Moduli of unramified irregular singular parabolic connections on a smooth projective curve

Michi-aki Inaba and Masa-Hiko Saito

To the memory of Professor Masaki Maruyama

Abstract In this paper we construct a coarse moduli scheme of stable unramified irregular singular parabolic connections on a smooth projective curve and prove that the constructed moduli space is smooth and has a symplectic structure. Moreover, we will construct the moduli space of generalized monodromy data coming from topological monodromies, formal monodromies, links, and Stokes data associated to the generic irregular connections. We will prove that for a generic choice of generalized local exponents, the generalized Riemann–Hilbert correspondence from the moduli space of the connections to the moduli space of the associated generalized monodromy data gives an analytic isomorphism. This shows that differential systems arising from (generalized) isomonodromic deformations of corresponding unramified irregular singular parabolic connections admit the geometric Painlevé property as in the regular singular cases proved generally.

0. Introduction

Let m,l be positive integers, and let ν be an element of $\mathbf{C}\,dw/w^{lm-l+1}+\cdots+\mathbf{C}\,dw/w$. We denote the $\mathbf{C}[[w]]$-module $\mathbf{C}[[w]]^{\oplus r}$ with the connection

$$\mathbf{C}[[w]]^{\oplus r} \longrightarrow \mathbf{C}[[w]]^{\oplus r}\otimes \frac{dw}{w^{lm-l+1}},$$
$$ae_j \;\mapsto\; dae_j+\nu ae_j+w^{-1}\,dwe_{j-1}$$

by $V(\nu,r)$. Here $e_1,\dots,e_r$ is the canonical basis of $\mathbf{C}[[w]]^{\oplus r}$ and $e_0=0$.

We have the following fundamental theorem.

THEOREM 0.1 (HUKUHARA AND TURRITTIN)

Let V be a free $\mathbf{C}[[z]]$-module of rank r, and let $\nabla: V\to V\otimes dz/z^m$ be a connection. Then there are positive integers $l,s,r_1,\dots,r_s$ such that for a variable w

Kyoto Journal of Mathematics, Vol. 53, No. 2 (2013), 433–482
DOI 10.1215/21562261-2081261,
Received February 24, 2012. Revised November 16, 2012. Accepted November 19, 2012.
2010 Mathematics Subject Classification: 14D20, 34M56, 34M55.
Saito's work supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research (S) 19104002, 24224001, and Challenging Exploratory Research 23654010.
Inaba's work supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research Young Scientists (B) 22740014.

with $w^l = z$, *there exist* $\nu_1, \dots, \nu_s \in \mathbf{C}\, dw/w^{ml-l+1} + \cdots + \mathbf{C}\, dw/w$ *such that*

$$(V, \nabla) \otimes_{\mathbf{C}[[z]]} \mathbf{C}\big((w)\big) \cong \big(V(\nu_1, r_1) \oplus \cdots \oplus V(\nu_s, r_s)\big) \otimes_{\mathbf{C}[[w]]} \mathbf{C}\big((w)\big).$$

For the proof of Theorem 0.1, see [27, Theorem 6.8.1], for example. Note that $\nu_1, \dots, \nu_s$ in Theorem 0.1 are unique modulo $\mathbf{Z}\, dw/w$. So we can take $\nu_1, \dots, \nu_s$ as invariants of a connection. In this paper we consider only the case $l = 1$.

Let C be a smooth projective irreducible curve over $\mathbf{C}$ of genus g, let $t_1, \dots, t_n$ be distinct points of C, and let $m_1, \dots, m_n$ be positive integers. Put $D := \sum_{i=1}^n m_i t_i$. Take $d \in \mathbf{Z}$ and $\boldsymbol{\nu} = (\nu_j^{(i)})_{0 \le j \le r-1}^{1 \le i \le n}$ such that $\nu_j^{(i)} \in \mathbf{C}\, dz_i/z_i^{m_i} + \cdots + \mathbf{C}\, dz_i/z_i$ and that $d + \sum_{i=1}^n \sum_{j=0}^{r-1} \mathrm{res}_{t_i}(\nu_j^{(i)}) = 0$, where z_i is a generator of the maximal ideal of $\mathcal{O}_{C,t_i}$. Let N_i be positive integers such that $N_i \ge m_i$ for $i = 1, \dots, n$, and set $N_i t_i := \mathrm{Spec}(\mathcal{O}_{C,t_i}/(z_i^{N_i}))$. $(E, \nabla, \{l_j^{(i)}\})$ is said to be an irregular singular $\boldsymbol{\nu}$-parabolic connection of parabolic depth (N_i) if E is a vector bundle on C of rank r and degree d, $\nabla : E \to E \otimes \Omega_C^1(D)$ is a connection, $E|_{N_i t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ is a filtration such that $l_j^{(i)}/l_{j+1}^{(i)} \cong \mathcal{O}_{N_i t_i}$, $\nabla|_{N_i t_i}(l_j^{(i)}) \subset l_j^{(i)} \otimes \Omega_C^1(D)$ for any i, j, and for the induced morphism $\overline{\nabla_j^{(i)}} : l_j^{(i)}/l_{j+1}^{(i)} \to (l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1(D)$, $(\overline{\nabla_j^{(i)}} - \nu_j^{(i)}\mathrm{id})(l_j^{(i)}/l_{j+1}^{(i)})$ is contained in the image of $(l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1 \to (l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1(D)$ for any i, j. We can define $\boldsymbol{\alpha}$-stability for a $\boldsymbol{\nu}$-parabolic connection (see Definition 2.2 for a precise definition of $\boldsymbol{\alpha}$-stability). We first show in Theorem 2.1 that there is a coarse moduli scheme $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (N_i))_{\boldsymbol{\nu}}$ of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections of parabolic depth (N_i). The main theorems of this paper are Theorems 2.2 and 4.1, which state that the moduli space $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))_{\boldsymbol{\nu}}$ of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \{l_j^{(i)}\})$ of parabolic depth (m_i) is smooth and has a symplectic structure.

However, there is a serious example (Remark 1.2) which states that for special $\boldsymbol{\nu}$, there is a member $(E, \nabla, \{l_j^{(i)}\}) \in M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))_{\boldsymbol{\nu}}$ such that the invariants $\nu_1, \dots, \nu_s$ for $(E, \nabla) \otimes \hat{\mathcal{O}}_{C,t_i}$ given in Theorem 0.1 are different from the data $\nu_0^{(i)}, \dots, \nu_{r-1}^{(i)}$ given by $\boldsymbol{\nu}$. So $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))_{\boldsymbol{\nu}}$ does not seem to be a good moduli space at a glance. On the other hand, assume that $N_i \ge r^2 m_i$ for any i and $0 \le \mathrm{Re}(\mathrm{res}_{t_i}(\nu_j^{(i)})) < 1$ for any i, j. Then Proposition 1.2 states that for any member $(E, \nabla, \{l_j^{(i)}\}) \in M_{D/C}^{\boldsymbol{\alpha}}(r, d, (N_i))_{\boldsymbol{\nu}}$, the data $\nu_1, \dots, \nu_s$ for $(E, \nabla) \otimes \hat{\mathcal{O}}_{C,t_i}$ given in Theorem 0.1 are the same as the data $\nu_0^{(i)}, \dots, \nu_{r-1}^{(i)}$ given by $\boldsymbol{\nu}$. So it seems that $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (N_i))_{\boldsymbol{\nu}}$ is a good moduli space. However, Remark 2.3 states that the moduli space $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (N_i))_{\boldsymbol{\nu}}$ is not smooth for special $\boldsymbol{\nu}$. So we cannot define isomonodromic deformations on the moduli space $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (N_i))$. After all the authors believe that the moduli space $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))$ of $\boldsymbol{\alpha}$-stable parabolic connections of parabolic depth (m_i) is a correct moduli space, although $\boldsymbol{\nu}$ does not necessarily reflect the invariants given in the Hukuhara–Turrittin theorem.

After we construct the good moduli space $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))_{\boldsymbol{\nu}}$ of the $\boldsymbol{\alpha}$-stable parabolic connections, we will investigate the Riemann–Hilbert correspondences for these moduli spaces and define the generalized isomonodromic flows or isomonodromic differential systems associated to them. For that purpose, it is necessary to construct a good moduli space of the generalized monodromy data for the parabolic $\boldsymbol{\nu}$-connection $(E, \nabla, \{l_j^{(i)}\}) \in M^{\boldsymbol{\alpha}}_{D/C}(r, d, (m_i))_{\boldsymbol{\nu}}$. However, for that purpose, we should fix the types of decompositions in the Hukuhara–Turrittin formal types at all irregular singular points. However, for some special $\boldsymbol{\nu}$, we cannot recover these formal types (see Remark 1.2). So in this paper, we will restrict ourselves to the case when the local exponent $\boldsymbol{\nu}$ is generic (cf. Definition 5.1). In this case, we can also construct the coarse moduli scheme $\mathcal{R}(\boldsymbol{\nu})$ of the data consisting of Stokes data, links, and global topological monodromy representation of $\pi_1(C \setminus \{t_1, \ldots, t_n\})$. Let us denote by $\boldsymbol{\nu}_{\mathrm{res}}$ the residue part of $\boldsymbol{\nu}$ and by $\mathbf{p} = \{\widehat{\gamma}_i\} = \mathbf{e}(\boldsymbol{\nu}_{\mathrm{res}})$ its exponential. Under the assumption that $\boldsymbol{\nu}$ is generic, nonresonant, and irreducible, we can see that the moduli space $\mathcal{R}(\boldsymbol{\nu})$ is a nonsingular affine scheme. Moreover for a fixed generic $\boldsymbol{\nu}$, we can define the Riemann–Hilbert correspondence $\mathrm{RH}_{(D/C),\boldsymbol{\nu}} : M^{\boldsymbol{\alpha}}_{D/C}(r, d, (m_i))_{\boldsymbol{\nu}} \longrightarrow \mathcal{R}(\boldsymbol{\nu})$, and in Theorem 5.1, we prove that $\mathrm{RH}_{(D/C),\boldsymbol{\nu}}$ is an analytic isomorphism under the assumption that $\boldsymbol{\nu}$ is generic, nonresonant, and irreducible. In Section 6, we will define the isomonodromic differential systems induced by the family of the Riemann–Hilbert correspondences and show that the corresponding differential systems have geometric Painlevé property when $\boldsymbol{\nu}$ is generic, simple, nonresonant, and irreducible (cf. Theorem 6.1). Then as a corollary we can obtain the geometric Painlevé property of the 5 types of classical Painlevé equations listed below when $\mathbf{p} = \mathbf{e}(\boldsymbol{\nu}_{\mathrm{res}})$ is nonresonant and irreducible (note that if rank E =2, $\boldsymbol{\nu}$ are always simple):

$$
\begin{aligned}
&P_{VI}(D_4^{(1)})_{\mathbf{p}}, \qquad P_V(D_5^{(1)})_{\mathbf{p}}, \qquad P_{III}(D_6^{(1)})_{\mathbf{p}}, \\
&P_{IV}(E_6^{(1)})_{\mathbf{p}}, \qquad P_{II}(E_7^{(1)})_{\mathbf{p}}.
\end{aligned}
\tag{1}
$$

For $P_{VI}(D_4^{(1)})_{\mathbf{p}}$, we showed the geometric Painlevé property for any $\mathbf{p}$ (see [10], [11], [8]). More generally, the geometric Painlevé property for isomonodromic differential systems associated to the regular singular parabolic connections for any $\mathbf{p}$ was proved completely in [8].

The rough plan of this paper is as follows. In Section 1, we will prepare some results on the formal parabolic connections and their reductions to the finite orders. In Section 2, we will construct the coarse moduli scheme $M^{\boldsymbol{\alpha}}_{D/C}(r, d, (N_i))_{\boldsymbol{\nu}}$ for $N_i \geq m_i$ as a quasi-projective scheme and show that $M^{\boldsymbol{\alpha}}_{D/C}(r, d, (m_i))_{\boldsymbol{\nu}}$ is smooth for any $\boldsymbol{\nu}$. In Section 3, we will show the existence of the smooth family of the moduli spaces of parabolic connections over the space of generalized exponents when we also vary the divisor $D = \sum_{i=1}^n m_i t_i$ in a product of Hilbert schemes of points (cf. Theorem 3.1). Theorem 3.1 seems important from the view point of confluence process of singular points. In Section 4, we will show the existence of the relative symplectic form ω on the family of moduli spaces of parabolic connections parameterized by $\boldsymbol{\nu}$. We will use Theorem 3.1 to reduce the proof of the closedness $d\omega = 0$ to the case of regular singular cases in [8]. In Section 5, we will review on the generalized monodromy data and construct the moduli space of them when $\boldsymbol{\nu}$ is generic. Moreover, we define the Riemann–Hilbert correspondence and show that it gives an analytic isomorphism for generic, nonresonant and irreducible $\boldsymbol{\nu}$. In Section 6, fixing a nonresonant and irreducible $\boldsymbol{\nu}_{\mathrm{res}}$, we will define the family of Riemann–Hilbert correspondences and define the isomonodromic flows on the phase space $\pi_{2,\boldsymbol{\nu}_{\mathrm{res}}} : M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}} \longrightarrow T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}$ which is the family of moduli spaces of $\boldsymbol{\alpha}$-stable parabolic connections over a certain space $T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}$ of parameters including generic, simple exponents $\boldsymbol{\nu}$ with the fixed residue part $\boldsymbol{\nu}_{\mathrm{res}}$ (see (26)). The isomonodromic flows define an isomonodromic foliation or an isomonodromic differential system on the phase space, and its geometric Painlevé property follows easily from the definition based on Theorem 5.1. The geometric Painlevé property gives a complete and clear proof of the analytic Painlevé property for the isomonodromic differential systems with nonresonant and irreducible exponents $\boldsymbol{\nu}_{\mathrm{res}}$ or $\mathbf{p} = \mathbf{e}(\boldsymbol{\nu}_{\mathrm{res}})$.

As explained in [9], it is important to construct the fibers of the phase space of the isomonodromic differential system as smooth algebraic schemes. One can use affine algebraic coordinates of the fibers over an open set of parameter spaces to write down the differential systems explicitly. Then the differential systems satisfy the analytic Painlevé property, which easily follows from the geometric Painlevé property.

We should mention that Malgrange [14] and [15] and Miwa [16] gave proofs of the analytic Painlevé property for isomonodromic differential systems for irregular connections on $\mathbf{P}^1$. However, to give a complete proof of the geometric Painlevé property, we believe that our algebro-geometric construction of the family of the moduli spaces of connections is indispensable (see also [10], [11], [8] for the regular singular cases).

Bremer and Sage [2] studied the moduli space of irregular singular connections on $\mathbf{P}^1$. They considered also the ramified case. However, they assumed that the bundle V is trivial, which means that their moduli space only covers a Zariski open set of our moduli space, which is not enough to prove the geometric Painlevé property even for generic unramified cases (see Remark 5.3).

1. Preliminary

As a corollary of Theorem 0.1, we obtain the following proposition.

PROPOSITION 1.1

Let V be a free $\mathbf{C}[[z]]$-module of rank r, and let $\nabla : V \to V \otimes dz/z^m$ be a connection. Then there is a positive integer l such that for a variable w with $w^l = z$, there exist $\nu_0, \dots, \nu_{r-1} \in \mathbf{C}\, dw/w^{lm-l+1} + \cdots + \mathbf{C}\, dw/w$ and a filtration $V \otimes \mathbf{C}[[w]] = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_{r-1} \supset V_r = 0$ by subbundles such that $\nabla(V_j) \subset V_j \otimes dw/w^{lm-l+1}$ and $V_j/V_{j+1} \cong V(\nu_j, 1)$ for any $j = 0, 1, \dots, r-1$.

Proof

We prove the proposition by induction on r. For $r = 1$, we take a basis e of V. Then we have $\nabla(e) = \nu e\, dz$ for some $\nu \in \mathbf{C}[[z]]z^{-m}$. We can write $\nu = \sum_{j \geq -m} a_j z^j$. We put $\nu_0 := \sum_{j\geq 0} a_j z^j$ and $\mu := \int \nu_0 = \sum_{j\geq 0} (j+1)^{-1} a_j z^{j+1}$. Then we have

$\exp(-\mu) \in \mathbf{C}[[z]]$ and

$$\frac{d}{dz}\exp(-\mu) = -\exp(-\mu)\frac{d\mu}{dz} = -\exp(-\mu)\nu_0.$$

We put $e' := \exp(-\mu)e$. Then e' is a basis of V and

$$\begin{aligned}\nabla(e') &= \nabla\bigl(\exp(-\mu)e\bigr) = \exp(-\mu)\nabla(e) + \frac{d\exp(-\mu)}{dz} e\,dz \\ &= \exp(-\mu)\nu e\,dz - \exp(-\mu)\nu_0 e\,dz \\ &= (\nu - \nu_0)\exp(-\mu)e\,dz = (\nu-\nu_0)\,dze'.\end{aligned}$$

Hence we have $V \cong V((\nu - \nu_0)\,dz, 1)$.

Now assume that $r > 1$. By Theorem 0.1, there is a positive integer l such that for a variable w with $w^l = z$, there exist $\mu_1, \dots, \mu_s \in \mathbf{C}\,dw/w^m + \cdots + \mathbf{C}\,dw/w$, positive integers $r_1, \dots, r_s$ and an isomorphism

$$\begin{aligned}&\varphi : V \otimes_{\mathbf{C}[[z]]} \mathbf{C}\bigl((w)\bigr) \\ &\quad \xrightarrow{\sim} \bigl(V(\mu_1, r_1) \otimes_{\mathbf{C}[[w]]} \mathbf{C}\bigl((w)\bigr)\bigr) \oplus \cdots \oplus \bigl(V(\mu_s, r_s) \otimes_{\mathbf{C}[[w]]} \mathbf{C}\bigl((w)\bigr)\bigr).\end{aligned}$$

We can take an element $e_{r-1} \in \varphi^{-1}(V(\mu_s, r_s))$ such that $\nabla(e_{r-1}) = \mu_s e_{r-1}$. Let m_{r-1} be the smallest integer such that $w^{m_{r-1}}e_{r-1} \in V \otimes_{\mathbf{C}[[z]]} \mathbf{C}[[w]]$. Then we have

$$\begin{aligned}\nabla(w^{m_{r-1}}e_{r-1}) &= m_{r-1}w^{m_{r-1}-1}\,dwe_{r-1} + \mu_s w^{m_{r-1}}e_{r-1} \\ &= (m_{r-1}w^{-1}\,dw + \mu_s)w^{m_{r-1}}e_{r-1}.\end{aligned}$$

If we put $V_{r-1} := \mathbf{C}[[w]]w^{m_{r-1}}e_{r-1}$ and $\mu'_{r-1} := m_{r-1}w^{-1}\,dw + \mu_s$, then $V_{r-1} \cong V(\nu_{r-1}, 1)$ and $W_{r-1} := (V \otimes_{\mathbf{C}[[z]]} \mathbf{C}[[w]])/V_{r-1}$ is a torsion-free $\mathbf{C}[[w]]$-module. So W_{r-1} is a free $\mathbf{C}[[w]]$-module of rank $r-1$, and ∇ induces a connection

$$\bar{\nabla} : W_{r-1} \longrightarrow W_{r-1} \otimes \frac{dw}{w^{ml-l+1}}.$$

Then by the induction assumption, there is a filtration $W_{r-1} = \bar{V}_0 \supset \bar{V}_1 \supset \cdots \supset \bar{V}_{r-2} \supset \bar{V}_{r-1} = 0$ by subbundles such that $\bar{\nabla}(\bar{V}_j) \subset \bar{V}_j \otimes dw/w^{lm-l+1}$ and $\bar{V}_j/\bar{V}_{j+1} \cong V(\mu'_j, 1)$ for $j = 0, \dots, r-2$ for some $\mu'_j \in \mathbf{C}\,dw/w^{ml-l+1} + \cdots + \mathbf{C}\,dw/w$ $(0 \le j \le r-2)$. Let V_j be the pull back of $\bar{V}_j$ by the homomorphism $V \otimes_{\mathbf{C}[[z]]} \mathbf{C}[[w]] \to W_{r-1}$ $(0 \le j \le r-1)$. Then $\nabla(V_j) \subset V_j \otimes dw/w^{lm-l+1}$ and $V_j/V_{j+1} \cong V(\mu'_j, 1)$ for $0 \le j \le r-1$. □

REMARK 1.1

By the proof of Proposition 1.1, we can easily see that $\{\nu_j \mod \mathbf{Z}\,dw/w\}$ in Proposition 1.1 are nothing but the invariants in the Hukuhara–Turrittin theorem (Theorem 0.1). We should remark that we cannot give a decomposition $(V, \nabla) \otimes_{\mathbf{C}[[z]]} \mathbf{C}[[w]] \cong \bigoplus_{j=0}^{r-1} V(\nu_j, 1)$ even if ν_j modulo $\mathbf{Z}\,dw/w$ are mutually distinct.

REMARK 1.2

Unfortunately, we cannot recover $\nu_0, \dots, \nu_{r-1}$ from $\nabla \otimes \mathbf{C}[[w]]/(w^{ml-l+1})$. Indeed, consider the connection $\nabla : \mathbf{C}[[z]]^{\oplus 2} \to \mathbf{C}[[z]]^{\oplus 2} \otimes dz/z^6$ given by

$$\nabla = d + \begin{pmatrix} z^{-6}\,dz + z^{-2}\,dz & z^{-4}\,dz \\ 0 & z^{-6}\,dz - z^{-2}\,dz \end{pmatrix}.$$

Let

$$\nabla \otimes \mathbf{C}[[z]]/(z^6) : \big(\mathbf{C}[[z]]/(z^6)\big)^{\oplus 2} \to \big(\mathbf{C}[[z]]/(z^6)\big)^{\oplus 2} \otimes \frac{dz}{z^6}$$

be the induced $\mathbf{C}[[z]]/(z^6)$-homomorphism. Then $\nabla \otimes \mathbf{C}[[z]]/(z^6)$ can be given by the matrix

$$A = \begin{pmatrix} z^{-6}\,dz + z^{-2}\,dz & z^{-4}\,dz \\ 0 & z^{-6}\,dz - z^{-2}\,dz \end{pmatrix}$$

with respect to the basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of $(\mathbf{C}[[z]]/(z^6))^{\oplus 2}$. So the "eigenvalues" of $\nabla \otimes \mathbf{C}[z]/(z^6)$ with respect to this basis are $z^{-6}\,dz + z^{-2}\,dz, z^{-6}\,dz - z^{-2}\,dz$.

On the other hand, take the basis

$$\begin{pmatrix} 1+z^4 \\ -z^2 \end{pmatrix}, \qquad \begin{pmatrix} -z^2 \\ 1+z^4 \end{pmatrix}$$

of $(\mathbf{C}[[z]]/(z^6))^{\oplus 2}$. Then we have

$$\begin{aligned} \big(\nabla \otimes \mathbf{C}[[z]]/(z^6)\big) \begin{pmatrix} 1+z^4 \\ -z^2 \end{pmatrix} &= \begin{pmatrix} z^{-6}\,dz + z^{-2}\,dz & z^{-4}\,dz \\ 0 & z^{-6}\,dz - z^{-2}\,dz \end{pmatrix} \begin{pmatrix} 1+z^4 \\ -z^2 \end{pmatrix} \\ &= \begin{pmatrix} z^{-6}\,dz + z^{-2}\,dz \\ -z^{-4}\,dz \end{pmatrix} = z^{-6}\,dz \begin{pmatrix} 1+z^4 \\ -z^2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \big(\nabla \otimes \mathbf{C}[[z]]/(z^6)\big) \begin{pmatrix} -z^2 \\ 1+z^4 \end{pmatrix} &= \begin{pmatrix} z^{-6}\,dz + z^{-2}\,dz & z^{-4}\,dz \\ 0 & z^{-6}\,dz - z^{-2}\,dz \end{pmatrix} \begin{pmatrix} -z^2 \\ 1+z^4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ z^{-6}\,dz \end{pmatrix} = z^{-4}\,dz \begin{pmatrix} 1+z^4 \\ -z^2 \end{pmatrix} + z^{-6}\,dz \begin{pmatrix} -z^2 \\ 1+z^4 \end{pmatrix}. \end{aligned}$$

Thus the representation matrix of $\nabla \otimes \mathbf{C}[[z]]/(z^6)$ with respect to the basis

$$\begin{pmatrix} 1+z^4 \\ -z^2 \end{pmatrix}, \qquad \begin{pmatrix} -z^2 \\ 1+z^4 \end{pmatrix}$$

is given by

$$\begin{pmatrix} z^{-6}\,dz & z^{-4}\,dz \\ 0 & z^{-6}\,dz \end{pmatrix}.$$

So the "eigenvalues" of $\nabla \otimes \mathbf{C}[z]/(z^6)$ with respect to this basis are $z^{-6}\,dz, z^{-6}\,dz$. Thus we conclude that the "eigenvalues" of $\nabla \otimes \mathbf{C}[z]/(z^6)$ can not be well defined. In other words, the eigenvalues $\nu_1, \ldots, \nu_s$ given in the Hukuhara–Turrittin theorem (Theorem 0.1) cannot be recovered from $\nabla \otimes \mathbf{C}[z]/(z^6)$.

