The classification of semistable plane sheaves supported on sextic curves

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Abstract We classify all Gieseker semistable sheaves on the complex projective plane that have dimension 1 and multiplicity 6. We decompose their moduli spaces into strata which occur naturally as quotients modulo actions of certain algebraic groups. In most cases we give concrete geometric descriptions of the strata.

1. Introduction and summary of results

Let $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ denote the moduli space of Gieseker semistable sheaves on $\mathbb{P}^2(\mathbb{C})$ with Hilbert polynomial $\mathrm{P}(m) = rm + \chi$, r and χ being fixed integers, $r \geq 1$. Le Potier [7] found that $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ is an irreducible projective variety of dimension r^2+1, smooth at points given by stable sheaves and rational if $\chi \equiv 1$ or $2 \mod r$. In [3] and [10] were classified all semistable sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(4,\chi)$ and $\mathrm{M}_{\mathbb{P}^2}(5,\chi)$, for all values of χ. These moduli spaces were shown to have natural stratifications given by cohomological conditions on the sheaves involved. In this paper we apply the same methods to the study of sheaves giving points in the moduli spaces $\mathrm{M}_{\mathbb{P}^2}(6,\chi)$, and we succeed in finding a complete classification for such sheaves. We refer to the introductory section of [3] for a motivation of the problem and for a brief historical context. We refer to [3, Section 2] for an account of the techniques we shall use. Section 2.4 of [3] contains a discussion about Kronecker modules and their moduli spaces.

In view of the obvious isomorphism $\mathrm{M}_{\mathbb{P}^2}(r,\chi) \simeq \mathrm{M}_{\mathbb{P}^2}(r,\chi+r)$ and of the duality isomorphism $\mathrm{M}_{\mathbb{P}^2}(r,\chi) \simeq \mathrm{M}_{\mathbb{P}^2}(r,-\chi)$ of [9], it is enough, when $r = 6$, to consider only the cases when $\chi = 1, 2, 3, 0$. These cases are dealt with in Sections 3, 4, 5, and 6, respectively. In Section 2 we gather some general results for later use and, for the convenience of the reader, we review the Beilinson monad and spectral sequences. In the remaining part of this section we summarize our classification results.

Kyoto Journal of Mathematics, Vol. 53, No. 4 (2013), 739–786
DOI 10.1215/21562261-2366086,
Received May 7, 2012. Accepted August 24, 2012.
2010 Mathematics Subject Classification: Primary 14D20; Secondary 14D22.
Author's work supported by Consiliul Naţional al Cercetării Ştiinţifice (Romania) grant PN II–RU 169/2010 PD–219.

1.1. Notations

$$\begin{aligned}
\mathrm{M}_{\mathbb{P}^2}(r,\chi) &= \text{the moduli space of Gieseker semistable sheaves on } \mathbb{P}^2\\
&\quad \text{with Hilbert polynomial } \mathrm{P}(m) = rm+\chi;\\
\mathrm{N}(n,p,q) &= \text{the Kronecker moduli space of semistable } (q\times p)\text{-matrices}\\
&\quad \text{with entries in } \mathbb{C}^n \text{ (cf. [3, Section 2.4]);}\\
\mathrm{Hilb}_{\mathbb{P}^2}(n) &= \text{the Hilbert scheme of } n \text{ points in } \mathbb{P}^2;\\
\mathrm{Hilb}_{\mathbb{P}^2}(d,n) &= \text{the flag Hilbert scheme of curves of degree } d \text{ in } \mathbb{P}^2\\
&\quad \text{containing } n \text{ points;}\\
V &= \text{a fixed vector space of dimension 3 over } \mathbb{C};\\
\mathbb{P}^2 &= \text{the projective plane of lines in } V;\\
\mathcal{O}(d) &= \text{the structure sheaf of } \mathbb{P}^2 \text{ twisted by } d;\\
n\mathcal{A} &= \text{the direct sum of } n \text{ copies of the sheaf } \mathcal{A};\\
\{X,Y,Z\} &= \text{basis of } V^*;\\
\{R,S,T\} &= \text{basis of } V^*;\\
[\mathcal{F}] &= \text{the stable equivalence class of a sheaf } \mathcal{F};\\
\mathcal{F}^{\mathrm{D}} &= \mathcal{E}xt^1(\mathcal{F},\omega_{\mathbb{P}^2}) \text{ if } \mathcal{F} \text{ is a one-dimensional sheaf on } \mathbb{P}^2;\\
X^{\mathrm{D}} &= \text{the image in } \mathrm{M}_{\mathbb{P}^2}(r,-\chi) \text{ or in } \mathrm{M}_{\mathbb{P}^2}(r,r-\chi)\text{, as may be the case,}\\
&\quad \text{of a set } X\subset \mathrm{M}_{\mathbb{P}^2}(r,\chi) \text{ under the duality morphism;}\\
X^{\mathrm{s}} &= \text{the open subset of points given by stable sheaves inside a set } X;\\
\mathrm{P}_{\mathcal{F}} &= \text{the Hilbert polynomial of a sheaf } \mathcal{F};\\
\mathrm{p}(\mathcal{F}) &= \chi/r\text{, the slope of a sheaf } \mathcal{F}\text{, where } \mathrm{P}_{\mathcal{F}}(m) = rm+\chi;\\
\mathbb{C}_x,\mathbb{C}_y,\mathbb{C}_z &= \text{the structure sheaves of closed points } x,y,z\in\mathbb{P}^2;\\
\mathcal{O}_L &= \text{the structure sheaf of a line } L\subset\mathbb{P}^2.
\end{aligned}$$

1.2. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6,1)$

This moduli space can be decomposed into five strata: an open stratum X_0; two locally closed irreducible strata X_1, X_2 of codimension 2, respectively, 4; a locally closed stratum that is the disjoint union of two irreducible locally closed subsets X_3 and X_4, each of codimension 6; and a closed irreducible stratum X_5 of codimension 8. The stratum X_0 is an open subset inside a fiber bundle with fiber $\mathbb{P}^{17}$ and base $\mathrm{N}(3,5,4)$; X_2 is an open subset inside a fiber bundle with fiber $\mathbb{P}^{21}$ and base $Y\times\mathbb{P}^2$, where Y is the smooth projective variety of dimension 10 constructed at Claim 3.2.1; X_3 is an open subset inside a fiber bundle with fiber $\mathbb{P}^{23}$ and base $\mathbb{P}^2\times\mathrm{N}(3,2,3)$. The closed stratum X_5 is isomorphic to $\mathrm{Hilb}_{\mathbb{P}^2}(6,2)$.

Table 1. Summary for $\mathrm{M}_{\mathbb{P}^2}(6,1)$

	Cohomological conditions	*Classification of sheaves $\mathcal{F}$ giving points in X_i*	
X_0	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=0$	$0\longrightarrow 5\mathcal{O}(-2)\xrightarrow{\varphi}4\mathcal{O}(-1)\oplus\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{11} is semistable as a Kronecker module	0
X_1	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=0$	$0\longrightarrow\mathcal{O}(-3)\oplus 2\mathcal{O}(-2)\xrightarrow{\varphi}\mathcal{O}(-1)\oplus 2\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ φ is not equivalent to a morphism of the form $\begin{bmatrix}\star&0&0\\\star&\star&\star\\\star&\star&\star\end{bmatrix},\begin{bmatrix}\star&\star&0\\\star&\star&0\\\star&\star&\star\end{bmatrix},\begin{bmatrix}\star&\star&\star\\\star&\star&\star\\\star&0&0\end{bmatrix},\begin{bmatrix}0&0&\star\\\star&\star&\star\\\star&\star&\star\end{bmatrix}$	2
X_2	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=1$	$0\to\mathcal{O}(-3)\oplus 2\mathcal{O}(-2)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}2\mathcal{O}(-1)\oplus 2\mathcal{O}\to\mathcal{F}\to 0$ $\varphi=\begin{bmatrix}q_1&l_{11}&l_{12}&0\\q_2&l_{21}&l_{22}&0\\f_1&q_{11}&q_{12}&l_1\\f_2&q_{21}&q_{22}&l_2\end{bmatrix}$ l_1,l_2 are linearly independent, $d=l_{11}l_{22}-l_{12}l_{21}\neq 0$, $\begin{vmatrix}q_1&l_{11}\\q_2&l_{21}\end{vmatrix},\begin{vmatrix}q_1&l_{12}\\q_2&l_{22}\end{vmatrix}$ are linearly independent modulo d	4
X_3	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=2$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=2$	$0\longrightarrow 2\mathcal{O}(-3)\oplus 2\mathcal{O}(-1)\xrightarrow{\varphi}\mathcal{O}(-2)\oplus 3\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{11} has linearly independent entries φ_{22} has linearly independent maximal minors	6
X_4	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=2$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=3$	$0\to 2\mathcal{O}(-3)\oplus\mathcal{O}(-2)\xrightarrow{\varphi}\mathcal{O}(-2)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(1)\to\mathcal{F}\to 0$ $\varphi=\begin{bmatrix}0&0&1\\q_1&q_2&0\\g_1&g_2&0\end{bmatrix}$, where q_1,q_2 have no common factor or $\varphi=\begin{bmatrix}l_1&l_2&0\\q_1&q_2&l\\g_1&g_2&h\end{bmatrix},\varphi\nsim\begin{bmatrix}\star&\star&0\\0&0&\star\\\star&\star&\star\end{bmatrix}$, where l_1,l_2 are linearly independent, $l\neq 0$	6
X_5	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=3$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=4$	$0\longrightarrow\mathcal{O}(-4)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}\mathcal{O}\oplus\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ $\varphi_{12}\neq 0$, $\varphi_{12}\nmid\varphi_{22}$	8

Each locally closed subset $X_i\subset\mathrm{M}_{\mathbb{P}^2}(6,1)$ is defined by the cohomological conditions listed in the second column of Table 1 above. We equip X_i with the canonical induced reduced structure. In the third column of Table 1 we describe, by means of locally free resolutions of length 1, all semistable sheaves $\mathcal{F}$ on $\mathbb{P}^2$ whose stable equivalence class is in X_i. Thus, for each X_i there are sheaves $\mathcal{A}_i$, $\mathcal{B}_i$ on $\mathbb{P}^2$ that are direct sums of line bundles, such that each sheaf

$\mathcal{F}$ giving a point in X_i is the cokernel of some morphism $\varphi \in \mathrm{Hom}(\mathcal{A}_i, \mathcal{B}_i)$. The linear algebraic group $G_i = (\mathrm{Aut}(\mathcal{A}_i) \times \mathrm{Aut}(\mathcal{B}_i))/\mathbb{C}^*$ acts by conjugation on the finite-dimensional vector space $\mathbb{W}_i = \mathrm{Hom}(\mathcal{A}_i, \mathcal{B}_i)$. Here $\mathbb{C}^*$ is identified with the subgroup of homotheties of $\mathrm{Aut}(\mathcal{A}_i) \times \mathrm{Aut}(\mathcal{B}_i)$. Let $W_i \subset \mathbb{W}_i$ be the locally closed subset of injective morphisms φ satisfying the conditions from the third column of the table. We equip W_i with the canonical induced reduced structure. In each case we shall prove that the map $W_i \to X_i$ defined by $\varphi \mapsto [\mathcal{C}oker(\varphi)]$ is a geometric quotient map for the action of G_i.

1.3. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6,2)$

This moduli space can also be decomposed into five strata: an open stratum X_0; a locally closed stratum that is the disjoint union of two irreducible locally closed subsets X_1 and X_2, each of codimension 3; a locally closed stratum that is the disjoint union of two irreducible locally closed subsets X_3 and X_4, each of codimension 5; an irreducible locally closed stratum X_5 of codimension 7; and a closed irreducible stratum X_6 of codimension 9. For some of these sets we have concrete geometric descriptions: X_1 is a certain open subset inside a fiber bundle with fiber $\mathbb{P}^{20}$ and base $\mathrm{N}(3,4,3) \times \mathbb{P}^2$; X_3 is an open subset of a fiber bundle with fiber $\mathbb{P}^{22}$ and base $\mathrm{Hilb}_{\mathbb{P}^2}(2) \times \mathrm{N}(3,2,3)$; X_5 is an open subset of a fiber bundle with fiber $\mathbb{P}^{24}$ and base $\mathbb{P}^2 \times \mathrm{Hilb}_{\mathbb{P}^2}(2)$; the closed stratum X_6 is isomorphic to the universal sextic in $\mathbb{P}^2 \times \mathbb{P}(\mathrm{S}^6 V^*)$. The classification of sheaves in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ is summarized in Table 2 below, which is organized in the same way as Table 1.

1.4. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6,3)$

Here we have seven strata, compare Table 3 below. The open stratum X_0 is isomorphic to an open subset of $\mathrm{N}(6,3,3)$. The locally closed stratum X_1 has codimension 1 and is birational to $\mathbb{P}^{36}$. The codimension 4 stratum is the union of three irreducible locally closed subsets X_2, X_3, X_3^{D}. Here X_2 is an open subset of a fiber bundle over $\mathrm{N}(3,3,2) \times \mathrm{N}(3,2,3)$ with fiber $\mathbb{P}^{21}$ and X_3^{s} isomorphic to an open subset of a fiber bundle over $\mathrm{N}(3,3,4)$ with fiber $\mathbb{P}^{21}$. The open subset X_4^{s} of the locally closed stratum X_4 of codimension 5 is isomorphic to an open subset of a tower of bundles with fiber $\mathbb{P}^{21}$ and base a fiber bundle over $\mathbb{P}^5$ with fiber $\mathbb{P}^6$. The locally closed stratum X_5 of codimension 6 is isomorphic to an open subset of a fiber bundle over $\mathrm{Hilb}_{\mathbb{P}^2}(2) \times \mathrm{Hilb}_{\mathbb{P}^2}(2)$ with fiber $\mathbb{P}^{23}$. The locally closed stratum X_6 is an open subset of a fiber bundle over $\mathbb{P}^2 \times \mathbb{P}^2$ with fiber $\mathbb{P}^{25}$ and has codimension 8. Finally, we have a closed stratum X_7 consisting of all sheaves of the form $\mathcal{O}_C(2)$ for $C \subset \mathbb{P}^2$ a sextic curve. Thus $X_7 \simeq \mathbb{P}^{27}$. The map $W_0 \to X_0$ is a good quotient map. The map $W_1 \to X_1$ is a categorical quotient map. The maps $W_3^{\mathrm{s}} \to X_3^{\mathrm{s}}$, $(W_3^{\mathrm{D}})^{\mathrm{s}} \to (X_3^{\mathrm{D}})^{\mathrm{s}}$, $W_4^{\mathrm{s}} \to X_4^{\mathrm{s}}$, and $W_i \to X_i$, $i = 2,5,6,7$, are geometric quotient maps. The sheaves in X_0, X_1, X_2, X_3, and X_3^{D} have been classified in [8].

1.5. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6,0)$

Here we have five strata: X_0, X_1, X_2, $X_3 \cup X_3^{\mathrm{D}}$, and X_4, of codimension given in Table 4. The map $W_0 \to X_0$ is a good quotient map. The maps $W_1 \to X_1$

Table 2. Summary for $\mathrm{M}_{\mathbb{P}^2}(6,2)$

	Cohomological conditions	*Classification of sheaves $\mathcal{F}$ giving points in X_i*	
X_0	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=0$	$0\longrightarrow 4\mathcal{O}(-2)\xrightarrow{\varphi}2\mathcal{O}(-1)\oplus 2\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ φ is not equivalent to a morphism of any of the forms $\begin{bmatrix} \star & 0 & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \begin{bmatrix} \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \end{bmatrix}$	0
X_1	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=1$	$0\longrightarrow 4\mathcal{O}(-2)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}3\mathcal{O}(-1)\oplus 2\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ $\varphi_{12}=0$, φ_{11} and φ_{22} are semistable as Kronecker modules	3
X_2	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=1$	$0\longrightarrow\mathcal{O}(-3)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}3\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ $(\varphi_{12},\varphi_{13})$ has linearly independent maximal minors	3
X_3	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=2$	$0\to\mathcal{O}(-3)\oplus\mathcal{O}(-2)\oplus 2\mathcal{O}(-1)\xrightarrow{\varphi}\mathcal{O}(-1)\oplus 3\mathcal{O}\to\mathcal{F}\to 0$ $\varphi_{13}=0$, $\varphi_{12}\neq 0$ and does not divide φ_{11} φ_{23} has linearly independent maximal minors	5
X_4	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=3$	$0\longrightarrow\mathcal{O}(-3)\oplus 2\mathcal{O}(-2)\xrightarrow{\varphi}2\mathcal{O}(-1)\oplus\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ φ is not equivalent to a morphism of any of the forms $\begin{bmatrix} \star & 0 & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}, \begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{bmatrix}, \begin{bmatrix} 0 & 0 & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}, \begin{bmatrix} 0 & \star & \star \\ 0 & \star & \star \\ \star & \star & \star \end{bmatrix}$	5
X_5	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=2$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=4$	$0\longrightarrow 2\mathcal{O}(-3)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}\mathcal{O}(-2)\oplus\mathcal{O}\oplus\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{11} has linearly independent entries $\varphi_{22}\neq 0$ and does not divide φ_{32}	7
X_6	$\mathrm{h}^0(\mathcal{F}(-1))=2$ $\mathrm{h}^1(\mathcal{F})=3$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=6$	$0\longrightarrow\mathcal{O}(-4)\oplus\mathcal{O}\longrightarrow 2\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{12} has linearly independent entries	9

and $W_2\to X_2$ are categorical quotient maps away from the points of the form $[\mathcal{O}_{C_1}\oplus\mathcal{O}_{C_2}]$, where C_1, C_2 are cubic curves. The maps $W_3\to X_3$, $W_3^{\mathrm{D}}\to X_3^{\mathrm{D}}$, and $W_4\to X_4$ are geometric quotient maps away from properly semistable points, that is, points of the form $[\mathcal{O}_L(-1)\oplus\mathcal{O}_Q(1)]$, where L is a line and Q is a quintic curve. Thus X_3^{s} and $(X_3^{\mathrm{D}})^{\mathrm{s}}$ are isomorphic to the open subset of $\mathrm{Hilb}_{\mathbb{P}^2}(6,3)$ of pairs (C,Z), where C is a sextic curve and $Z\subset C$ is a zero-dimensional subscheme of length 3 that is not contained in a line. Moreover, X_4^{s} is isomorphic

Table 3. Summary for $\mathrm{M}_{\mathbb{P}^2}(6,3)$

	Cohomological conditions	*Classification of sheaves $\mathcal{F}$ giving points in X_i*	
X_0	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=0$	$0\longrightarrow 3\mathcal{O}(-2)\xrightarrow{\varphi}3\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$	0
X_1	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=1$	$0\longrightarrow 3\mathcal{O}(-2)\oplus\mathcal{O}(-1)\xrightarrow{\varphi}\mathcal{O}(-1)\oplus 3\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ $\varphi_{12}=0$ φ is not equivalent to a morphism of any of the forms $\begin{bmatrix} \star & 0 & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \begin{bmatrix} \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \end{bmatrix}$	1
X_2	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=2$	$0\longrightarrow 3\mathcal{O}(-2)\oplus 2\mathcal{O}(-1)\xrightarrow{\varphi}2\mathcal{O}(-1)\oplus 3\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ $\varphi_{12}=0$ φ_{11} and φ_{22} are semistable as Kronecker modules	4
X_3	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=3$	$0\longrightarrow\mathcal{O}(-3)\oplus 3\mathcal{O}(-1)\xrightarrow{\varphi}4\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{12} is semistable as a Kronecker module	4
X_3^{D}	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=3$	$0\longrightarrow 4\mathcal{O}(-2)\xrightarrow{\varphi}3\mathcal{O}(-1)\oplus\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{11} is semistable as a Kronecker module	4
X_4	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=3$	$0\longrightarrow\mathcal{O}(-3)\oplus\mathcal{O}(-2)\xrightarrow{\varphi}\mathcal{O}\oplus\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ $\varphi_{12}\neq 0$	5
X_5	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=4$	$\mathcal{O}(-3)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(-1)\overset{\varphi}{\hookrightarrow}\mathcal{O}(-1)\oplus\mathcal{O}\oplus\mathcal{O}(1)\twoheadrightarrow\mathcal{F}$ $\varphi_{13}=0$, $\varphi_{12}\neq 0$, $\varphi_{23}\neq 0$, $\varphi_{12}\nmid\varphi_{11}$, $\varphi_{23}\nmid\varphi_{33}$	6
X_6	$\mathrm{h}^0(\mathcal{F}(-1))=2$ $\mathrm{h}^1(\mathcal{F})=2$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=6$	$0\longrightarrow 2\mathcal{O}(-3)\oplus\mathcal{O}\xrightarrow{\varphi}\mathcal{O}(-2)\oplus 2\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{11} has linearly independent entries φ_{22} has linearly independent entries	8
X_7	$\mathrm{h}^0(\mathcal{F}(-1))=3$ $\mathrm{h}^1(\mathcal{F})=3$ $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=8$	$0\longrightarrow\mathcal{O}(-4)\xrightarrow{\varphi}\mathcal{O}(2)\longrightarrow\mathcal{F}\longrightarrow 0$	10

to the locally closed subscheme of $\mathrm{Hilb}_{\mathbb{P}^2}(6,3)$ given by the condition that Z be contained in a line L that is not a component of C.

