d')^k g(z_1,z_2)$$

for any $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in PSL(2,\mathcal{O})$, we call g a *Hilbert modular form of weight* k for $\mathbb{Q}(\sqrt{5})$. If $g(z_2,z_1)=g(z_1,z_2)$, g is called a *symmetric modular form.* If $g(z_2,z_1)=-g(z_1,z_2)$, g is called an *alternating modular form.*

If a meromorphic function f on $\mathbb{H}\times\mathbb{H}$ satisfies

$$f\Big(\frac{az_1+b}{cz_1+d},\frac{a'z_2+b'}{c'z_2+d'}\Big)=f(z_1,z_2)$$

for any $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in PSL(2,\mathcal{O})$, we call f a *Hilbert modular function* for $\mathbb{Q}(\sqrt{5})$.

Set

$$W=\begin{pmatrix}1 & 1\\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2}\end{pmatrix}.$$

It holds that

$$A=U\oplus\begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix}=U\oplus WU^tW.$$

The correspondence

$$j:(z_1,z_2)\to(z_1z_2:-1:z_1:z_2)(I_2\oplus {}^tW^{-1})$$

defines a biholomorphic mapping

$$(\mathbb{H}_+\times\mathbb{H}_+)\cup(\mathbb{H}_-\times\mathbb{H}_-)\to\mathcal{D}.$$

The group $PSL(2,\mathcal{O})$ is generated by three elements:

$$g_1=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\qquad g_2=\begin{pmatrix}1 & \frac{1+\sqrt{5}}{2}\\ 0 & 1\end{pmatrix},\qquad g_3=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We have an isomorphism:

$$\begin{array}{cccc}\tilde{j} & : & \langle PSL(2,\mathcal{O}),\tau\rangle & \to & PO^+(A,\mathbb{Z})\\ & ; & g & \mapsto & j\circ g\circ j^{-1}=\tilde{j}(g)=:\tilde{g}.\end{array}$$

Especially, we see

$$
\begin{cases}
\tilde{g_1} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{g_2} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \\
\tilde{g_3} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{\tau} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{cases} \tag{1.6}
$$

So, the above j gives a modular isomorphism:

$$
j : \big(\mathbb{H} \times \mathbb{H}, \langle \mathrm{PSL}(2, \mathcal{O}), \tau \rangle\big) \simeq \big(\mathcal{D}_+, \mathrm{PO}^+(A, \mathbb{Z})\big). \tag{1.7}
$$

Recall $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$ and the period mapping Φ for $\mathcal{F}_0$. The mapping $j^{-1} \circ \Phi : \Lambda \to \mathbb{H} \times \mathbb{H}$ gives an explicit transcendental correspondence between Λ and $\mathbb{H} \times \mathbb{H}$.

Hirzebruch [H] studied the Hilbert modular orbifold $\overline{(\mathbb{H} \times \mathbb{H})/\langle \mathrm{PSL}(2, \mathcal{O}), \tau \rangle}$. Here, we survey his results.

The Klein icosahedral polynomials are

$$
\begin{cases}
\mathfrak{A}(\zeta_0 : \zeta_1 : \zeta_2) = \zeta_0^2 + \zeta_1 \zeta_2, \\
\mathfrak{B}(\zeta_0 : \zeta_1 : \zeta_2) = 8\zeta_0^4 \zeta_1 \zeta_2 - 2\zeta_0^2 \zeta_1^2 \zeta_2^2 + \zeta_1^3 \zeta_2^3 - \zeta_0(\zeta_1^5 + \zeta_2^5), \\
\mathfrak{C}(\zeta_0 : \zeta_1 : \zeta_2) = 320\zeta_0^6 \zeta_1^2 \zeta_2^2 - 160\zeta_0^4 \zeta_1^3 \zeta_2^3 + 20\zeta_0^2 \zeta_1^4 \zeta_2^4 + 6\zeta_1^5 \zeta_2^5 \\
\qquad\qquad - 4\zeta_0(\zeta_1^5 + \zeta_2^5)(32\zeta_0^4 - 20\zeta_0^2 \zeta_1 \zeta_2 + 5\zeta_1^2 \zeta_2^2) + \zeta_1^{10} + \zeta_2^{10}, \\
12\mathfrak{D}(\zeta_0 : \zeta_1 : \zeta_2) = (\zeta_1^5 - \zeta_2^5)(-1024\zeta_0^{10} + 3840\zeta_0^8 \zeta_1 \zeta_2 \\
\qquad\qquad - 3840\zeta_0^6 \zeta_1^2 \zeta_2^2 + 1200\zeta_0^4 \zeta_1^3 \zeta_2^3 - 100\zeta_0^2 \zeta_1^4 \zeta_2^4 + \zeta_1^5 \zeta_2^5) \\
\qquad\qquad + \zeta_0(\zeta_1^{10} - \zeta_2^{10})(352\zeta_0^4 - 160\zeta_0^2 \zeta_1 \zeta_2 + 10\zeta_1^2 \zeta_2^2) \\
\qquad\qquad + (\zeta_1^{15} - \zeta_2^{15}).
\end{cases} \tag{1.8}
$$

We have the following relation:

$$
\begin{aligned}
R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) := 144\mathfrak{D}^2 - \big(&-1728\mathfrak{B}^5 + 720\mathfrak{A}\mathfrak{C}\mathfrak{B}^3 - 80\mathfrak{A}^2\mathfrak{C}^2\mathfrak{B} \\
&+ 64\mathfrak{A}^3(5\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C})^2 + \mathfrak{C}^3\big) = 0.
\end{aligned} \tag{1.9}
$$

Set

$$
X = \frac{\mathfrak{B}}{\mathfrak{A}^3}, \qquad Y = \frac{\mathfrak{C}}{\mathfrak{A}^5}. \tag{1.10}
$$

Now, set

$$
\Gamma(\sqrt{5}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, \alpha \equiv \delta \equiv 1, \beta \equiv \delta \equiv 0 \pmod{\sqrt{5}} \right\}.
$$

We note that the group $\mathrm{PSL}(2, \mathcal{O})/\Gamma(\sqrt{5})$ is isomorphic to the alternating group $\mathcal{A}_5$. Hirzebruch [H] studied the canonical bundle of the orbifold $\overline{(\mathbb{H} \times \mathbb{H})/\Gamma(\sqrt{5})}$ by an algebrogeometric method. He proved the following.

PROPOSITION 1.1 ([H, PP. 307–310])

(1) *The nonsingular model of* $\overline{(\mathbb{H}\times\mathbb{H})/\langle\Gamma(\sqrt{5}),\tau\rangle}$ *is* $\mathbb{P}^2(\mathbb{C})=\{(\zeta_0;\zeta_1;\zeta_2)\}$ *by adding six points. A homogeneous polynomial of degree* k *in* ζ_0,ζ_1*, and* ζ_2 *defines a modular form for* $\Gamma(\sqrt{5})$ *of weight* k*.*

(2) *The ring of symmetric modular forms for* $\mathrm{PSL}(2,\mathcal{O})$ *is isomorphic to the ring*

$$\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}]/\big(R(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})=0\big),$$

where $R(\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D})$ *is the Klein relation (1.9).* $\mathfrak{A}$ *(resp.,* $\mathfrak{B},\mathfrak{C},\mathfrak{D}$*) gives a symmetric modular form for* $\mathrm{PSL}(2,\mathcal{O})$ *of weight* 2 *(resp.,* $6,10,15$*).*

(3) *There exists an alternating modular form* $\mathfrak{c}$ *of weight* 5 *such that* $\mathfrak{c}^2=\mathfrak{C}$*. The ring of Hilbert modular forms for* $\mathrm{PSL}(2,\mathcal{O})$ *is isomorphic to the ring*

$$\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{c},\mathfrak{D}]/\big(R(\mathfrak{A},\mathfrak{B},\mathfrak{c}^2,\mathfrak{D})=0\big).$$

Let $c'\in\mathbb{C}-\{0\}$. We consider the action $(\zeta_0,\zeta_1,\zeta_2)\mapsto(c'\zeta_0,c'\zeta_1,c'\zeta_2)$. Because $\mathfrak{A}$ (resp., $\mathfrak{B},\mathfrak{C}$) is a homogeneous polynomial of degree 2 (resp., $6,10$) in ζ_0,ζ_1, and ζ_2, we have the action $(\mathfrak{A},\mathfrak{B},\mathfrak{C})\mapsto(c'^2\mathfrak{A},c'^6\mathfrak{B},c'^{10}\mathfrak{C})$. Therefore, we regard $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$-space as the weighted projective space $\mathbb{P}(1,3,5)$. Especially, the pair

$$(X,Y)=\Big(\frac{\mathfrak{B}}{\mathfrak{A}^3},\frac{\mathfrak{C}}{\mathfrak{A}^5}\Big) \tag{1.11}$$

gives a system of affine coordinates on $\{\mathfrak{A}\neq 0\}$.

By the arguments of Klein [Kl], Hirzebruch [H], and Kobayashi, Kushibiki and Naruki [KKN], we know the following properties of the action of $\mathcal{A}_5$ on $\overline{(\mathbb{H}\times\mathbb{H})/\langle\Gamma(\sqrt{5}),\tau\rangle}=\mathbb{P}^2(\mathbb{C})=\{\zeta_0:\zeta_1:\zeta_2\}$.