On the other hand, we have the following proposition, which will be possible to improve more generally according to the referee's valuable comment.

PROPOSITION 1.2

Let V,W be free $\mathbf{C}[[z]]/(z^{r^2m})$-modules of rank r with connections

$$\nabla^V : V \longrightarrow V \otimes \frac{dz}{z^m},$$

$$\nabla^W : W \longrightarrow W \otimes \frac{dz}{z^m}$$

and filtrations

$$V = V_0 \supset V_1 \supset \cdots V_{r-1} \supset V_r = 0,$$

$$W = W_0 \supset W_1 \supset \cdots W_{r-1} \supset W_r = 0,$$

such that $V_i/V_{i+1} \cong \mathbf{C}[[z]]/(z^{r^2m})$, $W_i/W_{i+1} \cong \mathbf{C}[[z]]/(z^{r^2m})$ and that $\nabla^V(V_i) \subset V_i \otimes dz/z^m$, $\nabla^W(W_i) \subset W_i \otimes dz/z^m$ for any i. Let $\nabla^V_i : V_i/V_{i+1} \to (V_i/V_{i+1}) \otimes dz/z^m$ and $\nabla^W_i : W_i/W_{i+1} \to (W_i/W_{i+1}) \otimes dz/z^m$ be the morphisms induced by ∇^V and ∇^W, respectively. Choose a basis e^V_i of V_i/V_{i+1} (resp., e^W_i of W_i/W_{i+1}) such that $\nabla^V_i(e^V_i) = \nu^V_i e^V_i$ and $\nabla^W_i(e^W_i) = \nu^W_i e^W_i$ with

$$\nu^V_i = (a^{(i)}_{-m} z^{-m} + a^{(i)}_{-m+1} z^{-m+1} + \cdots + a^{(i)}_{-1} z^{-1})\, dz,$$

$$\nu^W_i = (b^{(i)}_{-m} z^{-m} + b^{(i)}_{-m+1} z^{-m+1} + \cdots + b^{(i)}_{-1} z^{-1})\, dz.$$

Assume that $0 \le \mathrm{Re}(a^{(i)}_{-1}) < 1$ and $0 \le \mathrm{Re}(b^{(i)}_{-1}) < 1$ for any i. If there is an isomorphism $\varphi : V \xrightarrow{\sim} W$ of $\mathbf{C}[[z]]/(z^{r^2m})$-modules such that $\nabla^W \circ \varphi = (\varphi \otimes \mathrm{id}) \circ \nabla^V$, then there is a permutation $\sigma \in S_r$ such that $\nu^V_i = \nu^W_{\sigma(i)}$ for any $i = 0, \ldots, r-1$.

Proof

We prove the proposition by induction on r. Assume that $r = 1$. We can write $\varphi(e^V_0) = c e^W_0$ with $c \in (\mathbf{C}[[z]]/(z^m))^\times$. Then we have

$$\begin{aligned}(dc)e^W_0 + c\nu^W_0 e^W_0 &= \nabla^W(ce^W_0) = \nabla^W \varphi(e^V_0) = (\varphi \otimes \mathrm{id})\nabla^V(e^V_0) \\ &= (\varphi \otimes \mathrm{id})(\nu^V_0 e^V_0) = \nu^V_0 \varphi(e^V_0) \\ &= c\nu^V_0 e^W_0.\end{aligned}$$

So we have

$$dc = c(\nu^V_0 - \nu^W_0).$$

If $\nu^V_0 \ne \nu^W_0$, we can write

$$\nu^V_0 - \nu^W_0 = \alpha_{-n} z^{-n}\, dz + \alpha_{-n+1} z^{-n+1}\, dz + \cdots + \alpha_{-1} z^{-1}\, dz$$

with $n \ge 1$ and $\alpha_{-n} \ne 0$. If we put $c = c_0 + c_1 z + c_2 z^2 + \cdots + c_{m-1} z^{m-1}$ with each $c_j \in \mathbf{C}$, then we have $c_0 \ne 0$. So we have

$$\begin{aligned} dc &= c(\nu_0^V - \nu_0^W) \\ &= (c_0 + c_1 z + \cdots + c_{m-1} z^{m-1})(\alpha_{-n} z^{-n}\, dz + \cdots + \alpha_{-1} z^{-1}\, dz) \\ &= c_0 \alpha_{-n} z^{-n}\, dz + \sum_{j>-n} \beta_j z^j\, dz \notin \mathbf{C}[[z]]/(z^m) \otimes dz, \end{aligned}$$

which is a contradiction. Thus we have $\nu_0^V = \nu_0^W$.

Next assume that $r > 1$. Consider the composite

$$\psi : V_{r-1} \hookrightarrow V \xrightarrow{\varphi} W \longrightarrow W/W_1.$$

There exists an element $c \in \mathbf{C}[[z]]/(z^{r^2 m})$ such that $\psi(e_{r-1}^V) = c e_0^W$ in W/W_1. Then we have

$$\begin{aligned} c \nu_{r-1}^V e_0^W &= (\psi \otimes \mathrm{id})(\nu_{r-1}^V e_{r-1}^V) = (\psi \otimes \mathrm{id}) \circ \nabla^V (e_{r-1}^V) \\ &= \nabla^W \circ \psi(e_{r-1}^V) = \nabla^W (c e_0^W) \\ &= (dc) e_0^W + c \nu_0^W e_0^W, \end{aligned}$$

and so we have

$$dc = c(\nu_{r-1}^V - \nu_0^W).$$

If ψ is an isomorphism, then we have $\nu_{r-1}^V = \nu_0^W$ and the composite

$$V \xrightarrow{\varphi} W \longrightarrow W/W_1 \xrightarrow{\psi^{-1}} V_{r-1}$$

gives a splitting of the exact sequence

$$0 \longrightarrow V_{r-1} \longrightarrow V \longrightarrow V/V_{r-1} \longrightarrow 0.$$

So we have $V = V_{r-1} \oplus V/V_{r-1}$. Similarly we have a splitting $W = W/W_1 \oplus W_1$, and we have an isomorphism $V/V_{r-1} \cong W_1$ which is compatible with the connections. So we obtain an isomorphism $(V/V_{r-1}) \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m}) \xrightarrow{\sim} W_1 \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m})$. By the induction hypotheses, there is a permutation $\sigma \in S_r$ such that $\sigma(r-1) = 0$ and $\nu_i^V = \nu_{\sigma(i)}^W$ for any i.

So assume that ψ is not an isomorphism. Then we can write $c = c_k z^k + c_{k+1} z^{k+1} + \cdots + c_{r^2 m} z^{r^2 m}$ with $c_k \neq 0$ and $k > 0$. If $k \leq (r^2 - 1)m$, we have

$$dc = k c_k z^{k-1}\, dz + (k+1) c_{k+1} z^k\, dz + \cdots + (r^2 m - 1) c_{r^2 m - 1} z^{r^2 m - 2}\, dz \neq 0.$$

So we have $\nu_{r-1}^V - \nu_0^W \neq 0$. Put $n := \max\{j \mid a_{-j}^{(r-1)} - b_{-j}^{(0)} \neq 0\}$. Then we have

$$\begin{aligned} dc = c(\nu_{r-1}^V - \nu_0^W) &= \Big(\sum_{j=k}^{r^2 m - 1} c_j z^j \Big) \sum_{j=-n}^{-1} (a_j^{(r-1)} - b_j^{(0)}) z^j dz \\ &= c_k (a_{-n}^{(r-1)} - b_{-n}^{(0)}) z^{k-n}\, dz + \sum_{j>k-n} \gamma_j z^j. \end{aligned}$$

Thus we have $k - 1 = k - n$ and $k c_k = c_k (a_{-n}^{(r-1)} - b_{-n}^{(0)})$. So $n = 1$ and $a_{-1}^{(r-1)} - b_{-1}^{(0)} = k \geq 1$, which contradicts the assumption that $0 \leq \mathrm{Re}(a_{-1}^{(r-1)}) < 1$, $0 \leq$

$\mathrm{Re}(b^{(0)}_{-1}) < 1$. Hence we have $k \geq (r^2-1)m+1$. Then $\mathrm{Im}\,\psi \in z^{(r^2-1)m+1}(W/W_1)$. So φ induces a morphism

$$V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) \longrightarrow W_1 \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}),$$

which also induces a morphism

$$\psi_1 : V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) \longrightarrow (W_1/W_2) \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}).$$

We define $c^{(1)} \in \mathbf{C}[[z]]/(z^{(r^2-1)m})$ by $\psi_1(e^V_{r-1}) = c^{(1)} e^W_1$. If ψ_1 is isomorphic, then

$$\begin{aligned}(W_1/W_2) \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) &\xrightarrow{\psi_1^{-1}} V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) \\ &\xrightarrow{\varphi} W_1 \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m})\end{aligned}$$

gives a splitting of the exact sequence

$$\begin{aligned}0 \longrightarrow W_2 \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) &\longrightarrow W_1 \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) \\ &\longrightarrow (W_1/W_2) \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) \longrightarrow 0.\end{aligned}$$

So we have

$$\begin{aligned}&W_1 \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m}) \\ &\quad = \big((W_1/W_2) \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m})\big) \oplus \big(W_2 \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m})\big)\end{aligned}$$

and $(\varphi \otimes \mathrm{id})(V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m})) = (W_1/W_2) \otimes \mathbf{C}[[z]]/(z^{(r^2-1)m})$. Then we have $\nu^V_{r-1} = \nu^W_1$ and an isomorphism

$$\begin{aligned}&(V/V_{r-1}) \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m}) \\ &\quad \xrightarrow{\sim} \big(W \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m})\big)/\big((W_1/W_2) \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m})\big).\end{aligned}$$

By the induction hypothesis, there exists a permutation $\sigma \in S_r$ such that $\sigma(r-1) = 1$ and $\nu^V_i = \nu^W_{\sigma(i)}$ for $i \neq r-1$. If ψ_1 is not an isomorphism, then we can see by a similar argument to the above that φ induces a homomorphism

$$\psi_2 : V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-2)m}) \longrightarrow (W_2/W_3) \otimes \mathbf{C}[[z]]/(z^{(r^2-2)m}).$$

We repeat this argument and we finally obtain an isomorphism

$$\psi_j : V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) \xrightarrow{\sim} (W_j/W_{j+1}) \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m})$$

for some j with $0 \leq j \leq r-1$. So there is a slitting

$$\begin{aligned}(W_j/W_{j+1}) \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) &\xrightarrow{\psi_j^{-1}} V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) \\ &\xrightarrow{\varphi} W_j \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m})\end{aligned}$$

of the exact sequence

$$\begin{aligned}0 \longrightarrow W_{j+1} \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) &\longrightarrow W_j \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) \\ &\longrightarrow (W_j/W_{j+1}) \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) \longrightarrow 0.\end{aligned}$$

Therefore we have

$$W_j \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) = \big(W_j/W_{j+1} \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m})\big) \oplus \big(W_{j+1} \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m})\big)$$

and $(\varphi \otimes \mathrm{id})(V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m})) \subset (W_j/W_{j+1}) \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m})$. Since $\varphi \otimes \mathrm{id}$ induces an isomorphism $V_{r-1} \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m}) \xrightarrow{\sim} (W_j/W_{j+1}) \otimes \mathbf{C}[[z]]/(z^{(r^2-j)m})$, it also induces an isomorphism

$$(V/V_{r-1}) \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m}) \xrightarrow{\sim} \big(W \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m})\big)/\big((W_j/W_{j+1}) \otimes \mathbf{C}[[z]]/(z^{(r-1)^2 m})\big).$$

So we have $\nu^V_{r-1} = \nu^W_j$, and by the induction hypothesis there exists a permutation $\sigma \in S_r$ such that $\sigma(r-1) = j$ and $\nu^V_k = \nu^W_{\sigma(k)}$ for any $k \neq r-1$. □

REMARK 1.3

Assume that $l = 1$, and assume that $0 \le \mathrm{Re}(\mathrm{res}(\nu_j)) < 1$ for any j in Proposition 1.1. Then the eigenvalues $\nu_i^{V \otimes \mathbf{C}[z]/(z^{r^2 m})}$ appearing in Proposition 1.2 are nothing but the eigenvalues given in the Hukuhara–Turrittin theorem (Theorem 0.1).

2. Moduli space of unramified irregular singular parabolic connections

Let C be a smooth projective irreducible curve over $\mathbf{C}$ of genus g, and let

$$D = \sum_{i=1}^{n} m_i t_i \quad (m_i > 0,\ t_i \neq t_j \text{ for } i \neq j)$$

be an effective divisor on C. Take a generator z_i of the maximal ideal of $\mathcal{O}_{C,t_i}$. Let E be a vector bundle of rank r on C, and let $\nabla : E \to E \otimes \Omega^1_C(D)$ be a connection. Take a positive integer N_i with $N_i \ge m_i$, and put $N_i t_i := \mathrm{Spec}(\mathcal{O}_{C,t_i}/(z_i^{N_i}))$. Then ∇ induces a morphism

$$\nabla|_{N_i t_i} : E \otimes \mathcal{O}_{C,t_i}/(z_i^{N_i}) \longrightarrow E \otimes \Omega^1_C(D) \otimes \mathcal{O}_{C,t_i}/(z_i^{N_i}).$$

Put

$$N_r^{(n)}(d,D) := \left\{ \boldsymbol{\nu} = (\nu_j^{(i)})^{0 \le j \le r-1}_{1 \le i \le n} \;\middle|\; \begin{array}{l} \nu_j^{(i)} \in \sum_{k=-m_i}^{-1} \mathbf{C} z_i^k dz_i, \quad \text{and} \\ d + \sum_{1 \le i \le n} \sum_{0 \le j \le r-1} \mathrm{res}_{t_i}(\nu_j^{(i)}) = 0 \end{array} \right\}. \tag{2}$$

DEFINITION 2.1

Take $\boldsymbol{\nu} \in N_r^{(n)}(d,D)$. We say that $(E, \nabla, \{l_j^{(i)}\})$ is an unramified irregular singular $\boldsymbol{\nu}$-parabolic connection of parabolic depth $(N_i)_{i=1}^n$ on C if

(1) E is a rank r vector bundle of degree d on C,

(2) $\nabla : E \to E \otimes \Omega_C(D)$ is a connection, and

(3) $E|_{N_i t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ is a filtration by free $\mathcal{O}_{N_i t_i}$-modules such that $l_j^{(i)}/l_{j+1}^{(i)} \cong \mathcal{O}_{N_i t_i}$ for any i, j, $\nabla|_{N_i t_i}(l_j^{(i)}) \subset l_j^{(i)} \otimes \Omega^1_C(D)$ for any

i, j, and for the induced morphism $\overline{\nabla}_j^{(i)} : l_j^{(i)}/l_{j+1}^{(i)} \to l_j^{(i)}/l_{j+1}^{(i)} \otimes \Omega_C^1(D)$, $\mathrm{Im}(\overline{\nabla}_j^{(i)} - \nu_j^{(i)} \mathrm{id}_{l_j^{(i)}/l_{j+1}^{(i)}})$ is contained in the image of $(l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1 \to (l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1(D)$.

We fix a sequence of rational numbers $\boldsymbol{\alpha} = (\alpha_j^{(i)})_{1 \le j \le r}^{1 \le i \le n}$ such that $0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1$ for any i and $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i, j) \neq (i', j')$.

DEFINITION 2.2

A $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ is said to be $\boldsymbol{\alpha}$-stable (resp., $\boldsymbol{\alpha}$-semistable) if for any subbundle $0 \neq F \subsetneq E$ with $\nabla(F) \subset F \otimes \Omega_C^1(D)$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \operatorname{length}((F|_{N_i t_i} \cap l_{j-1}^{(i)})/(F|_{N_i t_i} \cap l_j^{(i)}))}{\operatorname{rank} F} \underset{(\text{resp., } \le)}{<} \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \operatorname{length}(l_{j-1}^{(i)}/l_j^{(i)})}{\operatorname{rank} E}$$

holds.

REMARK 2.1

O. Biquard and P. Boalch [3, Section 8] considered a stability condition for a meromorphic connection with the assumption that the restriction of the connection to each singular point is equivalent to a diagonal one. For a parabolic weight $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ with $0 < \alpha_j^{(i)} < 1/N_i$, the $\boldsymbol{\alpha}$-stability in our definition for a parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ is equivalent to the $(\alpha_j^{(i)} N_i)$-stability in [3] for (E, ∇) under the main assumption in [3].

REMARK 2.2

Take a parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ with parabolic depth (m_i). Fix $l_{j'}^{(i')}$, and put $E' := \ker(E \to E|_{m_{i'} t_{i'}}/l_{j'}^{(i')})$. Then ∇ induces a connection $\nabla' : E' \to E' \otimes \Omega_C^1(D)$. We define a parabolic structure $\{(l')_j^{(i)}\}$ on E' by $(l')_j^{(i)} := l_j^{(i)}$ for $i \neq i'$, $(l')_j^{(i')} := \ker(E'|_{m_{i'} t_{i'}} \to E|_{m_{i'} t_{i'}}/l_{j+j'}^{(i')})$ for $0 \le j \le r - j'$, and $(l')_j^{(i')} := \mathrm{im}(l_{j-r+j'}^{(i')} \otimes \mathcal{O}_C(-m_{i'} t_{i'}) \hookrightarrow E|_{m_{i'} t_{i'}} \otimes \mathcal{O}_C(-m_{i'} t_{i'}) \to E'|_{m_{i'} t_{i'}})$ for $r - j' \le j \le r$. Then we obtain a new parabolic connection $(E', \nabla', \{(l')_j^{(i)}\})$. We call this the elementary transform of $(E, \nabla, \{l_j^{(i)}\})$ along $l_{j'}^{(i')}$. We put $(\alpha')_j^{(i)} := \alpha_j^{(i)}$ for $i \neq i'$, $(\alpha')_j^{(i')} := \alpha_{j+j'}^{(i')}$ for $1 \le j \le r - j'$, and $(\alpha')_j^{(i')} := \alpha_{j-r+j'}^{(i')} + 1$ for $r - j' + 1 \le j \le r$. Then $(E, \nabla, \{l_j^{(i)}\})$ is $\boldsymbol{\alpha}$-stable if and only if $(E', \nabla', \{(l')_j^{(i)}\})$ satisfies the following stability condition: for any subbundle $F' \subset E'$ with $\nabla'(F') \subset F' \otimes \Omega_C^1(D)$,

$$\frac{\deg F' + \sum_{i=1}^n \sum_{j=1}^r (\alpha')_j^{(i)} \operatorname{length}((F'|_{m_i t_i} \cap (l')_{j-1}^{(i)})/(F'|_{m_i t_i} \cap (l')_j^{(i)}))}{\operatorname{rank} F'} < \frac{\deg E' + \sum_{i=1}^n \sum_{j=1}^r (\alpha')_j^{(i)} \operatorname{length}((l')_{j-1}^{(i)}/(l')_j^{(i)})}{\operatorname{rank} E'}$$

holds. So we can consider a stability of a parabolic connection with respect to a weight $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ without the condition $0 < \alpha_j^{(i)} < 1$.

Let S be an algebraic scheme over $\mathbf{C}$, and let $\mathcal{C}$ be a projective flat scheme over S, such that each geometric fiber $\mathcal{C}_s$ of $\mathcal{C}$ over S is a smooth irreducible curve of genus g. Let $\tilde{t}_1, \dots, \tilde{t}_n \subset \mathcal{C}$ be closed subschemes such that the composite $\tilde{t}_i \hookrightarrow \mathcal{C} \to S$ is an isomorphism for any i and that $\tilde{t}_i \cap \tilde{t}_j = \emptyset$ for any $i \neq j$. We put $D := \sum_{i=1}^n m_i \tilde{t}_i$. Then D is an effective Cartier divisor on $\mathcal{C}$ flat over S. Let $\mathcal{N}_r^{(n)}(d, D)$ be the scheme over S such that for any $T \to S$,

$$(3) \qquad \mathcal{N}_r^{(n)}(d, D)(T) = \left\{ \boldsymbol{\nu} = (\nu_j^{(i)}) \;\middle|\; \begin{array}{l} \nu_j^{(i)} \in H^0(T, \Omega^1_{\mathcal{C}}(m_i \tilde{t}_i)_T / (\Omega^1_{\mathcal{C}})_T) \\ d + \sum_{i,j} \mathrm{res}_{\tilde{t}_i}(\nu_j^{(i)}) = 0. \end{array} \right\}.$$

THEOREM 2.1

There exists a relative coarse moduli scheme $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i)) \xrightarrow{\pi} \mathcal{N}_r^{(n)}(d, D)$ of $\boldsymbol{\alpha}$-stable unramified irregular singular $\boldsymbol{\nu}$-parabolic connections ($\boldsymbol{\nu}$ moves around in $\mathcal{N}_r^{(n)}(d, D)$) on $\mathcal{C}$ over S of parabolic depth $(N_i)_{i=1}^n$. Moreover, $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i))$ is quasi-projective over $\mathcal{N}_r^{(n)}(d, D)$.

Proof

Fix a weight $\boldsymbol{\alpha}$ which determines the stability of irregular singular parabolic connections. We take positive integers β_1, β_2, γ and rational numbers $0 < \tilde{\alpha}_1^{(i)} < \tilde{\alpha}_2^{(i)} < \cdots < \tilde{\alpha}_r^{(i)} < 1$ satisfying $(\beta_1 + \beta_2)\alpha_j^{(i)} = \beta_1 \tilde{\alpha}_j^{(i)}$ for any i, j. We assume that $\gamma \gg 0$. We can take an increasing sequence $0 < \alpha'_1 < \alpha'_2 < \cdots < \alpha'_{nr} < 1$ such that $\{\alpha'_p | p = 1, \dots, nr\} = \{\tilde{\alpha}_j^{(i)} | 1 \le i \le n, 1 \le j \le r\}$.