Table 4. Summary for $\mathrm{M}_{\mathbb{P}^2}(6,0)$

	Cohomological conditions	*Classification of sheaves $\mathcal{F}$ giving points in X_i*	
X_0	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=0$ $\mathrm{h}^1(\mathcal{F}(1))=0$	$0\longrightarrow 6\mathcal{O}(-2)\xrightarrow{\varphi}6\mathcal{O}(-1)\longrightarrow\mathcal{F}\longrightarrow 0$	0
X_1	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=1$ $\mathrm{h}^1(\mathcal{F}(1))=0$	$0\longrightarrow\mathcal{O}(-3)\oplus 3\mathcal{O}(-2)\xrightarrow{\varphi}3\mathcal{O}(-1)\oplus\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{12} is semistable as a Kronecker module	1
X_2	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=2$ $\mathrm{h}^1(\mathcal{F}(1))=0$	$2\mathcal{O}(-3)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(-1)\overset{\varphi}{\hookrightarrow}\mathcal{O}(-2)\oplus\mathcal{O}(-1)\oplus 2\mathcal{O}\twoheadrightarrow\mathcal{F}$ see the conditions on φ at Proposition 6.1.3	4
X_3	$\mathrm{h}^0(\mathcal{F}(-1))=0$ $\mathrm{h}^1(\mathcal{F})=3$ $\mathrm{h}^1(\mathcal{F}(1))=1$	$0\longrightarrow\mathcal{O}(-4)\oplus 2\mathcal{O}(-1)\xrightarrow{\varphi}3\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{12} has linearly independent maximal minors	7
X_3^{D}	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=3$ $\mathrm{h}^1(\mathcal{F}(1))=0$	$0\longrightarrow 3\mathcal{O}(-3)\xrightarrow{\varphi}2\mathcal{O}(-2)\oplus\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ φ_{11} has linearly independent maximal minors	7
X_4	$\mathrm{h}^0(\mathcal{F}(-1))=1$ $\mathrm{h}^1(\mathcal{F})=3$ $\mathrm{h}^1(\mathcal{F}(1))=1$	$0\longrightarrow\mathcal{O}(-4)\oplus\mathcal{O}(-2)\xrightarrow{\varphi}\mathcal{O}(-1)\oplus\mathcal{O}(1)\longrightarrow\mathcal{F}\longrightarrow 0$ $\varphi_{12}\neq 0$	8

2. Preliminaries

2.1. The Beilinson monad and spectral sequences

In this subsection $\mathcal{F}$ will be a coherent sheaf on $\mathbb{P}^2$ with support of dimension 1. The E^1-term of the Beilinson spectral sequence I converging to $\mathcal{F}$ has display diagram

$$(2.1.1)\qquad \begin{array}{ccccc} \mathrm{H}^1\big(\mathcal{F}(-2)\big)\otimes\mathcal{O}(-1) & \xrightarrow{\varphi_1} & \mathrm{H}^1\big(\mathcal{F}(-1)\big)\otimes\Omega^1(1) & \xrightarrow{\varphi_2} & \mathrm{H}^1(\mathcal{F})\otimes\mathcal{O} \\ \mathrm{H}^0\big(\mathcal{F}(-2)\big)\otimes\mathcal{O}(-1) & \xrightarrow{\varphi_3} & \mathrm{H}^0\big(\mathcal{F}(-1)\big)\otimes\Omega^1(1) & \xrightarrow{\varphi_4} & \mathrm{H}^0(\mathcal{F})\otimes\mathcal{O}. \end{array}$$

The spectral sequence degenerates at E^3, which shows that φ_2 is surjective and that we have the exact sequences

$$(2.1.2)\quad 0\longrightarrow\mathrm{H}^0\big(\mathcal{F}(-2)\big)\otimes\mathcal{O}(-1)\xrightarrow{\varphi_3}\mathrm{H}^0\big(\mathcal{F}(-1)\big)\otimes\Omega^1(1)$$
$$\xrightarrow{\varphi_4}\mathrm{H}^0(\mathcal{F})\otimes\mathcal{O}\longrightarrow\mathcal{C}oker(\varphi_4)\longrightarrow 0,$$

$$(2.1.3)\quad 0\longrightarrow\mathcal{K}er(\varphi_1)\xrightarrow{\varphi_5}\mathcal{C}oker(\varphi_4)\longrightarrow\mathcal{F}\longrightarrow\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)\longrightarrow 0.$$

The E^1-term of the Beilinson spectral sequence II converging to $\mathcal{F}$ has display diagram

$$\text{(2.1.4)}\quad \begin{array}{ccccc} \mathrm{H}^1\big(\mathcal{F}(-1)\big)\otimes\mathcal{O}(-2) & \xrightarrow{\varphi_1} & \mathrm{H}^1\big(\mathcal{F}\otimes\Omega^1(1)\big)\otimes\mathcal{O}(-1) & \xrightarrow{\varphi_2} & \mathrm{H}^1(\mathcal{F})\otimes\mathcal{O} \\ \mathrm{H}^0\big(\mathcal{F}(-1)\big)\otimes\mathcal{O}(-2) & \xrightarrow{\varphi_3} & \mathrm{H}^0\big(\mathcal{F}\otimes\Omega^1(1)\big)\otimes\mathcal{O}(-1) & \xrightarrow{\varphi_4} & \mathrm{H}^0(\mathcal{F})\otimes\mathcal{O}. \end{array}$$

As above, this spectral sequence degenerates at E^3 and yields the exact sequences

$$\text{(2.1.5)}\quad 0\longrightarrow \mathrm{H}^0\big(\mathcal{F}(-1)\big)\otimes\mathcal{O}(-2)\xrightarrow{\varphi_3}\mathrm{H}^0\big(\mathcal{F}\otimes\Omega^1(1)\big)\otimes\mathcal{O}(-1)$$
$$\xrightarrow{\varphi_4}\mathrm{H}^0(\mathcal{F})\otimes\mathcal{O}\longrightarrow \mathcal{C}oker(\varphi_4)\longrightarrow 0,$$

$$\text{(2.1.6)}\quad 0\longrightarrow \mathcal{K}er(\varphi_1)\xrightarrow{\varphi_5}\mathcal{C}oker(\varphi_4)\longrightarrow\mathcal{F}\longrightarrow\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)\longrightarrow 0.$$

The Beilinson free monad associated to $\mathcal{F}$ is a sequence

$$\text{(2.1.7)}\quad \begin{gathered} 0\longrightarrow\mathcal{C}^{-2}\longrightarrow\mathcal{C}^{-1}\longrightarrow\mathcal{C}^{0}\longrightarrow\mathcal{C}^{1}\longrightarrow\mathcal{C}^{2}\longrightarrow 0,\\ \mathcal{C}^p=\bigoplus_{i+j=p}\mathrm{H}^j\big(\mathcal{F}\otimes\Omega^{-i}(-i)\big)\otimes\mathcal{O}(i), \end{gathered}$$

which is exact, except at $\mathcal{C}^0$, where the cohomology is $\mathcal{F}$. Note that $\mathcal{C}^2=0$ because $\mathcal{F}$ is assumed to have dimension 1. The maps

$$\mathrm{H}^0\big(\mathcal{F}\otimes\Omega^{-i}(-i)\big)\otimes\mathcal{O}(i)\longrightarrow\mathrm{H}^1\big(\mathcal{F}\otimes\Omega^{-i}(-i)\big)\otimes\mathcal{O}(i),$$

$i=0,-1,-2$, occurring in the monad are zero (cf., for instance, [9, Lemma 1]).

2.2. Cohomology bounds

PROPOSITION 2.2.1

(i) *Let* $\mathcal{F}$ *give a point in* $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$, *where* $0\le\chi<r$. *Assume that* $\mathrm{h}^1(\mathcal{F})>0$. *Then* $\mathrm{h}^1(\mathcal{F}(1))>2\mathrm{h}^1(\mathcal{F})-\mathrm{h}^1(\mathcal{F}(-1))$.

(ii) *Let* $\mathcal{F}$ *give a point in* $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$, *where* $0<\chi\le r$. *Assume that* $\mathrm{h}^0(\mathcal{F}(-1))>0$. *Then* $\mathrm{h}^0(\mathcal{F}(-2))>2\mathrm{h}^0(\mathcal{F}(-1))-\mathrm{h}^0(\mathcal{F})$.

Proof

Part (ii) is equivalent to (i) by duality, so we concentrate on (i). Write $p=\mathrm{h}^1(\mathcal{F})$, $q=\mathrm{h}^0(\mathcal{F}(-1))$, $m=\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))$. The Beilinson free monad (2.1.7) for $\mathcal{F}$ takes the form

$$0\longrightarrow q\mathcal{O}(-2)\xrightarrow{\psi}(q+r-\chi)\mathcal{O}(-2)\oplus m\mathcal{O}(-1)$$
$$\longrightarrow(m+r-2\chi)\mathcal{O}(-1)\oplus(p+\chi)\mathcal{O}\xrightarrow{\eta}p\mathcal{O}\longrightarrow 0$$

and yields a resolution

$$0\longrightarrow(q+r-\chi)\mathcal{O}(-2)\oplus\mathcal{C}oker(\psi_{21})\xrightarrow{\varphi}\mathcal{K}er(\eta_{11})\oplus(p+\chi)\mathcal{O}\longrightarrow\mathcal{F}\longrightarrow 0$$

in which $\varphi_{12}=0$. Since $\mathcal{F}$ maps surjectively to $\mathcal{C}oker(\varphi_{11})$ we have the inequality

$$m+r-2\chi-p=\operatorname{rank}\big(\mathcal{K}er(\eta_{11})\big)\le q+r-\chi.$$

If the inequality is not strict, then $\mathcal{C}oker(\varphi_{11})$ has negative slope, contradicting the semistability of $\mathcal{F}$. Thus $m<p+q+\chi$. We have

$$\begin{aligned}
\mathrm{h}^0\big(\mathcal{F}(1)\big) &= \mathrm{h}^0\big((p+\chi)\mathcal{O}(1)\big)+\mathrm{h}^0\big(\mathcal{K}er(\eta_{11})(1)\big)-\mathrm{h}^0\big(\mathcal{C}oker(\psi_{21})(1)\big)\\
&\ge \mathrm{h}^0\big((p+\chi)\mathcal{O}(1)\big)-\mathrm{h}^0\big(\mathcal{C}oker(\psi_{21})(1)\big)\\
&= 3p+3\chi-m\\
&> 2p+2\chi-q,\\
\mathrm{h}^1\big(\mathcal{F}(1)\big) &= \mathrm{h}^0\big(\mathcal{F}(1)\big)-r-\chi>2p+\chi-q-r=2\mathrm{h}^1(\mathcal{F})-\mathrm{h}^1\big(\mathcal{F}(-1)\big).
\end{aligned}$$

□

COROLLARY 2.2.2

There are no sheaves $\mathcal{F}$ giving points

(i) *in* $\mathrm{M}_{\mathbb{P}^2}(6,1)$ *and satisfying* $\mathrm{h}^0(\mathcal{F}(-1))\le 1$, $\mathrm{h}^1(\mathcal{F})\ge 3$, $\mathrm{h}^1(\mathcal{F}(1))=0$;
(ii) *in* $\mathrm{M}_{\mathbb{P}^2}(6,1)$ *and satisfying* $\mathrm{h}^0(\mathcal{F}(-1))=1$, $\mathrm{h}^1(\mathcal{F})=1$;
(iii) *in* $\mathrm{M}_{\mathbb{P}^2}(6,1)$ *and satisfying* $\mathrm{h}^0(\mathcal{F}(-1))=2$, $\mathrm{h}^1(\mathcal{F}(1))=0$;
(iv) *in* $\mathrm{M}_{\mathbb{P}^2}(6,2)$ *and satisfying* $\mathrm{h}^0(\mathcal{F}(-1))\le 1$, $\mathrm{h}^1(\mathcal{F})\ge 3$, $\mathrm{h}^1(\mathcal{F}(1))=0$;
(v) *in* $\mathrm{M}_{\mathbb{P}^2}(6,2)$ *and satisfying* $\mathrm{h}^0(\mathcal{F}(-1))=0$, $\mathrm{h}^1(\mathcal{F})=2$, $\mathrm{h}^1(\mathcal{F}(1))=0$;
(vi) *in* $\mathrm{M}_{\mathbb{P}^2}(6,3)$ *and satisfying* $\mathrm{h}^0(\mathcal{F}(-1))\le 1$, $\mathrm{h}^1(\mathcal{F})\ge 2$, $\mathrm{h}^1(\mathcal{F}(1))=0$;
(vii) *in* $\mathrm{M}_{\mathbb{P}^2}(6,0)$ *and satisfying* $\mathrm{h}^0(\mathcal{F}(-1))=0$, $\mathrm{h}^1(\mathcal{F})\ge 3$, $\mathrm{h}^1(\mathcal{F}(1))=0$.

Proof

Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$. According to [3, Proposition 2.1.3], $\mathrm{h}^0(\mathcal{F}(-2))=0$. In view of Proposition 2.2.1(ii) we have $\mathrm{h}^0(\mathcal{F})>2\mathrm{h}^0(\mathcal{F}(-1))$. This proves (ii). Assume now that $\mathcal{F}$ satisfies the conditions from (iii). Then $\mathrm{h}^1(\mathcal{F})=\mathrm{h}^0(\mathcal{F})-1\ge 4$. On the other hand, by Proposition 2.2.1(i), we have $7=\mathrm{h}^1(\mathcal{F}(-1))>2\mathrm{h}^1(\mathcal{F})$. This yields a contradiction and proves (iii). All other parts of the corollary are direct applications of Proposition 2.2.1(i). □

2.3. Stability criteria

PROPOSITION 2.3.1

Let n be a positive integer, and let $d_1\le\cdots\le d_n$, $e_1\le\cdots\le e_n$ be integers satisfying the relations

(i) $$e_1-d_1\ge e_2+\cdots+e_n-d_2-\cdots-d_n,$$

(ii) $$e_1+d_1\le\frac{e_2^2+\cdots+e_n^2-d_2^2-\cdots-d_n^2}{e_2+\cdots+e_n-d_2-\cdots-d_n}.$$

Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^2$ having resolution

$$0\longrightarrow\mathcal{O}(d_1)\oplus\cdots\oplus\mathcal{O}(d_n)\xrightarrow{\varphi}\mathcal{O}(e_1)\oplus\cdots\oplus\mathcal{O}(e_n)\longrightarrow\mathcal{F}\longrightarrow 0.$$

Assume that the maximal minors of the restriction of φ to $\mathcal{O}(d_2)\oplus\cdots\oplus\mathcal{O}(d_n)$ have no common factor and that none of them has degree zero. Then $\mathcal{F}$ is stable, unless the ratio

$$r=\frac{e_1^2+\cdots+e_n^2-d_1^2-\cdots-d_n^2}{e_1+\cdots+e_n-d_1-\cdots-d_n}$$

is an integer and $\mathcal{F}$ has a subsheaf $\mathcal{S}$ given by a resolution

$$0\longrightarrow\mathcal{O}(d_1)\longrightarrow\mathcal{O}(r-d_1)\longrightarrow\mathcal{S}\longrightarrow 0.$$

In this case $\mathrm{p}(\mathcal{S})=\mathrm{p}(\mathcal{F})$ and $\mathcal{F}$ is properly semistable. Note that condition (ii) can be replaced by the requirement that $e_i\ge d_i$ for $2\le i\le n$.