PROPOSITION 1.2

(1) *The correspondence* $(\zeta_0:\zeta_1:\zeta_2)\mapsto(\mathfrak{A}(\zeta_0:\zeta_1:\zeta_2):\mathfrak{B}(\zeta_0:\zeta_1:\zeta_2):\mathfrak{C}(\zeta_0:\zeta_1:\zeta_2))$ *gives an identification between* $\overline{\mathbb{P}^2(\mathbb{C})/\mathcal{A}_5}$ *and* $\mathbb{P}(1,3,5)$*. Then, the Hilbert modular orbifold* $\overline{(\mathbb{H}\times\mathbb{H})/\langle\mathrm{PSL}(2,\mathcal{O}),\tau\rangle}$ *is identified with* $\mathbb{P}(1,3,5)$*. The cusp* $(\sqrt{-1}\infty,\sqrt{-1}\infty)\in\overline{(\mathbb{H}\times\mathbb{H})/\langle\mathrm{PSL}(2,\mathcal{O}),\tau\rangle}$ *is given by the point* $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})=(1:0:0)$*. So, the quotient space* $(\mathbb{H}\times\mathbb{H})/\langle\mathrm{PSL}(2,\mathcal{O}),\tau\rangle$ *corresponds to* $\mathbb{P}(1,3,5)-\{(1:0:0)\}$*.*

(2) *The divisor* $\{\mathfrak{D}=0\}$ *consists of fifteen lines in* $\mathbb{P}^2(\mathbb{C})$*. These fifteen lines of* $\{\mathfrak{D}=0\}$ *are the reflection lines of fifteen involutions of* $\mathcal{A}_5$*. (Note that* $\mathcal{A}_5$ *is generated by three involutions.)*

(3) *The involution* τ *induces an involution on the orbifold* $\overline{(\mathbb{H}\times\mathbb{H})/\mathrm{PSL}(2,\mathcal{O})}$*. The branch locus of the canonical projection* $\overline{(\mathbb{H}\times\mathbb{H})/\mathrm{PSL}(2,\mathcal{O})}\to\mathbb{P}(1,3,5)$ *is given by* $\{\mathfrak{C}=0\}$*.*

Set

$$\mathfrak{X}=\big\{(X,Y)\in\mathbb{C}^2\mid Y\big(1728X^5-720X^3Y+80XY^2-64(5X^2-Y)^2-Y^3\big)\neq 0\big\}. \tag{1.12}$$

In [N1, Section 6], we obtained the birational mapping $\Lambda \to \mathfrak{X}$ given by

$$(\lambda,\mu) \mapsto (X,Y) = \Big(\frac{25\mu}{2(\lambda-1/4)^3}, \frac{-3125\mu^2}{(\lambda-1/4)^5}\Big). \tag{1.13}$$

THEOREM 1.2 ([N1, THEOREM 6.3])

By the correspondence (1.13), the period differential equation for the family $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$ *is transformed to the system of differential equations*

$$\begin{cases} u_{XX} = L_1 u_{XY} + A_1 u_X + B_1 u_Y + P_1 u, \\ u_{YY} = M_1 u_{XY} + C_1 u_X + D_1 u_Y + Q_1 u \end{cases} \tag{1.14}$$

with

$$\begin{cases} L_1 = \frac{-20(4X^2+3XY-4Y)}{36X^2-32X-Y}, \qquad M_1 = \frac{-2(54X^3-50X^2-3XY+2Y)}{5Y(36X^2-32X-Y)}, \\ A_1 = \frac{-2(20X^3-8XY+9X^2Y+Y^2)}{XY(36X^2-32X-Y)}, \qquad B_1 = \frac{10Y(-8+3X)}{X(36X^2-32X-Y)}, \\ C_1 = \frac{-2(-25X^2+27X^3+2Y-3XY)}{5Y^2(36X^2-32X-Y)}, \qquad D_1 = \frac{-2(-120X^2+135X^3-2Y-3XY)}{5XY(36X^2-32X-Y)}, \\ P_1 = \frac{-2(8X-Y)}{X^2(36X^2-32X-Y)}, \qquad Q_1 = \frac{-2(-10+9X)}{25XY(36X^2-32X-Y)}. \end{cases}$$

REMARK 1.1

In [N1], we saw that (1.14) is a uniformizing differential equation of the Hilbert modular orbifod $\overline{(\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle}$. In other words, the solutions of (1.14) define the developing map of the canonical projection $\mathbb{H}\times\mathbb{H} \to (\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle$. This gives an alternative proof of Theorem 1.1(2).

2. The period of the family $\mathcal{F}$

2.1. The family $\mathcal{F}$ of $K3$ surfaces

We obtain a new family $\mathcal{F}$ of $K3$ surfaces with explicit defining equations from the family $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$.

PROPOSITION 2.1

The family of $K3$ surfaces $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$ for $(\lambda,\mu)\in\Lambda$ is transformed to the family $\mathcal{F} = \{S(X,Y)\}$ for $(X,Y)\in\mathfrak{X}$:

$$S(X,Y) : z^2 = x^3 - 4y^2(4y-5)x^2 + 20Xy^3x + Yy^4. \tag{2.1}$$

Proof

By the transformation (1.13) and the birational transformation given by

$$\begin{cases} x_0 = \frac{Yy}{10Xx_1}, \\ y_0 = \frac{4Y^2x_1y_1^2}{-50X^2Yx_1y_1 - 5XY^2y_1^2 + 5XYz_1}, \\ z_0 = -\frac{10XYx_1y_1 + Y^2y_1^2 - Yz_1}{20XYx_1y_1}, \end{cases}$$

the family $\mathcal{F}_0 = \{S_0(\lambda,\mu)\}$ is transformed to the family $\mathcal{F}_1 = \{S_1(X,Y)\}$ given by

$$S_1(X,Y) : z_1^2 = Y\big(x_1^3 - 4y_1^2(4y_1-5)x_1^2 + 20Xy_1^3x_1 + Yy_1^4\big)$$

over $\mathfrak{X}$. Then, by the correspondence $(x_1,y_1,z_1)\mapsto(x,y,z)=(x_1,y_1,\frac{1}{\sqrt{Y}}z_1)$, we have the family $\mathcal{F}=\{S(X,Y)\}$ given by (2.1). □

From (1.3), we obtain the multivalued analytic period mapping

$$\Phi_1:\mathfrak{X}\to\mathcal{D}_+;(X,Y)\mapsto\Big(\int_{\Gamma_1}\omega:\int_{\Gamma_2}\omega:\int_{\Gamma_3}\omega:\int_{\Gamma_4}\omega\Big), \tag{2.2}$$

where $\omega=\frac{dx\wedge dy}{z}$ is the unique holomorphic 2-form on $S(X,Y)$ up to a constant factor and $\Gamma_1,\dots,\Gamma_4$ are certain 2-cycles on $S(X,Y)$. (This period mapping is stated in detail at the beginning of Section 2.2.)

REMARK 2.1

The correspondence $(x_1,y_1,z_1)\mapsto(x,y,z)=(x_1,y_1,\frac{1}{\sqrt{Y}}z_1)$ in the proof of Proposition 2.1 induces the double covering $\mathfrak{X}'\to\mathfrak{X}$ given by $(X,Y')\mapsto(X,Y)=(X,Y'^2)$. However, (X,Y') and $(X,-Y')\in\mathfrak{X}'$ define mutually isomorphic P-marked $K3$ surfaces (see Definition 2.1). So, we obtain the above period mapping Φ_1 on $\mathfrak{X}$.

Hence, from Theorem 1.1, for a generic point $(X,Y)\in\mathfrak{X}$, the intersection matrix of the Néron–Severi lattice $\mathrm{NS}(S(X,Y))$ is given by (1.4), and that of the transcendental lattice $\mathrm{Tr}(S(X,Y))$ is given by A in (1.5). The projective monodromy group of Φ_1 is isomorphic to $\mathrm{PO}^+(A,\mathbb{Z})$. From Theorem 1.2, the period differential equation for the family $\mathcal{F}=\{S(X,Y)\}$ is given by (1.14).

PROPOSITION 2.2

Under the correspondence (1.11), the surface $S(X,Y)$ is birationally equivalent to

$$S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}):z^2=x^3-4(4y^3-5\mathfrak{A}y^2)x^2+20\mathfrak{B}y^3x+\mathfrak{C}y^4. \tag{2.3}$$

Proof

Putting $X=\frac{\mathfrak{B}}{\mathfrak{A}^3},Y=\frac{\mathfrak{C}}{\mathfrak{A}^5}$ to (2.1), we have

$$\mathfrak{A}^5z^2=\mathfrak{A}^5x^3+(20y^2-16y^3)\mathfrak{A}^5x^2+20\mathfrak{A}^2\mathfrak{B}y^3x+\mathfrak{C}y^4.$$

Then, by the correspondence

$$x\mapsto\frac{x}{\mathfrak{A}^3},\qquad y\mapsto\frac{y}{\mathfrak{A}},\qquad z\mapsto\frac{z}{\sqrt{\mathfrak{A}^9}},$$

we obtain (2.3). □

REMARK 2.2

For two surfaces

$$\begin{cases} S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}):z^2=x^3-4(4y^3-5\mathfrak{A}y^2)x^2+20\mathfrak{B}y^3x+\mathfrak{C}y^4,\\ S(k^2\mathfrak{A}:k^6\mathfrak{B}:k^{10}\mathfrak{C}):z^2=x^3-4(4y^3-5k^2\mathfrak{A}y^2)x^2+20k^6\mathfrak{B}y^3x+k^{10}\mathfrak{C}y^4,\end{cases}$$

we have an isomorphism $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \to S(k^2\mathfrak{A}:k^6\mathfrak{B}:k^{10}\mathfrak{C})$ given by $(x,y,z)\mapsto (k^6x,k^2y,k^9z)$ as elliptic surfaces. Therefore, $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})\in\mathbb{P}(1:3:5)$ gives an isomorphism class of these elliptic $K3$ surfaces.

We set $K_1=\{Y=0\}$ and $K_2=\{1728X^5-720X^3Y+80XY^2-64(5X^2-Y)^2-Y^3=0\}$.

THEOREM 2.1

The $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$*-space* $\mathbb{P}(1,3,5)$ *gives a compactification of the parameter space* $\mathfrak{X}$ *of the family* $\mathcal{F}=\{S(X,Y)\}$ *of* $K3$ *surfaces given by (2.1). Namely, if* $(1:0:0)\neq(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})\in\mathbb{P}(1,3,5)$*, then the corresponding surface* $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ *is a* $K3$ *surface. On the other hand,* $S(1:0:0)$ *is a rational surface.*

Proof

First, we prove the case $\mathfrak{A}\neq 0$. In this case, we consider $S(X,Y)$ in (2.1). We have the Kodaira normal form of (2.1):

$$z_1^2=x_1^3-g_2(y)x-g_3(y)\quad (y\neq\infty), \tag{2.4}$$

with

$$\begin{cases} g_2(y)=-(20Xy^3-\frac{16}{3}y^4(4y-5)^2),\\ g_3(y)=-(Yy^4+\frac{80}{3}y^5(4y-5)X-\frac{128}{27}y^6(4y-5)^3),\end{cases}$$

and

$$z_2^2=x_2^3-h_2(y_1)x_2-h_3(y_1)\quad (y\neq 0), \tag{2.5}$$

with

$$\begin{cases} h_2(y_1)=-(20Xy_1^5-\frac{256}{3}y_1^2+\frac{640}{3}y_1^3-\frac{400}{3}y_1^4),\\ h_3(y_1)=-(Yy_1^8+\frac{320}{3}Xy_1^6-\frac{400}{3}Xy_1^7-\frac{8192}{27}y_1^3+\frac{10240}{9}y_1^4-\frac{12800}{9}y_1^5+\frac{16000}{27}y_1^6),\end{cases}$$

where $y_1=\frac{1}{y}$. The discriminant D_0 (resp., D_∞) of the right-hand side of (2.4) (resp., (2.5)) is given by

$$\begin{cases} D_0=y^8(27Y^2+32000X^3y-7200XYy\\ \qquad -160000X^2y^2+32000Yy^2+5760XYy^2\\ \qquad +256000X^2y^3-76800Yy^3-102400X^2y^4+61440Yy^4-16384Yy^5),\\ D_\infty=y_1^{11}(-16384Y-102400X^2y_1+61440Yy_1\\ \qquad +256000X^2y_1^2-76800Yy_1^2-160000X^2y_1^3\\ \qquad +32000Yy_1^3+5760XYy_1^3+32000X^3y_1^4-7200XYy_1^4+27Y^2y_1^5).\end{cases}$$

If $(X,Y)\in\mathfrak{X}$, then we have

$$\mathrm{ord}_y(D_0)=8,\qquad \mathrm{ord}_y(g_2)=3,\qquad \mathrm{ord}_y(g_3)=4,$$

so $\pi^{-1}(0)$ is the singular fiber of type IV^* (for details, see [Ko] or [Sh]). Similarly, we have

$$\mathrm{ord}_y(D_\infty) = 11, \qquad \mathrm{ord}_y(h_2) = 2, \qquad \mathrm{ord}_y(h_3) = 3,$$

so $\pi^{-1}(\infty) = I_5^*$. We have 5 other singular fibers of type I_1. Therefore, for $(X,Y) \in \mathfrak{X}$, $S(X,Y)$ is an elliptic $K3$ surface whose singular fibers are of type $IV^* + 5I_1 + I_5^*$.