Take any member $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i))(T)$, where $\mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i))$ is the moduli functor of $\boldsymbol{\alpha}$-stable unramified irregular singular parabolic connections of parabolic depth (N_i). We define subsheaves $F_p(E) \subset E$ inductively as follows. First we put $F_1(E) := E$. Inductively we define $F_{p+1}(E) := \ker(F_p(E) \to (E|_{N_i(\tilde{t}_i)_T}) / l_j^{(i)})$, where (i, j) is determined by $\alpha'_p = \alpha_j^{(i)}$. We also put $d_p := \mathrm{length}((E/F_{p+1}(E)) \otimes k(x))$ for $p = 1, \dots, rn$ and $x \in T$. Then $(E, \nabla, \{l_j^{(i)}\}) \mapsto (E, E, \mathrm{id}_E, \nabla, F_*(E))$ determines a morphism

$$\iota : \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\big(r, d, (N_i)\big) \longrightarrow \overline{\mathcal{M}^{D', \boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma}_{\mathcal{C} \times_S \mathcal{N}_r^{(n)}(d,D) / \mathcal{N}_r^{(n)}(d,D)}}\big(r, d, \{d_i\}_{1 \le i \le rn}\big),$$

where $\overline{\mathcal{M}^{D', \boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma}_{\mathcal{C} \times_S \mathcal{N}_r^{(n)}(d,D) / \mathcal{N}_r^{(n)}(d,D)}}(r, d, \{d_i\}_{1 \le i \le rn})$ is the moduli functor of $(\boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma)$-stable parabolic $\Lambda^1_{D'}$-triples whose coarse moduli scheme $\overline{M^{D', \boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma}_{\mathcal{C} \times_S \mathcal{N}_r^{(n)}(d,D) / \mathcal{N}_r^{(n)}(d,D)}}(r, d, \{d_i\}_{1 \le i \le rn})$ exists by [10, Theorem 5.1]. Here we put $D' := \sum_{i=1}^n N_i \tilde{t}_i$. We can check that ι is representable by an immersion. So we can prove in the same way as [10, Theorem 2.1] that a certain locally closed subscheme $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i))$ of $\overline{M^{D', \boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma}_{\mathcal{C} \times_S \mathcal{N}_r^{(n)}(d,D) / \mathcal{N}_r^{(n)}(d,D)}}(r, d, \{d_i\}_{1 \le i \le rn})$ is just the coarse moduli scheme of $\mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i))$. By construction, we can see that $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i))$ represents the étale sheafification of $\mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (N_i))$. $\square$

There is also a coarse moduli scheme $\tilde{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(N_i))$ of $\boldsymbol{\nu}$-parabolic connections $(E,\nabla,\{l^{(i)}_j\})$ of parabolic depth (N_i) such that $(E,\nabla,\{l^{(i)}_j\otimes\mathbf{C}[z_i]/(z_i^{m_i})\})$ is $\boldsymbol{\alpha}$-stable. Indeed we can construct $\tilde{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(N_i))$ as a quasi-projective scheme over $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))$.

THEOREM 2.2

$M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))$ is smooth over $\mathcal{N}^{(n)}_r(d,D)$, and

$$\dim\bigl(M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\bigl(r,d,(m_i)\bigr)_{\boldsymbol{\nu}}\bigr)=2r^2(g-1)+\sum_{i=1}^{n}m_ir(r-1)+2$$

for any $\boldsymbol{\nu}\in\mathcal{N}^{(n)}_r(d,D)$ if $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}$ is not empty.

We will prove Theorem 2.2 in several steps.

We can canonically define a morphism

$$\det : M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\bigl(r,d,(m_i)\bigr)\longrightarrow M_{D/\mathcal{C}/S}\bigl(1,d,(m_i)\bigr)\times_{\mathcal{N}^{(n)}_1(d,D)}\mathcal{N}^{(n)}_r(d,D)$$

by

$$\det\bigl(E,\nabla,\{l^{(i)}_j\}\bigr):=\bigl(\det(E),\det(\nabla),\pi\bigl(E,\nabla,\{l^{(i)}_j\}\bigr)\bigr).$$

Here $M_{D/\mathcal{C}/S}(1,d,(m_i))$ is the moduli space of pairs (L,∇^L) of a line bundle L on $\mathcal{C}_s$ and a connection $\nabla^L : L\to L\otimes\Omega^1_{\mathcal{C}_s}(D_s)$. Note that we put

$$\det(\nabla):=(\nabla\wedge\mathrm{id}\wedge\cdots\wedge\mathrm{id})+(\mathrm{id}\wedge\nabla\wedge\cdots\wedge\mathrm{id})+\cdots+(\mathrm{id}\wedge\cdots\wedge\mathrm{id}\wedge\nabla)$$

and that the morphism $\mathrm{Tr}:\mathcal{N}^{(n)}_r(d,D)\to\mathcal{N}^{(n)}_1(d,D)$ is given by $\mathrm{Tr}((\nu^{(i)}_j))=(\sum_{j=0}^{r-1}\nu^{(i)}_j)_{i=1}^n$.

PROPOSITION 2.1

The morphism

$$\det : M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\bigl(r,d,(m_i)\bigr)\longrightarrow M_{D/\mathcal{C}/S}\bigl(1,d,(m_i)\bigr)\times_{\mathcal{N}^{(n)}_1(d,D)}\mathcal{N}^{(n)}_r(d,D)$$

defined above is smooth.

Proof

We can see by an easy argument that it is sufficient to show that the morphism of moduli functors

$$\det : \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\bigl(r,d,(m_i)\bigr)\longrightarrow M_{D/\mathcal{C}/S}\bigl(1,d,(m_i)\bigr)\times_{\mathcal{N}^{(n)}_1(d,D)}\mathcal{N}^{(n)}_r(d,D)$$

is formally smooth. Let A be an Artinian local ring with maximal ideal m and residue field $k=A/m$. Take an ideal I of A such that $mI=0$. Let

$$\begin{array}{ccc}\mathrm{Spec}(A/I) & \xrightarrow{f} & \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\bigl(r,d,(m_i)\bigr)\\ \downarrow & & \downarrow{\scriptstyle\det}\\ \mathrm{Spec}(A) & \xrightarrow{g} & M_{D/\mathcal{C}/S}\bigl(1,d,(m_i)\bigr)\times_{\mathcal{N}^{(n)}_1(d,D)}\mathcal{N}^{(n)}_r(d,D)\end{array}$$

be a commutative diagram. Here g corresponds to a line bundle L on $\mathcal{C}_A$ with a connection $\nabla^L : L \to L \otimes \Omega^1_{\mathcal{C}_A/A}(D_A)$ and $\boldsymbol{\nu} = (\nu^{(i)}_j) \in \mathcal{N}^{(n)}_r(d, D)(A)$ such that $\nabla^L|_{m_i(\tilde{t}_i)_A}(a) = (\sum_{j=0}^{r-1} \nu^{(i)}_j)a$ for any $a \in L|_{m_i(\tilde{t}_i)_A}$ and $i = 1, \dots, n$. Moreover f corresponds to an element $(E, \nabla, \{l^{(i)}_j\}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (m_i))(A/I)$. Put $(\overline{E}, \overline{\nabla}, \{\bar{l}^{(i)}_j\}) := (E, \nabla, \{l^{(i)}_j\}) \otimes A/m$. We set

$$\begin{aligned}
\mathcal{F}^0_0 &:= \left\{ a \in \mathcal{E}nd(\overline{E}) \,\middle|\, \mathrm{Tr}(a) = 0 \text{ and } a|_{m_i(\tilde{t}_i)_k}(\bar{l}^{(i)}_j) \subset \bar{l}^{(i)}_j \text{ for any } i, j \right\}, \\
\mathcal{F}^1_0 &:= \Big\{ b \in \mathcal{E}nd(\overline{E}) \otimes \Omega^1_{\mathcal{C}/S}(D) \,\Big|\, \mathrm{Tr}(b) = 0 \text{ and} \\
&\qquad b|_{m_i(\tilde{t}_i)_k}(\bar{l}^{(i)}_j) \subset \bar{l}^{(i)}_{j+1} \otimes \Omega^1_{\mathcal{C}/S}(D) \text{ for any } i, j \Big\}, \\
\nabla_{\mathcal{F}^\bullet_0} &: \mathcal{F}^0_0 \ni a \mapsto \overline{\nabla} a - a \overline{\nabla} \in \mathcal{F}^1_0.
\end{aligned}$$

Let $\mathcal{C}_A = \bigcup_\alpha U_\alpha$ be an affine open covering such that $E|_{U_\alpha \otimes A/I} \cong \mathcal{O}^{\oplus r}_{U_\alpha \otimes A/I}$, $\sharp\{(\tilde{t}_i)_A \mid (\tilde{t}_i)_A \in U_\alpha\} \le 1$ for any α and $\sharp\{\alpha \mid (\tilde{t}_i)_A \in U_\alpha\} = 1$ for any $(\tilde{t}_i)_A$. Take a free $\mathcal{O}_{U_\alpha}$-module E_α with isomorphisms $\varphi_\alpha : \det(E_\alpha) \xrightarrow{\sim} L|_{U_\alpha}$ and $\phi_\alpha : E_\alpha \otimes A/I \xrightarrow{\sim} E|_{U_\alpha \otimes A/I}$ such that

$$\varphi_\alpha \otimes A/I = \det(\phi_\alpha) : \det(E_\alpha) \otimes A/I \xrightarrow{\sim} \det(E)|_{U_\alpha \otimes A/I} = (L \otimes A/I)|_{U_\alpha \otimes A/I}.$$

If $(\tilde{t}_i)_A \in U_\alpha$, we may assume that the parabolic structure $\{l^{(i)}_j\}$ is given by

$$l^{(i)}_{r-j} = \langle e_1|_{m_i(\tilde{t}_i)_{A/I}}, \dots, e_j|_{m_i(\tilde{t}_i)_{A/I}} \rangle,$$

where $e_1, \dots, e_r$ is the standard basis of E_α. We define a parabolic structure $\{(l_\alpha)^{(i)}_j\}$ on E_α by

$$(l_\alpha)^{(i)}_{r-j} := \langle e_1|_{m_i(\tilde{t}_i)_A}, \dots, e_j|_{m_i(\tilde{t}_i)_A} \rangle.$$

The connection $\phi_\alpha^{-1} \circ (\nabla|_{U_\alpha}) \circ \phi_\alpha : E_\alpha \otimes A/I \to E_\alpha \otimes \Omega^1_{\mathcal{C}/S}(D) \otimes A/I$ is given by a connection matrix $B_\alpha \in H^0((E_\alpha)^\vee \otimes E_\alpha \otimes \Omega^1_{\mathcal{C}/S}(D) \otimes A/I)$. Then we have

$$B_\alpha|_{(\tilde{t}_i)_{A/I}} = \begin{pmatrix} \nu^{(i)}_{r-1} \otimes A/I & * & \cdots & * \\ 0 & \nu^{(i)}_{r-2} \otimes A/I & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu^{(i)}_0 \otimes A/I \end{pmatrix}.$$

We can take a lift $\tilde{B}_\alpha \in H^0(E^\vee_\alpha \otimes E_\alpha \otimes \Omega^1_{\mathcal{C}/S}(D))$ of B_α such that

$$\tilde{B}_\alpha|_{(\tilde{t}_i)_A} = \begin{pmatrix} \nu^{(i)}_{r-1} & * & \cdots & * \\ 0 & \nu^{(i)}_{r-2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu^{(i)}_0 \end{pmatrix}$$

and such that $\mathrm{Tr}(\tilde{B}_\alpha)(e_1\wedge\cdots\wedge e_r)=(\varphi_\alpha\otimes\mathrm{id})^{-1}(\nabla_L|_{U_\alpha}(\varphi_\alpha(e_1\wedge\cdots\wedge e_r)))$. Consider the connection $\nabla_\alpha : E_\alpha \to E_\alpha\otimes\Omega^1_{\mathcal{C}/S}(D)$ defined by

$$\nabla_\alpha \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \tilde{B}_\alpha \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}.$$

Then we obtain a local parabolic connection $(E_\alpha,\nabla_\alpha,\{(l_\alpha)^{(i)}_j\})$. If $(\tilde{t}_i)_A\notin U_\alpha$ for any i, we can easily define a local parabolic connection $(E_\alpha,\nabla_\alpha,\{(l_\alpha)^{(i)}_j\})$. (In this case the parabolic structure $\{(l_\alpha)^{(i)}_j\}$ is nothing.)

We put $U_{\alpha\beta}:=U_\alpha\cap U_\beta$ and $U_{\alpha\beta\gamma}:=U_\alpha\cap U_\beta\cap U_\gamma$. Take an isomorphism

$$\theta_{\beta\alpha} : E_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} E_\beta|_{U_{\alpha\beta}}$$

such that $\theta_{\beta\alpha}\otimes A/I=\phi_\beta^{-1}\circ\phi_\alpha$ and $\varphi_\beta\circ\det(\theta_{\beta\alpha})=\varphi_\alpha$. We put

$$u_{\alpha\beta\gamma}:=\phi_\alpha\circ(\theta^{-1}_{\gamma\alpha}|_{U_{\alpha\beta\gamma}}\circ\theta_{\gamma\beta}|_{U_{\alpha\beta\gamma}}\circ\theta_{\beta\alpha}|_{U_{\alpha\beta\gamma}}-\mathrm{id}_{E_\alpha|_{U_{\alpha\beta\gamma}}})\circ\phi_\alpha^{-1}$$

and

$$v_{\alpha\beta}:=\phi_\alpha\circ(\nabla_\alpha|_{U_{\alpha\beta}}-\theta^{-1}_{\beta\alpha}\circ\nabla_\beta|_{U_{\alpha\beta}}\circ\theta_{\beta\alpha})\circ\phi_\alpha^{-1}.$$

Then we have $\{u_{\alpha\beta\gamma}\}\in C^2(\{U_\alpha\},\mathcal{F}^0_0\otimes I)$ and $\{v_{\alpha\beta}\}\in C^1(\{U_\alpha\},\mathcal{F}^1_0\otimes I)$. We can easily see that

$$d\{u_{\alpha\beta\gamma}\}=0 \qquad \text{and} \qquad \nabla_{\mathcal{F}^\bullet_0}\{u_{\alpha\beta\gamma}\}=-d\{v_{\alpha\beta}\}.$$

So we can define an element

$$\omega\bigl(E,\nabla,\{l^{(i)}_j\}\bigr):=\bigl[\bigl(\{u_{\alpha\beta\gamma}\},\{v_{\alpha\beta}\}\bigr)\bigr]\in\mathbf{H}^2(\mathcal{F}^\bullet_0)\otimes_k I.$$

Then we can check that $\omega(E,\nabla,\{l^{(i)}_j\})=0$ if and only if $(E,\nabla,\{l^{(i)}_j\})$ can be lifted to an element $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ of $\mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))(A)$ such that

$$\det\bigl(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\}\bigr)=g.$$

From the spectral sequence $H^q(\mathcal{F}^p_0)\Rightarrow\mathbf{H}^{p+q}(\mathcal{F}^\bullet_0)$, there is an isomorphism

$$\mathbf{H}^2(\mathcal{F}^\bullet_0)\cong\mathrm{coker}\bigl(H^1(\mathcal{F}^0_0)\xrightarrow{H^1(\nabla_{\mathcal{F}^\bullet_0})}H^1(\mathcal{F}^1_0)\bigr).$$

Since $(\mathcal{F}^0_0)^\vee\otimes\Omega^1_{\mathcal{C}_k/k}\cong\mathcal{F}^1_0$ and $(\mathcal{F}^1_0)^\vee\otimes\Omega^1_{\mathcal{C}_k/k}\cong\mathcal{F}^0_0$, we have

$$\begin{aligned}
\mathbf{H}^2(\mathcal{F}^\bullet_0)&\cong\mathrm{coker}\bigl(H^1(\mathcal{F}^0_0)\xrightarrow{H^1(\nabla_{\mathcal{F}^\bullet_0})}H^1(\mathcal{F}^1_0)\bigr)\\
&\cong\ker\bigl(H^1(\mathcal{F}^1_0)^\vee\xrightarrow{H^1(\nabla_{\mathcal{F}^\bullet_0})^\vee}H^1(\mathcal{F}^0_0)^\vee\bigr)^\vee\\
&\cong\ker\bigl(H^0\bigl((\mathcal{F}^1_0)^\vee\otimes\Omega^1_{\mathcal{C}_k/k}\bigr)\xrightarrow{-H^0(\nabla_{(\mathcal{F}^\bullet_0)^\vee})}H^0((\mathcal{F}^0_0)^\vee\otimes\Omega^1_{\mathcal{C}_k/k})\bigr)^\vee\\
&\cong\ker\bigl(H^0(\mathcal{F}^0_0)\xrightarrow{-H^0(\nabla_{\mathcal{F}^\bullet_0})}H^0(\mathcal{F}^1_0)\bigr)^\vee.
\end{aligned}$$

Take any element $a \in \ker(H^0(\mathcal{F}_0^0) \xrightarrow{-H^0(\nabla_{\mathcal{F}_0^\bullet})} H^0(\mathcal{F}_0^1))$. Then we have $a \in \mathrm{End}(\overline{E}, \overline{\nabla}, \{\bar{l}_j^{(i)}\})$. Since $(\overline{E}, \overline{\nabla}, \{\bar{l}_j^{(i)}\})$ is $\boldsymbol{\alpha}$-stable, we have $a = c \cdot \mathrm{id}_{\overline{E}}$ for some $c \in \mathbf{C}$. So we have $a = 0$, because $\mathrm{Tr}(a) = 0$. Thus we have $\ker(H^0(\mathcal{F}_0^0) \xrightarrow{-H^0(\nabla_{\mathcal{F}_0^\bullet})} H^0(\mathcal{F}_0^1)) = 0$, and so we have $\mathbf{H}^2(\mathcal{F}_0^\bullet) = 0$. In particular, we have $\omega(E, \nabla, \{l_j^{(i)}\}) = 0$. Thus $(E, \nabla, \{l_j^{(i)}\})$ can be lifted to a member $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (m_i))(A)$ such that $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \otimes A/I \cong (E, \nabla, \{l_j^{(i)}\})$ and $\det(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) = g$. Hence det is a smooth morphism. $\square$

We can see that the moduli space $M_{D/\mathcal{C}/S}(1, d, (m_i))$ is an affine space bundle over $\mathrm{Pic}^d_{\mathcal{C}/S} \times \mathcal{N}_1^{(n)}(d, D)$ with fibers $H^0(\Omega^1_{\mathcal{C}_s})$ $(s \in S)$. So $M_{D/\mathcal{C}/S}(1, d, (m_i))$ is smooth over $\mathcal{N}_1^{(n)}(d, D)$. Combined with Proposition 2.1, we can see that $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (m_i))$ is smooth over $\mathcal{N}_r^{(n)}(d, D)$.

PROPOSITION 2.2

For any $\boldsymbol{\nu} \in \mathcal{N}_r^{(n)}(d, D)$, the fiber $\pi^{-1}(\boldsymbol{\nu}) = M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (m_i))_{\boldsymbol{\nu}}$ is equidimensional of dimension $2r^2(g-1) + 2 + r(r-1)\sum_{i=1}^n m_i$ if it is not empty.

Proof

Since $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (m_i))_{\boldsymbol{\nu}}$ is smooth over $\mathbf{C}$ for any $\boldsymbol{\nu} \in \mathcal{N}_r^{(n)}(d, D)(\mathbf{C})$, it is sufficient to show that the dimension of the tangent space $\Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}}(x)$ of $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (m_i))_{\boldsymbol{\nu}}$ at any point $x = (E, \nabla, \{l_j^{(i)}\}) \in M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d, (m_i))_{\boldsymbol{\nu}}$ is equal to

$$2r^2(g-1) + 2 + r(r-1)\sum_{i=1}^n m_i.$$

We define a complex $\mathcal{F}^\bullet$ on $\mathcal{C}_x$ by

$$\mathcal{F}^0 := \left\{ a \in \mathcal{E}nd(E) \;\middle|\; a|_{m_i(\tilde{t}_i)_x}(l_j^{(i)}) \subset l_j^{(i)} \text{ for any } i, j \right\},$$

$$\mathcal{F}^1 := \left\{ b \in \mathcal{E}nd(E) \otimes \Omega^1_{\mathcal{C}/S}(D) \;\middle|\; b|_{m_i(\tilde{t}_i)_x}(l_j^{(i)}) \subset l_{j+1}^{(i)} \otimes \Omega^1_{\mathcal{C}/S}(D) \text{ for any } i, j \right\},$$

$$\nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 \ni a \mapsto \nabla \circ a - a \circ \nabla \in \mathcal{F}^1.$$

Take a tangent vector $v \in \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}}(x)$. Then v corresponds to a member

$$\left(E^v, \nabla^v, \left\{(l^v)_j^{(i)}\right\}\right) \in M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\big(r, d, (m_i)\big)_{\boldsymbol{\nu}}\big(\mathbf{C}[\epsilon]\big)$$

such that $(E^v, \nabla^v, \{(l^v)_j^{(i)}\}) \otimes \mathbf{C}[\epsilon]/(\epsilon) \cong (E, \nabla, \{l_j^{(i)}\})$, where $\epsilon^2 = 0$. Take an affine open covering $\mathcal{C}_x = \bigcup_\alpha U_\alpha$ such that $E|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$, $\sharp\{i \mid (\tilde{t}_i)_x \in U_\alpha\} \leq 1$ for any α and $\sharp\{\alpha | (\tilde{t}_i)_x \in U_\alpha\} = 1$ for any i. We can take an isomorphism

$$\varphi_\alpha : E^v|_{U_\alpha \times \mathrm{Spec}\,\mathbf{C}[\epsilon]} \xrightarrow{\sim} \big(E \otimes_{\mathbf{C}} \mathbf{C}[\epsilon]\big)|_{U_\alpha \times \mathrm{Spec}\,\mathbf{C}[\epsilon]}$$

such that $\varphi_\alpha \otimes \mathbf{C}[\epsilon]/(\epsilon) : E^v \otimes \mathbf{C}[\epsilon]/(\epsilon)|_{U_\alpha} \xrightarrow{\sim} (E \otimes \mathbf{C}[\epsilon]/(\epsilon))|_{U_\alpha} = E|_{U_\alpha}$ is the given isomorphism. We put

$$u_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} - \mathrm{id}_{(E\otimes\mathbf{C}[\epsilon])|_{U_{\alpha\beta}\times\mathrm{Spec}\,\mathbf{C}[\epsilon]}},$$
$$v_\alpha := (\varphi_\alpha \otimes \mathrm{id}) \circ \nabla^v|_{U_\alpha\times\mathrm{Spec}\,\mathbf{C}[\epsilon]} \circ \varphi_\alpha^{-1} - \nabla \otimes \mathbf{C}[\epsilon]|_{U_\alpha\times\mathrm{Spec}\,\mathbf{C}[\epsilon]}.$$

Then we have $\{u_{\alpha\beta}\} \in C^1(\{U_\alpha\}, (\epsilon)\otimes\mathcal{F}^0)$, $\{v_\alpha\} \in C^0(\{U_\alpha\}, (\epsilon)\otimes\mathcal{F}^1)$, and

$$d\{u_{\alpha\beta}\} = \{u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}\} = 0, \qquad \nabla_{\mathcal{F}^\bullet}\{u_{\alpha\beta}\} = \{v_\beta - v_\alpha\} = d\{v_\alpha\}.$$

So $[(\{u_{\alpha\beta}\}, \{v_\alpha\})]$ determines an element $\sigma_x(v) \in \mathbf{H}^1(\mathcal{F}^\bullet)$. We can easily check that the correspondence $v \mapsto \sigma_x(v)$ gives an isomorphism

$$\Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}}(x) \xrightarrow{\sim} \mathbf{H}^1(\mathcal{F}^\bullet).$$

From the spectral sequence $H^q(\mathcal{F}^p) \Rightarrow \mathbf{H}^{p+q}(\mathcal{F}^\bullet)$, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{C} \longrightarrow H^0(\mathcal{F}^0) \longrightarrow H^0(\mathcal{F}^1) \longrightarrow \mathbf{H}^1(\mathcal{F}^\bullet) \\ \longrightarrow H^1(\mathcal{F}^0) \longrightarrow H^1(\mathcal{F}^1) \longrightarrow \mathbf{C} \longrightarrow 0. \end{aligned}$$

So we have

$$\begin{aligned} \dim \mathbf{H}^1(\mathcal{F}^\bullet) &= \dim H^0(\mathcal{F}^1) + \dim H^1(\mathcal{F}^0) - \dim H^0(\mathcal{F}^0) \\ &\quad - \dim H^1(\mathcal{F}^1) + 2\dim_{\mathbf{C}} \mathbf{C} \\ &= \dim H^0\big((\mathcal{F}^0)^\vee \otimes \Omega^1_{\mathcal{C}_x}\big) + \dim H^1(\mathcal{F}^0) - \dim H^0(\mathcal{F}^0) \\ &\quad - \dim H^1\big((\mathcal{F}^0)^\vee \otimes \Omega^1_{\mathcal{C}_x}\big) + 2 \\ &= \dim H^1(\mathcal{F}^0)^\vee + \dim H^1(\mathcal{F}^0) - \dim H^0(\mathcal{F}^0) - \dim H^0(\mathcal{F}^0)^\vee + 2 \\ &= 2 - 2\chi(\mathcal{F}^0). \end{aligned}$$