Proof

Let $C\subset\mathbb{P}^2$ be the curve given by the equation $\det(\varphi)=0$. Its degree is $d=e_1+\cdots+e_n-d_1-\cdots-d_n$. Let ψ denote the restriction of φ to $\mathcal{O}(d_2)\oplus\cdots\oplus\mathcal{O}(d_n)$, and let ζ_i be the maximal minor of the matrix representing ψ obtained by deleting row i. We have an exact sequence

$$0\longrightarrow\mathcal{O}(d_2)\oplus\cdots\oplus\mathcal{O}(d_n)\stackrel{\psi}{\longrightarrow}\mathcal{O}(e_1)\oplus\cdots\oplus\mathcal{O}(e_n)\stackrel{\zeta}{\longrightarrow}\mathcal{O}(e)\longrightarrow\mathcal{C}\longrightarrow 0,$$

$$\zeta=\begin{bmatrix}\zeta_1 & -\zeta_2 & \cdots & (-1)^{n+1}\zeta_n\end{bmatrix},\qquad e=d+d_1.$$

The Hilbert polynomial of $\mathcal{C}$ is a constant, namely, $\frac{d^2}{2}+dd_1+\sum_{i=1}^n\frac{d_i^2-e_i^2}{2}$, showing that $\mathcal{C}$ is the structure sheaf of a zero-dimensional scheme $Z\subset\mathbb{P}^2$ and that $\mathcal{C}oker(\psi)\simeq\mathcal{I}_Z(e)$ and $\mathcal{F}\simeq\mathcal{J}_Z(e)$, where $\mathcal{J}_Z\subset\mathcal{O}_C$ is the ideal sheaf of Z in C. Clearly $\mathcal{F}$ has no zero-dimensional torsion. Let $\mathcal{S}\subset\mathcal{F}$ be a subsheaf of multiplicity at most $d-1$. According to [8, Lemma 6.7] there is a sheaf $\mathcal{A}$ such that $\mathcal{S}\subset\mathcal{A}\subset\mathcal{O}_C(e)$, $\mathcal{A}/\mathcal{S}$ is supported on finitely many points, and $\mathcal{O}_C(e)/\mathcal{A}\simeq\mathcal{O}_S(e)$ for a curve $S\subset\mathbb{P}^2$ of degree s, $1\le s\le d-1$. We have the relations

$$\begin{aligned}\mathrm{P}_{\mathcal{S}}(m)&=\mathrm{P}_{\mathcal{O}_C(e)}(m)-\mathrm{P}_{\mathcal{O}_S(e)}(m)-\mathrm{h}^0(\mathcal{A}/\mathcal{S})\\&=dm+de-\frac{d(d-3)}{2}-sm-se+\frac{s(s-3)}{2}-\mathrm{h}^0(\mathcal{A}/\mathcal{S}),\\\mathrm{p}(\mathcal{S})&=e+\frac{3}{2}-\frac{d+s}{2}-\frac{\mathrm{h}^0(\mathcal{A}/\mathcal{S})}{d-s},\\\mathrm{P}_{\mathcal{F}}(m)&=dm+\frac{3d}{2}+\sum_{i=1}^n\frac{e_i^2-d_i^2}{2},\\\mathrm{p}(\mathcal{F})&=\frac{3}{2}+\sum_{i=1}^n\frac{e_i^2-d_i^2}{2d}.\end{aligned}$$

In order to show that $\mathcal{F}$ is semistable we will prove that $\mathrm{p}(\mathcal{S})\le\mathrm{p}(\mathcal{F})$. This is equivalent to the inequality

$$d_1+\frac{d}{2}+\sum_{i=1}^n\frac{d_i^2-e_i^2}{2d}\le\frac{s}{2}+\frac{\mathrm{h}^0(\mathcal{A}/\mathcal{S})}{d-s}.$$

Assume that

$$\frac{s}{2} < d_1 + \frac{d}{2} + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d}$$

and

$$\mathrm{h}^0(\mathcal{A}/\mathcal{S}) \le (d-s)\Big(d_1 + \frac{d}{2} - \frac{s}{2} + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d}\Big).$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}/\mathcal{S} & \longrightarrow & \mathcal{O}_C(e)/\mathcal{S} & \longrightarrow & \mathcal{O}_S(e) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ & & \mathcal{A}/\mathcal{S} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

in which Y is a subscheme of Z of length at least

$$\frac{d^2}{2} + dd_1 + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2} - (d-s)\Big(d_1 + \frac{d}{2} - \frac{s}{2} + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d}\Big)$$
$$= s\Big(d + d_1 - \frac{s}{2} + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d}\Big).$$

We claim that $\mathrm{length}(Y) > s\deg(\zeta_1) = s(d - e_1 + d_1)$. This follows from the equivalent inequalities

$$d + d_1 - \frac{s}{2} + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d} > d - e_1 + d_1,$$

$$e_1 + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d} > \frac{s}{2},$$

which follow from the inequality

$$e_1 + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d} \ge d_1 + \frac{d}{2} + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{2d}.$$

The latter is equivalent to condition (i) from the hypothesis. This proves the claim. Since Y is a subscheme of S and also of the curve given by the equation $\zeta_1 = 0$, we can apply Bézout's theorem to deduce that S and the curve given by the equation $\zeta_1 = 0$ have a common component. Since $\gcd(\zeta_1, \dots, \zeta_n) = 1$, we may perform elementary row operations on the matrix representing φ to ensure that ζ_1 is irreducible. Thus ζ_1 divides the equation defining S. In particular, $\deg(\zeta_1) \le s$. It follows that

$$d - e_1 + d_1 = \deg(\zeta_1) < 2d_1 + d + \sum_{i=1}^{n} \frac{d_i^2 - e_i^2}{d},$$

$$\sum_{i=1}^{n}(e_i^2-d_i^2)<d(d_1+e_1)=e_1^2-d_1^2+(d_1+e_1)\sum_{i=2}^{n}(e_i-d_i),$$

$$\sum_{i=2}^{n}(e_i^2-d_i^2)<(d_1+e_1)\sum_{i=2}^{n}(e_i-d_i).$$

The last inequality contradicts condition (ii) from the hypothesis. The above discussion shows that $\mathrm{p}(\mathcal{S})<\mathrm{p}(\mathcal{F})$ unless $\mathcal{S}=\mathcal{A}$ and

$$\frac{s}{2}=d_1+\frac{d}{2}+\sum_{i=1}^{n}\frac{d_i^2-e_i^2}{2d},$$

in which case $\mathrm{p}(\mathcal{S})=\mathrm{p}(\mathcal{F})$ and $\mathcal{F}$ is semistable but not stable. Clearly we have an exact sequence

$$0\longrightarrow\mathcal{O}(e-d)\longrightarrow\mathcal{O}(e-s)\longrightarrow\mathcal{S}\longrightarrow 0.$$

Note that $e-d=d_1$, $e-s=r-d_1$. □

COROLLARY 2.3.2

Let $d_1\leq d_2<e_1\leq e_2$ be integers satisfying the condition $e_1-d_1\geq e_2-d_2$. Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^2$ having resolution

$$0\longrightarrow\mathcal{O}(d_1)\oplus\mathcal{O}(d_2)\stackrel{\varphi}{\longrightarrow}\mathcal{O}(e_1)\oplus\mathcal{O}(e_2)\longrightarrow\mathcal{F}\longrightarrow 0.$$

Assume that φ_{12} and φ_{22} have no common factor. Then $\mathcal{F}$ is stable, unless $e_1-d_1=e_2-d_2$ and $\varphi\sim\left[\begin{smallmatrix}0&\star\\ \star&\star\end{smallmatrix}\right]$, in which case $\mathcal{F}$ is semistable but not stable.

Proof

According to the proposition above, $\mathcal{F}$ is stable unless the ratio

$$r=\frac{e_1^2+e_2^2-d_1^2-d_2^2}{e_1+e_2-d_1-d_2}$$

is an integer and $\mathcal{F}$ has a subsheaf $\mathcal{S}$ given by a certain resolution. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(d_1) & \longrightarrow & \mathcal{O}(r-d_1) & \longrightarrow & \mathcal{S} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(d_1)\oplus\mathcal{O}(d_2) & \stackrel{\varphi}{\longrightarrow} & \mathcal{O}(e_1)\oplus\mathcal{O}(e_2) & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}$$

in which α and β are injective. Thus $r-d_1\leq e_2$; that is,

$$e_1^2+e_2^2-d_1^2-d_2^2\leq(e_1+e_2-d_1-d_2)(d_1+e_2),$$

$$(e_1-d_2)(e_1+d_2)\leq(e_1-d_2)(d_1+e_2),$$

$$e_1+d_2\leq d_1+e_2.$$

Thus $e_1-d_1=e_2-d_2$, $r-d_1=e_2$, and φ has the special form given above. □

3. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6,1)$

3.1. Classification of sheaves

PROPOSITION 3.1.1

Every sheaf $\mathcal{F}$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and satisfying the condition $\mathrm{h}^1(\mathcal{F})=0$ also satisfies the condition $\mathrm{h}^0(\mathcal{F}(-1))=0$. These sheaves are precisely the sheaves having a resolution of the form

$$0 \longrightarrow 5\mathcal{O}(-2) \xrightarrow{\varphi} 4\mathcal{O}(-1)\oplus\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} is semistable as a Kronecker module.

Proof

The statement follows by duality from [8, Claim 4.2]. □

CLAIM 3.1.2

Consider an exact sequence of sheaves on $\mathbb{P}^2$:

$$0 \longrightarrow \mathcal{O}(-3)\oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1)\oplus\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q_1 & l_{11} & l_{12} \\ q_2 & l_{21} & l_{22} \\ f & g_1 & g_2 \end{bmatrix},$$

where $l_{11}l_{22}-l_{12}l_{21}\neq 0$ and the images of $q_1l_{21}-q_2l_{11}$ and $q_1l_{22}-q_2l_{12}$ in $\mathrm{S}^3V^*/(l_{11}l_{22}-l_{12}l_{21})V^*$ are linearly independent. Then $\mathcal{F}$ gives a stable point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$.

Proof

By hypothesis the maximal minors of the matrix

$$\psi = \begin{bmatrix} q_1 & l_{11} & l_{12} \\ q_2 & l_{21} & l_{22} \end{bmatrix}$$

cannot have a common quadratic factor. If they have no common factor, then the claim follows by duality from Proposition 2.3.1. Assume that they have a common linear factor. Then $\mathcal{K}er(\psi)\simeq\mathcal{O}(-4)$ and $\mathcal{C}oker(\psi)$ is supported on a line L. From the snake lemma we get an extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\psi) \longrightarrow 0,$$

where C is a quintic curve. Because of the conditions on ψ it is easy to check that $\mathcal{C}oker(\psi)$ has zero-dimensional torsion of length at most 1. Assume that $\mathcal{C}oker(\psi)$ has no zero-dimensional torsion, that is, $\mathcal{C}oker(\psi)\simeq\mathcal{O}_L(1)$. Let $\mathcal{F}'\subset\mathcal{F}$ be a nonzero subsheaf of multiplicity at most 5. Denote by $\mathcal{C}$ its image in $\mathcal{O}_L(1)$, and put $\mathcal{K}=\mathcal{F}'\cap\mathcal{O}_C(1)$. If $\mathcal{C}=0$, then $\mathrm{p}(\mathcal{F}')\le 0$ because $\mathcal{O}_C$ is stable. Assume that $\mathcal{C}\neq 0$; that is, $\mathcal{C}$ has multiplicity 1. If $\mathcal{K}=0$ and $\mathcal{F}'$ destabilizes $\mathcal{F}$, then $\mathcal{F}'\simeq\mathcal{O}_L$ or $\mathcal{F}'\simeq\mathcal{O}_L(1)$. Both situations can be ruled out using diagrams analogous to diagram (8) at Proposition 3.1.3 below. Thus we may assume that $1\le\mathrm{mult}(\mathcal{K})\le 4$. According to [8, Lemma 6.7], there is a sheaf $\mathcal{A}$ such that $\mathcal{K}\subset\mathcal{A}\subset\mathcal{O}_C(1)$, $\mathcal{A}/\mathcal{K}$

is supported on finitely many points, and $\mathcal{O}_C(1)/\mathcal{A} \simeq \mathcal{O}_S(1)$ for a curve $S \subset \mathbb{P}^2$ of degree s, $1 \leq s \leq 4$. Thus

$$\begin{aligned}
\mathrm{P}_{\mathcal{F}'}(m) &= \mathrm{P}_{\mathcal{K}}(m) + \mathrm{P}_{\mathcal{C}}(m) \\
&= \mathrm{P}_{\mathcal{A}}(m) - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) + \mathrm{P}_{\mathcal{O}_L(1)}(m) - \mathrm{h}^0\big(\mathcal{O}_L(1)/\mathcal{C}\big) \\
&= (5-s)m + \frac{s^2-5s}{2} + m + 2 - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) - \mathrm{h}^0\big(\mathcal{O}_L(1)/\mathcal{C}\big), \\
\mathrm{p}(\mathcal{F}') &= \frac{1}{6-s}\Big(\frac{s^2-5s}{2} + 2 - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) - \mathrm{h}^0\big(\mathcal{O}_L(1)/\mathcal{C}\big)\Big) \\
&\leq \frac{s^2-5s+4}{2(6-s)} < \frac{1}{3} = \mathrm{p}(\mathcal{F}).
\end{aligned}$$

We see that in this case $\mathcal{F}$ is stable. Assume next that $\mathcal{C}oker(\psi)$ has a zero-dimensional subsheaf $\mathcal{T}$ of length 1. Let $\mathcal{E}$ be the preimage of $\mathcal{T}$ in $\mathcal{F}$. According to [10, Proposition 3.1.5], $\mathcal{E}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$. Let $\mathcal{F}'$ and $\mathcal{C}$ be as above. If $\mathcal{C} \subset \mathcal{T}$, then $\mathcal{F}' \subset \mathcal{E}$, and hence $\mathrm{p}(\mathcal{F}') \leq \mathrm{p}(\mathcal{E}) < \mathrm{p}(\mathcal{F})$. If $\mathcal{C}$ is not a subsheaf of $\mathcal{T}$, then we can estimate $\mathrm{p}(\mathcal{F}')$ as above, concluding again that it is less than the slope of $\mathcal{F}$. □

PROPOSITION 3.1.3

The sheaves $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,1)$ *and satisfying the conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 1$, $\mathrm{h}^1(\mathcal{F}(1)) = 0$ *are precisely the sheaves having a resolution of the form*

(i) $$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q & l_1 & l_2 \\ f_1 & q_{11} & q_{12} \\ f_2 & q_{21} & q_{22} \end{bmatrix},$$

where φ *is not equivalent to a morphism represented by a matrix of one of the following four forms:*

$$\varphi_1 = \begin{bmatrix} \star & 0 & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}, \qquad \varphi_2 = \begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{bmatrix},$$

$$\varphi_3 = \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & 0 & 0 \end{bmatrix}, \qquad \varphi_4 = \begin{bmatrix} 0 & 0 & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix},$$

or the sheaves having a resolution of the form

(ii) $$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q_1 & l_{11} & l_{12} & 0 \\ q_2 & l_{21} & l_{22} & 0 \\ f_1 & q_{11} & q_{12} & l_1 \\ f_2 & q_{21} & q_{22} & l_2 \end{bmatrix},$$

where l_1, l_2 are linearly independent one-forms, $l_{11}l_{22} - l_{12}l_{21} \neq 0$, and the images of $q_1 l_{21} - q_2 l_{11}$ and $q_1 l_{22} - q_2 l_{12}$ in $\mathrm{S}^3 V^*/(l_{11}l_{22} - l_{12}l_{21})V^*$ are linearly independent.

Proof
Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and satisfy the above cohomological conditions. Display diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}(1)$ reads

$$\begin{array}{ccccc} 5\mathcal{O}(-1) & \xrightarrow{\varphi_1} & \Omega^1(1) & & 0 \\ & & & & \\ 0 & & 2\Omega^1(1) & \xrightarrow{\varphi_4} & 7\mathcal{O}. \end{array}$$

Resolving $\Omega^1(1)$ yields the exact sequence

$$0 \longrightarrow \mathcal{K}er(\varphi_1) \longrightarrow \mathcal{O}(-2) \oplus 5\mathcal{O}(-1) \xrightarrow{\sigma} 3\mathcal{O}(-1) \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0.$$

Notice that $\mathcal{F}(1)$ maps surjectively to $\mathcal{C}oker(\varphi_1)$. Thus $\mathrm{rank}(\sigma_{12}) = 3$; otherwise $\mathcal{C}oker(\varphi_1)$ would have positive rank or would be isomorphic to $\mathcal{O}_L(-1)$ violating the semistability of $\mathcal{F}(1)$. We have shown that $\mathcal{C}oker(\varphi_1) = 0$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-2) \oplus 2\mathcal{O}(-1)$. Combining the exact sequences (2.1.2) and (2.1.3) we obtain the resolution

$$0 \longrightarrow \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \oplus 2\Omega^1(1) \longrightarrow 7\mathcal{O} \longrightarrow \mathcal{F}(1) \longrightarrow 0,$$

hence a resolution

$$0 \longrightarrow \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \oplus 6\mathcal{O} \xrightarrow{\rho} 7\mathcal{O} \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F}(1) \longrightarrow 0.$$

Notice that $\mathrm{rank}(\rho_{13}) \geq 5$; otherwise $\mathcal{F}(1)$ would map surjectively to the cokernel of a morphism $\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \to 3\mathcal{O}$, in violation of semistability. Canceling $5\mathcal{O}$ and tensoring with $\mathcal{O}(-1)$ we arrive at the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

From this we get resolution (i) or (ii), depending on whether $\varphi_{13} \neq 0$ or $\varphi_{13} = 0$.

Conversely, we assume that $\mathcal{F}$ has resolution (i) and we need to show that there are no destabilizing subsheaves $\mathcal{E}$. We argue by contradiction; that is, we assume that there is such a subsheaf $\mathcal{E}$. We may assume that $\mathcal{E}$ is semistable. As $\mathrm{h}^0(\mathcal{E}) \leq 2$, $\mathcal{E}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(r,1)$ or $\mathrm{M}_{\mathbb{P}^2}(r,2)$ for some r, $1 \leq r \leq 5$. The cohomology groups $\mathrm{H}^0(\mathcal{E}(-1))$ and $\mathrm{H}^0(\mathcal{E} \otimes \Omega^1(1))$ vanish because the corresponding cohomology groups for $\mathcal{F}$ vanish. From the description of $\mathrm{M}_{\mathbb{P}^2}(r,1)$ and $\mathrm{M}_{\mathbb{P}^2}(r,2)$, $1 \leq r \leq 5$, found in [3] and [10], we see that $\mathcal{E}$ may have one of the following resolutions:

(1) $$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0,$$

(2) $$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0,$$

(3) $$0 \longrightarrow 3\mathcal{O}(-2) \longrightarrow 2\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0,$$

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow 4\mathcal{O}(-2) \longrightarrow 3\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow 3\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0. \tag{7}$$

Resolution (1) must fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-2) & \xrightarrow{\psi} & \mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) & \xrightarrow{\varphi} & \mathcal{O}(-1) \oplus 2\mathcal{O} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array} \tag{8}$$

in which α is injective (being injective on global sections). Thus β is injective, too, and $\varphi \sim \varphi_2$, contradicting our hypothesis on φ. Similarly, every other resolution must fit into a commutative diagram in which α and $\alpha(1)$ are injective on global sections. This rules out resolution (7) because in that case α must be injective; hence $\mathcal{K}er(\beta) = 0$, which is absurd. If $\mathcal{E}$ has resolution (5), then α is equivalent to a morphism represented by a matrix having one of the following two forms:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 & u_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where u_1, u_2, u_3 are linearly independent one-forms. In the first case $\mathcal{K}er(\beta) \simeq \mathcal{O}(-2)$, and in the second case $\mathcal{K}er(\beta) \simeq \Omega^1$. Both situations are absurd. Assume that $\mathcal{E}$ has resolution (3). Since β cannot be injective, we see that α is equivalent to a morphism represented by a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ u_1 & u_2 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

hence $\mathcal{K}er(\alpha) \simeq \mathcal{O}(-2)$, and hence $\varphi \sim \varphi_1$, which is a contradiction. For resolutions (2), (4), and (6), α and β must be injective, and we get the contradictory conclusions that $\varphi \sim \varphi_3$, $\varphi \sim \varphi_1$, or $\varphi \sim \varphi_4$.

Assume now that $\mathcal{F}$ has resolution (ii). The sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$ is the cokernel of the transpose of φ. From the snake lemma we have an extension

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{G}'$ is the cokernel of a morphism $\psi\colon \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \to 2\mathcal{O} \oplus \mathcal{O}(1)$,

$$\psi = \begin{bmatrix} \star & l_{22} & l_{12} \\ \star & l_{21} & l_{11} \\ \star & q_2 & q_1 \end{bmatrix}.$$

From Claim 3.1.2 we know that $\mathcal{G}'$ gives a stable point in $\mathrm{M}_{\mathbb{P}^2}(6,4)$. It is now straightforward to check that any destabilizing subsheaf $\mathcal{E}$ of $\mathcal{G}$ must give a point in $\mathrm{M}_{\mathbb{P}^2}(1,1)$ or $\mathrm{M}_{\mathbb{P}^2}(2,2)$. The existence of such sheaves can be ruled out as above using diagrams analogous to diagram (8). □

CLAIM 3.1.4
Let $\mathcal{F}$ be a sheaf having a resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

in which ψ_{12} has linearly independent maximal minors. Then $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,0)$. If the maximal minors of ψ_{12} have no common factor, then $\mathcal{F}$ is stable. If they have a common linear factor l, then $\mathcal{O}_L(-1) \subset \mathcal{F}$ is the unique proper subsheaf of slope zero, where $L \subset \mathbb{P}^2$ is the line with equation $l = 0$.

Proof
When the maximal minors of ψ_{12} have no common factor the claim follows from Proposition 2.3.1. Assume that the maximal minors of ψ_{12} have a common linear factor l. We have an extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(1) \longrightarrow 0,$$

where L is the line with equation $l = 0$ and C is a quintic curve. Thus $\mathcal{F}$ is semistable and $\mathcal{O}_L(-1)$, $\mathcal{O}_C(1)$ are its stable factors. The latter cannot be a subsheaf of $\mathcal{F}$ because $\mathrm{H}^0(\mathcal{F}(-1))$ vanishes. □

PROPOSITION 3.1.5
The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 2$, $\mathrm{h}^1(\mathcal{F}(1)) = 0$ are precisely the sheaves having a resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

in which φ_{11} has linearly independent entries and φ_{22} has linearly independent maximal minors.