By the same way, we know the structure of the elliptic surface $S(X,Y)$ for $(X,Y) \notin \mathfrak{X}$. If $X \neq 0$ and $Y = 0$ (namely, $(X,Y) \in K_1 - \{(0,0)\}$), then $S(X,0)$ is an elliptic $K3$ surface with the singular fibers of type $III^* + 3I_1 + I_6^*$. If $(X,Y) \in K_2 - \{(0,0)\}$, $S(X,Y)$ is an elliptic $K3$ surface with the singular fibers of type $IV^* + 3I_1 + I_2 + I_5^*$. However, we see easily that $S(0,0)$ is not a $K3$ surface, but a rational surface.

Next, we consider the case $\mathfrak{A} = 0$. In this case, note that $(\mathfrak{B},\mathfrak{C}) \neq (0,0)$. We have the equation of $S(0:\mathfrak{B}:\mathfrak{C})$: $z^2 = x^3 - 16y^3x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4$. On $\{\mathfrak{A} = 0\} \subset \mathbb{P}(1,3,5)$, we use the parameter $l = \frac{\mathfrak{C}^3}{\mathfrak{B}^5}$. By the correspondence $x = \frac{\mathfrak{C}^3}{\mathfrak{B}^4}x', y = \frac{\mathfrak{C}^2}{\mathfrak{B}^3}y'$, and $z = \frac{\sqrt{\mathfrak{C}^9}}{\mathfrak{B}^6}z'$, we have

$$S(l) : z'^2 = x'^3 - 16ly'^3x'^2 + 20y'^3x' + y'^4.$$

The discriminant of the right-hand side is given by $y'^8(27 + 32000y' + 5760ly'^2 - 102400l^2y'^4 - 16384l^3y'^5)$. From this, we can see that $S(l)$ is an elliptic $K3$ surface with the singular fibers of type $IV^* + 5I_1 + I_5^*$. □

Hence, we obtain the extended family $\{S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \mid (\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \in \mathbb{P}(1,3,5) - \{(1:0:0)\}\}$ of $K3$ surfaces. For simplicity, let $\mathcal{F}$ denotes this extended family.

2.2. The extension Φ of the period mapping Φ_1

Set $c_0 = (1:0:0) \in \mathbb{P}(1,3,5)$. In this subsection, we extend the period mapping $\Phi_1 : \mathfrak{X} \to \mathcal{D}_+$ in (2.2) to $\Phi : \mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+$.

First, we recall the S-marking on $\mathfrak{X}$. According to Theorem 2.1 and its proof, we have the elliptic $K3$ surface

$$\pi_{(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})} : S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \to \mathbb{P}^1(\mathbb{C}) = (y\text{-sphere})$$

for any $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C}) \in \mathbb{P}(1,3,5) - \{c_0\}$.

Take a generic point $(X_0,Y_0) \in \mathfrak{X}$. The elliptic $K3$ surface $\check{S} = S(X_0,Y_0)$ given by (2.4) and (2.5) has the singular fibers of type $IV^* + 5I_1 + I_5^*$. Let F be a general fiber of this elliptic fibration, and let O be the zero of the Mordell–Weil group of sections. We have two irreducible components of the divisor C given by $\{x = 0, z^2 = Yy^4\}$. We take the section R given by $y \mapsto (x,y,z) = (0,y,\sqrt{Y}y^2)$. This gives a component of the divisor C. Let us consider the irreducible decomposition $\bigcup_{j=0}^6 a_j$ (resp., $\bigcup_{j=0}^9 b_j$) of the singular fiber $\pi_{(X,Y)}^{-1}(0)$ (resp., $\pi_{(X,Y)}^{-1}(\infty)$) of type IV^* (resp., I_5^*). These curves are illustrated in Figure 2. Note that $a_0 \cap O \neq \phi$, $b_0 \cap O \neq \phi$, $a_6 \cap R \neq \phi$, and $b_9 \cap R \neq \phi$.

We set $\Gamma_5 = F, \Gamma_6 = O, \Gamma_7 = R$, $\Gamma_{8+k} = a_{k+1}$ $(0 \leq k \leq 5)$, $\Gamma_{14+l} = b_{l+1}$ $(0 \leq l \leq 8)$. We have the lattice $\check{L} = \langle \Gamma_5, \ldots, \Gamma_{22} \rangle_{\mathbb{Z}} \subset H_2(\check{S},\mathbb{Z})$. We can check that

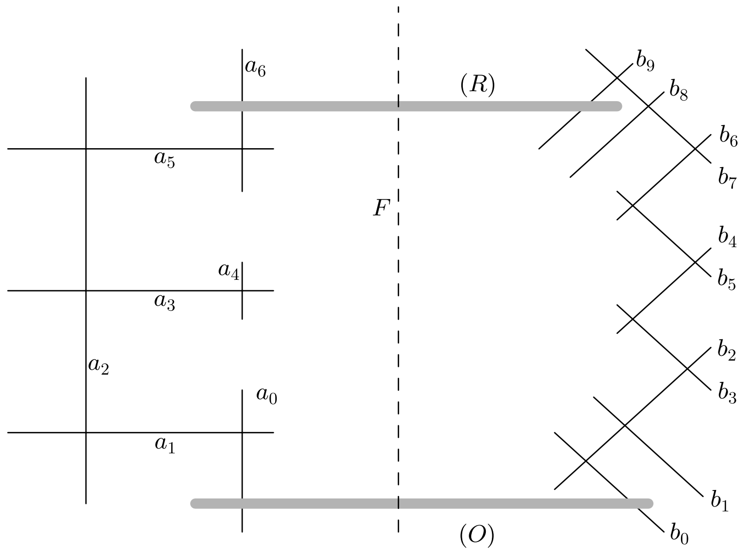

Figure 2. The elliptic fibration given by (2.3).

$|\det(\check{L})| = 5$. Hence, we have

$$\check{L} = \mathrm{NS}(\check{S}).$$

Since $\check{L}$ is a primitive lattice, there exists $\Gamma_1, \dots, \Gamma_4 \in H_2(\check{S}, \mathbb{Z})$ such that $\langle \Gamma_1, \dots, \Gamma_4, \Gamma_5, \dots, \Gamma_{22} \rangle_{\mathbb{Z}} = H_2(\check{S}, \mathbb{Z})$. Let $\{\Gamma_1^*, \dots, \Gamma_{22}^*\}$ be the dual basis of $\{\Gamma_1, \dots, \Gamma_{22}\}$ in $H_2(\check{S}, \mathbb{Z})$. Then, we see that $\langle \Gamma_1^*, \dots, \Gamma_4^* \rangle_{\mathbb{Z}}$ is the transcendental lattice. We may assume that its intersection matrix is

$$(\Gamma_j^* \cdot \Gamma_k^*)_{1 \le j,k \le 4} = A, \tag{2.6}$$

where A is given by (1.2). We define the period of $\check{S}$ by

$$\Phi_1(X_0, Y_0) = \left(\int_{\Gamma_1} \omega : \cdots : \int_{\Gamma_4} \omega \right).$$

Take a small connected neighborhood V_0 of (X_0, Y_0) in $\mathfrak{X}$ so that we have a local topological trivialization:

$$\tau : \left\{ S(p) \,\middle|\, p \in V_0 \right\} \to \check{S} \times V_0. \tag{2.7}$$

Let $\varpi : \check{S} \times V_0 \to \check{S}$ be the canonical projection. Set $r = \varpi \circ \tau$. Then,

$$r'_p = r|_{S(p)}$$

gives a $\mathcal{C}^\infty$-isomorphism of surfaces. For any $p \in V_0$, we have an isometry $\psi_p : H_2(S(p), \mathbb{Z}) \to H_2(\check{S}, \mathbb{Z})$ given by

$$\psi_p = {r'_p}_*.$$

We call this isometry the S-marking on V_0. By an analytic continuation along an arc $\alpha \subset \mathfrak{X}$, we define the S-marking on $\mathfrak{X}$. This depends on the choice of α.

The S-marking preserves the Néron–Severi lattice. We define the period mapping $\Phi_1 : \mathfrak{X} \to \mathcal{D}_+$ by

$$p \mapsto \Big(\int_{\psi_p^{-1}(\Gamma_1)} \omega : \cdots : \int_{\psi_p^{-1}(\Gamma_4)} \omega \Big).$$

This is equal to the period mapping in (2.2).

Here, we recall the P-marking for $K3$ surfaces, which is defined in [N1, Section 5].

DEFINITION 2.1

Let S be an algebraic $K3$ surface. An isometry

$$\psi : H_2(S,\mathbb{Z}) \to H_2(\check{S},\mathbb{Z})$$

is called the *P-marking* if

(i) $\psi^{-1}(\mathrm{NS}(\check{S})) \subset \mathrm{NS}(S)$,

(ii) $\psi^{-1}(F), \psi^{-1}(O), \psi^{-1}(R), \psi^{-1}(a_j)$ $(1 \le j \le 6)$, and $\psi^{-1}(b_j)$ $(1 \le j \le 9)$ are all effective divisors,

(iii) $(\psi^{-1}(F) \cdot C) \ge 0$ for any effective class C; namely, $\psi^{-1}(F)$ is nef.

A pair (S,ψ) is called a *P-marked* $K3$ *surface*.