Here we used the isomorphisms $\mathcal{F}^1 \cong (\mathcal{F}^0)^\vee \otimes \Omega^1_{\mathcal{C}_x}$, $\mathcal{F}^0 \cong (\mathcal{F}^1)^\vee \otimes \Omega^1_{\mathcal{C}_x}$ and Serre duality. We define a subsheaf $\mathcal{E}_1 \subset \mathcal{E}nd(E)$ by the exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}nd(E) \longrightarrow \bigoplus_{i=1}^n \mathcal{H}om_{\mathcal{O}_{m_i(\tilde{t}_i)_x}}(l_1^{(i)}, l_0^{(i)}/l_1^{(i)}) \longrightarrow 0.$$

Inductively we define a subsheaf $\mathcal{E}_k \subset \mathcal{E}nd(E)$ by the exact sequence

$$0 \longrightarrow \mathcal{E}_k \longrightarrow \mathcal{E}_{k-1} \longrightarrow \bigoplus_{i=1}^n \mathcal{H}om_{\mathcal{O}_{m_i(\tilde{t}_i)_x}}(l_k^{(i)}, l_{k-1}^{(i)}/l_k^{(i)}) \longrightarrow 0.$$

Then we have $\mathcal{E}_{r-1} = \mathcal{F}^0$, and we have

$$\begin{aligned} \chi(\mathcal{F}^0) = \chi(\mathcal{E}_{r-1}) &= \chi\big(\mathcal{E}nd(E)\big) - \sum_{i=1}^n \sum_{j=1}^{r-1} \mathrm{length}\big(\mathcal{H}om_{\mathcal{O}_{m_i(\tilde{t}_i)_x}}(l_j^{(i)}, l_{j-1}^{(i)}/l_j^{(i)})\big) \\ &= r^2(1-g) - \sum_{i=1}^n \sum_{j=1}^{r-1} m_i(r-j) \\ &= r^2(1-g) - r(r-1)\sum_{i=1}^n m_i/2. \end{aligned}$$

Thus we have

$$\dim \mathbf{H}^1(\mathcal{F}^\bullet) = 2 - 2\chi(\mathcal{F}^0) = 2 + 2r^2(g-1) + r(r-1)\sum_{i=1}^{n} m_i,$$

and the statement of the proposition follows. □

Proof of Theorem 2.2

By Proposition 2.1, we can see that $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))$ is smooth over $\mathcal{N}^{(n)}_r(d,D)$, and by Proposition 2.2, every fiber $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}$ over $\boldsymbol{\nu} \in \mathcal{N}^{(n)}_r(d,D)$ is smooth of equidimension $2r^2(g-1)+2+r(r-1)\sum_{i=1}^n m_i$. So we obtain Theorem 2.2. □

PROPOSITION 2.3

Take $\boldsymbol{\nu} = (\nu^{(i)}_j) \in \mathcal{N}^{(n)}_r(d,D)$, and write $\nu^{(i)}_j = \sum_{k=-m_i}^{-1} a^{(i,j)}_k z_i^k dz_i$, where $(C,t_1, \ldots, t_n) := (\mathcal{C}, \tilde{t}_1, \ldots, \tilde{t}_n)_{\boldsymbol{\nu}}$ and z_i is a generator of the maximal ideal of $\mathcal{O}_{C,t_i}$. Assume that $m_i > 1$ and $a^{(i,j)}_{-m_i} \neq a^{(i,j')}_{-m_i}$ for any i and any $j \neq j'$. Then the canonical morphism

$$p : \tilde{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\bigl(r,d,(N_i)\bigr)_{\boldsymbol{\nu}} \longrightarrow M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\bigl(r,d,(m_i)\bigr)_{\boldsymbol{\nu}},$$
$$\bigl(E,\nabla,\{l^{(i)}_j\}\bigr) \mapsto \bigl(E,\nabla,\bigl\{l^{(i)}_j \otimes \mathcal{O}_{C,t_i}/(z_i^{m_i})\bigr\}\bigr)$$

is an isomorphism.

Proof

Take any member $(E,\nabla,\{l^{(i)}_j\}) \in M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}(\mathbf{C})$. We can see the following claim by the Hukuhara–Turrittin theorem (see [27, Theorem 6.1.1]).

CLAIM

We have $(E,\nabla) \otimes \mathbf{C}[[z_i]] \cong V(\nu^{(i)}_{r-1},1) \oplus \cdots \oplus V(\nu^{(i)}_0,1)$.

By the above claim, we have $l^{(i)}_j = V(\nu^{(i)}_{r-1},1)|_{m_i t_i} \oplus \cdots \oplus V(\nu^{(i)}_j,1)|_{m_i t_i}$. If we set $\tilde{l}^{(i)}_j := V(\nu^{(i)}_{r-1},1)|_{N_i t_i} \oplus \cdots \oplus V(\nu^{(i)}_j,1)|_{N_i t_i}$, then $(E,\nabla,\{\tilde{l}^{(i)}_j\}) \in M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d,(N_i))_{\boldsymbol{\nu}}$ and $p(E,\nabla,\{\tilde{l}^{(i)}_j\}) = (E,\nabla,\{l^{(i)}_j\})$. Thus p is surjective.

Take any member $(E,\nabla,\{\bar{l}^{(i)}_j\}) \in M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}(U)$ and two members $(E,\nabla,\{l^{(i)}_j\})$, $(E,\nabla,\{(l')^{(i)}_j\}) \in p^{-1}(E,\nabla,\{\bar{l}^{(i)}_j\})$, where U is a scheme over $\mathbf{C}$. Take any point $x \in U$ and a local section $e'_{r-1} \in ((l')^{(i)}_{r-1})_x$ such that $(\mathcal{O}_{N_i t_i} \otimes \mathcal{O}_{U,x})e'_{r-1} = ((l')^{(i)}_{r-1})_x$, and $\nabla(e'_{r-1}) = \nu^{(i)}_{r-1} e'_{r-1}$. Let c_1 be the image of e'_{r-1} by the homomorphism

$$\pi_1 : E|_{N_i t_i \times U} \longrightarrow E|_{N_i t_i \times U}/l^{(i)}_1 \cong \mathcal{O}_{N_i t_i \times U}.$$

Then we have

$$c_1 \nu^{(i)}_{r-1} = \pi_1(\nu^{(i)}_{r-1} e'_{r-1}) = \pi_1 \nabla(e'_{r-1}) = \nabla \pi_1(e'_{r-1}) = \nabla(c_1) = dc_1 + c_1 \nu^{(i)}_1.$$

So we have

$$c_1(\nu^{(i)}_{r-1}-\nu^{(i)}_1)=dc_1.$$

Since

$$\nu^{(i)}_{r-1}-\nu^{(i)}_1=(a^{(i,r-1)}_{-m_i}-a^{(i,1)}_{-m_i})z_i^{-m_i}\,dz_i+(a^{(i,r-1)}_{-m_i+1}-a^{(i,1)}_{-m_i+1})z_i^{-m_i+1}\,dz_i+\cdots$$

and $a^{(i,r-1)}_{-m_i}-a^{(i,1)}_{-m_i}\in\mathbf{C}\setminus\{0\}$, we have $c_1=0$. Similarly, the projection of e'_{r-1} to $E|_{N_it_i\times U}/l^{(i)}_j$ is zero for $j=1,\dots,r-1$. So we have $e'_{r-1}\in(l^{(i)}_{r-1})_x$ and so $(l')^{(i)}_{r-1}\subset l^{(i)}_{r-1}$. Similarly we have $l^{(i)}_{r-1}\subset(l')^{(i)}_{r-1}$ and $l^{(i)}_{r-1}=(l')^{(i)}_{r-1}$. By induction on r, we have $l^{(i)}_j=(l')^{(i)}_j$ for $j=1,\dots,r-1$. So we have $(E,\nabla,\{l^{(i)}_j\})=(E,\nabla,\{(l')^{(i)}_j\})$. Thus p is a monomorphism.

Finally we will show that p is smooth. Let A be an Artinian local ring with maximal ideal m, and let I be an ideal of A such that $mI=0$. Assume that a commutative diagram

$$\begin{array}{ccc}
\mathrm{Spec}(A/I) & \xrightarrow{\ f\ } & \tilde{\mathcal{M}}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\big(r,d,(N_i)\big)_{\boldsymbol{\nu}} \\
\Big\downarrow & & \Big\downarrow p \\
\mathrm{Spec}(A) & \xrightarrow{\ g\ } & \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\big(r,d,(m_i)\big)
\end{array}$$

is given. Then g corresponds to a member $(E,\nabla,\{l^{(i)}_j\})\in\mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}(A)$, and f corresponds to a member $(E\otimes A/I,\nabla\otimes A/I,\{\bar{l}^{(i)}_j\})\in\tilde{\mathcal{M}}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(N_i))_{\boldsymbol{\nu}}(A/I)$. Note that $l^{(i)}_j=\bigoplus_{k=j}^{r-1}\ker(\nabla|_{m_it_i}-\nu^{(i)}_k)$ and that $\bar{l}^{(i)}_j=\bigoplus_{k=j}^{r-1}\ker((\nabla\otimes A/I)|_{N_it_i}-\nu^{(i)}_k)$. We can easily check that a canonical homomorphism $\ker(\nabla|_{N_it_i}-\nu^{(i)}_j)\to\ker(\nabla|_{m_it_i}-\nu^{(i)}_j)$ is surjective. So the canonical homomorphism $\varphi:\bigoplus_{j=0}^{r-1}\ker(\nabla|_{N_it_i}-\nu^{(i)}_j)\to E|_{N_it_i}$ is surjective by Nakayama's lemma. We can easily check that φ is also injective. If we put $\tilde{l}^{(i)}_j:=\bigoplus_{k=j}^{r-1}\ker(\nabla|_{N_it_i}-\nu^{(i)}_k)$, then $(E,\nabla,\{\tilde{l}^{(i)}_j\})\in\tilde{\mathcal{M}}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(N_i))(A)$, $p(E,\nabla,\{\tilde{l}^{(i)}_j\})=(E,\nabla,\{l^{(i)}_j\})$ and $(E,\nabla,\{\tilde{l}^{(i)}_j\})\otimes A/I=(E\otimes A/I,\nabla\otimes A/I,\{\bar{l}^{(i)}_j\})$. Thus p is a smooth morphism.

By the above proof, p becomes bijective and étale. Hence p is an isomorphism. $\square$

REMARK 2.3

In general the moduli space $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(N_i))$ is not smooth over $\mathcal{N}^{(n)}_r(d,D)$ if $N_i>m_i$. For example, assume that $m_i>1$ for any i and $g\ge 1$. Then a general fiber $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(N_i))_{\boldsymbol{\nu}}$ over $\boldsymbol{\nu}\in\mathcal{N}^{(n)}_r(d,D)$ is smooth of dimension $2r^2(g-1)+2+r(r-1)\sum_{i=1}^n m_i$ by Proposition 2.3 and Theorem 2.2. Take $x\in S$, and put $C:=\mathcal{C}_x$, $t_i:=(\tilde{t}_i)_x$ and $E:=\mathcal{O}_C(-t_1)\oplus\mathcal{O}_C$. Take a nonzero section $\omega\in H^0(C,\Omega^1_C((m_1+1)t_1))$, and consider the connection

$$\nabla:E\longrightarrow E\otimes\Omega^1_C\Big(\sum_{i=1}^n m_it_i\Big),$$

$$\nabla \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} df_1 \\ df_2 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \bigl(f_1 \in \mathcal{O}_C(-t_1), f_2 \in \mathcal{O}_C\bigr).$$

Then there is a canonical extension $0 \to \mathcal{O}_C(-t_1) \to E \to \mathcal{O}_C \to 0$ which is compatible with the connections. We take the parabolic structure $l_1^{(i)} := \mathcal{O}_C(-t_1)|_{N_i t_i}$. If we take $\alpha_j^{(i)} \ll 1$ for any i, j, $(E, \nabla, \{l_j^{(i)}\})$ becomes $\boldsymbol{\alpha}$-stable. We define a complex $\mathcal{F}^\bullet$ as follows:

$$\mathcal{F}^0 := \bigl\{ a \in \mathcal{E}nd(E) \bigm| a|_{N_i t_i}(l_j^{(i)}) \subset l_j^{(i)} \text{ for any } i, j \bigr\},$$

$$\mathcal{F}^1 := \left\{ b \in \mathcal{E}nd(E) \otimes \Omega_C^1\Bigl(\sum_{i=1}^n m_i t_i\Bigr) \;\middle|\; \begin{array}{l} b|_{N_i t_i}(l_j^{(i)}) \subset l_j^{(i)} \otimes \Omega_C^1(\sum_{i=1}^n m_i t_i) \\ \text{for any } i, j, \\ \overline{b_j^{(i)}}(l_j^{(i)}/l_{j+1}^{(i)}) \text{ is contained in the image of} \\ (l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1 \to (l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1(m_i t_i) \\ \text{for any } i, j \end{array} \right\},$$

$$\nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 \ni a \mapsto \nabla a - a \nabla \in \mathcal{F}^1,$$

where $\overline{b_j^{(i)}} : l_j^{(i)}/l_{j+1}^{(i)} \to (l_j^{(i)}/l_{j+1}^{(i)}) \otimes \Omega_C^1(m_i t_i)$ is the homomorphism induced by $b|_{N_i t_i}$. We can see that the relative tangent space $\Theta_{M^{\boldsymbol{\alpha}}_{D/C/S}(2,-1,(m_i))/\mathcal{N}_r^{(n)}(d,D)} \otimes k(x)$ at the point $x = (E, \nabla, \{l_j^{(i)}\})$ is isomorphic to $\mathbf{H}^1(\mathcal{F}^\bullet)$. From the spectral sequence $H^q(\mathcal{F}^p) \Rightarrow \mathbf{H}^{p+q}(\mathcal{F}^\bullet)$, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{C} \longrightarrow H^0(\mathcal{F}^0) \longrightarrow H^0(\mathcal{F}^1) \longrightarrow \mathbf{H}^1(\mathcal{F}^\bullet) \\ \longrightarrow H^1(\mathcal{F}^0) \longrightarrow H^1(\mathcal{F}^1) \longrightarrow \mathbf{H}^2(\mathcal{F}^\bullet) \longrightarrow 0. \end{aligned}$$

So we have

$$\begin{aligned} \dim \mathbf{H}^1(\mathcal{F}^\bullet) &= \dim H^0(\mathcal{F}^1) + \dim H^1(\mathcal{F}^0) - \dim H^0(\mathcal{F}^0) \\ &\quad - \dim H^1(\mathcal{F}^1) + 1 + \dim \mathbf{H}^2(\mathcal{F}^\bullet) \\ &= \chi(\mathcal{F}^1) - \chi(\mathcal{F}^0) + 1 + \dim \mathbf{H}^2(\mathcal{F}^\bullet) \\ &= \Bigl(2^2(1-g) + 2^2\Bigl(2g - 2 + \sum_{i=1}^n m_i\Bigr) - \sum_{i=1}^n (N_i + 2m_i)\Bigr) \\ &\quad - \Bigl(2^2(1-g) - \sum_{i=1}^n N_i\Bigr) + 1 + \dim \mathbf{H}^2(\mathcal{F}^\bullet) \\ &= 8(g-1) + 2 + 2\sum_{i=1}^n m_i + \bigl(\dim \mathbf{H}^2(\mathcal{F}^\bullet) - 1\bigr). \end{aligned}$$

If we put

$$(\mathcal{F}')^0 := \left\{ a \in \mathcal{E}nd(E) \otimes \mathcal{O}_C\Bigl(\sum_{i=1}^n (N_i - m_i) t_i\Bigr) \;\middle|\; \begin{array}{l} a|_{N_i t_i}(l_j^{(i)}) \subset l_j^{(i)} \otimes \mathcal{O}_C((N_i - m_i)t_i) \\ \text{for any } i, j \text{ and } \overline{a_j^{(i)}}(l_j^{(i)}/l_{j+1}^{(i)}) \\ \text{is contained in the image of} \\ (l_j^{(i)}/l_{j+1}^{(i)}) \to (l_j^{(i)}/l_{j+1}^{(i)}) \\ \otimes \mathcal{O}_C((N_i - m_i)t_i) \text{ for any } i, j \end{array} \right\},$$

$$(\mathcal{F}')^1 := \left\{ b \in \mathcal{E}nd(E) \otimes \Omega^1_C\Big(\sum_{i=1}^n N_i t_i\Big) \;\Big|\; b|_{N_i t_i}(l^{(i)}_j) \subset l^{(i)}_{j+1} \otimes \Omega^1_C(N_i t_i) \text{ for any } i,j \right\},$$

$$\nabla_{(\mathcal{F}')^\bullet} : (\mathcal{F}')^0 \ni a \mapsto \nabla a - a\nabla \in (\mathcal{F}')^1,$$

then we have $(\mathcal{F}^1)^\vee \otimes \Omega^1_C \cong (\mathcal{F}')^0$ and $(\mathcal{F}^0)^\vee \otimes \Omega^1_C \cong (\mathcal{F}')^1$. We have

$$\begin{aligned}
\mathbf{H}^2(\mathcal{F}^\bullet) &\cong \operatorname{coker}\big(H^1(\mathcal{F}^0) \xrightarrow{H^1(\nabla_{\mathcal{F}^\bullet})} H^1(\mathcal{F}^1)\big) \\
&\cong \ker\big(H^1(\mathcal{F}^1)^\vee \xrightarrow{H^1(\nabla_{\mathcal{F}^\bullet})^\vee} H^1(\mathcal{F}^0)^\vee\big)^\vee \\
&\cong \ker\big(H^0\big((\mathcal{F}^1)^\vee \otimes \Omega^1_C\big) \xrightarrow{H^1(\nabla_{\mathcal{F}^\bullet})^\vee} H^0\big((\mathcal{F}^0)^\vee \otimes \Omega^1_C\big)\big)^\vee \\
&\cong \ker\big(H^0\big((\mathcal{F}')^0\big) \xrightarrow{-H^0(\nabla_{(\mathcal{F}')^\bullet})} H^0\big((\mathcal{F}')^1\big)\big)^\vee .
\end{aligned}$$

Note that $\mathbf{C} \cdot \mathrm{id}_E \subset \ker(H^0((\mathcal{F}')^0) \xrightarrow{-H^0(\nabla_{(\mathcal{F}')^\bullet})} H^0((\mathcal{F}')^1))$. Let $f : E \to E(t_1)$ be the composite

$$f : E \longrightarrow \mathcal{O}_C \hookrightarrow E(t_1).$$

Since f is compatible with the connections, we have

$$f \in \ker\big(H^0\big((\mathcal{F}')^0\big) \xrightarrow{-H^0(\nabla_{(\mathcal{F}')^\bullet})} H^0\big((\mathcal{F}')^1\big)\big).$$

Note that $0 \neq f \notin \mathbf{C} \cdot \mathrm{id}_E$. So we have $\dim \mathbf{H}^2(\mathcal{F}^\bullet) \geq 2$, which means that $\dim \mathbf{H}^1(\mathcal{F}^\bullet) \geq 8(g-1) + 3 + 2\sum_{i=1}^n m_i$. So $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(2, -1, (N_i))$ is not smooth over $\mathcal{N}^{(n)}_r(d, D)$ at x.

3. Smoothness of the family of the moduli spaces over configuration space

Take any point $x \in S$. If we put $t_i := (\tilde{t}_i)_x$, we have $D_x = \sum_{i=1}^n m_i t_i$. Consider the Hilbert scheme $H_i := \mathrm{Hilb}^{m_i}_{\mathcal{C}_x}$. Put $H := \mathrm{Hilb}^{m_1}_{\mathcal{C}_x} \times \cdots \times \mathrm{Hilb}^{m_n}_{\mathcal{C}_x}$, and let $D_i \subset \mathcal{C}_x \times H$ be the universal divisors for $i = 1, \ldots, n$. Note that H is smooth over $\mathbf{C}$. Let $H' \subset H$ be the open subscheme such that $H' = \{h \in H \mid (D_i)_h \cap (D_j)_h = \emptyset \text{ for } i \neq j\}$. Consider the affine space bundle

$$\mathcal{N} := \prod_{i=1}^n \mathbf{V}_*\big((\pi_i)_*\big(\Omega^1_{\mathcal{C}_x \times H'}\big((D_i)_{H'}\big)|_{(D_i)_{H_i}}\big)\big)$$

over H', where $\pi_i : (D_i)_{H'} \to H'$ is the projection. Take the universal family $(\tilde{\nu}^{(i)}_j)$, where $\tilde{\nu}^{(i)}_j \in H^0((D_i)_{\mathcal{N}}, \Omega^1_{\mathcal{C}_x}((D_i)_{\mathcal{N}})|_{(D_i)_{\mathcal{N}}})$.

Assume that $\boldsymbol{\nu} = (\nu^{(i)}_j) \in \mathcal{N}$ is given. Let $h \in H'$ be the corresponding point, and write $(D_i)_h = \sum_k m'_k t'_k$ with $t'_k \neq t'_j$ for $k \neq j$ and $\nu^{(i)}_j = \sum_k \nu'_k$ with $\nu'_k \in H^0(m'_k t'_k, \Omega^1_{\mathcal{C}_x}((D_i)_h)|_{m'_k t'_k})$. Then we define $f^{(i)}_j(\boldsymbol{\nu}) = \sum_k \mathrm{res}_{t'_k}(\nu'_k)$.

Though it is obvious that the function $f^{(i)}_j$ defined above is an algebraic function on $\mathcal{N}$, we give a proof by way of precaution. We can take a disk $\Delta_k \subset \mathcal{C}_x$ containing t'_k such that $\overline{\Delta_k} \cap \overline{\Delta_{k'}} = \emptyset$ for $k \neq k'$. Taking a sufficiently small analytic open neighborhood U of h in H', we can write $(D_i)_U = \sum_k D'_k$ with D'_k an effective Cartier divisor on $\mathcal{C}_x \times U$ flat over U, $(D'_k)_h = m'_k t'_k$ and $(D'_k)_g \subset \Delta_k$ for any $g \in U$. Then we can write $(\tilde{\nu}^{(i)}_j)_{\mathcal{N}_U} = \sum_k \tilde{\nu}'_k$ with $\tilde{\nu}'_k \in H^0((D'_k)_{\mathcal{N}_U}, \Omega^1_{\mathcal{C}_x}(D'_k) \otimes \mathcal{O}_{(D'_k)_{\mathcal{N}_U}})$. By shrinking U if necessary, we can take an open subset $W'_k \subset \mathcal{C}_x \times \mathcal{N}_U$ such that $\overline{\Delta_k} \times \mathcal{N}_U \subset W'_k$ and a section $\tilde{\omega}'_k \in H^0(W'_k, \Omega^1_{\mathcal{C}_x \times \mathcal{N}_U/\mathcal{N}_U}(D'_k)|_{W'_k})$ such that $\tilde{\omega}'_k|_{D'_k \times_U \mathcal{N}_U} = \tilde{\nu}'_k$. Then we have

$$(f^{(i)}_j)_U = \frac{1}{2\pi\sqrt{-1}} \sum_k \int_{\partial \Delta_k} \tilde{\omega}'_k.$$

So $(f^{(i)}_j)_U$ becomes a holomorphic function on $\mathcal{N}_U$. We can glue $(f^{(i)}_j)_U$ and obtain a holomorphic function $f^{(i)}_j$ on $\mathcal{N}$. Note that $f^{(i)}_j|_{\mathcal{N}_{H^\circ}}$ is an algebraic function on $\mathcal{N}_{H^\circ}$ by its definition, where H° is the Zariski-open subset of H' defined by

$$H^\circ := \big\{ h \in H' | (D_i)_h \text{ has no multiple component} \big\}.$$

So $f^{(i)}_j$ is a rational function on $\mathcal{N}$, which is holomorphic on $\mathcal{N}$, and hence $f^{(i)}_j$ becomes an algebraic function on $\mathcal{N}$.