Proof
Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and satisfy the above cohomological conditions. Display diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}(1)$ reads

$$\begin{array}{ccccc} 5\mathcal{O}(-1) & \xrightarrow{\varphi_1} & 2\Omega^1(1) & & 0 \\ & & & & \\ 0 & & 3\Omega^1(1) & \xrightarrow{\varphi_4} & 7\mathcal{O}. \end{array}$$

Resolving $2\Omega^1(1)$ yields the exact sequence

$$0 \longrightarrow \mathcal{K}er(\varphi_1) \longrightarrow 2\mathcal{O}(-2) \oplus 5\mathcal{O}(-1) \xrightarrow{\sigma} 6\mathcal{O}(-1) \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0.$$

Arguing as in the proof of Proposition 3.1.3, we see that $\operatorname{rank}(\sigma_{12}) = 5$; therefore, $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$, and $\mathcal{C}oker(\varphi_1) \simeq \mathbb{C}_x$. From (2.1.2) we get the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \oplus 3\Omega^1(1) \longrightarrow 7\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow 0,$$

hence the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 9\mathcal{O} \longrightarrow 7\mathcal{O} \oplus 3\mathcal{O}(1) \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow 0.$$

From (2.1.3) we get the extension

$$0 \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow \mathcal{F}(1) \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

We apply the horseshoe lemma to the above extension, to the above resolution of $\mathcal{C}oker(\varphi_5)$, and to the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(-1)$. We obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus 9\mathcal{O} \longrightarrow \mathcal{O}(-1) \oplus 7\mathcal{O} \oplus 3\mathcal{O}(1) \longrightarrow \mathcal{F}(1) \longrightarrow 0.$$

The map $\mathcal{O}(-3) \to \mathcal{O}(-3)$ is nonzero because $\mathrm{h}^1(\mathcal{F}(1)) = 0$. Canceling $\mathcal{O}(-3)$ and tensoring with $\mathcal{O}(-1)$ yields the resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 9\mathcal{O}(-1) \stackrel{\rho}{\longrightarrow} \mathcal{O}(-2) \oplus 7\mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Notice that $\operatorname{rank}(\rho_{22}) = 7$; otherwise $\mathcal{F}$ would map surjectively to the cokernel of a morphism $2\mathcal{O}(-3) \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)$, in violation of semistability. Canceling $7\mathcal{O}(-1)$ we arrive at a resolution as in the proposition.

Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we need to show that there are no destabilizing subsheaves. From the snake lemma we get an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{F}'$ has a resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-1) \stackrel{\psi}{\longrightarrow} 3\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0$$

in which $\psi_{12} = \varphi_{22}$. According to Claim 3.1.4, $\mathcal{F}'$ is semistable and the only possible subsheaf of $\mathcal{F}'$ of slope zero must be of the form $\mathcal{O}_L(-1)$. It follows that for every subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $\mathrm{p}(\mathcal{E}) \leq 0$ excepting, possibly, subsheaves that fit into an extension of the form

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{E} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

In this case $\mathcal{E} \simeq \mathcal{O}_L$ because $\mathcal{E}$ has no zero-dimensional torsion, and we have a diagram similar to diagram (8), leading to a contradiction. □

PROPOSITION 3.1.6

The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 1$, $\mathrm{h}^1(\mathcal{F}) = 2$ are precisely the sheaves having a resolution of the form

(i) $$0 \longrightarrow 2\mathcal{O}(-3) \stackrel{\varphi}{\longrightarrow} \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q_1 & q_2 \\ g_1 & g_2 \end{bmatrix},$$

where q_1, q_2 have no common factor, or the sheaves having a resolution of the form

(ii) $$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} l_1 & l_2 & 0 \\ q_1 & q_2 & l \\ g_1 & g_2 & h \end{bmatrix}, \quad \textit{where } \varphi \nsim \begin{bmatrix} \star & \star & 0 \\ 0 & 0 & \star \\ \star & \star & \star \end{bmatrix},$$

l_1, l_2 are linearly independent one-forms, and $l \neq 0$.

Proof
Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and satisfy the above cohomological conditions. Denote $m = \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson tableau (2.1.4) for the sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$ reads

$$3\mathcal{O}(-2) \xrightarrow{\varphi_1} m\mathcal{O}(-1) \xrightarrow{\varphi_2} \mathcal{O}$$

$$2\mathcal{O}(-2) \xrightarrow{\varphi_3} (m+4)\mathcal{O}(-1) \xrightarrow{\varphi_4} 6\mathcal{O}.$$

Since φ_2 is surjective, $m \geq 3$. Since $\mathcal{G}$ maps surjectively to $\mathcal{C} = \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)$, $m \leq 4$. If $m = 4$, then $\mathrm{p}(\mathcal{C}) = -1/2$, violating the semistability of $\mathcal{G}$. Thus $m = 3$. As at [10, Proposition 2.2.4], we have $\mathcal{K}er(\varphi_2) = \mathcal{I}m(\varphi_1)$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. As at [10, Proposition 3.2.5], it can be shown that $\mathcal{C}oker(\varphi_3) \simeq 2\Omega^1(1) \oplus \mathcal{O}(-1)$. Combining the exact sequences (2.1.5) and (2.1.6) we obtain the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\Omega^1(1) \oplus \mathcal{O}(-1) \longrightarrow 6\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Dualizing and resolving $2\Omega^1$ leads to the resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 6\mathcal{O}(-2) \xrightarrow{\rho} 6\mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Note that $\mathrm{rank}(\rho_{12}) \geq 5$; otherwise $\mathcal{F}$ would map surjectively to the cokernel of a morphism $2\mathcal{O}(-3) \to 2\mathcal{O}(-2)$, in violation of semistability. When $\mathrm{rank}(\rho) = 5$ we get resolution (ii). When $\mathrm{rank}(\rho) = 6$ we get resolution (i).

Conversely, if $\mathcal{F}$ has resolution (i), then, in view of Corollary 2.3.2, $\mathcal{F}$ is stable. Assume now that $\mathcal{F}$ has resolution (ii). We examine first the case when l does not divide h. From the snake lemma we have an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{F}'$ is the cokernel of a morphism $\psi \colon \mathcal{O}(-4) \oplus \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(1)$ for which ψ_{12} does not divide ψ_{22}. In view of Corollary 2.3.2, $\mathcal{F}'$ is semistable and the only possible subsheaf of $\mathcal{F}'$ of slope zero must be of the form $\mathcal{O}_C(1)$, for a quintic curve $C \subset \mathbb{P}^2$. It follows that every proper subsheaf of $\mathcal{F}$ has non-positive slope except, possibly, extensions $\mathcal{E}$ of $\mathbb{C}_x$ by $\mathcal{O}_C(1)$. According to [10, Proposition 3.1.5], we have a resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{E} \longrightarrow 0.$$

This forms part of a diagram analogous to diagram (8), leading to a contradiction.

Assume now that l divides h. We may assume that $h = 0$. Let L be the line given by the equation $l = 0$. From the snake lemma we get a nonsplit extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{E}$ is as above. According to [10, Proposition 3.1.5], $\mathcal{E}$ is stable. It is easy to see now that $\mathcal{F}$ is stable as well. □

PROPOSITION 3.1.7

(i) *The sheaves* $\mathcal{G}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,4)$ *and satisfying the condition* $\mathrm{h}^0(\mathcal{G}(-2)) > 0$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow 2\mathcal{O}(-3) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(2) \longrightarrow \mathcal{G} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} l_1 & l_2 \\ f_1 & f_2 \end{bmatrix},$$

where l_1, l_2 *are linearly independent one-forms.*

(ii) *By duality, the sheaves* $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,2)$ *and satisfying the condition* $\mathrm{h}^1(F(1)) > 0$ *are precisely the sheaves having resolution*

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O} \xrightarrow{\varphi^{\mathrm{T}}} 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

These are precisely the sheaves of the form $\mathcal{J}_x(2)$, *where* $\mathcal{J}_x \subset \mathcal{O}_C$ *is the ideal sheaf of a closed point* x *inside a sextic curve* $C \subset \mathbb{P}^2$.

Proof

The argument is entirely analogous to the argument at [10, Proposition 3.1.5]. □

PROPOSITION 3.1.8

The sheaves $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,1)$ *and satisfying the condition* $\mathrm{h}^1(\mathcal{F}(1)) > 0$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} h & l \\ g & q \end{bmatrix},$$

where $l \neq 0$ *and* l *does not divide* q. *These are precisely the sheaves of the form* $\mathcal{J}_Z(2)$, *where* $\mathcal{J}_Z \subset \mathcal{O}_C$ *is the ideal sheaf of a zero-dimensional subscheme* Z *of length* 2 *inside a sextic curve* $C \subset \mathbb{P}^2$.

Proof

Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and satisfy the condition $\mathrm{h}^1(\mathcal{F}(1)) > 0$. Denote $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$. According to [9], $\mathcal{G}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,5)$ and $\mathrm{h}^0(\mathcal{G}(-2)) > 0$. As in

[3, Proposition 2.1.3], there is an injective morphism $\mathcal{O}_C \to \mathcal{G}(-2)$, where $C \subset \mathbb{P}^2$ is a curve. Clearly C has degree 6; otherwise $\mathcal{O}_C$ would destabilize $\mathcal{G}(-2)$. The quotient sheaf $\mathcal{C} = \mathcal{G}/\mathcal{O}_C(2)$ has support of dimension zero and length 2. Write $\mathcal{C}$ as an extension of $\mathcal{O}_{\mathbb{P}^2}$-modules of the form

$$0 \longrightarrow \mathbb{C}_x \longrightarrow \mathcal{C} \longrightarrow \mathbb{C}_y \longrightarrow 0.$$

Let $\mathcal{G}'$ be the preimage of $\mathbb{C}_x$ in $\mathcal{G}$. This subsheaf has no zero-dimensional torsion and is an extension of $\mathbb{C}_x$ by $\mathcal{O}_C(2)$; hence, in view of Proposition 3.1.7, it has a resolution of the form

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(2) \longrightarrow \mathcal{G}' \longrightarrow 0.$$

We construct a resolution of $\mathcal{G}$ from the above resolution of $\mathcal{G}'$ and from the standard resolution of $\mathbb{C}_y$ tensored with $\mathcal{O}(-1)$:

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(2) \longrightarrow \mathcal{G} \longrightarrow 0.$$

If the morphism $\mathcal{O}(-3) \to 2\mathcal{O}(-3)$ were zero, then it could be shown, as in the proof of [10, Proposition 2.3.2], that $\mathbb{C}_y$ is a direct summand of $\mathcal{G}$. This would contradict our hypothesis. Thus we may cancel $\mathcal{O}(-3)$ to get the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(2) \longrightarrow \mathcal{G} \longrightarrow 0.$$

If the morphism $2\mathcal{O}(-2) \to \mathcal{O}(-2)$ were zero, then $\mathcal{G}$ would have a destabilizing quotient sheaf of the form $\mathcal{O}_L(-2)$. Thus we may cancel $\mathcal{O}(-2)$ to get a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \stackrel{\psi}{\longrightarrow} \mathcal{O}(-1) \oplus \mathcal{O}(2) \longrightarrow \mathcal{G} \longrightarrow 0,$$

$$\psi = \begin{bmatrix} q & l \\ g & h \end{bmatrix},$$

in which $l \neq 0$ and l does not divide q. Dualizing, we get a resolution for $\mathcal{F}$ as in the proposition. The converse follows from Corollary 2.3.2. □

In the remaining part of this subsection we shall prove that there are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ besides the sheaves we have discussed so far. In view of Proposition 3.1.8 we may restrict our attention to the case when $\mathrm{H}^1(\mathcal{F}(1)) = 0$. Assume that $\mathrm{h}^0(\mathcal{F}(-1)) \leq 1$. According to Corollary 2.2.2(i), (ii) and Proposition 3.1.1 the pair $(\mathrm{h}^0(\mathcal{F}(-1)), \mathrm{h}^1(\mathcal{F}))$ may be one of the following: $(0,0)$, $(0,1)$, $(0,2)$, $(1,2)$. Each of these situations has already been examined. The following concludes the classification of sheaves in $\mathrm{M}_{\mathbb{P}^2}(6,1)$.

PROPOSITION 3.1.9

Let $\mathcal{F}$ be a sheaf giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$. Then $\mathrm{h}^0(\mathcal{F}(-1)) = 0$ or 1.

Proof

Assume that $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$ and $\mathrm{h}^0(\mathcal{F}(-1)) > 0$. As in the proof of [3, Proposition 2.1.3], there is an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for a curve $C \subset \mathbb{P}^2$. From the semistability of $\mathcal{F}$ we see that C has degree 5 or 6. In the first case $\mathcal{F}(-1)/\mathcal{O}_C$ has Hilbert polynomial $\mathrm{P}(m) = m$ and has no zero-dimensional torsion. Indeed, the pullback in $\mathcal{F}(-1)$ of any nonzero subsheaf of $\mathcal{F}(-1)/\mathcal{O}_C$ supported on finitely many points would destabilize $\mathcal{F}(-1)$. We deduce that $\mathcal{F}(-1)/\mathcal{O}_C$ is isomorphic to $\mathcal{O}_L(-1)$; hence $\mathrm{h}^0(\mathcal{F}(-1)) = 1$.

Assume now that C is a sextic curve and $\mathrm{H}^1(\mathcal{F}(1)) = 0$. The quotient sheaf $\mathcal{C} = \mathcal{F}(-1)/\mathcal{O}_C$ has support of dimension zero and length 4. Assume that $\mathrm{h}^0(\mathcal{F}(-1)) > 1$. Then, in view of Corollary 2.2.2(iii), we have $\mathrm{h}^0(\mathcal{F}(-1)) \geq 3$. We claim that there is a global section s of $\mathcal{F}(-1)$ such that its image in $\mathcal{C}$ generates a subsheaf isomorphic to $\mathcal{O}_Z$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 1, 2, or 3. Indeed, as $\mathrm{h}^0(\mathcal{O}_C) = 1$ and $\mathrm{h}^0(\mathcal{F}(-1)) \geq 3$, there are global sections s_1 and s_2 of $\mathcal{F}(-1)$ such that their images in $\mathcal{C}$ are linearly independent. Consider a subsheaf $\mathcal{C}' \subset \mathcal{C}$ of length 3. Choose $c_1, c_2 \in \mathbb{C}$, not both zero, such that the image of $c_1 s_1 + c_2 s_2$ under the composite map $\mathcal{F}(-1) \to \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ is zero. Then $s = c_1 s_1 + c_2 s_2$ satisfies our requirements.

Let $\mathcal{F}' \subset \mathcal{F}(-1)$ be the preimage of $\mathcal{O}_Z$. Assume first that Z is not contained in a line, so, in particular, it has length 3. According to [1, Proposition 4.5], we have a resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow 3\mathcal{O}(-2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Combining this with the standard resolution of $\mathcal{O}_C$ we obtain the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-6) \oplus 3\mathcal{O}(-2) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

As the map $2\mathcal{O}(-3) \to \mathcal{O}(-6)$ in the above complex is zero and as $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O})$ vanishes, we can show, as in the proof of [10, Proposition 2.3.2], that $\mathcal{O}_Z$ is a direct summand of $\mathcal{F}'$. This is absurd; by hypothesis $\mathcal{F}(-1)$ has no zero-dimensional torsion. The same argument applies if Z is contained in a line and has length 3, except that this time we use the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

The cases when $\mathrm{length}(Z) = 1$ or 2 are analogous. Thus $\mathrm{h}^0(\mathcal{F}(-1)) = 1$. □

3.2. The strata as quotients

In Section 3.1 we classified all sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6,1)$, namely, we showed that this moduli space can be decomposed into six subsets $X_0, \ldots, X_5$ (cf. Table 1). Recall the notations $\mathbb{W}_i$, W_i, G_i, $0 \leq i \leq 5$, from Section 1.2. The fibers of the canonical maps $\rho_i \colon W_i \to X_i$ are precisely the G_i-orbits. Given $[\mathcal{F}] \in X_i$, we constructed $\varphi \in \rho_i^{-1}[\mathcal{F}]$ starting from the Beilinson spectral sequence I or II associated to $\mathcal{F}$ or some twist of this sheaf and performing algebraic operations. This construction is local in the sense that it can be done for flat families of sheaves that are in a sufficiently small neighborhood of $[\mathcal{F}]$. This allows us to deduce, as at [3, Theorem 3.1.6], that the maps ρ_i are categorical quotient maps. Applying [11, Remark 2, p. 5], it follows that X_i is normal. From [12, Theorem 4.2], we conclude that each ρ_i is a geometric quotient map.

Some of these quotients have concrete descriptions. The quotient W_5/G_5 is isomorphic to the flag Hilbert scheme of pairs (C, Z), where $C \subset \mathbb{P}^2$ is a curve of degree 6 and $Z \subset C$ is a zero-dimensional scheme of length 2. Let $W_0' \subset \mathbb{W}_0$ be the set of morphisms φ for which φ_{11} is semistable as a Kronecker module and $\varphi_{21} \neq v\varphi_{11}$ for any $v \in \mathrm{Hom}(4\mathcal{O}(-1), \mathcal{O})$. Clearly $W_0 \subsetneqq W_0'$, being the subset of injective morphisms. According to [4, Section 9.3], the geometric quotient W_0'/G_0 exists and is the projectivization of a certain vector bundle over $\mathrm{N}(3,5,4)$ of rank 18. Clearly W_0/G_0 is a proper open subset of W_0'/G_0.

The quotient W_3/G_3 can be constructed as at [10, Proposition 2.2.2]. Let $W_3' \subset \mathbb{W}_3$ be the subset given by the following conditions: $\varphi_{12} = 0$, φ_{11} has linearly independent entries, φ_{22} has linearly independent maximal minors, $\varphi_{21} \neq \varphi_{22}u + v\varphi_{11}$ for any $u \in \mathrm{Hom}(2\mathcal{O}(-3), 2\mathcal{O}(-1))$, and $v \in \mathrm{Hom}(\mathcal{O}(-2), 3\mathcal{O})$. Clearly $W_3 \subsetneqq W_3'$, being the subset of injective morphisms. Let U_3 be the set of pairs $(\varphi_{11}, \varphi_{22})$ satisfying the above properties, and let Γ_3 be the canonical group acting on U_3. Applying the method of [10, Proposition 2.2.2] one can show that the quotient W_3'/G_3 exists and is the projectivisation of a vector bundle of rank 24 over $U_3/\Gamma_3 \simeq \mathbb{P}^2 \times \mathrm{N}(3,2,3)$. Thus W_3/G_3 is a proper open subset of W_3'/G_3. Analogously one can construct the quotient W_2/G_2 except that this time one has to pay special attention to the fact that the canonical group acting on the space of triples $(\varphi_{11}, \varphi_{12}, \varphi_{23})$ satisfying the properties of Proposition 3.1.3(ii) is nonreductive.

CLAIM 3.2.1

Let $\mathbb{U} = \mathrm{Hom}(\mathcal{O}(-3) \oplus 2\mathcal{O}(-2), 2\mathcal{O}(-1))$, *and let* $U \subset \mathbb{U}$ *be the set of morphisms*

$$\psi = \begin{bmatrix} q_1 & l_{11} & l_{12} \\ q_2 & l_{21} & l_{22} \end{bmatrix}$$

that satisfy the conditions of Proposition 3.1.3(ii). Let G be the canonical group acting by conjugation on U. Then there exists a geometric quotient U/G, which is a smooth projective variety of dimension 10.