DEFINITION 2.2

Two P-marked $K3$ surfaces (S_1,ψ_1) and (S_2,ψ_2) are said to be isomorphic if there is a biholomorphic mapping $f : S_1 \to S_2$ with

$$\psi_2 \circ f_* \circ \psi_1^{-1} = \mathrm{id}_{H_2(\check{S},\mathbb{Z})} .$$

Two P-marked $K3$ surfaces (S_1,ψ_1) and (S_2,ψ_2) are said to be equivalent if there is a biholomorphic mapping $f : S_1 \to S_2$ with

$$(\psi_2 \circ f_* \circ \psi_1^{-1})|_{\mathrm{NS}(\check{S})} = \mathrm{id}_{\mathrm{NS}(\check{S})} .$$

REMARK 2.3

The other connected component R' of the divisor C given by the section $y \mapsto (x, y, -\sqrt{Y}y^2)$ intersects a_4 (resp., b_8) at $y = 0$ (resp., $y = \infty$). Letting q be the involution of $S(X,Y)$ given by $(x,y,z) \mapsto (x,y,-z)$, we have $q_*(R') = R$, $q_*(a_4) = a_6$, $q_*(a_3) = a_5$, and $q_*(b_8) = b_9$. Then, we can see that P-marked $K3$ surfaces $(\check{S}, \mathrm{id})$ and $(\check{S}, q_*)$ are isomorphic by q. This shows that our argument does not depend on the choice of the curves R or R'.

The period of a P-marked $K3$ surface (S,ψ) is given by

$$\tilde{\Phi}'(S,\psi) = \Big(\int_{\psi^{-1}(\Gamma_1)} \omega : \cdots : \int_{\psi^{-1}(\Gamma_4)} \omega \Big). \tag{2.8}$$

It is a point in $\mathcal{D}$. Let $\mathbb{X}$ be the isomorphism classes of P-marked $K3$ surfaces, and let

$$[\mathbb{X}] = \mathbb{X}/(P\text{-marked equivalence}).$$

By the Torelli theorem for $K3$ surfaces, the period mapping $\tilde{\Phi}' : \mathbb{X} \to \mathcal{D}$ for P-marked $K3$ surfaces defined by (2.8) gives an identification between $\mathbb{X}$ and $\mathcal{D}$. Moreover, a P-marked $K3$ surface (S_1, ψ_1) is equivalent to a P-marked $K3$ surface (S_2, ψ_2) if and only if

$$\tilde{\Phi}'(S_1, \psi_1) = g \circ \tilde{\Phi}'(S_2, \psi_2)$$

for some $g \in \mathrm{PO}(A, \mathbb{Z})$ (see [N1, Lemma 5.1]). Therefore, we identify $[\mathbb{X}]$ with

$$\mathcal{D}/\,\mathrm{PO}(A,\mathbb{Z}) = \mathcal{D}_+/\,\mathrm{PO}^+(A,\mathbb{Z}) \simeq (\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}), \tau\rangle. \tag{2.9}$$

Recall that the above isomorphism is given by the modular isomorphism j in (1.7).

We note that $\mathfrak{X}$ is embedded in $[\mathbb{X}]$ (see [N1, Remark 5.3]). Then, an S-marked $K3$ surface is a P-marked $K3$ surface, and the period mapping for P-marked $K3$ surfaces is an extension of the period mapping for S-marked $K3$ surfaces. From $\tilde{\Phi}' : \mathbb{X} \to \mathcal{D}$, we obtain a multivalued mapping $\Phi' : [\mathbb{X}] \to \mathcal{D}_+$. We have

$$\Phi'|_{\mathfrak{X}} = \Phi_1, \tag{2.10}$$

where Φ_1 is the period mapping in (2.2) for S-marked $K3$ surfaces.

Now, we extend the period mapping $\Phi_1 : \mathfrak{X} \to \mathcal{D}_+$ in (2.2) to $\Phi : \mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+$. We recall that $(\mathbb{P}(1,3,5) - \{c_0\}) - \mathfrak{X} = (K_1 \cup K_2 \cup \{\mathfrak{A} = 0\}) - \{c_0\}$.

First, since the local topological trivialization on $\mathfrak{X}$ in (2.7) is naturally extended to $\{\mathfrak{A} = 0\}$, there exist S-markings on $\{\mathfrak{A} = 0\}$ and the period mapping (2.2) on $\mathfrak{X}$ is extended to $\mathbb{P}(1,3,5) - (K_1 \cup K_2 \cup \{c_0\}) \to \mathcal{D}_+$.

Let us recall that the projective monodromy group of Φ_1 is isomorphic to $\mathrm{PO}^+(A, \mathbb{Z})$. According to (2.9) and Proposition 1.2(3) (resp., Proposition 1.2(2)), the local monodromy of the period mapping Φ_1 in (2.2) around K_1 (resp., K_2) is locally finite. Hence, the period mapping $\mathbb{P}(1,3,5) - (K_1 \cup K_2 \cup \{c_0\}) \to \mathcal{D}_+$ can be extended to $\mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+$. We note that this extension is assured by Griffiths [Gr, Theorem (9.5)].

Therefore, we have the extended period mapping

$$\Phi : \mathbb{P}(1,3,5) - \{c_0\} \to \mathcal{D}_+ \tag{2.11}$$

with

$$\Phi|_{\mathfrak{X}} = \Phi_1. \tag{2.12}$$

Since we have (2.9) and Proposition 1.2(1), the P-marked equivalence class $[\mathbb{X}]$ is identified with $\mathbb{P}(1,3,5) - \{c_0\}$. Because we have (2.10), (2.12), and $\mathfrak{X}$ is a Zariski-open set in $\mathbb{P}(1,3,5) - \{c_0\}$, Φ in (2.11) is equal to the period mapping Φ' on $[\mathbb{X}]$.

Let $[\Phi(p)] \in \mathcal{D}_+/\,\mathrm{PO}^+(A,\mathbb{Z})$ be the equivalence class of $\Phi(p) \in \mathcal{D}_+$. From the above argument, we have the following proposition.

PROPOSITION 2.3

The period mapping $\Phi' : [\mathbb{X}] \to \mathcal{D}_+$ for P-marked $K3$ surfaces is given by the period mapping Φ in (2.11) for the family $\mathcal{F} = \{S(p) \mid p \in \mathbb{P}(1,3,5) - \{c_0\}\}$ of

$K3$ surfaces. This is an extension of the period mapping in (2.2) for S-marked $K3$ surfaces. Especially, if $[\Phi(p_1)] = [\Phi(p_2)]$ in $\mathcal{D}_+/\mathrm{PO}^+(A,\mathbb{Z})$, then $p_1 = p_2$.

For $p \in \mathbb{P}(1,3,5) - \{c_0\}$, let

$$\psi_p : H_2\big(S(p),\mathbb{Z}\big) \to H_2(\check{S},\mathbb{Z})$$

be a P-marking naturally induced by the above proposition. The period of $S(p)$ is given by

$$\Phi(p) = \Big(\int_{\psi_p^{-1}(\Gamma_1)} \omega : \int_{\psi_p^{-1}(\Gamma_2)} \omega : \int_{\psi_p^{-1}(\Gamma_3)} \omega : \int_{\psi_p^{-1}(\Gamma_4)} \omega\Big). \tag{2.13}$$

According to Remark 1.1, the multivalued analytic mapping $(j^{-1}\circ\Phi)|_{\mathfrak{X}} : \mathfrak{X} \to \mathbb{H}\times\mathbb{H}$ gives a developing map of the canonical projection $\Pi : \mathbb{H}\times\mathbb{H} \to (\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle$. Hence, by Proposition 2.3, $(j^{-1}\circ\Phi)|_{\mathfrak{X}}$ is extended to the analytic mapping

$$j^{-1}\circ\Phi : \mathbb{P}(1,3,5) - \{c_0\} \to \mathbb{H}\times\mathbb{H}.$$

This gives a developing map of Π.

REMARK 2.4

Sato [Sa] showed that the system of differential equations on $\mathfrak{X}$,

$$\begin{cases} u_{XX} = Lu_{XY} + Au_X + Bu_Y + Pu, \\ u_{YY} = Mu_{XY} + Cu_X + Du_Y + Qu \end{cases}$$

with $L = \frac{-20(4X^2+3XY-4Y)}{36X^2-32X-Y}, M = \frac{-2(54X^3-50X^2-3XY+2Y)}{5Y(36X^2-32X-Y)}$ is a uniformizing differential equation of $\overline{(\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle}$. Namely, taking linearly independent solutions y_0, y_1, y_2, and y_3, the mapping $p \mapsto (y_0(p) : \cdots : y_3(p))$ gives a developing map $\mathfrak{X} \to \mathcal{D}_+$. Of course, our equation (1.14) is also a uniformizing differential equation in this sense. But, note that we do not know whether we can extend it to the singular locus applying the theory of the uniformizing differential equations. Since we regard $\mathbb{P}(1,3,5) - \{c_0\}$ as the parameter space of $\mathcal{F}$ and $p \mapsto (y_0(p) : \cdots : y_3(p))$ is the period mapping for $\mathcal{F}$, we obtain the extension of the solutions of (1.14) to the singular locus.

Hence, we obtain the following theorem.

THEOREM 2.2

The multivalued mapping $j^{-1}\circ\Phi : \mathbb{P}(1,3,5) - \{c_0\} \to \mathbb{H}\times\mathbb{H}$ gives the developing map of Π. Namely, the inverse mapping of $\Pi : \mathbb{H}\times\mathbb{H} \to (\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle$ is given by $j^{-1}\circ\Phi$ through the identification $(\mathbb{H}\times\mathbb{H})/\langle \mathrm{PSL}(2,\mathcal{O}),\tau\rangle \simeq \mathbb{P}(1,3,5) - \{c_0\}$ given by Proposition 1.2(1).

Let Δ be the diagonal

$$\Delta = \big\{(z_1,z_2) \in \mathbb{H}\times\mathbb{H} \,\big|\, z_1 = z_2\big\}.$$

From the above theorem and Proposition 1.2(3), we have the following.

COROLLARY 2.1
It holds that

$$\Pi(\Delta) = \big\{(\mathfrak{A}:\mathfrak{B}:0)\big\} - \{c_0\}$$

through the identification $(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle \simeq \mathbb{P}(1,3,5)-\{c_0\}$ *given by Proposition 1.2(1).*

Due to Theorem 2.2, we obtain the system of coordinates (z_1,z_2) of $\mathbb{H}\times\mathbb{H}$ coming from the multivalued period mapping (2.13) for the family of $K3$ surfaces $\{S(p)\}$:

$$\big(z_1(p),z_2(p)\big) = \Big(-\frac{\int_{\Gamma_3}\omega + ((1-\sqrt{5})/2)\int_{\Gamma_4}\omega}{\int_{\Gamma_2}\omega}, -\frac{\int_{\Gamma_3}\omega + ((1+\sqrt{5})2)\int_{\Gamma_4}\omega}{\int_{\Gamma_2}\omega}\Big). \tag{2.14}$$

Here, for simplicity, let Γ_j denote the 2-cycle $\psi_p^{-1}(\Gamma_j)$ on $S(p)$ for $j\in\{1,2,3,4\}$.