We define

$$\mathcal{N}^{(n)}_r\big(d, (D_i)\big) := \Big\{ \boldsymbol{\nu} \in \mathcal{N} \;\Big|\; d + \sum_{i=1}^{n} \sum_{j=0}^{r-1} f^{(i)}_j(\boldsymbol{\nu}) = 0 \Big\}.$$

Then we can easily see that $\mathcal{N}^{(n)}_r(d,(D_i))$ is smooth over H'. We put $\tilde{D} := \sum_{i=1}^n (D_i)_{\mathcal{N}^{(n)}_r(d,(D_i))}$ and define a moduli functor $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i)) : (\mathrm{Sch}/\mathcal{N}^{(n)}_r(d, (D_i))) \to (\mathrm{Sets})$ by

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}\big(r,d,(D_i)\big)(T)$$
$$:= \left\{ \big(E, \nabla, \{l^{(i)}_j\}\big) \;\middle|\; \begin{array}{l} E \text{ is a vector bundle on } \mathcal{C}_x \times T \text{ of rank } r, \\ \nabla : E \to E \otimes \Omega^1_{\mathcal{C}_x \times T/T}(\tilde{D}_T) \text{ is a relative connection}, \\ E|_{(D_i)_T} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_{r-1} \supset l^{(i)}_r = 0 \text{ is a filtration} \\ \text{such that for any } i,j,\ l^{(i)}_j / l^{(i)}_{j+1} \text{ is a line bundle on } (D_i)_T, \\ (\nabla|_{(D_i)_T} - (\tilde{\nu}^{(i)}_j)_T \mathrm{id}_{E|_{(D_i)_T}})(l^{(i)}_j) \subset l^{(i)}_{j+1} \otimes \Omega^1_{\mathcal{C}_x \times T/T}(\tilde{D}_T), \\ (E, \nabla, \{l^{(i)}_j\}) \otimes k(y) \text{ satisfies the } \boldsymbol{\alpha}\text{-stability } (\dagger) \text{ below} \\ \text{for any geometric point } y \text{ of } T, \end{array} \right\} \Big/ \sim,$$

where T is a locally Noetherian scheme over $\mathcal{N}^{(n)}_r(d,(D_i))$ and $(E, \nabla, \{l^{(i)}_j\}) \sim (E', \nabla', \{(l')^{(i)}_j\})$ if there is a line bundle $\mathcal{L}$ on T such that $(E, \nabla, \{l^{(i)}_j\}) \cong (E', \nabla', \{(l')^{(i)}_j\}) \otimes \mathcal{L}$. We have that $(E, \nabla, \{l^{(i)}_j\}) \otimes k(y)$ is $\boldsymbol{\alpha}$-stable if

for any subbundle $0 \neq F \subsetneq E \otimes k(y)$ with $(\nabla \otimes k(y))(F) \subset F \otimes \Omega^1_{\mathcal{C}_y}(\tilde{D}_y)$,

$$(\dagger) \quad \frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha^{(i)}_j \operatorname{length}((F|_{(D_i)_y} \cap (l^{(i)}_{j-1} \otimes k(y)))/(F|_{(D_i)_y} \cap (l^{(i)}_j \otimes k(y))))}{\operatorname{rank} F}$$
$$< \frac{\deg(E \otimes k(y)) + \sum_{i=1}^n \sum_{j=1}^r \alpha^{(i)}_j \operatorname{length}((l^{(i)}_{j-1} \otimes k(y))/(l^{(i)}_j \otimes k(y)))}{\operatorname{rank} E}.$$

THEOREM 3.1

There exists a relative coarse moduli scheme

$$\pi : M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}\big(r,d,(D_i)\big) \longrightarrow \mathcal{N}^{(n)}_r\big(d,(D_i)\big)$$

of $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$. *Moreover,* π *is a smooth morphism.*

Proof

We can see by the same argument as in Theorem 2.1 that there exists a relative coarse moduli scheme

$$\pi : M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}\big(r,d,(D_i)\big) \longrightarrow \mathcal{N}^{(n)}_r\big(d,(D_i)\big)$$

of $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$. More precisely, $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$ represents the étale sheafification of $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$. We can define a morphism

$$\det : M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}\big(r,d,(D_i)\big) \longrightarrow M_{\mathcal{C}_x}\big(1,d,(D_i)\big) \times_{\mathcal{N}^{(n)}_1(d,(D_i))} \mathcal{N}^{(n)}_r\big(d,(D_i)\big),$$
$$\big(E,\nabla,\{l^{(i)}_j\}\big) \mapsto \Big(\big(\det(E),\det(\nabla)\big),\pi\big(E,\nabla,\{l^{(i)}_j\}\big)\Big).$$

Here $M_{\mathcal{C}_x}(1,d,(D_i))$ is the moduli space of line bundles with a connection. We can construct $M_{\mathcal{C}_x}(1,d,(D_i))$ as an affine space bundle over $\mathrm{Pic}^d_{\mathcal{C}_x} \times \mathcal{N}^{(n)}_1(d,(D_i))$ whose fiber is isomorphic to $H^0(\Omega^1_{\mathcal{C}_x})$. So $M_{\mathcal{C}_x}(1,d,(D_i))$ is smooth over $\mathcal{N}^{(n)}_1(d,(D_i))$. Let A be an Artinian local ring with maximal ideal m and residue field $k=A/m$. Assume that an ideal I of A such that $mI=0$ and a commutative diagram

$$\begin{array}{ccc}
\mathrm{Spec}(A/I) & \xrightarrow{f} & \mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}\big(r,d,(D_i)\big) \\
\Big\downarrow & & \Big\downarrow \det \\
\mathrm{Spec}(A) & \xrightarrow{g} & M_{\mathcal{C}_x}\big(1,d,(D_i)\big) \times_{\mathcal{N}^{(n)}_1(d,(D_i))} \mathcal{N}^{(n)}_r\big(d,(D_i)\big)
\end{array}$$

are given. Here f corresponds to an A/I-valued point $(E,\nabla,\{l^{(i)}_j\}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))(A/I)$. Put $(\overline{E},\overline{\nabla},\{\bar{l}^{(i)}_j\}) := (E,\nabla,\{l^{(i)}_j\}) \otimes A/m$. Set

$$\mathcal{F}^0 := \big\{a \in \mathcal{E}nd(\overline{E}) \;\big|\; \mathrm{Tr}(a)=0 \text{ and } a|_{(D_i)_k}(\bar{l}^{(i)}_j) \subset \bar{l}^{(i)}_j \text{ for any } i,j\big\},$$
$$\mathcal{F}^1 := \big\{b \in \mathcal{E}nd(\overline{E}) \otimes \Omega^1_{(\mathcal{C}_x)_k}(\tilde{D}_k) \;\big|\; \mathrm{Tr}(b)=0 \text{ and}$$
$$b|_{(D_i)_k}(\bar{l}^{(i)}_j) \subset \bar{l}^{(i)}_{j+1} \otimes \Omega^1_{(\mathcal{C}_x)_k}(\tilde{D}_k) \text{ for any } i,j\big\},$$
$$\nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 \ni a \mapsto \overline{\nabla}a - a\overline{\nabla} \in \mathcal{F}^1.$$

Then we can see by the same argument as that of Proposition 2.1 that there is an obstruction class $\omega(E,\nabla,\{l^{(i)}_j\}) \in \mathbf{H}^2(\mathcal{F}^\bullet)\otimes I$ such that $\omega(E,\nabla,\{l^{(i)}_j\})=0$ if and only if $(E,\nabla,\{l^{(i)}_j\})$ can be lifted to an A-valued point $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))(A)$ such that $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\}) \otimes A/I \cong (E,\nabla,\{l^{(i)}_j\})$ and $\det(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\}) = g$. Since $(\overline{E},\overline{\nabla},\{\bar{l}^{(i)}_j\})$ is $\boldsymbol{\alpha}$-stable, we can see by the proof of Proposition 2.1 that $\mathbf{H}^2(\mathcal{F}^\bullet)=0$. So det is a smooth morphism. Since $M_{\mathcal{C}_x}(1,d,(D_i))$

is smooth over $\mathcal{N}_1^{(n)}(d,(D_i))$, we can see that $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$ is smooth over $\mathcal{N}_r^{(n)}(d,(D_i))$. $\square$

4. Relative symplectic form on the moduli space

THEOREM 4.1

There exists a relative symplectic form

$$\omega \in H^0\big(M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}\big(r,d,(m_i)\big),\Omega^2_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}_r^{(n)}(d,D)}\big).$$

We prove Theorem 4.1 in several steps.

PROPOSITION 4.1

There exists a skew-symmetric nondegenerate pairing

$$\omega : \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}_r^{(n)}(d,D)} \times \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}_r^{(n)}(d,D)} \longrightarrow \mathcal{O}_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))}.$$

Proof

There are an affine scheme U and an étale surjective morphism $p : U \to M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r, d,(m_i))$ which factors through $\mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))$; namely, there is a universal family $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ on $\mathcal{C}\times_S U$. We define a complex $\mathcal{F}^\bullet$ on $\mathcal{C}\times_S U$ by

$$\mathcal{F}^0 := \big\{a \in \mathcal{E}nd(\tilde{E}) \,\big|\, a|_{m_i(\tilde{t}_i)_U}(\tilde{l}^{(i)}_j) \subset \tilde{l}^{(i)}_j \text{for any } i,j\big\},$$
$$\mathcal{F}^1 := \big\{b \in \mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathcal{C}/S}(D) \,\big|\, b|_{m_i(\tilde{t}_i)_U}(\tilde{l}^{(i)}_j) \subset \tilde{l}^{(i)}_{j+1}\otimes\Omega^1_{\mathcal{C}/S}(D) \text{ for any } i,j\big\},$$
$$\nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 \ni a \mapsto \tilde{\nabla}\circ a - a\circ\tilde{\nabla} \in \mathcal{F}^1.$$

Let $\pi_U : \mathcal{C}\times_S U \to U$ be the projection. Then we have

$$\Theta_{U/\mathcal{N}_r^{(n)}(d,D)} \cong p^*(\Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}_r^{(n)}(d,D)}) \cong \mathbf{R}^1(\pi_U)_*(\mathcal{F}^\bullet).$$

Take an affine open covering $\mathcal{C}\times_S U = \bigcup_\alpha U_\alpha$ and a member $v \in H^0(U, \mathbf{R}^1(\pi_U)_*(\mathcal{F}^\bullet)) = \mathbf{H}^1(\mathcal{C}\times_S U,\mathcal{F}^\bullet_U)$; v is given by $[(\{u_{\alpha\beta}\},\{v_\alpha\})]$, where $\{u_{\alpha\beta}\} \in C^1(\{U_\alpha\},\mathcal{F}^0_U)$, $\{v_\alpha\} \in C^0(\{U_\alpha\},\mathcal{F}^1_U)$, and

$$d\{u_{\alpha\beta}\} = \{u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}\} = 0, \qquad \nabla_{\mathcal{F}^\bullet}\big(\{u_{\alpha\beta}\}\big) = \{v_\beta - v_\alpha\} = d\{v_\alpha\}.$$

We define a pairing

$$\begin{aligned}\omega_U : \mathbf{H}^1(\mathcal{C}\times_S U,\mathcal{F}^\bullet) &\times \mathbf{H}^1(\mathcal{C}\times_S U,\mathcal{F}^\bullet)\\ &\longrightarrow \mathbf{H}^2(\mathcal{C}\times_S U,\Omega^\bullet_{\mathcal{C}\times_S U/U}) \cong H^0(U,\mathcal{O}_U)\end{aligned}$$

by

$$\begin{aligned}&\omega_U\big(\big[(\{u_{\alpha\beta}\},\{v_\alpha\})\big],\big[(\{u'_{\alpha\beta}\},\{v'_\alpha\})\big]\big)\\ &\quad := \big[\big\{\mathrm{Tr}(u_{\alpha\beta}\circ u'_{\beta\gamma})\big\},-\big\{\mathrm{Tr}(u_{\alpha\beta}\circ v'_\beta) - \mathrm{Tr}(v_\alpha\circ u'_{\alpha\beta})\big\}\big].\end{aligned}$$

By construction, ω_U is functorial in U. So ω_U descends to a pairing

$$\omega : \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}^{(n)}_r(d,D)} \times \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}^{(n)}_r(d,D)} \longrightarrow \mathcal{O}_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))}.$$

Take any $\mathbf{C}$-valued point $x=(E,\nabla,\{l^{(i)}_j\})\in M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))(\mathbf{C})$, and put $\boldsymbol{\nu} := \pi(x)$. Then a tangent vector $v \in \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}^{(n)}_r(d,D)}(x) = \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}}(x)$ corresponds to a $\mathbf{C}[t]/(t^2)$-valued point $(E^v,\nabla^v,\{(l^v)^{(i)}_j\})\in \mathcal{M}^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}(\mathbf{C}[t]/(t^2))$ such that $(E^v,\nabla^v,\{(l^v)^{(i)}_j\})\otimes \mathbf{C}[t]/(t) \cong (E, \nabla,\{l^{(i)}_j\})$. We can check that $\omega(v,v)$ is nothing but the obstruction class for the lifting of $(E^v,\nabla^v,\{(l^v)^{(i)}_j\})$ to a member of $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}(\mathbf{C}[t]/(t^3))$. Since $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_{\boldsymbol{\nu}}$ is smooth, we have $\omega(v,v)=0$. So ω is a skew-symmetric bilinear pairing. Let $\xi : \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}^{(n)}_r(d,D)} \to \Theta^{\vee}_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}^{(n)}_r(d,D)}$ be the homomorphism induced by ω. For any $\mathbf{C}$-valued point $x\in M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d, (m_i))(\mathbf{C})$,

$$\xi(x) : \mathbf{H}^1\big(\mathcal{F}^{\bullet}(x)\big) = \Theta_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}^{(n)}_r(d,D)}(x) \longrightarrow \Theta^{\vee}_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))/\mathcal{N}^{(n)}_r(d,D)}(x) = \mathbf{H}^1\big(\mathcal{F}^{\bullet}(x)\big)^{\vee}$$

induces a commutative diagram

$$\begin{tikzcd}
H^0(\mathcal{F}^0(x)) \arrow[r] \arrow[d, "b_1"'] & H^0(\mathcal{F}^1(x)) \arrow[r] \arrow[d, "b_2"'] & \mathbf{H}^1(\mathcal{F}^{\bullet}(x)) \arrow[r] \arrow[d, "\xi"'] & H^1(\mathcal{F}^0(x)) \arrow[r] \arrow[d, "b_3"'] & H^1(\mathcal{F}^1(x)) \arrow[d, "b_4"'] \\
H^1(\mathcal{F}^1(x))^{\vee} \arrow[r] & H^1(\mathcal{F}^0(x))^{\vee} \arrow[r] & \mathbf{H}^1(\mathcal{F}^1(x))^{\vee} \arrow[r] & H^0(\mathcal{F}^1(x))^{\vee} \arrow[r] & H^1(\mathcal{F}^0(x))^{\vee}
\end{tikzcd}$$

where b_1,b_2,b_3,b_4 are isomorphisms induced by $\mathcal{F}^0(x)\cong \mathcal{F}^1(x)^{\vee}\otimes\Omega^1_{\mathcal{C}_x}$, $\mathcal{F}^1(x)\cong \mathcal{F}^0(x)^{\vee}\otimes\Omega^1_{\mathcal{C}_x}$ and the Serre duality. Thus ξ becomes an isomorphism by the five lemma. □

PROPOSITION 4.2

For the 2-form ω constructed in Proposition 4.1, we have $d\omega=0$.

Proof

Take any point $x\in S$. We will show that $d\omega|_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_x}=0$. We use the notation in Theorem 3.1. Note that the relative moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$ is smooth over $\mathcal{N}^{(n)}_r(d,(D_i))$. There is an affine scheme U and an étale surjective morphism $p: U\to M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$ which factors through $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))$; namely, there exists a universal family $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ on $\mathcal{C}_x\times U$. Set

$$\tilde{\mathcal{F}}^0 := \big\{a\in \mathcal{E}nd(\tilde{E}) \,\big|\, a|_{(D_i)_U}(\tilde{l}^{(i)}_j)\subset \tilde{l}^{(i)}_j \text{ for any } i,j\big\},$$

$$\tilde{\mathcal{F}}^1 := \big\{b\in \mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathcal{C}_x\times U/U}(\tilde{D}_U) \,\big|\, b|_{(D_i)_U}(\tilde{l}^{(i)}_j)\subset \tilde{l}^{(i)}_{j+1} \text{ for any } i,j\big\},$$

$$\nabla_{\tilde{\mathcal{F}}^{\bullet}} : \tilde{\mathcal{F}}^0 \ni a\mapsto \tilde{\nabla}a - a\tilde{\nabla}\in\tilde{\mathcal{F}}^1.$$

Then we have a canonical isomorphism $H^0(U, p^*(\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))/\mathcal{N}^{(n)}_r(d,(D_i))})) \cong \mathbf{H}^1(\tilde{\mathcal{F}}^\bullet)$. We can define a skew-symmetric pairing

$$\begin{aligned}
&\tilde{\omega}_U : \mathbf{H}^1(\tilde{\mathcal{F}}^\bullet) \times \mathbf{H}^1(\tilde{\mathcal{F}}^\bullet) \longrightarrow \mathbf{H}^2(U, \Omega^1_{\mathcal{C}_x \times U/U}) \cong \mathcal{O}_U,\\
&\big(\big[\big(\{u_{\alpha\beta}\}, \{v_\alpha\}\big)\big], \big[\big(\{u'_{\alpha\beta}\}, \{v'_\alpha\}\big)\big]\big)\\
&\quad \mapsto \big[\big(\big\{\mathrm{Tr}(u_{\alpha\beta} \circ u'_{\beta\gamma})\big\}, -\big\{\mathrm{Tr}(u_{\alpha\beta} \circ v'_\beta) - \mathrm{Tr}(v_\alpha \circ u'_{\alpha\beta})\big\}\big)\big].
\end{aligned}$$

Since $\tilde{\omega}_U$ is functorial in U, it descends to a 2-form

$$\tilde{\omega} \in H^0\big(M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}\big(r, d, (D_i)\big), \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))/\mathcal{N}^{(n)}_r(d,(D_i))}\big).$$

By construction, the restriction $\tilde{\omega}|_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))_{\overline{h}}}$ is nothing but the restriction $\omega|_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_x}$ of the 2-form ω defined in Proposition 4.1. On the other hand, for generic $\boldsymbol{\nu} \in \mathcal{N}^{(n)}_r(d, (D_i))$, the fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r, d, (D_i))_{\boldsymbol{\nu}}$ is nothing but the moduli space of regular singular parabolic connections considered in [8]. Note that for generic $\boldsymbol{\nu}$, every $\boldsymbol{\nu}$-parabolic connection is irreducible and automatically stable. Moreover, the restriction $\tilde{\omega}|_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))_{\boldsymbol{\nu}}}$ is nothing but the restriction of the relative 2-form considered in [8, Proposition 7.2]. By [8, Proposition 7.3], we have $d\tilde{\omega}|_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))_{\boldsymbol{\nu}}} = 0$. Since $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r, d, (D_i))$ is smooth over $\mathcal{N}^{(n)}_r(d, (D_i))$, we have $d\tilde{\omega} = 0$. So we have $d\omega|_{M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/S}(r,d,(m_i))_x} = d\tilde{\omega}|_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(r,d,(D_i))_{\overline{h}}} = 0$. Hence we have $d\omega = 0$. □

5. Moduli spaces of generalized monodromy data and Riemann–Hilbert correspondence

5.1. Fixing the formal type

Fix a nonsingular projective curve C and a divisor $D = \sum_{i=1}^n m_i t_i$ on C such that $m_i > 0$, $t_i \neq t_j$ for $i \neq j$. At each point t_i, we take a generator z_i of the maximal ideal $\mathfrak{m}_{t_i}$ of $\mathcal{O}_{C,t_i}$; then we have the formal completion $\widehat{\mathcal{O}_{C,t_i}} = \lim_k \mathcal{O}_{C,t_i}/\mathfrak{m}^k_{t_i} \simeq \mathbf{C}[[z_i]]$.

For given integers $r > 0, d$, let us fix generalized exponents $\boldsymbol{\nu} = (\nu^{(i)}_j)^{0 \leq j \leq r-1}_{1 \leq i \leq n} \in N^{(n)}_r(d, D)$ (cf. (2)). In Theorems 2.1 and 2.2, we have constructed a smooth quasi-projective moduli scheme $M^{\boldsymbol{\alpha}}_{D/C}(r, d, (m_i))_{\boldsymbol{\nu}}$ of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections on C of parabolic depth $(m_i)^r_{i=1}$, with rank r, deg d.

For each fixed $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l^{(i)}_j\}) \in M^{\boldsymbol{\alpha}}_{D/C}(r, d, (m_i))_{\boldsymbol{\nu}}$, we can define a formal connection by

$$\widehat{E}_{t_i} = E \otimes_{\mathcal{O}_{C,t_i}} \mathbf{C}[[z_i]], \qquad \widehat{\nabla}_{t_i} : \widehat{E}_{t_i} \longrightarrow \widehat{E}_{t_i} \otimes \mathbf{C}[[z_i]] \frac{dz_i}{(z_i)^{m_i}}.$$

In this section, we assume that $(\widehat{E}_{t_i}, \widehat{\nabla}_{t_i})$ is unramified for each $i, 1 \leq i \leq n$, that is, in the Hukuhara–Turrittin decomposition in Theorem 0.1, $l = 1$.

By Proposition 1.1, there exists a filtration by $\mathbf{C}[[z_i]]$-submodules

$$\widehat{E}_{t_i} = \widehat{l}^{(i)}_0 \supset \widehat{l}^{(i)}_1 \supset \widehat{l}^{(i)}_2 \supset \cdots \supset \widehat{l}^{(i)}_{r-1} \supset \widehat{l}^{(i)}_r = 0$$

such that $\widehat{\nabla}_{t_i}(\widehat{l}_j^{(i)}) \subset \widehat{l}_j^{(i)} \otimes dz_i/z_i^{m_i}$ and $\widehat{l}_j^{(i)}/\widehat{l}_{j+1}^{(i)} \simeq V(\tilde{\nu}_j^{(i)}, 1)$, where $\tilde{\boldsymbol{\nu}} = (\tilde{\nu}_j^{(i)})_{1\leq i\leq n}^{0\leq j\leq r-1} \in N_r^{(n)}(d,D)$.

The isomorphism class of $(\widehat{E}_{t_i}, \widehat{\nabla}_{t_i}, \{\widehat{l}_j^{(i)}\})$ at each t_i as $\mathbf{C}[[z_i]]$-connection is called the *formal type of the connection* (E,∇) at t_i. For each i, the data $\tilde{\boldsymbol{\nu}}^{(i)} := (\tilde{\nu}_j^{(i)})^{0\leq j\leq r-1}$ are called *formal generalized exponents* of $(\widehat{E}_{t_i}, \widehat{\nabla}_{t_i}, \{\widehat{l}_j^{(i)}\})$. Note that the original parabolic structure $\{l_j^{(i)}\}$ is a filtration of $E\otimes_{\mathcal{O}_{C,t_i}} \mathbf{C}[z_i]/(z_i^{m_i})$. Moreover, as we see in Remark 1.2, $\boldsymbol{\nu}$ may not be equal to the formal generalized exponents $\tilde{\boldsymbol{\nu}}$.

The main purpose of this section is to define the Riemann–Hilbert correspondence from the moduli space $M^{\boldsymbol{\alpha}}_{D/C}(r,d,(m_i))_{\boldsymbol{\nu}}$ of $\boldsymbol{\nu}$-parabolic connections to the moduli space of *generalized monodromy data* consisting of the *monodromy representation* of fundamental group $\pi_1(C\setminus\{t_1,\dots,t_n\},*)$, *links (or connection matrices)*, *formal monodromies,* and *Stokes data*. Moreover, we may expect that the Riemann–Hilbert correspondence is a proper bimeromorphic surjective analytic morphism for any $\boldsymbol{\nu}$ as we proved in the case of at most regular singularities, that is, the case when $m_i = 1$ for all $i, 1\leq i\leq n$ (cf. [8], [10], [11]).