Proof

It is straightforward to check that the conditions defining U are equivalent to saying that ψ is not equivalent to a morphism represented by a matrix having one of the following forms:

$$\begin{bmatrix} \star & 0 & 0 \\ \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \star \\ \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} 0 & \star & \star \\ 0 & \star & \star \end{bmatrix}.$$

This allows us to interpret U as the set of semistable points in the sense of [4]. Adopting the notations of [4], let $\Lambda = (\lambda_1, \lambda_2, \mu_1)$ be a polarization for the action of G on $\mathbb{U}$ satisfying the condition $1/4 < \lambda_2 < 1/2$. Using King's criterion of semistability (see [6]) and the above alternate description of U we deduce that U is the set $\mathbb{U}^{\mathrm{s}}(\Lambda)$ of morphisms that are stable relative to Λ (cf. [4]). According to [4, Propositions 6.1.1, 7.2.2, 8.1.3], there exists a geometric quotient $\mathbb{U}^{\mathrm{s}}(\Lambda)/G$, which is a smooth quasi-projective variety, provided $3/7 < \lambda_2 < 1/2$, which we assume to be the case. This quotient is projective because $\mathbb{U}^{\mathrm{s}}(\Lambda)$ coincides with the set of semistable points in $\mathbb{U}$ relative to Λ. □

The quotient W_2/G_2 is an open subset of the projectivization of a vector bundle over $(U/G) \times \mathbb{P}^2$ of rank 22.

3.3. Generic sheaves

Let $C \subset \mathbb{P}^2$ denote an arbitrary smooth sextic curve, and let P_i denote distinct points on C. According to [1, Propositions 4.5, 4.6], the cokernels of morphisms $4\mathcal{O}(-5) \to 5\mathcal{O}(-4)$ whose maximal minors have no common factor are precisely the ideal sheaves $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ of zero-dimensional schemes $Z \subset \mathbb{P}^2$ of length 10 that are not contained in a cubic curve. It follows that the generic sheaves giving points in X_0 are of the form $\mathcal{O}_C(P_1 + \cdots + P_{10})$, where P_i, $1 \le i \le 10$, are not contained in a cubic curve.

According to [1, Propositions 4.5 and 4.6], the cokernels of morphisms $2\mathcal{O}(-3) \to 3\mathcal{O}(-2)$ whose maximal minors have no common factor are precisely the ideal sheaves $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ of zero-dimensional schemes $Z \subset \mathbb{P}^2$ of length 3 that are not contained in a line. It follows that the generic sheaves in X_3 have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4)$, where P_1, P_2, P_3 are non-colinear.

Obviously, the generic sheaves in X_4 have the form $\mathcal{O}_C(1)(P_1 + P_2 + P_3 + P_4)$, where no three points among P_1, P_2, P_3, P_4 are colinear. Also, the generic sheaves in X_5 are of the form $\mathcal{O}_C(2)(-P_1 - P_2)$. According to Claim 3.3.1 below, the generic sheaves in X_1 have the form $\mathcal{O}_C(3)(-P_1 - \cdots - P_8)$, where no four points among $P_1, \ldots, P_8$ are colinear and no seven of them lie on a conic curve. By Claim 3.3.2, the generic sheaves in X_2 have the form $\mathcal{O}_C(1)(P_1 + \cdots + P_5 - P_6)$, where no three points among $P_1, \ldots, P_5$ are colinear.

CLAIM 3.3.1

Let $U \subset \mathrm{Hom}(2\mathcal{O}(-2), \mathcal{O}(-1) \oplus 2\mathcal{O})$ be the set of morphisms represented by matrices

$$\begin{bmatrix} l_1 & l_2 \\ q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

for which the maximal minors $l_1 q_{12} - l_2 q_{11}$ and $l_1 q_{22} - l_2 q_{21}$ have no common factor. The cokernels of the morphisms in U are precisely the sheaves of the form $\mathcal{I}_Z(3)$, where $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of a zero-dimensional subscheme $Z \subset \mathbb{P}^2$ of length 8, no subscheme of length 4 of which is contained in a line and no subscheme of length 7 of which is contained in a conic curve.

Proof

Let $\psi \in U$, and let ζ_i denote the maximal minor of ψ obtained by deleting row i. Since $\zeta_1, \zeta_2, \zeta_3$ have no common factor, there is an exact sequence of the form

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-1) \oplus 2\mathcal{O} \xrightarrow{\zeta} \mathcal{O}(3) \longrightarrow \mathcal{C} \longrightarrow 0,$$

$$\zeta = \begin{bmatrix} \zeta_1 & -\zeta_2 & \zeta_3 \end{bmatrix}.$$

The Hilbert polynomial of $\mathcal{C}$ is 8; hence $\mathcal{C}$ is the structure sheaf of a zero-dimensional scheme Z of length 8 and $\mathcal{C}oker(\psi) \simeq \mathcal{I}_Z(3)$. If four of the points of Z were on the line with equation $l = 0$, then, by Bézout's theorem, l would divide ζ_2 and ζ_3, contrary to our hypothesis. Similarly, if seven of the points of Z lay on the irreducible conic curve with equation $q = 0$, then q would divide ζ_2 and ζ_3.

For the converse we use the method of [1, Proposition 4.5]. Assume that $Z \subset \mathbb{P}^2$ is a subscheme as in the proposition. The Beilinson spectral sequence I with E^1-term

$$\mathrm{E}^1_{ij} = \mathrm{H}^j\big(\mathcal{I}_Z(2) \otimes \Omega^{-i}(-i)\big) \otimes \mathcal{O}(i)$$

converges to $\mathcal{I}_Z(2)$. By hypothesis $\mathrm{H}^0(\mathcal{I}_Z(2)) = 0$, hence also $\mathrm{H}^0(\mathcal{I}_Z(3) \otimes \Omega^1) = 0$ and $\mathrm{H}^0(\mathcal{I}_Z(1)) = 0$. Using Serre duality we can show that $\mathrm{H}^2(\mathcal{I}_Z(2))$, $\mathrm{H}^2(\mathcal{I}_Z(1))$ and $\mathrm{H}^2(\mathcal{I}_Z(3) \otimes \Omega^1)$ vanish. The middle row of the display diagram for the Beilinson spectral sequence yields a monad

$$0 \longrightarrow 5\mathcal{O}(-2) \xrightarrow{\alpha} 8\mathcal{O}(-1) \xrightarrow{\beta} 2\mathcal{O} \longrightarrow 0$$

with middle cohomology $\mathcal{I}_Z(2)$. Denote $\mathcal{B} = \mathcal{H}om(\mathcal{K}er(\beta), \mathcal{O}(-1))$. Applying the functor $\mathcal{H}om(\_\_, \mathcal{O}(-1))$ we get the exact sequences

$$0 \longrightarrow 2\mathcal{O}(-1) \xrightarrow{\beta^{\mathrm{T}}} 8\mathcal{O} \longrightarrow \mathcal{B} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{H}om\big(\mathcal{I}_Z(2), \mathcal{O}(-1)\big) \longrightarrow \mathcal{B} \longrightarrow 5\mathcal{O}(1).$$

From the first exact sequence we see that $\mathrm{h}^0(\mathcal{B}) = 8$, and from the second exact sequence we see that $\mathcal{B}$ is torsion-free. It follows that the morphism $8\mathcal{O} \to \mathcal{B}$ cannot factor through $7\mathcal{O} \oplus \mathbb{C}_x$. This allows us to deduce, as at [10, Proposition 2.1.4], that any matrix representing β^{T} has at least three linearly independent entries on each column; in other words, that β^{T} has one of the following canonical forms:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ X & R \\ Y & S \\ Z & T \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ X & 0 \\ Y & R \\ Z & S \\ 0 & T \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ X & 0 \\ Y & 0 \\ Z & R \\ 0 & S \\ 0 & T \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ X & 0 \\ Y & 0 \\ Z & 0 \\ 0 & X \\ 0 & Y \\ 0 & Z \end{bmatrix}.$$

Moreover, the morphism $8\mathcal{O} \to \mathcal{B}$ cannot factor through $6\mathcal{O} \oplus \mathcal{O}_L(1)$. This allows us to deduce, as at [10, Proposition 3.1.3], that the first three canonical forms are unfeasible. Thus $\mathcal{K}er(\beta) \simeq 2\Omega^1 \oplus 2\mathcal{O}(-1)$, so we have a resolution

$$0 \longrightarrow 5\mathcal{O}(-2) \longrightarrow 2\Omega^1 \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{I}_Z(2) \longrightarrow 0,$$

hence a resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 5\mathcal{O}(-2) \xrightarrow{\rho} 6\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{I}_Z(2) \longrightarrow 0.$$

Notice that $\operatorname{rank}(\rho_{12}) \geq 3$; otherwise $\mathcal{I}_Z(2)$ would map surjectively onto the cokernel of a morphism $2\mathcal{O}(-3) \to 4\mathcal{O}(-2)$, which is impossible, because $\operatorname{rank}(\mathcal{I}_Z(2)) = 1$. Assume that $\operatorname{rank}(\rho_{12}) = 3$. We get a resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\eta} 3\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{I}_Z(2) \longrightarrow 0$$

with $\eta_{12} = 0$. Clearly η_{22} is injective and $\mathcal{C}oker(\eta_{22})$ maps injectively to $\mathcal{I}_Z(2)$. This is absurd: $\mathcal{I}_Z(2)$ is a torsion-free sheaf, whereas $\mathcal{C}oker(\eta_{22})$ is a torsion sheaf. Assume that $\operatorname{rank}(\rho_{12}) = 4$. We have a resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\eta} 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{I}_Z(2) \longrightarrow 0$$

with $\eta_{12} = 0$. The entries of η_{22} are linearly independent; otherwise $\mathcal{I}_Z(2)$ would have a subsheaf of the form $\mathcal{O}_L(-1)$, which is absurd. Let x be the common zero of the entries of η_{22}. The points of Z distinct from x lie on the conic curve with equation $\det(\eta_{11}) = 0$, contradicting our hypothesis on Z. We conclude that $\operatorname{rank}(\rho_{12}) = 5$, and we arrive at the resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \xrightarrow{\psi} \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{I}_Z(2) \longrightarrow 0,$$

$$\psi = \begin{bmatrix} l_1 & l_2 \\ q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}.$$

We will show that ψ satisfies the conditions defining U. Assume that $\gcd(\zeta_2, \zeta_3)$ is a linear form l. By hypothesis, at least five points of Z do not lie on the line given by the equation $l = 0$. These points must be then in the common zero-set of ζ_2/l and ζ_3/l, which, by Bézout's theorem, is impossible. Likewise, $\gcd(\zeta_2, \zeta_3)$ cannot be a quadratic form. If ζ_2 divided ζ_3, then, performing possibly row operations on ψ, we may assume that $\zeta_3 = 0$. It would follow that

$$\psi \sim \begin{bmatrix} \star & \star \\ 0 & 0 \\ \star & \star \end{bmatrix} \qquad \text{or} \qquad \psi \sim \begin{bmatrix} \star & 0 \\ \star & 0 \\ \star & \star \end{bmatrix}.$$

In each case $\mathcal{I}_Z(2)$ would have a torsion subsheaf, which is absurd. □

CLAIM 3.3.2

Let $U \subset \operatorname{Hom}(2\mathcal{O}(-1), 2\mathcal{O} \oplus \mathcal{O}(1))$ *be the set of morphisms represented by matrices*

$$\begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ q_1 & q_2 \end{bmatrix}$$

such that $\zeta_3 = l_{11}l_{22} - l_{12}l_{21}$ *is irreducible and does not divide any of the other maximal minors. The cokernels of the morphisms in* U *are precisely the sheaves of the form* $\mathcal{I}_Z(3)$*, where* $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ *is the ideal sheaf of a zero-dimensional subscheme* $Z \subset \mathbb{P}^2$ *of length* 5*, no subscheme of length* 3 *of which is contained in a line.*

Proof
The argument is analogous to the argument at Claim 3.3.1. □

4. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6,2)$

4.1. Classification of sheaves

PROPOSITION 4.1.1
Every sheaf $\mathcal{F}$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the condition $\mathrm{h}^1(\mathcal{F})=0$ also satisfies the condition $\mathrm{h}^0(\mathcal{F}(-1))=0$. For these sheaves $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1))=0$ or 1. The sheaves from the first case are precisely the sheaves having a resolution of the form

$$\text{(i)} \qquad 0 \longrightarrow 4\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1)\oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ is not equivalent, modulo the action of the group of automorphisms, to a morphism represented by a matrix having one of the following forms:

$$\begin{bmatrix} \star & 0 & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \end{bmatrix}.$$

The sheaves in the second case are precisely the sheaves with resolution of the form

$$\text{(ii)} \qquad 0 \longrightarrow 4\mathcal{O}(-2)\oplus\mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}(-1)\oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12}=0$, φ_{11} is semistable as a Kronecker module and φ_{22} has linearly independent entries.

Proof
The first statement follows from [8, Claim 6.4]. The rest of the proposition follows by duality from [8, Claim 4.3]. □

PROPOSITION 4.1.2
The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1))=0$, $\mathrm{h}^1(\mathcal{F})=1$, $\mathrm{h}^1(\mathcal{F}(1))=0$ are precisely the sheaves having a resolution of the form

$$\text{(i)} \qquad 0 \longrightarrow \mathcal{O}(-3)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ is not equivalent to a morphism of any of the following forms:

$$\begin{bmatrix} \star & \star & \star \\ \star & \star & 0 \\ \star & \star & 0 \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star \\ \star & 0 & \star \\ \star & 0 & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & 0 & 0 \end{bmatrix},$$

or the sheaves having a resolution of the form

$$\text{(ii)} \qquad 0 \longrightarrow \mathcal{O}(-3)\oplus\mathcal{O}(-2)\oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1)\oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12} \neq 0$, $\varphi_{13} = 0$, φ_{11} is not divisible by φ_{12} and φ_{23} has linearly independent maximal minors.

Proof

Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the above cohomological conditions. Display diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}(1)$ reads

$$\begin{array}{ccccc} 4\mathcal{O}(-1) & \xrightarrow{\varphi_1} & \Omega^1(1) & & 0 \\ & & & & \\ 0 & & 3\Omega^1(1) & \xrightarrow{\varphi_4} & 8\mathcal{O}. \end{array}$$

As in Proposition 3.1.3, we have $\mathcal{C}oker(\varphi_1) = 0$, $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-1)$. Performing the same steps as at Proposition 3.1.3 we arrive at the resolution

$$0 \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus 9\mathcal{O} \xrightarrow{\rho} 8\mathcal{O} \oplus 3\mathcal{O}(1) \longrightarrow \mathcal{F}(1) \longrightarrow 0.$$

Notice that $\mathrm{rank}(\rho_{13}) \geq 7$; otherwise $\mathcal{F}(1)$ would map surjectively to the cokernel of a morphism $\mathcal{O}(-2) \oplus \mathcal{O}(-1) \to 2\mathcal{O}$, in violation of semistability. From here on we get resolution (i) or (ii), depending on whether $\mathrm{rank}(\rho_{13}) = 8$ or 7.

Conversely, if $\mathcal{F}$ has resolution (i), then we can argue as in Proposition 3.1.3 to show that $\mathcal{F}$ is semistable. Assume now that $\mathcal{F}$ has resolution (ii). Assume that there is a destabilizing subsheaf $\mathcal{E} \subset \mathcal{F}$ that is itself semistable. We have an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where Z is the zero-dimensional scheme of length 2 given by the ideal $(\varphi_{11}, \varphi_{12})$ and $\mathcal{F}'$ has a resolution as at Claim 3.1.4, so $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,0)$, and the only subsheaf of $\mathcal{F}'$ of slope zero, if there is one, must be of the form $\mathcal{O}_L(-1)$. It follows that $\mathcal{E}$ must have Hilbert polynomial $\mathrm{P}_{\mathcal{E}}(m) = 2m+1$, $m+2$, or $m+1$. In each case we have a diagram analogous to diagram (8) at Proposition 3.1.3, leading to a contradiction. □

PROPOSITION 4.1.3

The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 1$ and $\mathrm{h}^1(\mathcal{F}) = 1$ are precisely the sheaves having a resolution of the form

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ satisfies the conditions of Claim 3.1.2.

Proof

Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the above cohomological conditions. Denote $m = \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson diagram (2.1.4) for the dual sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,4)$ reads

$$3\mathcal{O}(-2) \xrightarrow{\varphi_1} m\mathcal{O}(-1) \xrightarrow{\varphi_2} \mathcal{O}$$

$$\mathcal{O}(-2) \xrightarrow{\varphi_3} (m+2)\mathcal{O}(-1) \xrightarrow{\varphi_4} 5\mathcal{O}.$$

Arguing as in the proof of Proposition 3.1.6, we can show that $m=3$, that $\mathcal{K}er(\varphi_2)=\mathcal{I}m(\varphi_1)$, and that $\mathcal{K}er(\varphi_1)\simeq\mathcal{O}(-3)$. Combining the exact sequences (2.1.5) and (2.1.6) we get the resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3)\oplus 5\mathcal{O}(-1) \longrightarrow 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

As in [10, Proposition 2.1.4], we have $\mathcal{C}oker(\psi)\simeq\mathcal{O}(-3)\oplus 2\mathcal{O}(-1)\oplus\Omega^1(1)$, and the cokernel of the induced morphism $\Omega^1(1)\to 5\mathcal{O}$ is isomorphic to $2\mathcal{O}\oplus\mathcal{O}(1)$. We finally arrive at the resolution dual to the resolution in the proposition:

$$0 \longrightarrow \mathcal{O}(-3)\oplus 2\mathcal{O}(-1) \longrightarrow 2\mathcal{O}\oplus\mathcal{O}(1) \longrightarrow \mathcal{G} \longrightarrow 0.$$

The converse is the object of Claim 3.1.2. □

PROPOSITION 4.1.4

The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1))=1$, $\mathrm{h}^1(\mathcal{F})=2$, $\mathrm{h}^1(\mathcal{F}(1))=0$ are precisely the sheaves having a resolution of the form

$$0 \longrightarrow 2\mathcal{O}(-3)\oplus\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2)\oplus\mathcal{O}\oplus\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} has linearly independent entries, $\varphi_{22}\neq 0$, and does not divide φ_{32}.

Proof

Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the above cohomological conditions. Display diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}(1)$ reads

$$5\mathcal{O}(-1) \xrightarrow{\varphi_1} 2\Omega^1(1) \qquad 0$$

$$\mathcal{O}(-1) \xrightarrow{\varphi_3} 4\Omega^1(1) \xrightarrow{\varphi_4} 8\mathcal{O}.$$

Arguing as in the proof of Proposition 3.1.5 we see that $\mathcal{K}er(\varphi_1)\simeq\mathcal{O}(-3)$ and $\mathcal{C}oker(\varphi_1)\simeq\mathbb{C}_x$. From (2.1.3) we have an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{F}'=\mathcal{C}oker(\varphi_5)(-1)$. From (2.1.2) we get the exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-4)\oplus 4\Omega^1 \longrightarrow 8\mathcal{O}(-1) \longrightarrow \mathcal{F}' \longrightarrow 0$$

and hence the resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-4)\oplus 12\mathcal{O}(-1) \xrightarrow{\rho} 8\mathcal{O}(-1)\oplus 4\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

If $\operatorname{rank}(\rho_{12}) \le 7$, then $\mathcal{F}'$ would have a subsheaf of slope 4/3 that would destabilize $\mathcal{F}$. Thus $\operatorname{rank}(\rho_{12}) = 8$, and we have the resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-4) \oplus 4\mathcal{O}(-1) \longrightarrow 4\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Arguing as at [10, Proposition 2.1.4], we can show that $\mathcal{C}oker(\psi_{21}) \simeq \mathcal{O}(-1) \oplus \Omega^1(1)$ and that the cokernel of the induced morphism $\Omega^1(1) \to 4\mathcal{O}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Combining this with the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(-2)$ we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The morphism $\mathcal{O}(-4) \to \mathcal{O}(-4)$ is nonzero because $\mathrm{h}^1(\mathcal{F}(1)) = 0$. Canceling $\mathcal{O}(-4)$ we obtain a resolution as in the proposition.