According to Proposition 1.2(1), by adding one cusp, we have the compactification $\overline{(\mathbb{H}\times\mathbb{H})/\langle PSL(2,\mathcal{O}),\tau\rangle}$. Then, putting $\Pi\circ j^{-1}\circ\Phi(c_0)=\overline{(\sqrt{-1}\infty,\sqrt{-1}\infty)}$, we obtain an extended mapping

$$\Pi\circ j^{-1}\circ\Phi : \mathbb{P}(1,3,5)\to\overline{(\mathbb{H}\times\mathbb{H})/\big\langle PSL(2,\mathcal{O}),\tau\big\rangle}, \tag{2.15}$$

where $\overline{(\sqrt{-1}\infty,\sqrt{-1}\infty)}$ stands for the $\langle PSL(2,\mathcal{O}),\tau\rangle$-orbit of $(\sqrt{-1}\infty,\sqrt{-1}\infty)$.

3. The family $\mathcal{F}_X$ and the period differential equation

In this section, we consider the family $\mathcal{F}_X=\{S(X,0)\}$ and the diagonal $\Delta=\{(z_1,z_2)\in\mathbb{H}\times\mathbb{H}\mid z_1=z_2\}$.

3.1. The family $\mathcal{F}_X$

In Section 2, we had the $K3$ surfaces $S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})$ for $(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})\in\mathbb{P}(1,3,5)-\{c_0\}$ and the period mapping (2.13). Restricting them to $\{\mathfrak{C}=0\}$, we obtain the family $\{S(\mathfrak{A}:\mathfrak{B}:0)\mid(\mathfrak{A}:\mathfrak{B}:0)\neq c_0\}$ of $K3$ surfaces with $S(\mathfrak{A}:\mathfrak{B}:0):z^2=x^3-4y^2(4y-5\mathfrak{A})x^2+20\mathfrak{B}y^3x$. Then, we have the family $\mathcal{F}_X=\{S(X,0)\}$ of $K3$ surfaces with

$$S(X,0): z^2 = x^3-4y^2(4y-5)x^2+20Xy^3x,$$

where $X(=\frac{\mathfrak{B}}{\mathfrak{A}^3})\in\mathbb{P}^1(\mathbb{C})-\{0\}$. In this section, we consider the family $\mathcal{F}_X$ and the period mapping for $\mathcal{F}_X$.

Set $\Sigma=(X\text{-sphere }\mathbb{P}^1(\mathbb{C}))-\{0,\frac{25}{27},\infty\}$. Because we have Proposition 2.3, we can prove the following theorem for the subfamily $\mathcal{F}'_X=\{S(X,0)\mid X\in\Sigma\}$ as in [N1].

THEOREM 3.1

(1) *For a generic point* $X\in\Sigma$, *the intersection matrix of the Néron–Severi lattice* $NS(S(X,0))$ *is given by*

$$E_8(-1)\oplus E_8(-1)\oplus U\oplus\langle -2\rangle$$

and that of the transcendental lattice $\mathrm{Tr}(S(X,0))$ *is given by*

$$U\oplus\langle 2\rangle =: A_X.$$

(2) *The projective monodromy group of the multivalued period mapping for* $\mathcal{F}'_X$ *is isomorphic to* $\mathrm{PO}^+(A_X,\mathbb{Z})$.

From the period mapping Φ in (2.13), the system of coordinates (z_1,z_2) in (2.14), Corollary 2.1, and the above theorem, we obtain a multivalued period mapping Φ_X for $\mathcal{F}_X$ such that

$$j^{-1}\circ\Phi_X:\{X \mid X\in\mathbb{P}^1(\mathbb{C})-\{0\}\}\to\Delta, \tag{3.1}$$

where Φ_X is given by $X\mapsto(\xi_1:\xi_2:\xi_3:\xi_4)=(\int_{\Gamma_1}\omega:\int_{\Gamma_2}\omega:\int_{\Gamma_3}\omega:0)\in\mathcal{D}_+$ satisfying the Riemann–Hodge relation $(\int_{\Gamma_1}\omega)(\int_{\Gamma_2}\omega)+(\int_{\Gamma_3}\omega)^2=0$. The fundamental group $\pi_1(\Sigma,*)$ induces the projective monodromy group M_X for Φ_X. According to Theorem 3.1(2), M_X is isomorphic to $\mathrm{PO}^+(A_X,\mathbb{Z})$. From (2.14), we have the coordinate z of $\Delta\simeq\mathbb{H}$:

$$z=-\frac{\int_{\Gamma_3}\omega}{\int_{\Gamma_2}\omega}. \tag{3.2}$$

Recalling (2.15), we obtain an extended mapping $\Pi\circ j^{-1}\circ\Phi_X:\mathbb{P}^1(\mathbb{C})\to\overline{\Delta/M_X}$. We note that $\Pi\circ j^{-1}\circ\Phi_X(0)$ is the M_X-orbit of $(\sqrt{-1}\infty,\sqrt{-1}\infty)$. The action of M_X on $\Delta(\subset\mathbb{H}\times\mathbb{H})$ induces the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}$, for we have the coordinate z in (3.2). Namely, there exist $\gamma_1,\gamma_2\in\pi_1(\Sigma,*)$ such that

$$\gamma_1(z)=z+1,\qquad \gamma_2(z)=-\frac{1}{z}. \tag{3.3}$$

So, $\overline{\Delta/M_X}$ is identified with the orbifold $\overline{\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})}\simeq\mathbb{P}^1(\mathbb{C})$.

REMARK 3.1

The projective monodromy group $M_X\simeq\mathrm{PO}^+(A_X,\mathbb{Z})$ of the period mapping Φ_X is generated by two elements:

$$\begin{pmatrix}1&-1&2\\0&1&0\\0&-1&1\end{pmatrix},\qquad\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix}. \tag{3.4}$$

These are induced by the monodromy matrices in (1.6).

3.2. The Gauss hypergeometric equation ${}_2E_1(\frac{1}{12},\frac{5}{12},1;t)$

We recall the Gauss hypergeometric equation

$$ {}_2E_1\Big(\frac{1}{12},\frac{5}{12},1;t\Big): t(1-t)\frac{d^2}{dt^2}u+\Big(1-\frac{3}{2}t\Big)\frac{d}{dt}u-\frac{5}{144}u=0. \tag{3.5}$$

The Riemann scheme of ${}_2E_1(\frac{1}{12}, \frac{5}{12}, 1; t)$ is given by

$$\begin{Bmatrix} t=0 & t=1 & t=\infty \\ 0 & 0 & 1/12 \\ 0 & 1/2 & 5/12 \end{Bmatrix}.$$

We can take the solutions $y_1(t)$ and $y_2(t)$ of ${}_2E_1(\frac{1}{12}, \frac{5}{12}, 1; t)$ such that the inverse mapping of the Schwarz mapping

$$\begin{array}{cccc} \sigma & : & \mathbb{C}-\{0,1\} & \rightarrow & \mathbb{H} \\ & ; & t & \mapsto & \sigma(t) = \frac{y_2(t)}{y_1(t)} = z_0 \end{array} \tag{3.6}$$

is given by

$$z_0 \mapsto \frac{1}{J(z_0)}, \tag{3.7}$$

where $J(z)$ is the elliptic J function with $J(\frac{1+\sqrt{-3}}{2}) = 0, J(\sqrt{-1}) = 1,$ and $J(\sqrt{-1}\infty) = \infty$.

REMARK 3.2

The above J-function is given by

$$J(z) = \frac{1}{1728}\Big(\frac{1}{q} + 744 + 196884q + \cdots\Big), \tag{3.8}$$

where $q = e^{2\pi\sqrt{-1}z}$.

Note that the Schwarz mapping σ is a multivalued analytic mapping. We can choose the single-valued branch of the Schwarz mapping σ on $(0,1) \subset \mathbb{R}$ such that $\sigma(t) \in \sqrt{-1}\mathbb{R}$ and

$$\lim_{t\to+0} \sigma(t) = \sqrt{-1}\infty, \qquad \lim_{t\to 1-0} \sigma(t) = \sqrt{-1}. \tag{3.9}$$

Then, the single-valued branch of the solutions $y_1(t)$ and $y_2(t)$ near $(0,1)(\subset \mathbb{R})$ is in the form

$$\begin{cases} y_1(t) = u_{11}(t), \\ y_2(t) = \log(t)\cdot u_{21}(t) + u_{22}(t), \end{cases} \tag{3.10}$$

where $u_{jk}(t)$ are unit holomorphic functions around $t=0$ and log stands for the principal value.

The projective monodromy group of ${}_2E_1(\frac{1}{12}, \frac{5}{12}, 1; t)$ is isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$. In other words, the action of the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}, *)$ on $\mathbb{H} = \{z_0 = \frac{y_2}{y_1}\}$ is generated by the two actions

$$z_0 \mapsto z_0 + 1, \qquad z_0 \mapsto -\frac{1}{z_0}, \tag{3.11}$$

if we normalize a basis y_1, y_2 of the solutions of ${}_2E_1(\frac{1}{12}, \frac{5}{12}, 1; t)$ around a base point.

REMARK 3.3

The projective monodromy group for the system $(y_2^2(t); -y_1^2(t); y_1(t)y_2(t))$ is generated by the two matrices in (3.4).

3.3. The period differential equation

In this subsection, we determine the period differential equation for the family $\mathcal{F}_X$. Then, considering the solutions of this period differential equation, we shall obtain the expression of X using the coordinate z in (3.2).