As explained in [13], [20], and [21], in order to construct the moduli space of generalized monodromy data and define the Riemann–Hilbert correspondence, we need to fix a formal type of the parabolic connection $(E,\nabla,\{l_j^{(i)}\})$ at each irregular or regular singular point t_i. However the counterexample in Remark 1.2 shows that for a special $\boldsymbol{\nu}$, one cannot determine the formal type of a connection $(E,\nabla,\{l_j^{(i)}\}) \in M^{\boldsymbol{\alpha}}_{D/C}(r,d,(m_i))_{\boldsymbol{\nu}}$, that is, the reductions up to the order m_i are not enough to determine the formal type for a special $\boldsymbol{\nu}$. Since we have Proposition 1.2, we may take deeper reductions of order $N_i = r^2 m_i > m_i$ to recover the formal type. However, in Remark 2.3, we see that the corresponding moduli space $M^{\boldsymbol{\alpha}}_{D/C}(r,d,(N_i))_{\boldsymbol{\nu}}$ is not smooth.

At this moment, we do not know how to handle these difficulties. For this reason, we impose the following genericity conditions on $\boldsymbol{\nu} = (\nu_j^{(i)})_{1\leq i\leq n}^{0\leq j\leq r-1} \in N_r^{(n)}(d,D)$.

Let us write $\nu_j^{(i)}(z_i)$ explicitly as

$$\begin{aligned}\nu_j^{(i)}(z_i) &= (a_{j,-m_i}^{(i)} z_i^{-m_i} + \cdots + a_{j,-1}^{(i)} z_i^{-1})\, dz_i \\ &= \sum_{k=-m_i}^{-1} (a_{j,k}^{(i)} z_i^k)\, dz_i \quad \text{for } 0\leq j\leq r-1.\end{aligned} \tag{4}$$

DEFINITION 5.1

Let $\boldsymbol{\nu} = \{\nu_j^{(i)}(z_i)\}_{1\leq i\leq n}^{0\leq j\leq r-1} \in N_r^{(n)}(d,D)$ be written as in (4). We have the following:

(1) $\boldsymbol{\nu}$ is *generic* if for every $(i,j_1),(i,j_2)$, $j_1\neq j_2$, the top terms are different, that is, $a_{j_1,-m_i}^{(i)} \neq a_{j_2,-m_i}^{(i)}$;

(2) $\boldsymbol{\nu}$ is *resonant* if for some $i, 1 \leq i \leq n$, with $m_i = 1$ there exists j_1, j_2, $j_1 \neq j_2$, such that

$$a^{(i)}_{j_1,-1} - a^{(i)}_{j_2,-1} \in \mathbf{Z};$$

moreover, $\boldsymbol{\nu}$ is called *nonresonant* if it is not resonant;

(3) $\boldsymbol{\nu}$ is *reducible* if for some $h, 1 \leq h < r$, there exist some choices of $j^{(i)}_1$, $\dots, j^{(i)}_h$, $0 \leq j^{(i)}_1 < j^{(i)}_2 < \cdots < j^{(i)}_h \leq r-1$ for each $i, 1 \leq i \leq n$, such that

$$\sum_{i=1}^{n} \sum_{k=1}^{h} a^{(i)}_{j^{(i)}_k,-1} \in \mathbf{Z}. \tag{5}$$

If $\boldsymbol{\nu}$ is not reducible, we call $\boldsymbol{\nu}$ *irreducible.*

Note that the genericity and resonance of $\boldsymbol{\nu}$ does not depend on the choice of the local coordinates z_i. An easy argument shows that if $\boldsymbol{\nu}$ is irreducible, every $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l^{(i)}_j\})$ is irreducible and hence $\boldsymbol{\alpha}$-stable for any choice of the weights $\boldsymbol{\alpha}$.

From now on, we assume that $\boldsymbol{\nu}$ is generic. From the Hukuhara–Turrittin theorem (see [27, Theorem 6.1.1]), it is easy to see the following lemma.

LEMMA 5.1

Let $(E, \nabla, \{l^{(i)}_j\})$ be an $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connection in $M^{\boldsymbol{\alpha}}_{D/C}(r, d, (m_i))_{\boldsymbol{\nu}}$. Assume that $\boldsymbol{\nu} = \{\nu^{(i)}_j(z_i)\}$ is generic. Then we have a direct sum decomposition of the formal connection

$$(\widehat{E}_{t_i}, \widehat{\nabla}_{t_i}) \simeq V(\nu^{(i)}_0, 1) \oplus V(\nu^{(i)}_1, 1) \oplus \cdots \oplus V(\nu^{(i)}_{r-1}, 1). \tag{6}$$

Here $V(\nu^{(i)}_j, 1) \simeq \mathbf{C}[[z_i]]e^{(i)}_j$ is a rank 1 $\mathbf{C}[[z_i]]$-module with a connection given by $e^{(i)}_j \mapsto \nu^{(i)}_j(z_i)e^{(i)}_j$. In particular, the formal type of $(\widehat{E}_{t_i}, \widehat{\nabla}_{t_i})$ is uniquely determined by generalized exponents $\{\nu^{(i)}_j\}_{0 \leq j \leq r-1}$. Moreover the decomposition (6) is compatible with the parabolic structure $\{l^{(i)}_j\}_{0 \leq i \leq r-1}$.

This lemma implies that there exists a free basis $e^{(i)}_0, \dots, e^{(i)}_{r-1}$ of $\widehat{E}_{t_i}$ as a $\mathbf{C}[[z_i]]$-module, such that

$$\widehat{\nabla}_{t_i} e^{(i)}_j = \nu^{(i)}_j(z_i) e^{(i)}_j.$$

Moreover, for $\widehat{E}_{t_i} \otimes \mathbf{C}[[z_i]]/(z_i^{m_i})$, the induced basis $\{\overline{e}^{(i)}_j\}$ gives a parabolic structure

$$l^{(i)}_k = \langle \overline{e}^{(i)}_k, \overline{e}^{(i)}_{k+1}, \dots, \overline{e}^{(i)}_{r-1} \rangle.$$

Let us take a generic $\boldsymbol{\nu} = \{\boldsymbol{\nu}^{(i)}\}_{1 \leq i \leq n} \in N^{(n)}_r(d, D)$. For each t_i, define *the space of formal solutions* at t_i by

$$V_{t_i} = \{\sigma \in \widehat{E}_{t_i} \otimes_{\mathbf{C}[[z_i]]} \mathrm{Univ}_{t_i} \mid \widehat{\nabla}_{t_i} \sigma = 0\} \tag{7}$$

where Univ_{t_i} denotes the differential ring extension of $\mathbf{C}[[z_i]]$ which is similarly defined as in [20, Section 1.2]. Under the isomorphism of (6), the space V_{t_i} is a $\mathbf{C}$-vector space of dimension r and has a natural decomposition

$$V_{t_i} = V_0^{(i)} \oplus \cdots \oplus V_{r-1}^{(i)}, \tag{8}$$

where $V_j^{(i)} = \mathbf{C}(f_j^{(i)}(z_i)e_j^{(i)})$ is a one-dimensional vector subspace and $f_j^{(i)}(z_i) = \exp(-\int \nu_j^{(i)}(z_i)) \in \mathrm{Univ}_{t_i}$. Note that we have $df_j^{(i)}(z_i) = -f_j^{(i)}(z_i)\nu_j^{(i)}(z_i)$.

5.2. Generalized monodromy data

As in Section 5.1, we fix a nonsingular projective curve C and a divisor $D = \sum_{i=1}^n m_i t_i$ on C such that $m_i > 0$, $t_i \neq t_j$ for $i \neq j$. Moreover, at each point t_i, we fix a generator z_i of the maximal ideal $\mathfrak{m}_{t_i}$ of $\mathcal{O}_{C,t_i}$ so that we have the formal completion $\widehat{\mathcal{O}_{C,t_i}} = \lim_k \mathcal{O}_{C,t_i}/\mathfrak{m}_{t_i}^k \simeq \mathbf{C}[[z_i]]$. Let us fix a generic element $\boldsymbol{\nu} \in N_r^{(n)}(d,D)$ written as in (4). Then Lemma 5.1 implies that the formal types of every $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l_j^{(i)}\}) \in M_{D/C}^{\boldsymbol{\alpha}}(r,d,(m_i))_{\boldsymbol{\nu}}$ at t_i can be fixed as in (6). Fixing these data, we will associate a generalized monodromy data to each $(E, \nabla, \{l_j^{(i)}\}) \in M_{D/C}^{\boldsymbol{\alpha}}(r,d,(m_i))_{\boldsymbol{\nu}}$ as follows. We will basically follow the formulations in [13] and [20] of the genus zero case, which is easily generalized to higher genus case (for the known facts on generalized monodromy data, see [1], [13], [22], [27], and [21]).

Local coordinates. For each $i, 1 \leq i \leq n$, we consider the fixed generator z_i of the maximal ideal of $\mathcal{O}_{C,t_i}$ as a local analytic coordinate around t_i of C.

Local neighborhoods. We have an analytic local neighborhood $\Delta_i \subset C$ of t_i which is identified with $\{z_i \mid |z_i| < \epsilon_i\}$ for a small positive number ϵ_i.

Singular directions and sectors. Let us identify $d, 0 \leq d < 2\pi$, as a ray starting from the origin $z_i = 0$ with an argument d. Fixing a generic $\boldsymbol{\nu} = \{\nu_j^{(i)}(z_i)\} \in N_r^{(n)}(d,D)$, we can define the singular directions $\{d_k^{(i)}\}_{1\leq k\leq s_i}^{1\leq i\leq n}$ such that $0 \leq d_1^{(i)} < d_2^{(i)} < \cdots < d_{s_i}^{(i)} < 2\pi$. A direction d at t_i is called *singular* if for some $j_1 \neq j_2$ the function $\exp(\int(\nu_{j_1}^{(i)} - \nu_{j_2}^{(i)}))$ has "maximal descent" along the ray $z_i = r_i e^{\sqrt{-1}d}$ for $r_i \to 0$. More explicitly, if

$$\nu_{j_1}^{(i)} - \nu_{j_2}^{(i)} = \left((a_{j_1,-m_i}^{(i)} - a_{j_2,-m_i}^{(i)})z_i^{-m_i} + \cdots\right) dz_i,$$

$(a_{j_1,-m_i}^{(i)} - a_{j_2,-m_i}^{(i)}) \neq 0$, then d is a singular direction if

$$-\left((a_{j_1,-m_i}^{(i)} - a_{j_2,-m_i}^{(i)})e^{-\sqrt{-1}(m_i-1)d} r_i^{-(m_i-1)}/(m_i-1)\right)$$

is a negative real number (for more detail, see [20, Section 1.3]). For each $i, 1 \leq i \leq n$, let $0 \leq d_1^{(i)} < d_2^{(i)} < \cdots < d_{s_i}^{(i)} < 2\pi$ be all the singular directions at t_i. To fix the order of Stokes data at t_i, we take a point $t_i^* \in \partial\Delta_i$ such that $d_{s_i}^{(i)} - 2\pi < \arg t_i^* < d_1^{(i)}$. (Later we will not impose this last condition for t_i^* when we will vary the associated data continuously.) We denote by $\gamma_i = \partial\Delta_i$ a closed counterclockwise

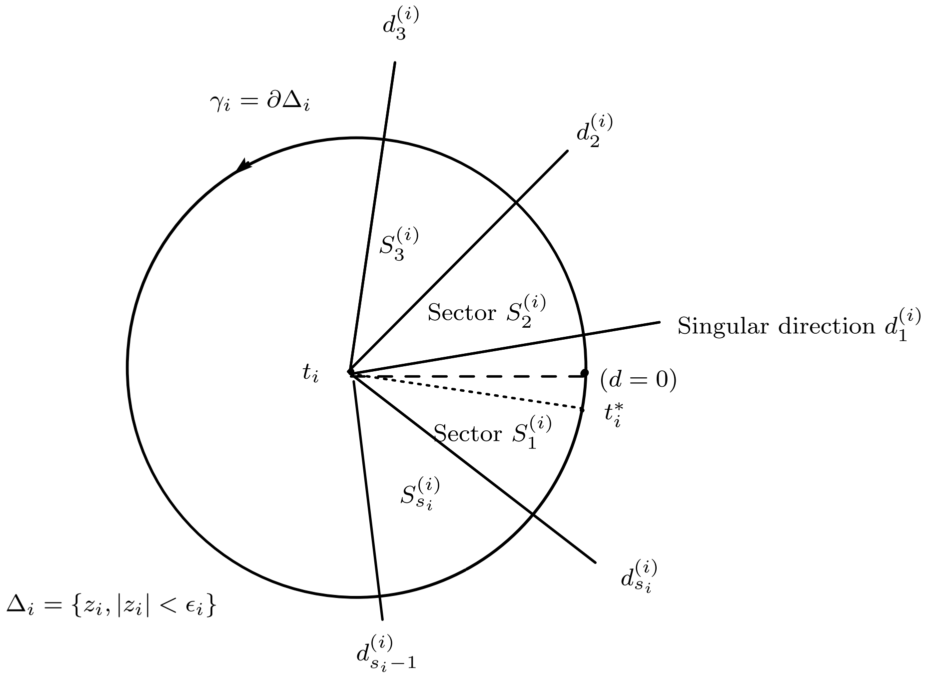

Figure 1. Local neighborhood of t_i.

loop starting from t_i^*. Moreover, we set $d_0^{(i)} = d_{s_i}^{(i)} - 2\pi < 0$. For each $1 \le k \le s_i$, we define a *sector* $S_k^{(i)}$ by

$$S_k^{(i)} = \left\{ z_i \in \Delta_i \;\middle|\; 0 < |z_i| < \epsilon_i, d_{k-1}^{(i)} < \arg z_i < d_k^{(i)} \right\} \tag{9}$$

(see Figure 1). For a singular direction d at t_i, let $\mathcal{J}(d,i)$ be the set of all pairs (j_1, j_2) such that a singular direction of $\nu_{j_1}^{(i)} - \nu_{j_2}^{(i)}$ is d. The number $\sharp\mathcal{J}(d,i)$ is called the multiplicity of d at t_i. It is easy to see that

$$\sum_{1 \le k \le s_i} \sharp\mathcal{J}(d_k^{(i)}, i) = (m_i - 1)r(r-1). \tag{10}$$

Note that if the multiplicity $\sharp\mathcal{J}(d_k^{(i)}, i)$ is one for all $1 \le k \le s_i$, the number of singular direction is equal to $(m_i - 1)r(r-1)$.

Paths and loops. We fix a point b on $C \setminus \{t_1, \ldots, t_n\}$ and a continuous path l_i from b to t_i^*. Let us set $\gamma_i^l := l_i \gamma_i l_i^{-1}$ for $1 \le i \le n$ and set the usual symplectic generators α_k, β_k, $1 \le k \le g$, of $\pi_1(C, b)$ so that the fundamental group $\pi_1(C \setminus \{t_1, \ldots, t_n\}, b)$ is generated by $\{\gamma_i^l, \alpha_k, \beta_k\}$. Moreover, we assume that our choice of paths l_i and loops γ_i α_k, β_l satisfies the conditions $\prod_{k=1}^{g} [\alpha_k, \beta_k] \prod_{i=1}^{n} \gamma_i^l = 1$, where $[\alpha_k, \beta_k] = \alpha_k \beta_k \alpha_k^{-1} \beta_k^{-1}$. Then we have the following presentation of the

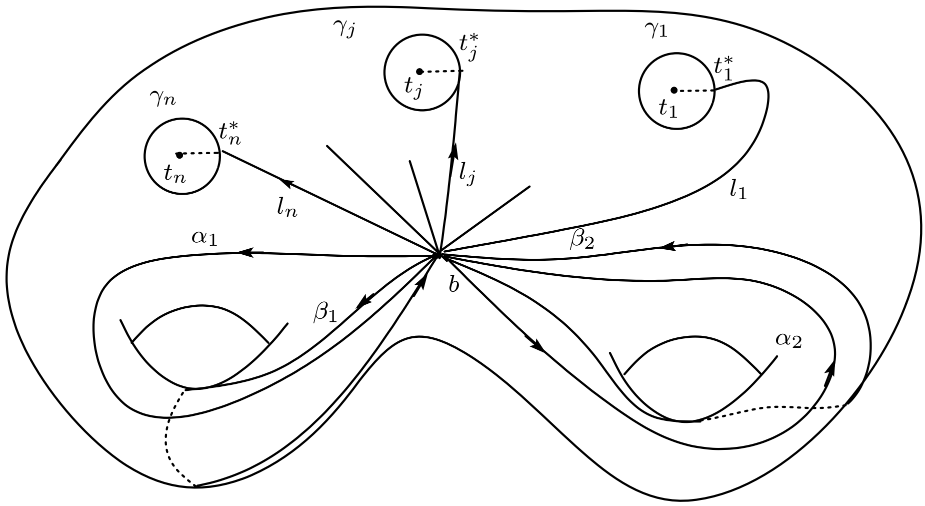

Figure 2. Paths and loops.

fundamental group (see Figure 2):

$$\pi_1\big(C \setminus \{t_1, \dots, t_n\}, b\big) = \Big\langle \gamma_i^l, \alpha_k, \beta_k \;\Big|\; \prod_{k=1}^{g} [\alpha_k, \beta_k] \prod_{i=1}^{n} \gamma_i^l = 1 \Big\rangle. \tag{11}$$

Spaces of formal solutions and analytic solutions. Since we assume that $\boldsymbol{\nu}$ is generic, we can fix a decomposition of the formal connection $(\widehat{E}_{t_i}, \widehat{\nabla}_{t_i}) \simeq V(\nu_0^{(i)}, 1) \oplus V(\nu_1^{(i)}, 1) \oplus \cdots \oplus V(\nu_{r-1}^{(i)}, 1)$ as in (6) and *the space of formal solutions* V_{t_i} as in (8). Moreover, we fix the space of analytic solutions V_b of (E, ∇) near b, which is a $\mathbf{C}$-vector space of dimension r.

Fixing these data, we can associate the following *generalized monodromy data* to each $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l_j^{(i)}\}) \in M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))_{\boldsymbol{\nu}}$.

Generalized monodromy data
Formal monodromy $\{\widehat{\gamma}_i\}$*.* For each $i, 1 \leq i \leq n$, we can define the formal monodromy $\widehat{\gamma}_i \in \mathrm{Aut}(V_{t_i})$ coming from a monodromy on formal solutions. The eigenvalues of $\widehat{\gamma}_i$ are determined by $\{a_{j,-1}^{(i)}\}_{0 \leq j \leq r-1}$ with some exponential maps. Since we fix the decomposition (8), $\widehat{\gamma}_i$ are fixed diagonal matrices.

Stokes data $\{St_{d_k^{(i)}}\}$*.* Let us consider a sector $S \subset \Delta_i \setminus \{0\}$, and let $V(S)$ denote the space of analytic (or convergent) solutions of $\nabla = 0$ on the sector S. Let $\{S_k^{(i)}\}_{1 \leq i \leq s_i}$ be the set of sectors defined in (9). For directions $d_{k,-} \in S_k^{(i)}$, $d_{k,+} \in S_{k+1}^{(i)}$, we have multisummation maps

$$\mathrm{mults}_{d_{k,-}} : V_{t_i} \longrightarrow V(S_k^{(i)}),$$

$$\mathrm{mults}_{d_{k,+}} : V_{t_i} \longrightarrow V(S_{k+1}^{(i)})$$

which are $\mathbf{C}$-linear isomorphisms between V_{t_i} and $V(S_k^{(i)})$ and $V(S_{k+1}^{(i)})$, respectively. The Stokes map $\mathrm{St}_{d_k^{(i)}}$ comes from an isomorphism

$$\mathrm{St}_{d_k^{(i)}} : V_{t_i} \longrightarrow V_{t_i} \tag{12}$$

which makes the following diagram commutative:

$$\begin{array}{lcc} \mathrm{mults}_{d_k,-} : V_{t_i} & \longrightarrow & V(S_k^{(i)}) \\ \qquad\quad \downarrow \mathrm{St}_{d_k^{(i)}} & & \| \\ \mathrm{mults}_{d_k,+} : V_{t_i} & \longrightarrow & V(S_{k+1}^{(i)}). \end{array}$$

The identification $V(S_k^{(i)}) = V(S_{k+1}^{(i)})$ comes from analytic continuations.

According to the decomposition (8) of $V_{t_i} = \bigoplus_{j=0}^{r-1} V_j^{(i)}$, each Stokes map $\mathrm{St}_{d_k^{(i)}}$ has the form

$$\mathrm{St}_{d_k^{(i)}} = \mathrm{Id} + \sum_{(j_1,j_2)\in \mathcal{J}(d_k^{(i)},i)} R_{j_1,j_2},$$

with $R_{j_1,j_2} = i_{j_1} \circ M_{j_1,j_2} \circ p_{j_2}$ where $0 \le j_1, j_2 \le r-1, j_1 \ne j_2$, $p_{j_1} : V_{t_i} \to V_{j_1}^{(i)}$ is the projection, and $i_{j_2} : V_{j_2}^{(i)} \to V_{t_i}$ is the canonical injection. Moreover, $M_{j_1,j_2} : V_{j_1}^{(i)} \to V_{j_2}^{(i)}$ is a linear map between one-dimensional spaces. So M_{j_1,j_2} is given by a scalar $c_{j_2,j_1} \in \mathbf{C}$. In the matrix form, one can write

$$\mathrm{St}_{d_k^{(i)}} = I_r + \sum_{(j_1,j_2)\in \mathcal{J}(d_k^{(i)},i)} c_{j_2,j_1} I_{j_2,j_1},$$

where I_{j_2,j_1} is the $(r \times r)$-matrix whose (i,k)-entry is zero except for $(i,k) = (j_2,j_1)$ and the (j_2,j_1)-entry is 1 (for this fact, see [21, Theorem 8.13] or [22, Lemma 6.5]).

The link $L_i \in \mathrm{Hom}_{\mathbf{C}}(V_b, V_{t_i})$. Analytic continuation along l_i gives a $\mathbf{C}$-linear isomorphism $V_b \longrightarrow V_{t_i^*}$. Composition of this isomorphism and the inverse of the multisummation map $V_{t_i^*} \longrightarrow V_{t_i}$ gives the linear map which is called a *link* (or a *connection matrix*)

$$L_i : V_b \xrightarrow{\simeq} V_{t_i^*} \xrightarrow{\simeq} V_{t_i}. \tag{13}$$

The topological monodromy $\mathrm{Top}_i \in \mathrm{Aut}(V_{t_i})$. Identifying $V_{t_i^*}$ with V_{t_i} by the multisummation map, an analytic continuation along the loop γ_i starting from t_i^* gives a topological monodromy $\mathrm{Top}_i \in \mathrm{Aut}(V_{t_i}) \simeq \mathrm{GL}_r(\mathbf{C})$. We have the following relation:

$$\mathrm{Top}_i = \widehat{\gamma}_i \circ \mathrm{St}_{d_{s_i}^{(i)}} \circ \cdots \circ \mathrm{St}_{d_2^{(i)}} \circ \mathrm{St}_{d_1^{(i)}}. \tag{14}$$

The global monodromy representation. We can consider the monodromy representation $\rho : \pi_1(C \setminus \{t_1, \dots, t_n\}, b) \longrightarrow \mathrm{Aut}(V_b) \simeq \mathrm{GL}_r(\mathbf{C})$. Moreover, $\rho(\gamma_i^l) =$

$L_i^{-1}\operatorname{Top}_i L_i$, and we set $A_k = \rho(\alpha_k), B_k = \rho(\beta_k)$. These data determine the monodromy representation $\rho : \pi_1(C\setminus\{t_1,\ldots,t_n\}, b) \longrightarrow \operatorname{Aut}(V_b) \simeq \mathrm{GL}_r(\mathbf{C})$ associated to analytic continuations of the space of solutions of $\nabla\sigma = 0$. We have the relation

$$\prod_{i=n}^{1} L_i^{-1}\operatorname{Top}_i L_i \prod_{k=g}^{1}(B_k^{-1}A_k^{-1}B_kA_k) = I_r. \tag{15}$$

(Note that in this notation, ρ becomes an antihomomorphism such that $\rho(\delta_1\delta_2) = \rho(\delta_2)\rho(\delta_1)$.) By the relation (14), we see that the formal monodromy $\widehat{\gamma}_i$, Stokes data $\{\mathrm{St}_{d_k^{(i)}}\}_{1\le k\le r(r-1)(m_i-1)}$, and the link L_i determine $\rho(\gamma_i^l)$.