Conversely, assume that $\mathcal{F}$ has a resolution as in the proposition. Then $\mathcal{F}$ is an extension of $\mathbb{C}_x$ by $\mathcal{F}'$, where, in view of Proposition 3.1.8, $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$. It follows that any possibly destabilizing subsheaf of $\mathcal{F}$ must be the structure sheaf of a line or of a conic curve. Each of these situations can be easily ruled out using diagrams similar to diagram (8) in Section 3.1. □

In the remaining part of this subsection we shall prove that there are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ beside the sheaves we have discussed in this subsection and the sheaves at Proposition 3.1.7(ii). In view of this result, we may restrict our attention to the case when $\mathrm{H}^1(\mathcal{F}(1)) = 0$. Assume that $\mathrm{h}^0(\mathcal{F}(-1)) \le 1$. According to Corollary 2.2.2(iv), (v), and Proposition 4.1.1 the pair $(\mathrm{h}^0(\mathcal{F}(-1)), \mathrm{h}^1(\mathcal{F}))$ may be one of the following: $(0,0)$, $(0,1)$, $(1,1)$, $(1,2)$. Each of these situations has already been examined. The following concludes the classification of sheaves in $\mathrm{M}_{\mathbb{P}^2}(6,2)$.

PROPOSITION 4.1.5

Let $\mathcal{F}$ be a sheaf giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the condition $\mathrm{h}^1(\mathcal{F}(1)) = 0$. Then $\mathrm{h}^0(\mathcal{F}(-1)) = 0$ or 1.

Proof

Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the condition $\mathrm{h}^0(\mathcal{F}(-1)) \ge 2$. As at [3, Proposition 2.1.3], there is an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for a curve $C \subset \mathbb{P}^2$. This curve has degree 5 or 6; otherwise $\mathcal{O}_C$ would destabilize $\mathcal{F}(-1)$. Assume that $\deg(C) = 5$. The quotient sheaf $\mathcal{C} = \mathcal{F}/\mathcal{O}_C(1)$ has Hilbert polynomial $\mathrm{P}(m) = m + 2$ and zero-dimensional torsion $\mathcal{T}$ of length at most 1. Indeed, the pullback in $\mathcal{F}$ of $\mathcal{T}$ would be a destabilizing subsheaf if $\operatorname{length}(\mathcal{T}) \ge 2$. If $\mathcal{T} = 0$, then $\mathcal{C} \simeq \mathcal{O}_L(1)$, forcing $\mathrm{h}^0(\mathcal{F}(-1)) = 2$. The morphism $\mathcal{O}(1) \to \mathcal{O}_L(1)$ lifts to a morphism $\mathcal{O}(1) \to \mathcal{F}$, which leads us to the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O} \longrightarrow 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Thus $\mathrm{h}^1(\mathcal{F}(1)) = 1$. Assume now that length$(\mathcal{T}) = 1$. Let $\mathcal{F}' \subset \mathcal{F}$ be the pullback of $\mathcal{T}$. According to [10, Proposition 3.1.5], we have $\mathrm{h}^0(\mathcal{F}'(-1)) = 1$. Since $\mathcal{F}/\mathcal{F}' \simeq \mathcal{O}_L$, we get $\mathrm{h}^0(\mathcal{F}(-1)) = 1$, contradicting our choice of $\mathcal{F}$.

Assume now that C is a sextic curve. The quotient sheaf $\mathcal{C} = \mathcal{F}/\mathcal{O}_C(1)$ is zero-dimensional of length 5. Let $\mathcal{C}' \subset \mathcal{C}$ be a subsheaf of length 4, and let $\mathcal{F}'$ be its preimage in $\mathcal{F}$. We claim that $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$. If this were not the case, then $\mathcal{F}'$ would have a destabilizing subsheaf $\mathcal{F}''$, which may be assumed to be semistable. We may assume, without loss of generality, that $\mathcal{F}$ is stable. Thus we have the inequalities $1/6 < \mathrm{p}(\mathcal{F}'') < 1/3$. This leaves only two possibilities: that $\mathcal{F}''$ give a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ or in $\mathrm{M}_{\mathbb{P}^2}(4,1)$. In the first case $\mathcal{F}/\mathcal{F}''$ is isomorphic to the structure sheaf of a line; hence $\mathrm{h}^0(\mathcal{F}(-1)) = \mathrm{h}^0(\mathcal{F}''(-1)) = 0$ or 1 (cf. [10]). This contradicts our choice of $\mathcal{F}$. In the second case $\mathcal{F}/\mathcal{F}''$ is easily seen to be semistable; hence it is isomorphic to the structure sheaf of a conic curve. We get $\mathrm{h}^0(\mathcal{F}(-1)) = \mathrm{h}^0(\mathcal{F}''(-1)) = 0$ (cf. [3]), contradicting our choice of $\mathcal{F}$. This proves the claim; that is, $\mathcal{F}'$ is semistable. We have $\mathrm{h}^0(\mathcal{F}'(-1)) \geq 1$ so, according to the results in Section 3.1, there are two possible resolutions for $\mathcal{F}'$:

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0$$

or

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Combining the first resolution with the standard resolution of $\mathbb{C}_x = \mathcal{C}/\mathcal{C}'$ tensored with $\mathcal{O}(1)$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O} \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From this it easily follows that $\mathbb{C}_x$ is a direct summand of $\mathcal{F}$, which violates semistability. Assume, finally, that $\mathcal{F}'$ has the second resolution. We can apply the horseshoe lemma as above, leading to the resolution

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{O} \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

We see from this that $\mathrm{h}^1(\mathcal{F}(1)) = 1$. □

4.2. The strata as quotients

In Section 4.1 we classified all sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$; namely, we showed that this moduli space can be decomposed into seven subsets $X_0, \ldots, X_6$ (cf. Table 2). For $1 \leq i \leq 6$, the sheaves giving points in X_i are stable. We will employ the notations $\mathbb{W}_i$, W_i, G_i, $0 \leq i \leq 6$, analogous to the notations from Section 3.2. For $1 \leq i \leq 6$, the fibers of the canonical maps $\rho_i \colon W_i \to X_i$ are precisely the G_i-orbits. It follows, as at Section 3.2, that these are geometric quotient maps. The semistable but not stable points of $\mathrm{M}_{\mathbb{P}^2}(6,2)$ are of the form $[\mathcal{F}_1 \oplus \mathcal{F}_2]$, where $\mathcal{F}_1$, $\mathcal{F}_2$ give points in $\mathrm{M}_{\mathbb{P}^2}(3,1)$, and they are all contained in X_0. Thus X_0 cannot be a geometric quotient. Instead, it is a good quotient.

PROPOSITION 4.2.1
There is a good quotient $W_0//G_0$, which is isomorphic to X_0.

Proof
Let $\mathbb{W}_0^{\mathrm{ss}}(\Lambda) \subset \mathbb{W}_0$ denote the set of morphisms that are semistable with respect to a polarization $\Lambda = (\lambda_1, \mu_1, \mu_2)$ satisfying the relation $1/8 < \mu_2 < 3/16$ (notation as at [4]). According to [2, Theorem 6.4], $\mathbb{W}_0^{\mathrm{ss}}(\Lambda)//G_0$ exists and is a projective variety. According to [8, Claim 4.3], W_0 is the subset of injective morphisms inside $\mathbb{W}_0^{\mathrm{ss}}(\Lambda)$. Thus $W_0//G_0$ exists and is a proper open subset of $\mathbb{W}_0^{\mathrm{ss}}(\Lambda)//G_0$.

Arguing as at [3, Section 4.2.1], we can easily see that two points of W_0 are in the same fiber of ρ_0 if and only if the relative closures in W_0 of their G_0-orbits intersect nontrivially. This allows us to apply the method of [3, Theorem 4.2.2] in order to show that ρ_0 is a categorical quotient map. We need to recover resolution (i) at Proposition 4.1.1 from the Beilinson spectral sequence. Fix $\mathcal{F}$ in X_0. Tableau (2.1.4) for the dual sheaf $\mathcal{F}^{\mathrm{D}}(1)$ reads

$$\begin{array}{ccc} 2\mathcal{O}(-2) & 0 & 0 \\ \\ 0 & 2\mathcal{O}(-1) & \xrightarrow{\varphi_4} 4\mathcal{O}. \end{array}$$

Combining the exact sequences (2.1.5) and (2.1.6) yields the dual to resolution (i) at Proposition 4.1.1. Thus $W_0 \to X_0$ is a categorical quotient map, and the isomorphism $W_0//G_0 \simeq X_0$ follows from the uniqueness of the categorical quotient. □

By analogy with [10, Proposition 2.2.2], the quotient W_1/G_1 is isomorphic to an open subset of the projectivization of a vector bundle over $\mathrm{N}(3,4,3) \times \mathbb{P}^2$ of rank 21. By analogy with [10, Proposition 3.2.3], the quotient W_3/G_3 is isomorphic to an open subset of the projectivization of a vector bundle over $\mathrm{Hilb}_{\mathbb{P}^2}(2) \times \mathrm{N}(3,2,3)$ of rank 23, and W_5/G_5 is isomorphic to an open subset of the projectivization of a vector bundle over $\mathbb{P}^2 \times \mathrm{Hilb}_{\mathbb{P}^2}(2)$ of rank 25. Recall the smooth projective variety U/G constructed at Claim 3.2.1. By analogy with [4, Section 9.3], W_4/G_4 is isomorphic to an open subset of the projectivization of a vector bundle over U/G of rank 23. The deepest stratum X_6 is isomorphic to $\mathrm{Hilb}_{\mathbb{P}^2}(6,1)$, that is, to the universal sextic curve in $\mathbb{P}^2 \times \mathbb{P}(\mathrm{S}^6 V^*)$.

4.3. Generic sheaves

Let $C \subset \mathbb{P}^2$ denote an arbitrary smooth sextic curve, and let P_i denote distinct points on C. According to [1, Propositions 4.5, 4.6], the cokernels of morphisms $3\mathcal{O}(-4) \to 4\mathcal{O}(-3)$ whose maximal minors have no common factor are precisely the ideal sheaves $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ of zero-dimensional schemes Z of length 6 that are not contained in a conic curve. It follows that the generic sheaves in X_1 have the form $\mathcal{O}_C(1)(P_1 + \cdots + P_6 - P_7)$, where $P_1, \ldots, P_6$ are not contained in a conic curve. Also from [1, Propositions 4.5, 4.6] we deduce that the generic sheaves in X_3 have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5)$, where P_1, P_2, P_3 are non-colinear. From Claim 3.3.2 we deduce, by duality, that the generic sheaves in X_4 are of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_5)$, where no three points among $P_1, \ldots, P_5$ are colinear. It is easy to see that the generic sheaves in X_5 are of the form $\mathcal{O}_C(2)(P_1 - P_2 - P_3)$. According to Claim 4.3.1 below, the generic sheaves in X_2 have the form $\mathcal{O}_C(3)(-P_1 - \cdots - P_7)$, where $P_1, \ldots, P_7$ do not lie on a conic curve and no four points among them are colinear.

CLAIM 4.3.1

Let $U \subset \mathrm{Hom}(\mathcal{O}(-2) \oplus \mathcal{O}(-1), 3\mathcal{O})$ be the set of morphisms whose maximal minors have no common factor. The cokernels of the morphisms in U are precisely the sheaves of the form $\mathcal{I}_Z(3)$, where $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of a zero-dimensional subscheme $Z \subset \mathbb{P}^2$ of length 7 that is not contained in a conic curve and no subscheme of length 4 of which is contained in a line.

Proof

The argument is analogous to the argument at Claim 3.3.1. □

5. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6, 3)$

5.1. Classification of sheaves

PROPOSITION 5.1.1

The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6, 3)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 0$, $\mathrm{h}^1(\mathcal{F}(1)) = 0$ are precisely the sheaves having one of the following resolutions:

(i) $$0 \longrightarrow 3\mathcal{O}(-2) \longrightarrow 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

(ii) $$0 \longrightarrow 3\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12} = 0$, the entries of φ_{11} span a subspace of V^* of dimension at least 2, the same is true for the entries of φ_{22}, and, moreover, φ is not equivalent to a morphism represented by a matrix of the form

$$\begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix},$$

(iii) $$0 \longrightarrow 3\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12} = 0$, φ_{11} has linearly independent maximal minors and the same for φ_{22}.

Proof

The proposition is a particular case of [8, Claims 4.6–4.8]. The above resolutions can also be easily obtained from the display diagram

$$\begin{array}{ccc} 3\mathcal{O}(-1) & 0 & 0 \\ \\ 0 & 3\Omega^1(1) \xrightarrow{\varphi_4} & 9\mathcal{O} \end{array}$$

of the Beilinson spectral sequence I converging to $\mathcal{F}(1)$. □

PROPOSITION 5.1.2

(i) *The sheaves* $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,3)$ *and satisfying the conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 1$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow \mathcal{O}(-3) \oplus 3\mathcal{O}(-1) \xrightarrow{\varphi} 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{12} *is semistable as a Kronecker module.*

(ii) *The sheaves* $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,3)$ *and satisfying the dual conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 1$, $\mathrm{h}^1(\mathcal{F}) = 0$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow 4\mathcal{O}(-2) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} *is semistable as a Kronecker module.*

Proof

Part (i) is a particular case of [8, Claim 5.3]. Part (ii) is equivalent to (i) by duality. □

PROPOSITION 5.1.3

The sheaves $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,3)$ *and satisfying the conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 1$, $\mathrm{h}^1(\mathcal{F}) = 1$, $\mathrm{h}^1(\mathcal{F}(1)) = 0$ *are precisely the sheaves having a resolution of the form*

(i) $$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12} \neq 0$, *or the sheaves having a resolution of the form*

(ii) $$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{13} = 0$, $\varphi_{23}, \varphi_{12} \neq 0$, φ_{12} *does not divide* φ_{11}, *and* φ_{23} *does not divide* φ_{33}.

Proof

Let $\mathcal{F}$ be a sheaf giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,3)$ and satisfying the above cohomological conditions. Diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}(1)$ takes the form

$$\begin{array}{ccc} 4\mathcal{O}(-1) \xrightarrow{\varphi_1} & \Omega^1(1) & 0 \\ \\ \mathcal{O}(-1) \xrightarrow{\varphi_3} & 4\Omega^1(1) \xrightarrow{\varphi_4} & 9\mathcal{O}. \end{array}$$

Arguing as at Proposition 3.1.3 we see that $\mathcal{C}oker(\varphi_1) = 0$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-1)$. Performing the same steps as at Proposition 3.1.3 we arrive at the resolution

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus 12\mathcal{O} \xrightarrow{\rho} 9\mathcal{O} \oplus 4\mathcal{O}(1) \longrightarrow \mathcal{F}(1) \longrightarrow 0.$$

Notice that $\operatorname{rank}(\rho_{13}) \geq 8$; otherwise, $\mathcal{F}(1)$ would map surjectively to the cokernel of a morphism $\mathcal{O}(-2) \oplus \mathcal{O}(-1) \to 2\mathcal{O}$, in violation of semistability. We arrive at a resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 4\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

in which $\psi_{11} = 0$, $\psi_{21} = 0$. Arguing as in the proof of [10, Proposition 2.1.4], we can show that $\mathcal{C}oker(\psi_{31}) \simeq \mathcal{O}(-1) \oplus \Omega^1(1)$ and that the cokernel of the induced map $\Omega^1(1) \to 4\mathcal{O}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$. We get the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Finally, we obtain resolutions (i) or (ii) depending on whether $\varphi_{13} \neq 0$ or $\varphi_{13} = 0$.

Conversely, assume that $\mathcal{F}$ has resolution (i). According to Corollary 2.3.2, if φ_{12} and φ_{22} have no common factor, then $\mathcal{F}$ is semistable. If φ_{12} divides φ_{22}, then $\mathcal{F}$ is stable-equivalent to $\mathcal{O}_C \oplus \mathcal{O}_Q(1)$, for a quartic curve Q and a conic curve C in $\mathbb{P}^2$. It remains to examine the case when $\gcd(\varphi_{12}, \varphi_{22})$ is a linear form l. In this case we have a nonsplit extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{E}$ has a resolution as at [10, Proposition 2.1.4], so it gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,3)$. It is easy to estimate the slope of any subsheaf of $\mathcal{F}$, showing that this sheaf is semistable.

Assume now that $\mathcal{F}$ has resolution (ii). From the snake lemma we get an extension

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where Z is the common zero-set of φ_{11} and φ_{12}, and $\mathcal{E}$ has a resolution as at Proposition 3.1.8, so it gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$. Assume that $\mathcal{F}' \subset \mathcal{F}$ is a destabilizing subsheaf. Since $\mathrm{p}(\mathcal{F}' \cap \mathcal{E}) \leq 0$, we see that $\mathcal{F}'$ has multiplicity at most 3. By duality, any destabilizing subsheaf of $\mathcal{F}^{\mathrm{D}}(1)$ has multiplicity at most 3; hence $\mathcal{F}'$ has multiplicity 3. Without loss of generality we may assume that $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(3,2)$, so there is a resolution

$$0 \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

This fits into a commutative diagram analogous to diagram (8) at Proposition 3.1.3, leading to a contradiction. □

PROPOSITION 5.1.4

The sheaves $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,3)$ *and satisfying the conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 2$, $\mathrm{h}^1(\mathcal{F}) = 2$, $\mathrm{h}^1(\mathcal{F}(1)) = 0$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O} \xrightarrow{\varphi} \mathcal{O}(-2) \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} has linearly independent entries, and the same is true for φ_{22}.

Proof
Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,3)$ and satisfy the above cohomological conditions. Display diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}(1)$ reads

$$\begin{array}{ccccc} 5\mathcal{O}(-1) & \xrightarrow{\varphi_1} & 2\Omega^1(1) & & 0 \\ \\ \\ 2\mathcal{O}(-1) & \xrightarrow{\varphi_3} & 5\Omega^1(1) & \xrightarrow{\varphi_4} & 9\mathcal{O}. \end{array}$$

Arguing as in the proof of Proposition 3.1.5, we see that $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$ and $\mathcal{C}oker(\varphi_1) \simeq \mathbb{C}_x$. From (2.1.3) we have an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{F}' = \mathcal{C}oker(\varphi_5)(-1)$. From (2.1.2) we get the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-4) \oplus 5\Omega^1 \longrightarrow 9\mathcal{O}(-1) \longrightarrow \mathcal{F}' \longrightarrow 0,$$

hence the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-4) \oplus 15\mathcal{O}(-1) \xrightarrow{\rho} 9\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

If $\operatorname{rank}(\rho_{12}) \le 8$, then $\mathcal{F}'$ would have a subsheaf of slope 5/3 that would destabilize $\mathcal{F}$. Thus $\operatorname{rank}(\rho_{12}) = 9$, and we have the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-4) \oplus 6\mathcal{O}(-1) \longrightarrow 5\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Arguing as at [10, Proposition 3.2.5], we can show that $\mathcal{C}oker(\psi_{21}) \simeq 2\Omega^1(1)$. The exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\Omega^1(1) \longrightarrow 5\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0$$

yields the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus 6\mathcal{O} \xrightarrow{\sigma} 5\mathcal{O} \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

If $\operatorname{rank}(\sigma_{12}) \le 4$, then $\mathcal{F}'$ would have a subsheaf of slope 2 that would destabilize $\mathcal{F}$. Thus $\operatorname{rank}(\sigma_{12}) = 5$, and we have a resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O} \longrightarrow 2\mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Combining this with the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(-2)$ we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \oplus \mathcal{O} \longrightarrow \mathcal{O}(-2) \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The morphism $\mathcal{O}(-4) \to \mathcal{O}(-4)$ is nonzero because $\mathrm{h}^1(\mathcal{F}(1)) = 0$. Canceling $\mathcal{O}(-4)$ we obtain a resolution as in the proposition.