PROPOSITION 3.1

On the locus $\{Y = 0\}$, the period differential equation (1.14) is restricted to the following ordinary differential equation of rank 4:

$$\begin{aligned} &\frac{d^4}{dX^4}u + \frac{3(243X^2 - 4060X + 2000)}{2X(81X^2 - 1155X + 1000)}\frac{d^3}{dX^3}u \\ &\quad + \frac{2034X^2 - 40680X + 8000}{8X^2(81X^2 - 1155X + 1000)}\frac{d^2}{dX^2}u \\ &\quad + \frac{15(3X - 80)}{8X^2(81X^2 - 1155X + 1000)}\frac{d}{dX}u = 0. \end{aligned} \tag{3.12}$$

Proof

Recalling the period differential equation (1.14), set

$$\begin{cases} E_1 u = L_1 u_{XY} + A_1 u_X + B_1 u_Y + P_1 u, \\ E_2 u = M_1 u_{XY} + C_1 u_X + D_1 u_Y + Q_1 u. \end{cases}$$

Deriving these equations, we have the system of equations

$$\begin{cases} u_{XX} = E_1 u, \qquad u_{XXX} = \frac{\partial}{\partial X} E_1 u, \qquad u_{XXY} = \frac{\partial}{\partial Y} E_1 u, \\ \quad u_{XXXX} = \frac{\partial^2}{\partial X^2} E_1 u, \qquad u_{XXXY} = \frac{\partial^2}{\partial X \partial Y} E_1 u, \\ u_{YY} = E_2 u, \qquad u_{XYY} = \frac{\partial}{\partial X} E_2 u, \qquad u_{YYY} = \frac{\partial}{\partial Y} E_2 u, \\ \quad u_{XXYY} = \frac{\partial^2}{\partial Y^2} E_1 u = \frac{\partial^2}{\partial X^2} E_2 u. \end{cases}$$

Our periods satisfy this system. From this system, canceling the terms u_Y, u_{XY}, $u_{YY}, u_{XXY}, u_{XYY}, u_{YYY}$, u_{XXXY}, and u_{XXYY}, we can obtain the differential equation

$$\begin{aligned} &a_4(X,Y)u_{XXXX} + a_3(X,Y)u_{XXX} + a_2(X,Y)u_{XX} \\ &\quad + a_1(X,Y)u_X + a_0(X,Y)u = 0, \end{aligned}$$

where $a_j(X,Y)$ $(j = 1,2,3,4)$ is a polynomial in X and Y. Putting $Y = 0$, we have (3.12). □

Set

$$\check{\eta}_j(X) = \int_{\Gamma_j} \omega \quad (j \in 1,2,3).$$

The equation (3.12) has the 4-dimensional space of solutions generated by $\check{\eta}_1(X)$, $\check{\eta}_2(X), \check{\eta}_3(X)$ and 1. The Riemann scheme of (3.12) is given by

$$\begin{Bmatrix} X=0 & X=25/27 & X=40/3 & X=\infty \\ 0 & 0 & 0 & 0 \\ 1 & 1/2 & 1 & -5/6 \\ 1 & 1 & 2 & -1/2 \\ 1 & 2 & 4 & -1/6 \end{Bmatrix}.$$

We set $X = \frac{25}{27}t$, and the equation (3.12) is transposed to

$$W_4 u = 0,$$

where

$$W_4 = \frac{d^4}{dt^4} + \frac{1620t^3 - 29232t^2 + 15552t}{72t^2(t-1)(5t-72)} \frac{d^3}{dt^3} + \frac{565t^2 - 12204t + 2592}{36t^2(t-1)(5t-72)} \frac{d^2}{dt^2} + \frac{25t - 720}{72t^2(t-1)(5t-72)} \frac{d}{dt}.$$

Straightforward calculation shows the following.

PROPOSITION 3.2

Set

$$W_3 = \frac{d^3}{dt^3} + \frac{3}{2(t-1)} \frac{d^2}{dt^2} + \frac{5t-36}{36t^2(t-1)} \frac{d}{dt} + \frac{72-5t}{72t^3(t-1)},$$

$$W_1 = \frac{d}{dt} + \frac{15t^2 - 298t + 216}{t(t-1)(5t-72)}.$$

It holds that

$$W_4 = W_1 \circ W_3. \tag{3.13}$$

Set $\eta_j(t) = \check{\eta}_j(\frac{25}{27}t)$ for $j \in \{1,2,3\}$.

PROPOSITION 3.3

The periods $\eta_1(t), \eta_2(t)$, and $\eta_3(t)$ are the solutions of

$$W_3 u = 0$$

satisfying

$$\eta_1\eta_2 + \eta_3^2 = 0. \tag{3.14}$$

Proof

Let $V = \langle \eta_1, \eta_2, \eta_3 \rangle_{\mathbb{C}}$ and $V' = \langle W_3\eta_1, W_3\eta_2, W_3\eta_3 \rangle_{\mathbb{C}}$. Since the linear mapping $W_3 : V \to V'$ given by $f \mapsto W_3 f$ is monodromy-equivalent and V is an irreducible representation, according to Schur's lemma, we have $V \simeq V'$ or $V' = \{0\}$. It follows from (3.13) that $V' \subset \mathrm{Ker}(W_1)$. Because $\dim(\mathrm{Ker}(W_1)) = 1$, we have $V' = \{0\}$.

For $t \mapsto (\eta_1(t):\eta_2(t):\eta_3(t))$ is the period mapping Φ_X, the relation (3.14) is clear. □

PROPOSITION 3.4

If u_1 and u_2 are solutions of ${}_2E_1(\frac{1}{12},\frac{5}{12},1;t)$, then $tu_1^2(t), tu_2^2(t)$, and $tu_1(t)u_2(t)$ are solutions of the period differential equation $W_3u=0$.

Proof

Take any solutions of ${}_2E_1(\frac{1}{12},\frac{5}{12},1;t)$ $u_1(t)$ and $u_2(t)$. For $j\in\{1,2\}$,

$$u_j'' = \frac{1-3t/2}{t(t-1)}u_j' - \frac{5}{144t(t-1)}u_j; \tag{3.15}$$

then

$$u_j^{(3)} = \frac{535t^2-715t+288}{144t^2(t-1)^2}u_j' + \frac{5(7t-4)}{288t^2(t-1)^2}u_j. \tag{3.16}$$

Here, by a straightforward calculation, we have

$$\begin{aligned} W_3(tu_1u_2) &= \frac{5}{72t(t-1)}u_1u_2 + \frac{113t-36}{36t(t-1)}(u_1'u_2+u_1u_2') + \frac{3(3t-2)}{t-1}u_1'u_2' \\ &\quad + \frac{3(3t-2)}{2(t-1)}(u_1''u_2+u_1u_2'') + 3t(u_1'u_2''+u_1''u_2') + t(u_1^{(3)}u_2+u_1u_2^{(3)}). \end{aligned} \tag{3.17}$$

Substituting (3.15) and (3.16) for (3.17), we have $W_3(tu_1u_2)=0$. □

REMARK 3.4

According to (3.12), the derivation $\frac{d}{dt}\eta_j$ $(j=1,2,3)$ of the period is a solution of the equation

$$\begin{aligned} &\frac{d^3}{dt^3}v + \frac{1620t^3-29232t^2+15552t}{72t^2(t-1)(5t-72)}\frac{d^2}{dt^2}v + \frac{1130t^2-24408t+5184}{72t^2(t-1)(5t-72)}\frac{d}{dt}v \\ &\quad + \frac{25t-720}{72t^2(t-1)(5t-72)}v = 0. \end{aligned} \tag{3.18}$$

Then, set

$$S(t) = {}_3F_2\Big(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;t\Big) + \frac{1}{5}{}_3F_2\Big(\frac{7}{6},\frac{1}{2},\frac{5}{6};1,1;t\Big),$$

where ${}_3F_2$ is the generalized hypergeometric series

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;t) = \sum_{t=0}^{\infty}\frac{(a_1,n)(a_2,n)(a_3,n)}{(b_1,n)(b_2,n)n!}t^n.$$

We see that $S(t)$ is a holomorphic solution of (3.18) around $t=0$. The indefinite integral of $S(t)$ with the integral constant 0 is given by

$$\begin{aligned} &t\cdot{}_3F_2\Big(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,2;t\Big) + \frac{1}{5}t\cdot{}_3F_2\Big(\frac{7}{6},\frac{1}{2},\frac{5}{6};1,2;t\Big) \\ &\quad = \frac{6}{5}t\cdot{}_3F_2\Big(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;t\Big) = \frac{6}{5}t\cdot\Big({}_2F_1\Big(\frac{1}{12},\frac{5}{12},1;t\Big)\Big)^2. \end{aligned}$$

Here, we applied Clausen's formula. From Proposition 3.4, this gives a holomorphic solution of $W_3u=0$ around $t=0$.

Let $y_1(t)$ and $y_2(t)$ be the single-valued branches of the solutions of ${}_2E_1(\frac{1}{12},\frac{5}{12},1;t)$ near $(0,1)\subset\mathbb{R}$ given in (3.9). Let

$$s_1(t)=ty_1^2(t),\qquad s_2(t)=ty_1(t)y_2(t),\qquad s_3(t)=ty_2^2(t).$$

Note that if $t\in(0,1)\subset\mathbb{R}$, we have

$$\begin{cases} s_1(t)=t\cdot v_{11}(t),\\ s_2(t)=t\cdot(\log(t)v_{21}(t)+v_{22}(t)),\\ s_3(t)=t\cdot(\log^2(t)v_{31}(t)+\log(t)v_{32}(t)+v_{33}(t)),\end{cases}\tag{3.19}$$

where $v_{jk}(t)$ are unit holomorphic functions around $t=0$. Moreover, they satisfy

$$-s_1(t)s_3(t)+s_2^2(t)=0.\tag{3.20}$$

LEMMA 3.1

A branch of the multivalued analytic mapping $t\mapsto(\eta_1(t):\eta_2(t):\eta_3(t))$ satisfies

$$\big(\eta_1(t):\eta_2(t):\eta_3(t)\big)=\big(s_3(t):-s_1(t):s_2(t)\big)\in\mathbb{P}^2(\mathbb{C}).$$

Proof

Because we have Proposition 1.2(1) and the coordinate z in (3.2), we take the single-valued branch of the multivalued period mapping $t\mapsto(\eta_1(t):\eta_2(t):\eta_3(t))$ on $t\in(0,1)\subset\mathbb{R}$ such that

$$\lim_{t\to+0}-\frac{\eta_3(t)}{\eta_2(t)}=\sqrt{-1}\infty.\tag{3.21}$$

In this proof, we consider $\eta_1(t),\eta_2(t)$, and $\eta_3(t)$ near $(0,1)(\subset\mathbb{R})$.