For a generic $\boldsymbol{\nu}\in N_r^{(n)}(d,D)$, we define the set $\tilde{\mathcal{R}}(\boldsymbol{\nu})$ of all tuples

$$\big\{\{\widehat{\gamma}_i\},\{\mathrm{St}_{d_k^{(i)}}\},\{L_i\},\{A_k,B_k\}\big\}$$

satisfying the following.

1. For each $1\le i\le n$, $\widehat{\gamma}_i \in \mathrm{GL}(V_{t_i})$ preserving the decomposition (8) whose eigenvalues are determined by $\{a_{j,-1}^{(i)}\}_{0\le j\le r-1}$. (Hence, $\widehat{\gamma}_i$ is a diagonal matrix with prescribed eigenvalues.)
2. For each $1\le i\le n$ and $1\le k\le s_i$, $\mathrm{St}_{d_k^{(i)}}\in\mathrm{GL}(V_{t_i})$ of the form $\mathrm{St}_{d_k^{(i)}} = \mathrm{Id}+\sum_{(j_1,j_2)\in\mathcal{J}(d_k^{(i)},i)} R_{j_1,j_2}$ where R_{j_1,j_2} corresponds to a one-dimensional homomorphism $c_{j_2,j_1} : V_{j_1}^{(i)} \longrightarrow V_{j_2}^{(i)}$.
3. We have linear bijections $L_i : V_b\longrightarrow V_{t_i}$ for $1\le i\le n$.
4. Define $\operatorname{Top}_i\in\mathrm{GL}(V_{t_i})$ by the formula (14). The set $\{\{\operatorname{Top}_i\}_{1\le i\le n}, \{A_k, B_k\in\mathrm{GL}(V_b)\}_{1\le k\le g}\}$ satisfies the relation (15).

DEFINITION 5.2

Two tuples $\{\{\widehat{\gamma}_i\},\{\mathrm{St}_{d_k^{(i)}}\},\{L_i\},\{A_k,B_k\}\}$ and $\{\{\widehat{\gamma}_i{}'\},\{\mathrm{St}'_{d_k^{(i)}}\},\{L_i'\},\{A_k',B_k'\}\}$ are called equivalent if there exist $\sigma^{(i)}\in\mathrm{GL}(V_{t_i})$ preserving the decomposition $V_{t_i} = \bigoplus_{j=0}^{r-1} V_j^{(i)}$ in (8) and $\sigma\in\mathrm{GL}(V_b)$ satisfying

$$\begin{aligned}
\sigma^{(i)}L_i &= L_i'\sigma \quad \text{for each } i, 1\le i\le n,\\
\sigma^{(i)}\widehat{\gamma}_i &= \widehat{\gamma}_i{}'\sigma^{(i)} \quad \text{for each } i, 1\le i\le n,\\
\sigma^{(i)}St_{d_k^{(i)}} &= St'_{d_k^{(i)}}\sigma^{(i)} \quad \text{for each } i, 1\le i\le n, 1\le k\le s_i,\\
A_k = \sigma^{-1}A_k'\sigma, &\qquad B_k = \sigma^{-1}B_k'\sigma \quad \text{for each } k, 1\le k\le g.
\end{aligned}$$

Note that under the assumption that $\boldsymbol{\nu}$ is generic we see that $\sigma^{(i)}\in\mathrm{GL}(V_{t_i})$ above is a diagonal matrix in $\prod_{j=0}^{r-1}\mathrm{GL}(V_j^{(i)})\simeq(\mathbf{C}^\times)^r$.

Since the set $\tilde{\mathcal{R}}(\boldsymbol{\nu})$ is an affine scheme with a natural action of the reductive group

$$G := \mathrm{GL}(V_b)\times\prod_{i=1}^{n}\prod_{j=0}^{r-1}\mathrm{GL}(V_j^{(i)}) \tag{16}$$

in Definition 5.2, we can construct the categorical quotient

$$\mathcal{R}(\boldsymbol{\nu}) = \tilde{\mathcal{R}}(\boldsymbol{\nu})//G \tag{17}$$

which is considered as the set of equivalence classes of the generalized monodromy data associated to $\boldsymbol{\nu}$. By definition of the categorical quotient, $\mathcal{R}(\boldsymbol{\nu})$ is an affine scheme.

PROPOSITION 5.1

Assume that $\boldsymbol{\nu} \in N_r^{(n)}(d,D)$ is generic, nonresonant, and irreducible (cf. Definition 5.1). Then moduli space $\mathcal{R}(\boldsymbol{\nu})$ is a nonsingular affine scheme and

$$\dim \mathcal{R}(\boldsymbol{\nu}) = 2r^2(g-1) + \sum_{i=1}^{n} m_i r(r-1) + 2$$

if $\mathcal{R}(\boldsymbol{\nu})$ is nonempty.

Proof

For a generic $\boldsymbol{\nu} \in N_r^{(n)}(d,D)$, consider the affine variety of tuples

$$\mathcal{S}(\boldsymbol{\nu}) = \big\{ \big\{ \{\widehat{\gamma}_i\}, \{\mathrm{St}_{d_k^{(i)}}\}, \{L_i\}, \{A_k, B_k\} \big\} \bigm| \text{without the relation (15)} \big\}.$$

Set $l_i = (m_i - 1)r(r-1)$, and recall the equality $\sum_{1 \le k \le s_i} \sharp\mathcal{J}(d_k^{(i)}, i) = l_i$ where $\sharp\mathcal{J}(d_k^{(i)}, i)$ is the multiplicity of the singular direction $d_k^{(i)}$. The set of Stokes matrices $\mathrm{St}_{d_k^{(i)}}$ is isomorphic to the affine variety $\mathbf{C}^{\sharp\mathcal{J}(d_k^{(i)},i)}$. Then we see that $\mathcal{S}(\boldsymbol{\nu}) \simeq \prod_{i=1}^{n} \mathbf{C}^{l_i} \times \mathrm{GL}_r(\mathbf{C})^n \times \mathrm{GL}_r(\mathbf{C})^{2g}$ (with $l_i = (m_i-1)r(r-1)$); hence $\mathcal{S}(\boldsymbol{\nu})$ is a smooth affine variety of dimension $\sum_{i=1}^{n}(m_i-1)r(r-1) + (n+2g)r^2$. Define the morphism

$$\mu : \mathcal{S}(\boldsymbol{\nu}) \longrightarrow \mathrm{SL}_r(\mathbf{C}) \tag{18}$$

by

$$\begin{aligned} &\mu\big(\big\{ \{\widehat{\gamma}_i\}, \{\mathrm{St}_{d_k^{(i)}}\}, \{L_i\}, \{A_k, B_k\} \big\}\big) \\ &\qquad = \prod_{i=n}^{1} L_i^{-1} \,\mathrm{Top}_i\, L_i \prod_{k=g}^{1} (B_k^{-1} A_k^{-1} B_k A_k) \end{aligned} \tag{19}$$

with $\mathrm{Top}_i = \widehat{\gamma}_i \circ St_{d_{s_i}^{(i)}} \circ \cdots \circ \mathrm{St}_{d_2^{(i)}} \circ \mathrm{St}_{d_1^{(i)}}$. Then we see that $\tilde{\mathcal{R}}(\boldsymbol{\nu}) = \mu^{-1}(I_r)$. As in [7, Theorem 2.2.5], in order to prove the smoothness of $\tilde{\mathcal{R}}(\boldsymbol{\nu})$, we only have to prove that the derivative $d\mu_s : T_{\mathcal{S},s} \longrightarrow T_{\mathrm{SL}_r(\mathbf{C}),I_r} \simeq \mathrm{sl}_r(\mathbf{C})$ is surjective at any point $s \in \mathcal{S}$. If $\boldsymbol{\nu}$ is nonresonant and irreducible, this can be shown by direct calculations of $d\mu_s$ as in the proof of [7, Theorem 2.2.5]. Therefore $\tilde{\mathcal{R}}(\boldsymbol{\nu})$ is a smooth affine scheme with

$$\begin{aligned} \dim \tilde{\mathcal{R}}(\boldsymbol{\nu}) &= \dim \mathcal{S}(\boldsymbol{\nu}) - (r^2-1) \\ &= \sum_{i=1}^{n} (m_i-1)r(r-1) + (n+2g)r^2 - (r^2-1). \end{aligned}$$

Recall that $G = \mathrm{GL}(V_b) \times \prod_{i=1}^n \prod_{j=0}^{r-1} \mathrm{GL}(V_j^{(i)}) \simeq \mathrm{GL}_r(\mathbf{C}) \times \prod_{i=1}^n (\mathbf{C}^\times)^r$ acts on $\tilde{\mathcal{R}}(\boldsymbol{\nu})$ as in Definition 5.2. Note that the subgroup $Z = \{(cI_r, (c,\dots,c)) \in G, c \in \mathbf{C}^\times\}$ acts on $\tilde{\mathcal{R}}(\boldsymbol{\nu})$ trivially. Then under the assumption on $\boldsymbol{\nu}$, it is also easy to see that the action of G/Z on $\tilde{\mathcal{R}}(\boldsymbol{\nu})$ is free. Hence $\mathcal{R}(\boldsymbol{\nu}) = \tilde{\mathcal{R}}(\boldsymbol{\nu})//G$ is a smooth affine scheme with

$$\begin{aligned}\dim \mathcal{R}(\boldsymbol{\nu}) &= \dim \tilde{\mathcal{R}}(\boldsymbol{\nu}) - (\dim G - 1)\\ &= \sum_{i=1}^n (m_i - 1) r(r-1) + (n+2g) r^2 - (r^2 - 1) - (r^2 + nr - 1)\\ &= 2r^2(g-1) + \sum_{i=1}^n m_i r(r-1) + 2.\end{aligned}$$

□

5.3. The generalized Riemann–Hilbert correspondence

Let us fix a data $(C, D = \sum_{i=1}^n m_i t_i)$, let z_i be a generator of $\mathfrak{m}_{t_i}$, and take a generic element $\boldsymbol{\nu} \in N_r^{(n)}(d, D)$. For these data, we can also fix an analytic neighborhood $\Delta_i = \{z_i \in \mathbf{C} \mid |z_i| < \epsilon_i\}$ of each t_i, singular directions $\{d_k^{(i)}\}$, sectors $\{S_k^{(i)}\}$, and $t_i^* \in \partial\Delta_i$ as in Section 5.2.

Moreover, we fix a base point $b \in C \setminus \{t_1, \dots, t_n\}$ and a continuous path l_i from b to t_i^* and loops $\{\gamma_i^l, \alpha_k, \beta_k\}$ with the condition (11).

Fixing these data, we can define the generalized Riemann–Hilbert correspondence as in Section 5.2:

$$\mathbf{RH}_{(D/C),\boldsymbol{\nu}} : M_{D/C}^{\boldsymbol{\alpha}}\big(r, d, (m_i)\big)_{\boldsymbol{\nu}} \longrightarrow \mathcal{R}(\boldsymbol{\nu}). \tag{20}$$

THEOREM 5.1

Under the notation above, assume further that $\boldsymbol{\nu}$ is nonresonant and irreducible. Then the generalized Riemann–Hilbert correspondence $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ (20) is an analytic isomorphism.

Proof

Under the assumption that $\boldsymbol{\nu}$ is generic, we can fix formal types of all singularities of $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \{l_j^{(i)}\}) \in M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))_{\boldsymbol{\nu}}$, and then we can define the Riemann–Hilbert correspondence $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ as we explained above. The fact that $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ is a holomorphic map can be proved as follows. All generalized monodromy data can be defined by a system of local fundamental solutions of $\nabla = 0$ defined in each open set including the sectors near the singular points t_i (cf. [27, Chapter VI]). If one has a holomorphic family of $\boldsymbol{\nu}$-parabolic connections, Sibuya [26] showed that at least locally in the parameter space there exists a family of a system of fundamental solutions depending on the parameter holomorphically. Hence, this shows that $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ is holomorphic.

Recall that $M_{D/C}^{\boldsymbol{\alpha}}(r, d, (m_i))_{\boldsymbol{\nu}}$ is a smooth quasi-projective scheme (see Theorem 2.2) and $\mathcal{R}(\boldsymbol{\nu})$ is a smooth affine algebraic scheme (see Proposition 5.1).

Since $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ is an analytic morphism between smooth analytic manifolds, we only have to prove that $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ is bijective.

First, we prove the surjectivity of $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$. Let us take a tuple

$$\{\{\widehat{\gamma}_i\},\{\mathrm{St}_{d_k^{(i)}}\},\{L_i\},\{A_k,B_k\}\}\in\tilde{\mathcal{R}}(\boldsymbol{\nu}).$$

By the Malgrange–Sibuya theorem formulated as in [1, Theorem 4.5.1] or in its original form (see [27, Theorem 6.11.1]), we see that the local analytic isomorphism class of the singular connection on each small neighborhood Δ_i with the fixed formal type $\boldsymbol{\nu}^{(i)}=\{\nu_j^{(i)}(z_i)\}$ has one-to-one correspondence with the set of the formal monodromy and Stokes data with the formal type determined by $\boldsymbol{\nu}^{(i)}$. So for each $i, 1\le i\le n$, we can take local analytic connections $(E^{(i)},\nabla^{(i)})$ on Δ_i whose formal types are given by $\boldsymbol{\nu}^{(i)}$ and whose local generalized monodromy data is isomorphic to $\{\{\widehat{\gamma}_i\},\{\mathrm{St}_{d_k^{(i)}}\}\}$.

Since we assume that $\boldsymbol{\nu}$ is generic and nonresonant, the parabolic structures $\{l_j^{(i)}\}$ of $(E^{(i)},\nabla^{(i)})$ at t_i can be uniquely determined, so we obtain local analytic $\boldsymbol{\nu}^{(i)}$-parabolic connections $(E^{(i)},\nabla^{(i)},\{l_j^{(i)}\})$.

The data $\{\mathrm{Top}_i,\{L_i\},A_k,B_k\}$ determine the monodromy data of a flat bundle $\mathbf{E}_1$ on $C_0:=C\setminus\{t_1,\dots,t_n\}$. Hence $E_1=\mathbf{E}_1\otimes\mathcal{O}_{C_0}$ is a locally free sheaf with a flat connection $\nabla:E_1\longrightarrow E_1\otimes\Omega^1_{C_0}$. Since by (15) the local monodromy data of (E_1,∇) and $(E^{(i)},\nabla^{(i)})$ is isomorphic over $\Delta_i\setminus\{0\}$, we can glue (E_1,∇_1) and $(E^{(i)},\nabla^{(i)})$ to obtain a holomorphic vector bundle E on C and a flat connection $\nabla:E\longrightarrow E\otimes\Omega^1_C(D)$. Then by GAGA, we obtain a $\boldsymbol{\nu}$-parabolic connection $(E,\nabla,\{l_j^{(i)}\})$ of degree d. (Note that by the Fuchs relation, the residue part of $\boldsymbol{\nu}$ determines the degree of E.) Since $\boldsymbol{\nu}$ is irreducible, this connection must be irreducible; hence it is $\boldsymbol{\alpha}$-stable for any weight $\boldsymbol{\alpha}$, so it is a member of $M^{\boldsymbol{\alpha}}_{D/C}(r,d,(m_i))_{\boldsymbol{\nu}}$. This shows that $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ is surjective. Now from this construction, the injectivity of $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ is obvious. Hence $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ is bijective. $\square$

REMARK 5.1

In the next section, we will vary the data $(C,\mathbf{t})$, the local generators $\{z_i\in\mathfrak{m}_{t_i}\}$ in a suitable moduli space, and $\boldsymbol{\nu}=\{\boldsymbol{\nu}^{(i)}(z_i)\}\in N_r^{(n)}(d,D)$, and we will construct the continuous analytic family of Riemann–Hilbert correspondences $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$. In order to do this, we first fix data $(C,\mathbf{t}),\{z_1,\dots,z_n\},\boldsymbol{\nu}$ as a base point in a connected component of the moduli space of such data. We will assume that $\boldsymbol{\nu}$ is generic and simple (see Definition 6.1) for a technical reason. We can also fix a small neighborhood Δ_i near t_i and the (simple) singular directions $\{d_j^{(i)}\}_{1\le j\le s_i}$ and the ordered sectors $\{S_k^{(i)}\}_{1\le k\le s_i}$. Moreover, we can fix t_i^* as before (see Figure 1). Fixing a base point $b\in C\setminus\{t_1,\dots,t_n\}$, we can also take and fix paths and loops as in Section 5.2. Once we fix these data, we can define the moduli space $\mathcal{R}(\boldsymbol{\nu})$ of generalized monodromy data (17) and the Riemann–Hilbert correspondence $\mathbf{RH}_{(D/C),\boldsymbol{\nu}}$ as in (20). Note that in order to define a data of $\mathcal{R}(\boldsymbol{\nu})$, we need to fix the paths $\{l_i\},\{\gamma^l,\alpha_k,\beta_k\}$ and the order of the sectors $\{S_k^{(i)}\}_{1\le k\le s_i}$

near each t_i which is determined by the singular directions determined by $\boldsymbol{\nu}^{(i)}$ as in Section 5.2. The closure of the first sector $S_1^{(i)}$ contains the end point t_i^* of the path l_i. If we vary $\boldsymbol{\nu}$ continuously from the original data in the connected component under the condition that $\boldsymbol{\nu}$ is generic and simple and fixing the data $(C,\mathbf{t}),\{z_i\}$, the singular directions and sectors are changing continuously. In this procedure, we need to keep the order of sectors; hence we need to change the point t_i^* and the path l_i continuously. It is easy to see that when we vary the data $(C,\mathbf{t})$, $\{z_i\}$, $\boldsymbol{\nu}$ continuously in the connected component of the moduli spaces (see Section 6.1) starting from the base data, we can vary continuously singular directions, sectors, and paths and loops starting from the original data. By this procedure, we can define the continuous analytic family of Riemann–Hilbert correspondences in each connected component of the moduli space.

REMARK 5.2

By using the result in [26], Jimbo, Miwa, and Ueno [13] discussed the analycity of the Riemann–Hilbert correspondence when one varies $\{(\mathbf{P}^1,D),\boldsymbol{\nu}\}\in M_{0,n}\times N_r^{(n)}(d,D)$ and discussed the isomonodromic deformations of linear connections. When $\boldsymbol{\nu}$ varies in the open set of $N_r^{(n)}(d,D)$ corresponding to generic exponents, one can define an analytic family of Riemann–Hilbert correspondences.

REMARK 5.3

The surjectivity part of Theorem 5.1 is related to the generalized Riemann–Hilbert problem with irregular singularities over $C=\mathbf{P}^1$ which has been investigated, for example, in [13], [5], and [20]. Usually, they would like to obtain singular connections (E,∇) with trivial bundle $E=\mathcal{O}_{\mathbf{P}^1}^{\oplus r}$. However from our viewpoint of global moduli spaces of the connections, even in the case of $C=\mathbf{P}^1$ and $d=\deg E=0$, it is not natural to assume that vector bundle E is always trivial, that is, $E=\mathcal{O}_{\mathbf{P}^1}^{\oplus r}$, for the set of such connections may correspond to a Zariski-dense open subset of the moduli space $M_{D/C}^{\boldsymbol{\alpha}}(r,d,(m_i))_{\boldsymbol{\nu}}$, but they may not cover all of the moduli space. The type of bundle E may jump, for example, as $E\simeq\mathcal{O}_{\mathbf{P}^1}(1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}^{\oplus(r-2)}$. The jumping phenomena of the bundle types in the moduli space of semistable bundles, which both of authors learned from Professor Maruyama, is one of keys of many moduli problems and makes the moduli theory interesting. In the case of the connections, divisors for jumping phenomena are corresponding to the τ-divisors.

REMARK 5.4

In [4], Boalch constructed the space of isomorphism classes of meromorphic connections on a degree zero bundles on $\mathbf{P}^1$ with compatible framing of fixed generic irregular type by an analytic method and showed that taking monodromy data induces the bijection between the space of meromorphic connections on degree zero bundles and the corresponding spaces of monodromy data (cf. [4, Corollary 4.9]). In [3], Biquard and Boalch generalized the analytic construction of the moduli spaces of the connections and showed that under a slightly weaker generic condition the space of meromorphic connections with fixed equivalence classes of polar part over a curve is a hyper-Kähler manifold. Despite these interesting analytic constructions, we believe that our algebro-geometric constructions of the moduli space of stable parabolic connections with fixed irregular singular types have some advantages, such as natural algebraic structures on the moduli spaces, which are crucial to write down the isomonodromic differential equations in some rational algebraic equations on the algebraic coordinates on the phase spaces.

REMARK 5.5

In [8], Inaba showed a stronger statement for the Riemann–Hilbert correspondence when all of the singularities are at most regular (that is, $m_i = 1$ for all i). See also [10], [11], and [9] for former results on the Riemann–Hilbert correspondences.