Conversely, assume that $\mathcal{F}$ has a resolution as in the proposition. Then $\mathcal{F}$ is an extension of $\mathbb{C}_x$ by $\mathcal{F}'$, where, in view of Proposition 3.1.7(ii), $\mathcal{F}'$ gives a stable point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$. It follows that any possibly destabilizing subsheaf of $\mathcal{F}$ must be the structure sheaf of a line. This situation, however, can be easily ruled out using a diagram analogous to diagram (8) in Section 3.1. □

PROPOSITION 5.1.5

The sheaves $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,3)$ *and satisfying the condition* $\mathrm{h}^1(\mathcal{F}(1)) > 0$ *are precisely the sheaves of the form* $\mathcal{O}_C(2)$, *where* $C \subset \mathbb{P}^2$ *is a sextic curve.*

Proof

The argument is entirely analogous to the argument at [10, Proposition 4.1.1]. □

PROPOSITION 5.1.6

Let $\mathcal{F}$ *give a point in* $\mathrm{M}_{\mathbb{P}^2}(6,3)$ *and satisfy the condition* $\mathrm{h}^0(\mathcal{F}(-1)) \geq 3$ *or the condition* $\mathrm{h}^1(\mathcal{F}) \geq 3$. *Then* $\mathcal{F} \simeq \mathcal{O}_C(2)$ *for some sextic curve* $C \subset \mathbb{P}^2$.

Proof

By Serre duality $\mathrm{h}^1(\mathcal{F}) = \mathrm{h}^0(\mathcal{F}^{\mathrm{D}})$, so it is enough to examine only the case when $\mathrm{h}^0(\mathcal{F}(-1)) \geq 3$. It is easy to see that $\mathcal{F}$ is stable (cf. the description in Section 5.2 of properly semistable sheaves). Arguing as at [3, Proposition 2.1.3], we see that there is an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for some curve $C \subset \mathbb{P}^2$ of degree at most 6. Since $\mathrm{p}(\mathcal{O}_C) < -1/2$, C has degree 5 or 6. Assume first that $\deg(C) = 6$. The quotient sheaf $\mathcal{C} = \mathcal{F}/\mathcal{O}_C(1)$ has length 6 and dimension zero. Let $\mathcal{C}' \subset \mathcal{C}$ be a subsheaf of length 5, and let $\mathcal{F}'$ be its preimage in $\mathcal{C}$. We have an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

We claim that $\mathcal{F}'$ is semistable. If this were not the case, then $\mathcal{F}'$ would have a destabilizing subsheaf $\mathcal{F}''$, which may be assumed to be stable. In fact, $\mathcal{F}''$ must give a point in $\mathrm{M}_{\mathbb{P}^2}(5,2)$ because $1/3 < \mathrm{p}(\mathcal{F}'') < 1/2$. According to [10, Section 2], we have the inequality $\mathrm{h}^0(\mathcal{F}''(-1)) \leq 1$. The quotient sheaf $\mathcal{F}/\mathcal{F}''$ has Hilbert polynomial $\mathrm{P}(m) = m+1$ and no zero-dimensional torsion, so $\mathcal{F}/\mathcal{F}'' \simeq \mathcal{O}_L$. Thus

$$\mathrm{h}^0\big(\mathcal{F}(-1)\big) \leq \mathrm{h}^0\big(\mathcal{F}''(-1)\big) + \mathrm{h}^0\big(\mathcal{O}_L(-1)\big) \leq 1,$$

contradicting our hypothesis. This proves that $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$. We have the relation $\mathrm{h}^0(\mathcal{F}'(-1)) \geq 2$, hence, according to the results in Section 4.1, there is a resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O} \longrightarrow 2\mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Combining this with the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(1)$ we get the exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}(-4) \oplus 3\mathcal{O} \longrightarrow 3\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From this we obtain the relation $\mathrm{h}^1(\mathcal{F}(1))=1$; hence, by Proposition 5.1.5, $\mathcal{F}$ is isomorphic to $\mathcal{O}_C(2)$.

Assume now that C has degree 5. The quotient sheaf $\mathcal{F}/\mathcal{O}_C(1)$ has Hilbert polynomial $\mathrm{P}(m)=m+3$. Let $\mathcal{T}$ denote its zero-dimensional torsion, and let $\mathcal{F}'$ be the preimage of $\mathcal{T}$ in $\mathcal{F}$. We have $\mathrm{length}(\mathcal{T})\le 2$; otherwise, $\mathcal{F}'$ would destabilize $\mathcal{F}$. If $\mathcal{T}=0$, then $\mathcal{F}/\mathcal{O}_C(1)\simeq\mathcal{O}_L(2)$. We apply the horseshoe lemma to the extension

$$0\longrightarrow\mathcal{O}_C(1)\longrightarrow\mathcal{F}\longrightarrow\mathcal{O}_L(2)\longrightarrow 0,$$

to the standard resolution of $\mathcal{O}_C(1)$, and to the resolution

$$0\longrightarrow\mathcal{O}(-1)\longrightarrow 3\mathcal{O}\longrightarrow 2\mathcal{O}(1)\longrightarrow\mathcal{O}_L(2)\longrightarrow 0.$$

We obtain again resolution $(*)$, leading to the conclusion of the proposition. Assume that $\mathrm{length}(\mathcal{T})=1$. According to [10, Proposition 3.1.5], we have $\mathrm{h}^0(\mathcal{F}'(-1))=1$. Since $\mathcal{F}/\mathcal{F}'\simeq\mathcal{O}_L(1)$, we see that $\mathrm{h}^0(\mathcal{F}(-1))\le 2$, contrary to our hypothesis. Assume that $\mathrm{length}(\mathcal{T})=2$. Since $\mathcal{F}$ is stable, it is easy to see that $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,2)$, so $\mathrm{h}^0(\mathcal{F}'(-1))\le 1$, forcing $\mathrm{h}^0(\mathcal{F}(-1))\le 1$, which contradicts our hypothesis. □

There are no other sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6,3)$ beside the sheaves we have discussed in this subsection. To see this we may, by virtue of Proposition 5.1.5, restrict our attention to the case when $\mathrm{H}^1(\mathcal{F}(1))=0$. According to Proposition 5.1.6 and Corollary 2.2.2(vi), the pair $(\mathrm{h}^0(\mathcal{F}(-1)),\mathrm{h}^1(\mathcal{F}))$ may be one of the following: $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(2,2)$. Each of these situations has been examined.

5.2. The strata as quotients

In Section 5.1 we classified all sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6,3)$; namely, we showed that this moduli space is the union of nine locally closed subsets, as in Table 3, which we will call, by an abuse of terminology, strata. As the notation suggests, the stratum X_3^{D} is the image of X_3 under the duality automorphism $[\mathcal{F}]\to[\mathcal{F}^{\mathrm{D}}(1)]$. The strata X_i, $0\le i\le 7$, $i\ne 3$, are invariant under this automorphism. We employ the notations $\mathbb{W}_i$, W_i, G_i, ρ_i, $0\le i\le 7$, analogous to the notations from Section 3.2. We denote $W_i^{\mathrm{s}}=\rho_i^{-1}(X_i^{\mathrm{s}})$. Adopting the notation of [3], let $\mathcal{E}_i$ denote an arbitrary sheaf giving a point in the codimension i stratum of $\mathrm{M}_{\mathbb{P}^2}(4,2)$, $i=0,1$. Let $C\subset\mathbb{P}^2$ denote an arbitrary conic curve; let $Q\subset\mathbb{P}^2$ denote an arbitrary quartic curve. It is easy to see that all points of the form $[\mathcal{O}_C\oplus\mathcal{E}_i]$ belong to X_i and to no other stratum. According to Corollary 2.3.2, the set $W_4\setminus W_4^{\mathrm{s}}$ consists of those morphisms φ such that φ_{12} divides φ_{11} or φ_{22}. The sheaves $\mathcal{F}=\mathcal{C}oker(\varphi)$, $\varphi\in W_4\setminus W_4^{\mathrm{s}}$, are precisely the extensions of $\mathcal{O}_C$ by $\mathcal{O}_Q(1)$ or of $\mathcal{O}_Q(1)$ by $\mathcal{O}_C$ satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1))=1$, $\mathrm{h}^1(\mathcal{F})=1$. Using the argument found at [10, Proposition 3.3.2], we can show that the extension sheaves

$$0\longrightarrow\mathcal{O}_Q(1)\longrightarrow\mathcal{F}\longrightarrow\mathcal{O}_C\longrightarrow 0$$

satisfying the condition $\mathrm{h}^1(\mathcal{F}) = 0$ are precisely the sheaves of the form $\mathcal{C}oker(\varphi)$, $\varphi \in W_3^{\mathrm{D}}$, such that the maximal minors of φ_{11} have a common quadratic factor. By duality, it follows that the extension sheaves

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Q(1) \longrightarrow 0$$

satisfying the condition $\mathrm{h}^0(\mathcal{F}(-1)) = 0$ are precisely the sheaves of the form $\mathcal{C}oker(\varphi)$, $\varphi \in W_3$, such that the maximal minors of φ_{12} have a common quadratic factor. This shows that the strata X_2, X_5, X_6, X_7 have only stable points and that the sets $X_3 \setminus X_3^{\mathrm{s}}$, $X_3^{\mathrm{D}} \setminus (X_3^{\mathrm{D}})^{\mathrm{s}}$, $X_4 \setminus X_4^{\mathrm{s}}$ coincide and consist of all points of the form $[\mathcal{O}_C \oplus \mathcal{O}_Q(1)]$.

The fibers of the canonical maps $\rho_i \colon W_i^{\mathrm{s}} \to X_i^{\mathrm{s}}$, $0 \le i \le 7$, are precisely the G_i-orbits; hence, by the argument found in Section 3.2, these are geometric quotient maps. Thus $X_i \simeq W_i/G_i$ for $i \in \{2,5,6,7\}$.

Assume that $i \in \{0,1\}$. Arguing as in [3, Section 4.2.1], we can easily see that two points of W_i are in the same fiber of ρ_i if and only if the relative closures in W_i of their G_i-orbits intersect nontrivially. This allows us to apply the method of [3, Theorem 4.2.2] in order to show that ρ_i is a categorical quotient map. Note that W_0 is a proper invariant open subset of the set of semistable Kronecker modules $3\mathcal{O}(-2) \to 3\mathcal{O}$, so there exists a good quotient $W_0/\!/G_0$ as an open subset of $\mathrm{N}(6,3,3)$. By the uniqueness of the categorical quotient we have an isomorphism $X_0 \simeq W_0/\!/G_0$. This shows that $\mathrm{M}_{\mathbb{P}^2}(6,3)$ and $\mathrm{N}(6,3,3)$ are birational. Let $W_{10} \subset W_1$ be the open invariant subset given by the condition that the entries of φ_{11} span V^* and the same for the entries of φ_{22}. Its image X_{10} is open in X_1. Since $W_{10} \subset W_1^{\mathrm{s}}$, the map $W_{10} \to X_{10}$ is a geometric quotient map. By analogy with [10, Proposition 2.2.2], the quotient W_{10}/G_1 is isomorphic to an open subset of the projectivisation of a vector bundle of rank 37 over $\mathrm{N}(3,3,1) \times \mathrm{N}(3,1,3)$. The base is isomorphic to a point, so X_{10} is an open subset of $\mathbb{P}^{36}$.

By analogy with [10, Proposition 2.2.2], the quotient W_2/G_2 is isomorphic to an open subset of the projectivization of a vector bundle of rank 22 over $\mathrm{N}(3,3,2) \times \mathrm{N}(3,2,3)$. Likewise, W_6/G_6 is isomorphic to an open subset of the projectivization of a vector bundle of rank 26 over $\mathbb{P}^2 \times \mathbb{P}^2$. By analogy with [10, Proposition 3.2.3], W_5/G_5 is isomorphic to an open subset of the projectivization of a vector bundle of rank 24 over $\mathrm{Hilb}_{\mathbb{P}^2}(2) \times \mathrm{Hilb}_{\mathbb{P}^2}(2)$. The stratum X_7 is isomorphic to $\mathbb{P}(\mathrm{S}^6 V^*)$.

By analogy with [4, Section 9.3], there exists a geometric quotient W_3/G_3, which is an open subset of the projectivization of a vector bundle of rank 22 over $\mathrm{N}(3,3,4)$. The induced map $W_3/G_3 \to X_3$ is an isomorphism over the set of stable points in X_3, as we saw above. One can easily see that the fiber of this map over any properly semistable point $[\mathcal{O}_C \oplus \mathcal{O}_Q(1)]$ is isomorphic to $\mathrm{S}^3 V^*$.

The linear algebraic group $G = \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(1))$ acts on the vector space $\mathbb{U} = \mathrm{Hom}(\mathcal{O}(-2), \mathcal{O} \oplus \mathcal{O}(1))$ by left multiplication. Consider the open G-invariant subset $U \subset \mathbb{U}$ of morphisms ψ for which ψ_{11} is nonzero and does not divide ψ_{21}. Consider the fiber bundle with base $\mathbb{P}(\mathrm{S}^2 V^*)$ and fiber $\mathbb{P}(\mathrm{S}^3 V^*/V^* q)$ at any point of the base represented by $q \in \mathrm{S}^2 V^*$. Clearly this fiber bundle is the geometric quotient of U modulo G. Consider the open G_4-invariant subset $W_4' \subset \mathbb{W}_4$ of morphisms φ whose restriction to $\mathcal{O}(-2)$ lies in U. Clearly W_4' is the trivial vector bundle over U with fiber $\mathrm{Hom}(\mathcal{O}(-3), \mathcal{O} \oplus \mathcal{O}(1))$. Consider the subbundle $\Sigma \subset W_4'$ given by the condition $(\varphi_{11}, \varphi_{21}) = (\varphi_{12}u, \varphi_{22}u)$, for some $u \in \mathrm{Hom}(\mathcal{O}(-3), \mathcal{O}(-2))$. As at [10, Proposition 2.2.5], the quotient bundle W_4'/Σ is G-linearized; hence it descends to a vector bundle E over U/G of rank 22. Its projectivization $\mathbb{P}(E)$ is the geometric quotient of $W_4' \setminus \Sigma$ modulo G_4. Notice that W_4^{s} is a proper open G_4-invariant subset of $W_4' \setminus \Sigma$. Thus $X_4^{\mathrm{s}} = W_4^{\mathrm{s}}/G_4$ is isomorphic to a proper open subset of $\mathbb{P}(E)$.

5.3. Generic sheaves

Let C denote an arbitrary smooth sextic curve in $\mathbb{P}^2$, and let P_i denote distinct points on C. By analogy with the case of the stratum $X_3 \subset \mathrm{M}_{\mathbb{P}^2}(6,1)$, we see that the generic sheaves in X_2 have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5 + P_6)$, where P_1, P_2, P_3 are non-colinear and the same for P_4, P_5, P_6. By analogy with the stratum $X_1 \subset \mathrm{M}_{\mathbb{P}^2}(6,2)$, we see that the generic sheaves in X_3 have the form $\mathcal{O}_C(3)(-P_1 - \cdots - P_6)$, where $P_1, \ldots, P_6$ are not contained in a conic curve. The generic sheaves in X_4 have the form $\mathcal{O}_C(3)(-P_1 - \cdots - P_6)$, where $P_1, \ldots, P_6$ lie on a conic curve and no four points among them are colinear. The generic sheaves in X_5 have the form $\mathcal{O}_C(2)(P_1 + P_2 - P_3 - P_4)$. The generic sheaves in X_6 have the form $\mathcal{O}_C(2)(P_1 - P_2)$.

6. The moduli space $\mathrm{M}_{\mathbb{P}^2}(6,0)$

6.1. Classification of sheaves

PROPOSITION 6.1.1

Let r be a positive integer. The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(r,0)$ and satisfying the condition $\mathrm{h}^1(\mathcal{F}) = 0$ are precisely the sheaves having a resolution of the form

$$0 \longrightarrow r\mathcal{O}(-2) \xrightarrow{\varphi} r\mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Moreover, $\mathcal{F}$ is properly semistable if and only if φ is properly semistable, viewed as a Kronecker module.

Proof

This is a generalization of [10, Proposition 4.1.2]. Assume that $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(r,0)$ and $\mathrm{h}^1(\mathcal{F}) = 0$. Diagram (2.1.4) for the Beilinson spectral sequence II converging to $\mathcal{F}$ reads

$$\begin{array}{ccccc} r\mathcal{O}(-2) & \xrightarrow{\varphi_1} & r\mathcal{O}(-1) & & 0 \\ & & & & \\ 0 & & 0 & & 0. \end{array}$$

The exact sequences (2.1.5) and (2.1.6) show that $\mathcal{F} \simeq \mathcal{C}oker(\varphi_1)$. The rest of the proof is exactly as at [10, Proposition 4.1.2]. □

PROPOSITION 6.1.2

Consider an integer $r \geq 3$. *The sheaves* $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(r,0)$ *and satisfying the conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 1$, $\mathrm{h}^1(\mathcal{F}(1)) = 0$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow \mathcal{O}(-3) \oplus (r-3)\mathcal{O}(-2) \xrightarrow{\varphi} (r-3)\mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{12} *is semistable as a Kronecker module.*

Proof

This is a generalization of [10, Proposition 4.1.3]. Assume that $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(r,0)$ and satisfies the above cohomological conditions. Diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}(1)$ reads

$$\begin{array}{ccccc} r\mathcal{O}(-1) & \xrightarrow{\varphi_1} & \Omega^1(1) & & 0 \\ & & & & \\ 0 & & \Omega^1(1) & \xrightarrow{\varphi_4} & r\mathcal{O}. \end{array}$$

Arguing as at Proposition 3.1.3, we can show that $\mathcal{C}oker(\varphi_1) = 0$ and that $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-2) \oplus (r-3)\mathcal{O}(-1)$. By duality, $\mathcal{C}oker(\varphi_4) \simeq (r-3)\mathcal{O} \oplus \mathcal{O}(1)$. The exact sequence (2.1.3) yields the resolution

$$0 \longrightarrow \mathcal{O}(-2) \oplus (r-3)\mathcal{O}(-1) \longrightarrow (r-3)\mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}(1) \longrightarrow 0.$$

The converse is exactly as at [10, Proposition 4.1.3]. □

PROPOSITION 6.1.3

The sheaves $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,0)$ *and satisfying the conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 2$ *are precisely the sheaves having one of the following resolutions:*

(i)
$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix},$$

(ii)
$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} l_1 & l_2 & 0 \\ f_{11} & f_{12} & q_1 \\ f_{21} & f_{22} & q_2 \end{bmatrix},$$

(iii)
$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q_1 & q_2 & 0 \\ f_{11} & f_{12} & l_1 \\ f_{21} & f_{22} & l_2 \end{bmatrix},$$

(iv)
$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi=\begin{bmatrix} l_1 & l_2 & 0 & 0\\ p_1 & p_2 & l & 0\\ f_{11} & f_{12} & p_1' & l_1'\\ f_{21} & f_{22} & p_2' & l_2'\end{bmatrix}.$$

Here q_1, q_2 are linearly independent, l_1, l_2 are linearly independent, l_1', l_2' are linearly independent, and $l \neq 0$.