According to Proposition 3.4, we have

$$\eta_j(t)=\sum_{k=1}^{3}a_{jk}s_k(t)\quad(j=1,2,3),$$

where a_{jk} $(j,k=1,2,3)$ are constants. Since we have (3.21), we obtain $a_{23}=0$. So, it follows that $\eta_2(t)=a_{21}s_1(t)+a_{22}s_2(t)$. From (3.19), we see that $\eta_1(t)\eta_2(t)$ does not contain $\log^4(t)$. Then, from (3.14), we have $a_{33}=0$. Recalling (3.21) again, we obtain $a_{22}=0$. Because we consider $y\mapsto(\eta_1(t):\eta_2(t):\eta_3(t))\in\mathbb{P}^2(\mathbb{C})$, we assume that $a_{21}=-1$. Then, the single-valued branches $\eta_j(t)$ $(j=1,2,3)$ are in the form

$$\begin{cases}\eta_1(t)=a_{11}s_1(t)+a_{12}s_2(t)+a_{13}s_3(t),\\ \eta_2(t)=-s_1(t),\\ \eta_3(t)=a_{31}s_1(t)+a_{32}s_2(t).\end{cases}$$

Hence, using (3.6), the coordinate z in (3.2) is given by

$$z = a_{32}\frac{s_2(z)}{s_1(z)} + a_{31} = a_{32}z_0 + a_{31}.$$

Considering the actions of $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\})$ on $z = -\frac{\eta_3}{\eta_2}$ space in (3.3) and $z_0 = \frac{y_2}{y_1}$ space in (3.11), we have $a_{31} = 0$ and $a_{32} = 1$.

Therefore, using (3.14) again, we obtain

$$\eta_1(t) = s_3(t), \qquad \eta_2(t) = -s_1(t), \qquad \eta_3(t) = s_2(t). \qquad \square$$

COROLLARY 3.1

A coordinate z in (3.2) of the diagonal Δ ($\simeq \mathbb{H}$) is equal to

$$z = \frac{y_2(t)}{y_1(t)}.$$

Proof

From the above lemma, this is clear. $\square$

THEOREM 3.2

The inverse of the multivalued period mapping $j^{-1} \circ \Phi_X : X \mapsto (z,z)$ in (3.1) is given by

$$X(z,z) = \frac{25}{27}\cdot\frac{1}{J(z)}.$$

Proof

From Corollary 3.1 and the inverse Schwarz mapping (3.7), we have $t(z) = \frac{1}{J(z)}$. Therefore, we obtain

$$X(z,z) = \frac{25}{27}\cdot t(z) = \frac{25}{27}\cdot\frac{1}{J(z)}. \qquad \square$$

4. The theta expressions of X and Y

First, we recall the classical elliptic functions. Let $z \in \mathbb{H}$.

The classical Eisenstein series are given by

$$G_2(z) = 60 \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^4}, \qquad G_3(z) = 140 \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^6}.$$

$G_2(z)$ (resp., $G_3(z)$) is a modular form of weight 4 (resp., 6) for $\mathrm{PSL}(2,\mathbb{Z})$. The ring of modular forms for $\mathrm{PSL}(2,\mathbb{Z})$ is $\mathbb{C}[G_2, G_3]$. We have $G_2(\sqrt{-1}\infty) = \frac{4\pi^4}{3}$ and $G_3(\sqrt{-1}\infty) = \frac{8\pi^6}{27}$. Let $E_4(z) = \frac{3}{4\pi^4}G_2(z)$ and $E_6(z) = \frac{27}{8\pi^6}G_3(z)$ be the normalized Eisenstein series. The discriminant form is

$$\Delta(z) = G_2^3(z) - 27G_3^2(z).$$

We have $\Delta(\sqrt{-1}\infty)=0$. This is a cusp form of weight 12. The cusp form of weight 12 is Δ up to a constant factor. The J-function in (3.8) is given by

$$J(z)=\frac{G_2^3(z)}{G_2^3(z)-27G_3^2(z)}=\frac{G_2^3(z)}{\Delta(z)}. \tag{4.1}$$

The field of modular functions for the modular group $\mathrm{PSL}(2,\mathbb{Z})$ is $\mathbb{C}(J(z))$.

For $a,b\in\{0,1\}$, the Jacobi theta constants are defined by

$$\vartheta_{ab}(z)=\sum_{n\in\mathbb{Z}}\exp\Big(\sqrt{-1}\pi\Big(n+\frac{a}{2}\Big)^2 z+2\sqrt{-1}\pi\Big(n+\frac{a}{2}\Big)\frac{b}{2}\Big)$$

for $(a,b)=(0,0),(0,1)$ and $(1,0)$. The functions $\vartheta_{00}^4(z),\vartheta_{01}^4(z)$, and $\vartheta_{10}^4(z)$ are the modular forms of weight 2 for the principal congruence subgroup $\Gamma(2)=\{\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\mid \alpha\equiv\delta\equiv 1,\beta\equiv\gamma\equiv 0 \pmod 2\}$. The ring of modular forms for $\Gamma(2)$ is

$$\mathbb{C}[\vartheta_{00}^4,\vartheta_{01}^4,\vartheta_{10}^4]/(\vartheta_{01}^4+\vartheta_{10}^4=\vartheta_{00}^4)=\mathbb{C}[\vartheta_{00}^4,\vartheta_{01}^4].$$

We note that

$$\frac{1}{1728}\Big(\frac{3}{4\pi^4}\Big)^3\Delta(z)=\frac{1}{2^8}\vartheta_{00}^8(z)\vartheta_{01}^8(z)\vartheta_{10}^8(z).$$

Next, we survey the theta constants for Hilbert modular forms for $\mathbb{Q}(\sqrt{5})$. They are introduced by Müller [M].

Set

$$\mathfrak{S}_2=\big\{Z\in\mathrm{Mat}(2,2)\,\big|\,{}^tZ=Z,\mathrm{Im}(Z)>0\big\}.$$

This is the Siegel upper half-plane consisting of (2×2)-complex matrices. For $a,b\in\{0,1\}^2$ with ${}^tab\equiv 0 \pmod 2$, set

$$\vartheta(Z;a,b)=\sum_{g\in\mathbb{Z}^2}\exp\Big(\pi\sqrt{-1}\Big({}^t\Big(g+\frac{1}{2}a\Big)Z\Big(g+\frac{1}{2}a\Big)+{}^tgb\Big)\Big).$$

We use the mapping $\psi:\mathbb{H}\times\mathbb{H}\to\mathfrak{S}_2$ given by

$$\begin{aligned}(z_1,z_2)=\zeta\mapsto&\begin{pmatrix}\mathrm{Tr}(\frac{\varepsilon\zeta}{\sqrt{5}}) & \mathrm{Tr}(\frac{\zeta}{\sqrt{5}})\\ \mathrm{Tr}(\frac{\zeta}{\sqrt{5}}) & \mathrm{Tr}(-\frac{\varepsilon'\zeta}{\sqrt{5}})\end{pmatrix}\\ &=\frac{1}{2\sqrt{5}}\begin{pmatrix}(1+\sqrt{5})z_1-(1-\sqrt{5})z_2 & 2(z_1-z_2)\\ 2(z_1-z_2) & (-1+\sqrt{5})z_1+(1+\sqrt{5})z_2\end{pmatrix},\end{aligned}$$

where $\varepsilon=\frac{1+\sqrt{5}}{2}$.

REMARK 4.1

Set

$$\mathcal{N}_5=\left\{\begin{pmatrix}\sigma_1&\sigma_2\\ \sigma_2&\sigma_3\end{pmatrix}\in\mathfrak{S}_2\,\middle|\,-\sigma_1+\sigma_2+\sigma_3=0\right\}.$$

Let p be the canonical projection $\mathfrak{S}_2\to\mathfrak{S}_2/\mathrm{Sp}(4,\mathbb{Z})$. Then, the Humbert surface $\mathcal{H}_5=p(\mathcal{N}_5)$ of invariant 5 gives the moduli space of principally polarized Abelian

Table 1. The correspondence between j and (a,b)

j	0	1	2	3	4	5	6	7	8	9
${}^t a$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$	$(1,0)$	$(0,0)$	$(0,1)$
${}^t b$	$(0,0)$	$(0,0)$	$(1,1)$	$(1,1)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$

surfaces A such that $\mathbb{Q}(\sqrt{5}) \subset \mathrm{End}(A) \otimes \mathbb{Q}$. We note that the above ψ is a mapping $\mathbb{H} \times \mathbb{H} \to \mathcal{N}_5$.

For $j \in \{0,1,\ldots,9\}$, we set

$$\theta_j(z_1,z_2) = \vartheta\big(\psi(z_1,z_2); a, b\big),$$

where the correspondence between j and (a,b) is given by Table 1. These theta constants are holomorphic functions on $\mathbb{H} \times \mathbb{H}$.

Let $a \in \mathbb{Z}$ and $j_1,\ldots,j_r \in \{0,\ldots,9\}$. We set $\theta^a_{j_1,\ldots,j_r} = \theta^a_{j_1} \cdots \theta^a_{j_r}$.

Set $s_5 = 2^{-6}\theta_{0123456789}$. This is an alternating modular form of weight 5. The following g_2 (resp., s_6, s_{10}, s_{15}) is a symmetric Hilbert modular form of weight 2 (resp., $6, 10, 15$) for $\mathbb{Q}(\sqrt{5})$:

$$\begin{cases} g_2 = \theta_{0145} - \theta_{1279} - \theta_{3478} + \theta_{0268} + \theta_{3569}, \\ s_6 = 2^{-8}(\theta^2_{012478} + \theta^2_{012569} + \theta^2_{034568} + \theta^2_{236789} + \theta^2_{134579}), \\ s_{10} = s_5^2 = 2^{-12}\theta^2_{0123456789}, \\ s_{15} = -2^{-18} \\ \qquad \times (\theta^9_{07}\theta^5_{18}\theta_{24} - \theta^9_{25}\theta^5_{16}\theta_{09} + \theta^9_{58}\theta^5_{03}\theta_{46} - \theta^9_{09}\theta^5_{25}\theta_{16} + \theta^9_{09}\theta^5_{16}\theta_{25} - \theta^9_{67}\theta^5_{23}\theta_{89} \\ \qquad + \theta^9_{18}\theta^5_{24}\theta_{07} - \theta^9_{24}\theta^5_{18}\theta_{07} - \theta^9_{46}\theta^5_{03}\theta_{58} - \theta^9_{24}\theta^5_{07}\theta_{18} - \theta^9_{89}\theta^5_{67}\theta_{23} - \theta^9_{07}\theta^5_{24}\theta_{18} \\ \qquad + \theta^9_{89}\theta^5_{23}\theta_{67} - \theta^9_{49}\theta^5_{13}\theta_{57} + \theta^9_{16}\theta^5_{09}\theta_{25} - \theta^9_{03}\theta^5_{46}\theta_{58} + \theta^9_{16}\theta^5_{25}\theta_{09} - \theta^9_{46}\theta^5_{58}\theta_{03} \\ \qquad - \theta^9_{25}\theta^5_{09}\theta_{16} - \theta^9_{57}\theta^5_{49}\theta_{13} + \theta^9_{67}\theta^5_{89}\theta_{23} + \theta^9_{58}\theta^5_{46}\theta_{03} + \theta^9_{57}\theta^5_{13}\theta_{49} - \theta^9_{23}\theta^5_{89}\theta_{67} \\ \qquad + \theta^9_{18}\theta^5_{07}\theta_{24} + \theta^9_{03}\theta^5_{58}\theta_{46} + \theta^9_{23}\theta^5_{67}\theta_{89} + \theta^9_{49}\theta^5_{57}\theta_{13} - \theta^9_{13}\theta^5_{57}\theta_{49} + \theta^9_{13}\theta^5_{49}\theta_{57}). \end{cases}$$