Proof

Diagram (2.1.1) for the Beilinson spectral sequence I converging to $\mathcal{F}^{\mathrm{D}}(2)$ has the form

$$\begin{array}{ccccc} 2\mathcal{O}(-1) & & 0 & & 0 \\ & & & & \\ 2\mathcal{O}(-1) & \xrightarrow{\varphi_3} & 6\Omega^1(1) & \xrightarrow{\varphi_4} & 12\mathcal{O}. \end{array}$$

Combining the exact sequences (2.1.2) and (2.1.3), we obtain the resolution

$$0 \longrightarrow 2\mathcal{O}(-1) \longrightarrow 2\mathcal{O}(-1) \oplus 6\Omega^1(1) \longrightarrow 12\mathcal{O} \longrightarrow \mathcal{F}^{\mathrm{D}}(2) \longrightarrow 0$$

and hence the resolution

$$0 \longrightarrow 2\mathcal{O}(-1) \longrightarrow 2\mathcal{O}(-1) \oplus 18\mathcal{O} \xrightarrow{\rho} 12\mathcal{O} \oplus 6\mathcal{O}(1) \longrightarrow \mathcal{F}^{\mathrm{D}}(2) \longrightarrow 0.$$

If $\mathrm{rank}(\rho_{12}) \leq 10$, then $\mathcal{F}^{\mathrm{D}}(2)$ would map surjectively to the cokernel of a morphism $2\mathcal{O}(-1) \to 2\mathcal{O}$, in violation of semistability. Thus $\mathrm{rank}(\rho_{12}) \geq 11$, which leads us to the resolution

$$0 \longrightarrow 2\mathcal{O}(-1) \xrightarrow{\psi} 2\mathcal{O}(-1) \oplus 7\mathcal{O} \longrightarrow \mathcal{O} \oplus 6\mathcal{O}(1) \longrightarrow \mathcal{F}^{\mathrm{D}}(2) \longrightarrow 0,$$

where $\psi_{11} = 0$. Arguing as at [10, Proposition 3.1.3], we see that $\mathcal{C}oker(\psi_{21})$ is isomorphic to $\mathcal{O} \oplus 2\Omega^1(2)$. The resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\Omega^1 \longrightarrow \mathcal{O}(-2) \oplus 6\mathcal{O}(-1) \longrightarrow \mathcal{F}^{\mathrm{D}} \longrightarrow 0$$

leads to the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 6\mathcal{O}(-1) \xrightarrow{\eta} \mathcal{O}(-2) \oplus 6\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F}^{\mathrm{D}} \longrightarrow 0.$$

If $\mathrm{rank}(\eta_{23}) \leq 4$, then $\mathcal{F}^{\mathrm{D}}$ would map surjectively to the cokernel of a morphism $2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \to \mathcal{O}(-2) \oplus 2\mathcal{O}(-1)$, in violation of semistability. Canceling $5\mathcal{O}(-1)$ and dualizing yields the resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

From this we get resolutions (i)–(iv) depending on whether $\varphi_{12} = 0$ and $\varphi_{23} = 0$.

Conversely, assume that $\mathcal{F}$ has resolution (i). If f_{12} and f_{22} have no common factor, then, by virtue of Corollary 2.3.2, $\mathcal{F}$ is semistable. Assume that $\gcd(f_{12}, f_{22})$ is a quadratic polynomial q. We get a nonsplit extension

$$0 \longrightarrow \mathcal{O}_C(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0,$$

where C is the conic curve given by the equation $q = 0$ and $\mathcal{F}'$ has a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi'} 2\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0$$

in which the entries of φ'_{12} are linearly independent. According to Corollary 2.3.2, $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(4,1)$. It is now easy to see that for any proper subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $\mathrm{p}(\mathcal{E}) \le 0$. Assume that $\gcd(f_{12}, f_{22})$ is a linear form l. We have an extension

$$0 \longrightarrow \mathcal{O}_L(-2) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0,$$

where L is the line given by the equation $l = 0$ and $\mathcal{F}'$ has a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi'} 2\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0$$

in which the entries of φ'_{12} have no common factor. According to Corollary 2.3.2, $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$. It is now easy to see that for any proper subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $\mathrm{p}(\mathcal{E}) \le 0$.

Assume now that $\mathcal{F}$ has resolution (ii) in which q_1, q_2 have no common factor. From the snake lemma we get an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where x is given by the equations $l_1 = 0$, $l_2 = 0$, and $\mathcal{F}'$ has a resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-2) \xrightarrow{\varphi'} 2\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0$$

in which the entries of φ'_{12} have no common factor. According to Corollary 2.3.2, $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,-1)$. It is now easy to see that for any proper subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $\mathrm{p}(\mathcal{E}) \le 0$. If q_1 and q_2 have a common linear factor, then we have an extension

$$(*) \qquad 0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0,$$

where $\mathcal{F}'$ has a resolution as at [10, Proposition 4.1.4], so it is semistable. Thus $\mathcal{F}$ is semistable.

Finally, we assume that $\mathcal{F}$ has resolution (iv). We have an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where x is given by the equations $l_1 = 0$, $l_2 = 0$, and $\mathcal{F}'$ has resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi'} \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0,$$

$$\varphi' = \begin{bmatrix} \star & l & 0 \\ \star & p'_1 & l'_1 \\ \star & p'_2 & l'_2 \end{bmatrix}. \quad \text{Assume first that } \varphi' \nsim \begin{bmatrix} \star & \star & 0 \\ \star & 0 & \star \\ \star & 0 & \star \end{bmatrix}.$$

Then, according to Proposition 3.1.6(ii), $\mathcal{F}'$ is semistable, showing that $\mathcal{F}$ is semistable. If φ' has the special form given above, then we have extension $(*)$, showing that $\mathcal{F}$ is semistable. □

PROPOSITION 6.1.4

(i) *The sheaves* $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,0)$ *and satisfying the conditions*

$\mathrm{h}^0(\mathcal{F}(-1)) > 0$, $\mathrm{h}^1(\mathcal{F}(1)) = 0$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow 3\mathcal{O}(-3) \xrightarrow{\varphi} 2\mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} *has linearly independent maximal minors.*

(ii) *The sheaves* $\mathcal{F}$ *giving points in* $\mathrm{M}_{\mathbb{P}^2}(6,0)$ *and satisfying the dual conditions* $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}(1)) > 0$ *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{12} *has linearly independent maximal minors.*

Proof

Part (ii) is equivalent to (i) by duality, so we concentrate on (i). Assume that $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,0)$ and satisfies the cohomological conditions from (i). There is an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for some curve $C \subset \mathbb{P}^2$. Note that $\deg(C) = 5$ or 6; otherwise, the semistability of $\mathcal{F}(-1)$ would be contradicted. Assume first that $\deg(C) = 5$. Let $\mathcal{T}$ denote the zero-dimensional torsion of $\mathcal{F}/\mathcal{O}_C(1)$. If $\mathcal{T} \neq 0$, then the pullback of $\mathcal{T}$ in $\mathcal{F}$ would be a destabilizing subsheaf. Thus $\mathcal{T} = 0$; hence $\mathcal{F}/\mathcal{O}_C(1) \simeq \mathcal{O}_L(-1)$. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L(-1) \longrightarrow 0,$$

to the standard resolution of $\mathcal{O}_C(1)$, and to the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow 3\mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}_L(-1) \longrightarrow 0.$$

We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus 3\mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Since $\mathrm{h}^1(\mathcal{F}(1)) = 0$, the map $\mathcal{O}(-4) \to \mathcal{O}(-4)$ is nonzero. Canceling $\mathcal{O}(-4)$ we arrive at a morphism as in the proposition.

Assume now that $\deg(C) = 6$. The quotient sheaf $\mathcal{C} = \mathcal{F}/\mathcal{O}_C(1)$ has dimension zero and length 3. Let $\mathbb{C}_x \subset \mathcal{C}$ be a subsheaf of length 1, and let $\mathcal{F}' \subset \mathcal{F}$ be its preimage. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F}' \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

to the standard resolution of $\mathcal{O}_C(1)$, and to the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(-3)$. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{O}(-5) \oplus 2\mathcal{O}(-4) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

The morphism $\mathcal{O}(-5) \to \mathcal{O}(-5)$ is nonzero; otherwise, arguing as at [10, Proposition 2.3.2], we would deduce that $\mathbb{C}_x$ is a direct summand of $\mathcal{F}'$, which is absurd. We have an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}' \longrightarrow 0,$$

where $\mathcal{C}'$ has length 2. Let $\mathbb{C}_y \subset \mathcal{C}'$ be a subsheaf of length 1, and let $\mathcal{F}'' \subset \mathcal{F}$ be its preimage. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow \mathbb{C}_y \longrightarrow 0,$$

to the standard resolution of $\mathbb{C}_y$ tensored with $\mathcal{O}(-2)$, and to the resolution

$$0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

We obtain a resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow 2\mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}'' \longrightarrow 0$$

in which, by the same argument as above, the morphism $\mathcal{O}(-4) \to 2\mathcal{O}(-4)$ is nonzero. Canceling $\mathcal{O}(-4)$ we obtain the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

The morphism $2\mathcal{O}(-3) \to \mathcal{O}(-3)$ is nonzero; otherwise, $\mathcal{F}$ would have a destabilizing subsheaf that is the cokernel of a morphism $2\mathcal{O}(-3) \to \mathcal{O}(-2) \oplus \mathcal{O}(1)$. Denote $\mathbb{C}_z = \mathcal{C}'/\mathbb{C}_y$. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_z \longrightarrow 0,$$

to the standard resolution of $\mathbb{C}_z$ tensored with $\mathcal{O}(-2)$, and to the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

We arrive at the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus 3\mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The morphism $\mathcal{O}(-4) \to \mathcal{O}(-4)$ is nonzero because $\mathrm{h}^1(\mathcal{F}(1)) = 0$. Canceling $\mathcal{O}(-4)$ we obtain a resolution as in the proposition.

Conversely, if $\mathcal{F}$ has resolution (i) or (ii), then, by virtue of Claim 3.1.4, $\mathcal{F}$ is semistable. □

PROPOSITION 6.1.5

The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,0)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) > 0$, $\mathrm{h}^1(\mathcal{F}(1)) > 0$ are precisely the sheaves having a resolution of the form

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0, \quad \textit{where } \varphi_{12} \neq 0.$$

Proof

Arguing as in the proof of Proposition 6.1.4, we see that there is an extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L(-1) \longrightarrow 0$$

for a quintic curve $C \subset \mathbb{P}^2$ or that there is a resolution

$$0 \longrightarrow \mathcal{O}(-4) \xrightarrow{\psi} \mathcal{O}(-4) \oplus 3\mathcal{O}(-3) \xrightarrow{\rho} 2\mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

In the first case we can combine the standard resolutions of $\mathcal{O}_C(1)$ and $\mathcal{O}_L(-1)$ to get a resolution as in the proposition. Indeed, by hypothesis $\mathrm{h}^0(\mathcal{F}(1)) \geq 7$, so $\mathcal{F}(1)$ has a section mapping to a nonzero section of $\mathcal{O}_L$.

In the second case $\psi_{11} = 0$ because $\mathrm{h}^1(\mathcal{F}(1)) > 0$. We claim that $\mathcal{C}oker(\psi_{21}) \simeq \Omega^1(-1)$; that is, the entries of ψ_{21} are linearly independent. Clearly they span a vector space of dimension at least 2. If

$$\psi_{21} \sim \begin{bmatrix} 0 \\ \star \\ \star \end{bmatrix}, \quad \text{then } \rho \sim \begin{bmatrix} \star & \star & \star & \star \\ \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \end{bmatrix}.$$

It would follow that $\mathcal{F}$ maps surjectively to the cokernel of an injective morphism $\mathcal{O}(-4) \oplus \mathcal{O}(-3) \to \mathcal{O}(-2) \oplus \mathcal{O}(1)$. This would contradict the semistability of $\mathcal{F}$. From the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \Omega^1(-1) \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0$$

we obtain the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus 3\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

If $\mathrm{rank}(\varphi_{12}) \leq 1$, then $\mathcal{F}$ would map surjectively to $\mathcal{O}_C(-2)$ for a conic curve $C \subset \mathbb{P}^2$, in violation of semistability. Thus $\mathrm{rank}(\varphi_{12}) = 2$, and canceling $2\mathcal{O}(-2)$ we obtain a resolution as in the proposition. □

In view of Corollary 2.2.2(vii) there are no other sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6,0)$ beside the sheaves we have discussed in this subsection.

6.2. The strata as quotients

In Section 6.1 we classified all sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6,0)$, namely, we showed that this moduli space is the union of six locally closed subsets as in Table 4. As the notation suggests, X_3^{D} is the image of X_3 under the duality automorphism $[\mathcal{F}] \to [\mathcal{F}^{\mathrm{D}}]$. The strata X_i, $i = 0,1,2,4$, are invariant under this automorphism. We employ the notations $\mathbb{W}_i$, W_i, G_i, ρ_i, $0 \leq i \leq 4$, analogous to the notations from Section 3.2. We denote $W_i^{\mathrm{s}} = \rho_i^{-1}(X_i^{\mathrm{s}})$.

From Proposition 6.1.1 we easily deduce that the points in X_0 given by properly semistable sheaves are of the form $[\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_\kappa]$, $\kappa \geq 2$, where $\mathcal{F}_i$ is stable and has resolution

$$0 \longrightarrow r_i\mathcal{O}(-2) \longrightarrow r_i\mathcal{O}(-1) \longrightarrow \mathcal{F}_i \longrightarrow 0,$$

$r_1 + \cdots + r_\kappa = 6$. In particular, we see that X_0 is disjoint from the other strata and that two points in W_0 are in the same fiber of ρ_0 if and only if the relative closures in W_0 of their orbits meet nontrivially. This allows us to apply the method of [4, Theorem 4.2.2] in order to show that ρ_0 is a categorical quotient map. Note that W_0 is a proper invariant open subset of the set of semistable Kronecker modules $6\mathcal{O}(-2) \to 6\mathcal{O}(-1)$, so there exists a good quotient $W_0//G_0$ as an open subset of $\mathrm{N}(3,6,6)$. By the uniqueness of the categorical quotient we have an isomorphism $X_0 \simeq W_0//G_0$. This shows that $\mathrm{M}_{\mathbb{P}^2}(6,0)$ and $\mathrm{N}(3,6,6)$ are birational.

According to Claim 3.1.4 and Corollary 2.3.2, the sets $X_3 \setminus X_3^{\mathrm{s}}$, $X_3^{\mathrm{D}} \setminus (X_3^{\mathrm{D}})^{\mathrm{s}}$, $X_4 \setminus X_4^{\mathrm{s}}$ coincide and consist of all points of the form $[\mathcal{O}_L(-1) \oplus \mathcal{O}_Q(1)]$, where $L \subset \mathbb{P}^2$ is a line and $Q \subset \mathbb{P}^2$ is a quintic curve. Moreover, from the proofs of

Propositions 6.1.4 and 6.1.5 it transpires that any sheaf stable-equivalent to $\mathcal{O}_L(-1) \oplus \mathcal{O}_Q(1)$ is the cokernel of some morphism in $W_3^{\mathrm{D}} \setminus W_3^{\mathrm{s}}$, $W_3 \setminus (W_3^{\mathrm{D}})^{\mathrm{s}}$, or $W_4^{\mathrm{D}} \setminus W_4^{\mathrm{s}}$. Thus X_3, X_3^{D}, X_4 are disjoint from the strata X_0, X_1, and X_2. For $i \in \{3,4\}$ the fibers of the map $W_i^{\mathrm{s}} \to X_i^{\mathrm{s}}$ are precisely the G_i-orbits; hence, as at Section 3.2, this is a geometric quotient map. According to [4, Section 9.3], W_3^{s}/G_3 is an open subset of a fiber bundle over $\mathrm{N}(3,2,3)$ with fiber $\mathbb{P}^{24}$. We can be more precise. According to Claim 3.1.4, W_3^{s} is the subset of W_3 of morphisms φ such that the maximal minors of φ_{12} have no common factor, hence, applying [1, Propositions 4.5, 4.6], we can show that the sheaves $\mathcal{C}oker(\varphi)$, $\varphi \in W_3^{\mathrm{s}}$, are precisely the sheaves of the form $\mathcal{J}_Z(2)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 3 that is not contained in a line, Z is contained in a sextic curve C, and $\mathcal{J}_Z \subset \mathcal{O}_C$ is the ideal of Z in C. Thus W_3^{s}/G_3 is isomorphic to the open subset of $\mathrm{Hilb}_{\mathbb{P}^2}(6,3)$ of pairs (C,Z) such that Z is not contained in a line. Similarly, the sheaves giving points in X_4^{s} are precisely the sheaves of the form $\mathcal{J}_Z(2)$, where Z is contained in a line L that is not a component of C. Thus X_4 is isomorphic to the locally closed subset $\{(C,Z), Z \subset L, L \nsubseteq C\}$ of $\mathrm{Hilb}_{\mathbb{P}^2}(6,3)$.

By the discussion above, the strata X_1 and X_2 are disjoint from X_0, X_3, X_3^{D}, X_4. Making a list of properly semistable sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6,0)$ we can show that

$$X_1 \cap X_2 = \big\{[\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}], C_1, C_2 \subset \mathbb{P}^2 \text{ cubic curves}\big\}.$$

Denote $X_{10} = X_1 \setminus X_2$, $X_{20} = X_2 \setminus X_1$, $W_{10} = \rho_1^{-1}(X_{10})$, $W_{20} = \rho_2^{-1}(X_{20})$. Assume that $i \in \{1,2\}$. Arguing as at [3, Section 4.2.1], we can easily see that two points of W_{i0} are in the same fiber of ρ_i if and only if the relative closures in W_{i0} of their G_i-orbits intersect nontrivially. This allows us to apply the method of [3, Theorem 4.2.2] in order to show that the maps $W_{i0} \to X_{i0}$ are categorical quotient maps.

6.3. Generic sheaves

Let $C \subset \mathbb{P}^2$ denote an arbitrary smooth sextic curve, and let P_i denote distinct points on C. According to [1, Propositions 4.5, 4.6], the cokernels of morphisms $5\mathcal{O}(-6) \to 6\mathcal{O}(-5)$ whose maximal minors have no common factor are precisely the ideal sheaves $\mathcal{I}_Z \subset \mathbb{P}^2$ of zero-dimensional schemes Z of length 15 that are not contained in a quartic curve. It follows that the generic sheaves in X_0 have the form $\mathcal{O}_C(4)(-P_1 - \cdots - P_{15})$, where $P_1, \ldots, P_{15}$ are not contained in a quartic curve. From Claim 6.3.1 below, it follows that the generic sheaves in X_1 have the form $\mathcal{O}_C(3)(-P_1 - \cdots - P_9)$, where $P_1, \ldots, P_9$ are contained in a unique cubic curve. The generic sheaves in X_2 have the form $\mathcal{O}_C(3)(-P_1 - \cdots - P_9)$, where $P_1, \ldots, P_9$ are contained in two cubic curves that have no common component. We saw in Section 6.2 that the generic sheaves in X_3 have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3)$, and the generic sheaves in X_3^{D} have the form $\mathcal{O}_C(1)(P_1 + P_2 + P_3)$, where P_1, P_2, P_3 are non-colinear. The generic sheaves in X_4 have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3)$, where P_1, P_2, P_3 are colinear.

CLAIM 6.3.1

Let $U \subset \mathrm{Hom}(3\mathcal{O}(-2), 3\mathcal{O}(-1) \oplus \mathcal{O})$ be the set of morphisms whose maximal minors have no common factor. The cokernels of morphisms in U are precisely the sheaves of the form $\mathcal{I}_Z(3)$, where $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of a zero-dimensional scheme Z of length 9, that is contained in a unique cubic curve.

Proof

The argument is analogous to the argument at Claim 3.3.1. □

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