(4.2)

PROPOSITION 4.1 ([M, SATZ 1])

(1) *The ring of the symmetric Hilbert modular forms for* $\mathbb{Q}(\sqrt{5})$ *is given by*

$$\mathbb{C}[g_2, s_6, s_{10}, s_{15}]/\big(M(g_2,s_6,s_{10},s_{15}) = 0\big),$$

where

$$\begin{aligned} &M(g_2,s_6,s_{10},s_{15}) \\ &= s_{15}^2 - \Big(5^5 s_{10}^3 - \frac{5^3}{2} g_2^2 s_6 s_{10}^2 + \frac{1}{2^4} g_2^5 s_{10}^2 + \frac{3^2\cdot 5^2}{2} g_2 s_6^3 s_{10} \\ &\quad - \frac{1}{2^3} g_2^4 s_6^2 s_{10} - 2\cdot 3^3 s_6^5 + \frac{1}{2^4} g_2^3 s_6^4\Big). \end{aligned} \tag{4.3}$$

(2) *The ring of the Hilbert modular forms for* $\mathbb{Q}(\sqrt{5})$ *is given by*

$$\mathbb{C}[g_2, s_5, s_6, s_{15}]/\big(M(g_2, s_5^2, s_6, s_{15}) = 0\big).$$

PROPOSITION 4.2 ([M, PP. 244–245])
Müller's modular forms satisfy

$$\begin{cases} g_2(i\infty, i\infty) = 1, \\ s_6(z,z) = \frac{2}{1728}(\frac{3}{4\pi^4})^3\Delta(z) = \frac{1}{2^7}\vartheta_{00}^8(z)\vartheta_{01}^8(z)\vartheta_{10}^8(z), \\ s_{10}(z,z) = 0. \end{cases}$$

Especially, the relations

$$\begin{cases} \frac{4\pi^4}{3}g_2(z,z) = \frac{4\pi^4}{3}E_4(z) = G_2(z), \\ 2^{11}\pi^{12}s_6(z,z) = G_2^3(z) - 27G_3^2(z) = \Delta(z) \end{cases}$$

hold.

Now, we obtain the theta expressions of the parameters X and Y for the family $\mathcal{F}$. According to Proposition 1.1, $\{X = \frac{\mathfrak{B}}{\mathfrak{A}^3}, Y = \frac{\mathfrak{C}}{\mathfrak{A}^5}\}$ gives a system of generators of symmetric Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$. From Theorem 2.2, the inverse correspondence $(z_1,z_2) \mapsto (X(z_1,z_2), Y(z_1,z_2))$ of the multivalued period mapping for $\mathcal{F}$ defines the pair of Hilbert modular functions of variables z_1 and z_2 in (2.14). In the following argument, we shall obtain the expression of $X(z_1,z_2)$ and $Y(z_1,z_2)$ as the quotients of Müller's modular forms.

For our argument, we set $Z = \frac{\mathfrak{D}^2}{\mathfrak{A}^{15}}$. This defines a symmertic Hilbert modular function for $\mathbb{Q}(\sqrt{5})$ also.

LEMMA 4.1
The modular functions $X(z_1,z_2), Y(z_1,z_2)$, and $Z(z_1,z_1)$ have the expressions

$$\begin{cases} X(z_1,z_2) = k_1\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}, \\ Y(z_1,z_2) = k_2\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}, \\ Z(z_1,z_2) = k_3\frac{s_{15}^2(z_1,z_2)}{g_2^{15}(z_1,z_2)}, \end{cases} \tag{4.4}$$

for some k_1, k_2, and $k_3 \in \mathbb{C}$.

Proof
Since $X = \frac{\mathfrak{B}}{\mathfrak{A}^3}$, the modular function X is given by the quotient of Hilbert modular forms of weight 6, and its denominator is the cube of a Hilbert modular form of weight 2. Note that a Hilbert modular form of weight 2 is equal to g_2 up to a constant factor. Then, we have

$$X(z_1,z_2) = \frac{k_{11}s_6(z_1,z_2) + k_{12}g_2^3(z_1,z_2)}{k_{13}g_2^3(z_1,z_2)},$$

where k_{11}, k_{12}, and k_{13} are constants. Recalling Proposition 1.2(1), we have $X(\sqrt{-1}\infty, \sqrt{-1}\infty) = 0$. Then, from Proposition 4.2, we obtain $k_{12} = 0$ and

$$X(z_1,z_2) = k_1\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}.$$

Since $Y = \frac{\mathfrak{C}}{\mathfrak{A}^5}$, the modular function Y is given by the quotient of Hilbert modular forms of weight 10. Its denominator is the 5th power of a modular form of weight 2. Then, we have

$$Y(z_1,z_2) = \frac{k_{21}s_{10}(z_1,z_2) + k_{22}g_2^5(z_1,z_2) + k_{23}g_2^2(z_1,z_2)s_6(z_1,z_2)}{k_{24}g_2^5(z_1,z_2)},$$

where k_{21}, k_{22}, k_{23}, and k_{24} are constants. By Proposition 1.2(3), we have $Y(z,z) = 0$. According to (4.2) and Proposition 4.2, if a modular form g of weight 10 vanishes on the diagonal Δ, then we have $g = \text{const} \cdot s_{10}$. So, it holds that $k_{22} = k_{23} = 0$. Therefore, we obtain

$$Y(z_1,z_2) = k_2 \frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}.$$

Recalling Proposition 1.1(2), we note that $\mathfrak{D}$ defines a symmetric Hilbert modular form of weight 15. Since $Z = \frac{\mathfrak{D}^2}{\mathfrak{A}^{15}}$, the modular function Z is given by the quotient of modular forms of weight 30. Its denominator is the 15th power of a modular form of weight 2, and its numerator is given by the square of a symmetric modular form of weight 15. According to Proposition 4.1(2), a symmetric modular form of weight 15 is given by $\text{const} \cdot s_{15}$. Then, we have

$$Z(z_1,z_2) = k_3 \frac{s_{15}^2(z_1,z_2)}{g_2^{15}(z_1,z_2)}. \qquad \square$$

THEOREM 4.1

The inverse correspondence of the multivalued period mapping $j^{-1} \circ \Phi : (X,Y) \mapsto (z_1,z_2)$ in (2.14) for the family $\mathcal{F}$ is given by the quotient of Müller's modular forms:

$$\begin{cases} X(z_1,z_2) = 2^5 \cdot 5^2 \cdot \frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}, \\ Y(z_1,z_2) = 2^{10} \cdot 5^5 \cdot \frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}. \end{cases}$$

Proof

First, we obtain the expression of X. To obtain it, we determine the constant k_1 in (4.4). Due to Theorem 3.2, (4.1), and Proposition 4.2, we have

$$X(z,z) = \frac{25}{27} \cdot \frac{1}{J(z)} = \frac{25}{27} \cdot \frac{2^{11}\pi^{12}s_6(z,z)}{(\frac{4\pi^4}{3})^3 g_2^3(z,z)} = 2^5 \cdot 5^2 \cdot \frac{s_6(z,z)}{g_2^3(z,z)}.$$

So, we obtain $k_1 = 2^5 \cdot 5^2$.

Next, we determine the constant k_3 in (4.4). By (1.9), we have

$$\begin{aligned} 144Z(z_1,z_2) = &-1728X^5(z_1,z_2) + 720X^3(z_1,z_2)Y(z_1,z_2) \\ &- 80X(z_1,z_2)Y^2(z_1,z_2) + 64\big(5X^2(z_1,z_2) \\ &- Y(z_1,z_2)\big)^2 + Y^3(z_1,z_2). \end{aligned} \tag{4.5}$$

Recalling that $Y(z,z) = 0$, we have

$$
\begin{aligned}
144Z(z,z) &= -1728X^5(z,z) + 64\cdot 25\cdot X^4(z,z) \\
&= -2^{26}\cdot 5^{10}\cdot\Big(2^5\cdot 3^3\cdot\frac{s_6(z,z)}{g_2^3(z,z)}-1\Big)\Big(\frac{s_6(z,z)}{g_2^3(z,z)}\Big)^4.
\end{aligned}
\tag{4.6}
$$

On the other hand, from (4.3), we have

$$
\begin{aligned}
\frac{s_{15}^2(z_1,z_2)}{g_2^{15}(z_1,z_2)} &= 5^5\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big)^3 - \frac{5^3}{2}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big)^2 \\
&\quad + \frac{3^2\cdot 5^2}{2}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^2\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big) \\
&\quad + \frac{1}{2^4}\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big)^2 - \frac{1}{2^3}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^2\Big(\frac{s_{10}(z_1,z_2)}{g_2^5(z_1,z_2)}\Big) \\
&\quad - 2\cdot 3^3\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^5 + \frac{1}{2^4}\Big(\frac{s_6(z_1,z_2)}{g_2^3(z_1,z_2)}\Big)^4.
\end{aligned}
\tag{4.7}
$$

So, because $s_{10}(z,z)=0$, we have

$$
\Big(\frac{s_{15}^2(z,z)}{g_2^{15}(z,z)}\Big) = \frac{1}{2^4}\Big(-2^5\cdot 3^3\frac{s_6(z,z)}{g_2^3(z,z)}+1\Big)\Big(\frac{s_6(z,z)}{g_2^3(z,z)}\Big)^4.
\tag{4.8}
$$

Since

$$Z(z,z) = k_3\frac{s_{15}^2(z,z)}{g_2^{15}(z,z)},$$

comparing (4.6), (4.8), we have $k_3 = 2^{26}\cdot 5^{10}\cdot 3^{-2}$.

Finally, from (4.5), (4.7), $k_1 = 2^5\cdot 5^2$, and $k_3 = 2^{26}\cdot 5^{10}\cdot 3^{-2}$, we have

$$k_2 = 2^{10}\cdot 5^5. \qquad \square$$

Thus, we obtain the explicit theta expression of the inverse correspondence $(z_1,z_2)\mapsto(X(z_1,z_2),Y(z_1,z_2))$ of the period mapping for our family $\mathcal{F}$ of $K3$ surfaces.

Acknowledgments. The author would like to thank Professor Hironori Shiga for helpful advises and valuable suggestions and also Professor Kimio Ueno and the members of his laboratory for kind encouragements. He is grateful to the referees for the valuable comments to improve the manuscript.

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