6. Geometric Painlevé property for generalized isomonodromy differential systems

6.1. Generalized isomonodromic differential systems and their geometric Painlevé property

Let us fix integers $g, n, d, r, (m_i)_{1\leq i\leq n}$ as in Section 5, and let $M_{g,n}$ be an algebraic scheme which is a smooth covering of the moduli stack of n-(distinct) pointed curves such that $M_{g,n}$ is smooth and has the universal family $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n) \longrightarrow M_{g,n}$. We put $D = \sum_{i=1}^n m_i \tilde{t}_i$. For each $(C, t_1, \dots, t_n) \in M_{g,n}$ and $i, 1 \leq i \leq n$, let $\Psi_i : \mathcal{O}_{C,t_i}/\mathfrak{m}_{t_i}^{m_i} \xrightarrow{\simeq} \mathbf{C}[z_i]/(z_i^{m_i})$ be ring isomorphisms. The moduli space $M_{g,n,(m_i)}$ of tuples $(C, t_1, \dots, t_n, \{\Psi_i\}_{1\leq i\leq n})$ is a smooth quasi-projective scheme over $M_{g,n}$. Let $M_{g,n,(m_i)} \longrightarrow M_{g,n}$ be the natural morphism, and consider the scheme $\mathcal{N}_r^{(n)}(d, D)$ over $M_{g,n}$ of generalized exponents defined in (3) in Section 2. Then by using the local coordinates z_i at $\tilde{t}_i$, we have a natural isomorphism

$$M_{g,n,(m_i)} \times_{M_{g,n}} \mathcal{N}_r^{(n)}(d, D) \simeq M_{g,n,(m_i)} \times N_r^{(n)}(d, D),$$

where $N_r^{(n)}(d, D)$ is defined in (2). This space is the parameter space of our moduli spaces, and for simplicity, from now on, we set

$$T = M_{g,n,(m_i)} \times_{M_{g,n}} \mathcal{N}_r^{(n)}(d, D) \simeq M_{g,n,(m_i)} \times N_r^{(n)}(d, D). \tag{21}$$

Let us take $\boldsymbol{\nu} = (\nu_j^{(i)})_{1\leq i\leq n}^{0\leq j\leq r-1} \in N_r^{(n)}(d, D)$ and write $\nu_j^{(i)}(z_i)$ as in (4):

$$\begin{aligned} \nu_j^{(i)}(z_i) &= \left(a_{j,-m_i}^{(i)} z_i^{-m_i} + \cdots + a_{j,-1}^{(i)} z_i^{-1}\right) dz_i \\ &= \sum_{k=-m_i}^{-1} (a_{j,k}^{(i)} z_i^k)\, dz_i \quad \text{for } 1 \leq i \leq n. \end{aligned} \tag{22}$$

Let us consider the following decomposition according to the order of expansions in (22):

$$N_r^{(n)}(d, D) = N_{\mathrm{top}} \times N_{\mathrm{mid}} \times N_{\mathrm{res}}$$

where we set $N_{\mathrm{top}} = \{(a^{(i)}_{j,-m_i}), m_i \geq 2\}, N_{\mathrm{mid}} = \{(a^{(i)}_{j,k}), -m_i < k < -1, m_i \geq 3\}$, $N_{\mathrm{res}} = \{(a^{(i)}_{j,-1})\}$. Using this decomposition, for $\boldsymbol{\nu} \in N^{(n)}_r(d,D)$, we can write as $\boldsymbol{\nu} = (\boldsymbol{\nu}_{\mathrm{top}}, \boldsymbol{\nu}_{\mathrm{mid}}, \boldsymbol{\nu}_{\mathrm{res}})$. Let us define

$$N^{\circ}_{\mathrm{top}} = \big\{(a^{(i)}_{j,-m_i}) \mid a^{(i)}_{j_1,-m_i} \neq a^{(i)}_{j_2,-m_i}, \text{if } j_1 \neq j_2\big\}.$$

Since the genericity condition on $\boldsymbol{\nu} \in N^{(n)}_r(d,D)$ depends on the part $\boldsymbol{\nu}_{\mathrm{top}}$ (cf. Definition 5.1),

$$N^{\circ} = N^{\circ}_{\mathrm{top}} \times N_{\mathrm{mid}} \times N_{\mathrm{res}} \subset N^{(n)}_r(d,D) \tag{23}$$

is the space of generic generalized exponents. Note that N° is an affine open subvariety of $N^{(n)}_r(d,D)$. Moreover, the conditions of resonance and reducibility on $\boldsymbol{\nu}$ depend just on $\boldsymbol{\nu}_{\mathrm{res}}$ (cf. Definition 5.1). Let us denote by $\mathcal{P}$ the set of formal monodromies $\{\widehat{\gamma}_i\}$ associated to $\boldsymbol{\nu}_{\mathrm{res}}$, which admits a surjective map by an exponential map

$$\mathbf{e} : N_{\mathrm{res}} \longrightarrow \mathcal{P}, \qquad \{a^{(i)}_{j,-1}\} \mapsto \{\widehat{\gamma}_i\}.$$

(Note that we have the Fuchs relation of $\boldsymbol{\nu}_{\mathrm{res}}$.)

Recall that for $\boldsymbol{\nu} \in N^{\circ}$, in the previous section, we define the moduli space of generalized monodromy data $\mathcal{R}(\boldsymbol{\nu})$ as in (17).

Now we will see the dependence of isomorphism classes of $\mathcal{R}(\boldsymbol{\nu})$ on $\boldsymbol{\nu} \in N^{\circ}$. In order to avoid technical difficulties coming from the multiplicity of the Stokes lines, we give the following definition.

DEFINITION 6.1

A generic local exponent $\boldsymbol{\nu} = (\nu^{(i)}_j(z_i)) \in N^{\circ}$ is called simple if all of the multiplicities of the singular directions of $\boldsymbol{\nu}$ are one. We denote by $N^{\circ,s}$ the set of all simple generic local exponents $\boldsymbol{\nu}$.

Since the singular directions for generic local exponents can be determined by $\boldsymbol{\nu}_{\mathrm{top}}$ as in Section 5.2, we have the following.

LEMMA 6.1

We can write

$$N^{\circ,s} = N^{\circ,s}_{\mathrm{top}} \times N_{\mathrm{mid}} \times N_{\mathrm{res}}, \tag{24}$$

where $N^{\circ,s}_{\mathrm{top}}$ consists of $\boldsymbol{\nu}_{\mathrm{top}} = (a^{(i)}_{j,-m_i}) \in N^{\circ}_{\mathrm{top}}$ with the conditions that for any i and $(j_1,j_2) \neq (k_1,k_2)$,

$$\arg(a^{(i)}_{j_1,-m_i} - a^{(i)}_{j_2,-m_i}) \not\equiv \arg(a^{(i)}_{k_1,-m_i} - a^{(i)}_{k_2,-m_i}) \mod 2\pi\mathbf{Z}.$$

Note that $N^{\circ,s}_{\mathrm{top}}$ is not a Zariski-open subset of N°_{top} and $N^{\circ,s}_{\mathrm{top}}$ may not be connected.

We constructed the moduli space $\mathcal{R}(\boldsymbol{\nu})$ of the generalized monodromy data associated to the formal type $\boldsymbol{\nu}$ as in (17). Since for $\boldsymbol{\nu} \in N^{\circ,s}$ every Stokes matrix associated to each singular direction is one-dimensional, we can easily see that the algebraic isomorphism class of the affine scheme $\mathcal{R}(\boldsymbol{\nu})$ only depends on $\boldsymbol{\nu}_{\mathrm{res}}$ or on $\mathbf{p}=\mathbf{e}(\boldsymbol{\nu}_{\mathrm{res}})$ for $\boldsymbol{\nu}\in N^{\circ,s}$. So we may write as $\mathcal{R}(\boldsymbol{\nu})=\mathcal{R}(\boldsymbol{\nu}_{\mathrm{res}})=\mathcal{R}(\mathbf{p})$ for $\boldsymbol{\nu}\in N^{\circ,s}$. Fix a base element in each connected component of $M_{g,n,(m_i)}\times N^{\circ,s}_{\mathrm{top}}\times N_{\mathrm{mid}}\times\mathcal{P}$, and fix the data of singular directions, sectors, paths, and loops for it as in the previous section. Varying the data continuously in each connected component of $M_{g,n,(m_i)}\times N^{\circ,s}_{\mathrm{top}}\times N_{\mathrm{mid}}\times\mathcal{P}$, we can construct the family of moduli spaces of generalized monodromy data

$$\pi_1:\mathcal{R}\longrightarrow M_{g,n,(m_i)}\times N^{\circ,s}_{\mathrm{top}}\times N_{\mathrm{mid}}\times\mathcal{P} \tag{25}$$

such that $\pi_1^{-1}((C,\mathbf{t},\{\Psi_i\}),(\boldsymbol{\nu}_{\mathrm{top}},\boldsymbol{\nu}_{\mathrm{mid}},\mathbf{e}(\boldsymbol{\nu}_{\mathrm{res}}))\simeq\mathcal{R}(\boldsymbol{\nu})=\mathcal{R}(\boldsymbol{\nu}_{\mathrm{res}})$. (Note that in order to construct the family (25), we need to consider the actions of the fundamental groups of the base spaces to singular directions and the homotopy classes of paths and loops in Section 5.2.) Let us fix $\boldsymbol{\nu}_{\mathrm{res}}\in N_{\mathrm{res}}$ and set $\mathbf{p}=\mathbf{e}(\boldsymbol{\nu}_{\mathrm{res}})=\{\widehat{\gamma}_i\}\in\mathcal{P}$. For simplicity, we set

$$\begin{aligned}T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}&=M_{g,n,(m_i)}\times N^{\circ,s}_{\mathrm{top}}\times N_{\mathrm{mid}}\times\{\boldsymbol{\nu}_{\mathrm{res}}\}\subset T^{\circ}\\&=M_{g,n,(m_i)}\times N^{\circ}_{\mathrm{top}}\times N_{\mathrm{mid}}\times N_{\mathrm{res}}.\end{aligned} \tag{26}$$

Since $T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}\simeq M_{g,n,(m_i)}\times N^{\circ,s}_{\mathrm{top}}\times N_{\mathrm{mid}}\times\{\mathbf{p}\}$, restricting the family π_1 (see (25)) to this space, we obtain the family of moduli spaces

$$\pi_{1,\mathbf{p}}:\mathcal{R}_{\mathbf{p}}\longrightarrow T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}} \tag{27}$$

which is analytically locally constant with the typical fiber $\mathcal{R}(\boldsymbol{\nu}_{\mathrm{res}})=\mathcal{R}(\mathbf{p})$. Considering the universal covering map

$$\begin{aligned}\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}&=\tilde{M}_{g,n,(m_i)}\times\tilde{N}^{\circ,s}_{\mathrm{top}}\times\tilde{N}_{\mathrm{mid}}\times\{\boldsymbol{\nu}_{\mathrm{res}}\}\longrightarrow T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}\\&=M_{g,n,(m_i)}\times N^{\circ,s}_{\mathrm{top}}\times N_{\mathrm{mid}}\times\{\boldsymbol{\nu}_{\mathrm{res}}\},\end{aligned} \tag{28}$$

we can pull back the family $\pi_{1,\mathbf{p}}$ (27) to the family over the universal covering which is isomorphic to the product fibration:

$$\tilde{\pi}_{1,\mathbf{p}}:\widetilde{\mathcal{R}}_{\mathbf{p}}\simeq\mathcal{R}(\mathbf{p})\times\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}\longrightarrow\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}$$

with the fixed fiber $\mathcal{R}(\boldsymbol{\nu}_{\mathrm{res}})=\mathcal{R}(\mathbf{p})$. On the other hand, by applying Theorems 2.1 and 2.2 to the family of n-pointed curves over $M_{g,n,(m_i)}$, there exists the quasi-projective smooth family of relative moduli spaces

$$\begin{aligned}&\pi_2:M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/M_{g,n,(m_i)}}\big(r,d,(m_i)\big)\\&\qquad\longrightarrow T=M_{g,n,(m_i)}\times_{M_{g,n}}\mathcal{N}^{(n)}_r(d,D)\cong M_{g,n,(m_i)}\times N^{(n)}_r(d,D).\end{aligned}$$

We denote by $M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}}$ the pullback of $T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}=M_{g,n,(m_i)}\times N^{\circ,s}_{\mathrm{top}}\times N_{\mathrm{mid}}\times\{\boldsymbol{\nu}_{\mathrm{res}}\}\subset T=M_{g,n,(m_i)}\times N^{(n)}_r(d,D)$ by the morphism π_2. Then there exists the quasi-projective smooth family of relative moduli spaces

$$\pi_{2,\boldsymbol{\nu}_{\mathrm{res}}}:M^{\boldsymbol{\alpha}}_{D/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}}\longrightarrow T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}. \tag{29}$$

Pulling back this family by the universal covering map (28), we obtain the family $\tilde{\pi}_{2,\boldsymbol{\nu}_{\rm res}} : M^{\boldsymbol{\alpha}}_{\tilde{D}/\tilde{\mathcal{C}}/\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}} \longrightarrow \widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}$ of moduli spaces.

Now take a base point $(C, t_1, \dots, t_n, \{z_i\}_{1 \leq i \leq n}, \boldsymbol{\nu})$ in each connected component of $T^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}$, and fix a small neighborhood Δ_i near each t_i and the (simple) singular directions $\{d^{(i)}_j\}_{1 \leq j \leq s_i}$ and the ordered sectors $\{S^{(i)}_k\}_{1 \leq k \leq s_i}$ for each i as in 5.2. Moreover, fixing $t^*_i \in S^{(i)}_1 \cap \partial \Delta_j$ and a base point $b \in C \setminus \{t_1, \dots, t_n\}$, we can fix paths $\{l_i\}, \{\gamma^l, \alpha_k, \beta_k\}$ as in 5.2.

As explained in Remark 5.1, when we vary the data in $T^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}$ or in $\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}$ starting from each base point, we can vary the choice of sectors, paths, and loops continuously. Hence we can define an analytic morphism

$$\mathbf{RH}_{\boldsymbol{\nu}_{\rm res}} : M^{\boldsymbol{\alpha}}_{\tilde{D}/\tilde{\mathcal{C}}/\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}} \longrightarrow \mathcal{R}(\boldsymbol{\nu}_{\rm res}) \times \widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}$$

which makes the following diagram commutative and induces the continuous analytic family of Riemann–Hilbert correspondences of fibers of $\tilde{\pi}_{2,\boldsymbol{\nu}_{\rm res}}$ and $\tilde{\pi}_{1,\mathbf{p}}$,

$$\begin{array}{ccc} M^{\boldsymbol{\alpha}}_{\tilde{D}/\tilde{\mathcal{C}}/\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}} & \stackrel{\mathrm{RH}_{\boldsymbol{\nu}_{\rm res}}}{\longrightarrow} & \mathcal{R}(\boldsymbol{\nu}_{\rm res}) \times \widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}} \\ \downarrow \tilde{\pi}_{2,\boldsymbol{\nu}_{\rm res}} & & \downarrow \tilde{\pi}_{1,\mathbf{p}} \\ \widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}} & = & \widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}. \end{array} \tag{30}$$

The analycity of $\mathbf{RH}_{\boldsymbol{\nu}_{\rm res}}$ also follows from the result in [26]. Since $\tilde{\pi}_{2,\boldsymbol{\nu}_{\rm res}}$ is smooth, we can consider the natural surjection of tangent sheaves

$$\varphi : \Theta_{M^{\boldsymbol{\alpha}}_{\tilde{D}/\tilde{\mathcal{C}}/\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}}} \longrightarrow \tilde{\pi}^*_{2,\boldsymbol{\nu}_{\rm res}}(\Theta_{\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}}) \longrightarrow 0. \tag{31}$$

Now one can introduce the (generalized) isomonodromic flows and isomonodromic differential systems as follows.

DEFINITION 6.2

Assume that $\boldsymbol{\nu}_{\rm res}$ is nonresonant and irreducible so that $\mathbf{RH}_{\boldsymbol{\nu}_{\rm res}}$ induces an analytic isomorphism between the closed fibers of $\tilde{\pi}_{1,\mathbf{p}}$ and $\tilde{\pi}_{2,\boldsymbol{\nu}_{\rm res}}$ over every closed point of $\widetilde{T}^{\circ}_{\boldsymbol{\nu}_{\rm res}}$ by Theorem 5.1. The pullback of the set of all constant sections of $\tilde{\pi}_{1,\mathbf{p}}$ over $\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}} (\subset \widetilde{T}^{\circ}_{\boldsymbol{\nu}_{\rm res}})$ via the Riemann–Hilbert correspondence $\mathbf{RH}_{\boldsymbol{\nu}_{\rm res}}$ gives the set of horizontal analytic sections of $\tilde{\pi}_{2,\boldsymbol{\nu}_{\rm res}}$ in (30) which we call the (*generalized*) *isomonodromic flows*. Then the isomonodromic flows define a splitting $\tilde{\Psi} : \tilde{\pi}^*_{2,\boldsymbol{\nu}_{\rm res}}(\Theta_{\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}}) \hookrightarrow \Theta_{M^{\boldsymbol{\alpha}}_{\tilde{D}/\tilde{\mathcal{C}}/\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}}}$ of the surjection (31) and define the subsheaf

$$\tilde{\boldsymbol{\theta}}_{\boldsymbol{\nu}_{\rm res}} = \tilde{\boldsymbol{\theta}}_{\mathbf{p}} := \tilde{\Psi}\big(\tilde{\pi}^*_{2,\boldsymbol{\nu}_{\rm res}}(\Theta_{\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}})\big) \subset \Theta_{M^{\boldsymbol{\alpha}}_{\tilde{D}/\tilde{\mathcal{C}}/\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}}}}, \tag{32}$$

which we call the isomonodromic foliation or the isomonodromic differential system. It is obvious that the isomonodromic flows become solution manifolds, or integral manifolds of the differential system $\tilde{\boldsymbol{\theta}}_{\boldsymbol{\nu}_{\rm res}}$. The differential system $\tilde{\boldsymbol{\theta}}_{\boldsymbol{\nu}_{\rm res}}$ in (32) is called *the isomonodromic differential system* associated to the moduli space of $\boldsymbol{\nu}$-parabolic connections. The parameter space $\widetilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\rm res}} = \tilde{M}_{g,n,(m_i)} \times \tilde{N}^{\circ,s}_{\rm top} \times$

$\tilde{N}_{\text{mid}} \times \{\boldsymbol{\nu}_{\text{res}}\}$ can be considered as the space of time variables, though some of parameters may be redundant.

Now from the diagram (30), we can descend $\mathrm{RH}_{\boldsymbol{\nu}_{\text{res}}}$ to obtain the following commutative diagram:

$$
\begin{array}{ccc}
M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}} & \stackrel{\mathrm{RH}_{\boldsymbol{\nu}_{\text{res}}}}{\longrightarrow} & \mathcal{R}_{\mathbf{p}} \\
\downarrow \pi_{2,\boldsymbol{\nu}_{\text{res}}} & & \downarrow \pi_{1,\mathbf{p}} \\
T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}} & = & T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}.
\end{array}
\tag{33}
$$

By the same reason, we can pull back the locally constant sections of $\pi_{1,\mathbf{p}}$ by $\mathrm{RH}_{\boldsymbol{\nu}_{\text{res}}}$ and define an isomonodromic flow on $\pi_{2,\boldsymbol{\nu}_{\text{res}}} : M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}} \longrightarrow T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}$.

Then we can also define the splitting

$$
\Psi : \pi^*_{2,\boldsymbol{\nu}_{\text{res}}}(\Theta_{T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}}) \hookrightarrow \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}}},
\tag{34}
$$

and we can define an analytic foliation

$$
\boldsymbol{\theta}_{\boldsymbol{\nu}_{\text{res}}} = \boldsymbol{\theta}_{\mathbf{p}} := \Psi\big(\pi^*_{2,\boldsymbol{\nu}_{\text{res}}}(\Theta_{T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}})\big) \subset \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}}}.
\tag{35}
$$

It is natural to consider both isomonodromic differential systems $\boldsymbol{\theta}_{\boldsymbol{\nu}_{\text{res}}}$ and $\tilde{\boldsymbol{\theta}}_{\boldsymbol{\nu}_{\text{res}}}$. Since their integral manifolds are the isomonodromic flows on the corresponding phase spaces, it is now almost trivial to see the following theorem, as is explained in [8] and [10].

THEOREM 6.1

Assume that $\boldsymbol{\nu}_{res}$ is nonresonant and irreducible. Then the isomonodromic differential system $\tilde{\boldsymbol{\theta}}_{\boldsymbol{\nu}_{\text{res}}}$ in (32) on the phase space $M^{\boldsymbol{\alpha}}_{\tilde{D}/\tilde{\mathcal{C}}/\tilde{T}^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}}$ satisfies the geometric Painlevé property. Moreover, the differential system $\boldsymbol{\theta}_{\boldsymbol{\nu}_{\text{res}}}$ in (35) on the phase space $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}}$ also satisfies the geometric Painlevé property.

Let us consider the affine variety $T^{\circ}_{\boldsymbol{\nu}_{\text{res}}}$ which contains $T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}$ as an analytic dense open set. Then we have the following diagram:

$$
\begin{array}{ccc}
M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}}} & \hookrightarrow & M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ}_{\boldsymbol{\nu}_{\text{res}}}} \\
\downarrow \pi_{2,\boldsymbol{\nu}_{\text{res}}} & & \downarrow {\pi'}_{2,\boldsymbol{\nu}_{\text{res}}} \\
T^{\circ,s}_{\boldsymbol{\nu}_{\text{res}}} & \subset & T^{\circ}_{\boldsymbol{\nu}_{\text{res}}}.
\end{array}
\tag{36}
$$

Since ${\pi'}_{2,\boldsymbol{\nu}_{\text{res}}}$ is smooth and algebraic, we have a natural surjective homomorphism

$$
\varphi : \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ}_{\boldsymbol{\nu}_{\text{res}}}}} \longrightarrow (\pi'_{2,\boldsymbol{\nu}_{\text{res}}})^*(\Theta_{T^{\circ}_{\boldsymbol{\nu}_{\text{res}}}}) \longrightarrow 0.
\tag{37}
$$

Over the phase space $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}}$, this is nothing but the surjection in (31). The following theorem says that the splitting Ψ in (34) can be extended to the algebraic splitting $\Psi : (\pi'_{2,\boldsymbol{\nu}_{\mathrm{res}}})^*(\Theta_{T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}}) \longrightarrow \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}}}$.

THEOREM 6.2

We can extend the splitting Ψ in (34) to the algebraic splitting

$$\Psi : (\pi'_{2,\boldsymbol{\nu}_{\mathrm{res}}})^*(\Theta_{T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}}) \hookrightarrow \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}}}. \tag{38}$$

Proof

Take an affine open subset $U \subset T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}$ and an algebraic vector field $v \in H^0(U, \Theta_{T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}})$; v corresponds to a morphism $\iota^v : \operatorname{Spec} \mathcal{O}_U[\epsilon] \to T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}$, where $\epsilon^2 = 0$. We denote the pullback to $\mathcal{C} \times \operatorname{Spec} \mathcal{O}_U[\epsilon]$ of the local defining equation of $\tilde{t}_i$ by $\tilde{g}_i$. We may assume that $\tilde{g}_i|_{m_i\tilde{t}_i}$ is the element given by v. Consider the composite

$$\begin{aligned} d_\epsilon : \mathcal{O}_{\mathcal{C}\times\operatorname{Spec}\mathcal{O}_U[\epsilon]} &\xrightarrow{d} \Omega^1_{\mathcal{C}\times\operatorname{Spec}\mathcal{O}_U[\epsilon]/U} = \mathcal{O}_{\mathcal{C}\times\operatorname{Spec}\mathcal{O}_U[\epsilon]}\, d\tilde{g}_i \oplus \mathcal{O}_{\mathcal{C}\times\operatorname{Spec}\mathcal{O}_U[\epsilon]}\, d\epsilon \\ &\to \mathcal{O}_{\mathcal{C}\times\operatorname{Spec}\mathcal{O}_U[\epsilon]}\, d\epsilon. \end{aligned}$$

Note that $\epsilon\, d\epsilon = 0$, and so $\mathcal{O}_{\mathcal{C}\times\operatorname{Spec}\mathcal{O}_U[\epsilon]}\, d\epsilon \cong \mathcal{O}_{\mathcal{C}_U}\, d\epsilon$. Let $(\nu^{(i)}_j) + \epsilon(\mu^{(i)}_j)$ be the pullback of the universal family on $T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}$ by ι^v, where $d_\epsilon(\nu^{(i)}_j) = 0$. There is an étale surjective morphism $V = \coprod_k V_k \to (\pi'_{2,\boldsymbol{\nu}_{\mathrm{res}}})^{-1}(U)$ such that V is an affine scheme and there is a universal family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\})$ on $\mathcal{C}_V$.

Take an affine open covering $\mathcal{C}_{V_k} = \bigcup_\alpha W_\alpha$. After shrinking V_k we may assume that $\sharp\{\alpha \mid (\tilde{t}_i)_{V_k} \subset W_\alpha\} = 1$ for any i and $\sharp\{i \mid (\tilde{t}_i)_{V_k} \cap W_\alpha \neq \emptyset\} \leq 1$ for any α. Take a free $\mathcal{O}_{W_\alpha}[\epsilon]$-module E_α with an isomorphism $E_\alpha \otimes \mathcal{O}_{W_\alpha}[\epsilon]/(\epsilon) \xrightarrow{\phi_\alpha} \tilde{E}|_{W_\alpha}$. Assume that $(\tilde{t}_i)_{V_k} \subset W_\alpha$. We can take a basis $e_0, \ldots, e_{r-1}$ of E_α and $A_\alpha \in \operatorname{End}(E_\alpha)$ such that $\tilde{\nabla}|_{W_\alpha}(e_j) = \tilde{g}_i^{-m_i}\, d\tilde{g}_i(A_\alpha \otimes \mathcal{O}_U[\epsilon]/(\epsilon))(e_j)$ and $A_\alpha|_{(2m_i-1)\tilde{t}_i}(e_j|_{(2m_i-1)\tilde{t}_i}) = (\tilde{g}_i^{m_i}\nu^{(i)}_j)e_j|_{(2m_i-1)\tilde{t}_i}$ for each $0 \leq j \leq r-1$. We may assume that $d_\epsilon(A_\alpha) = 0$. We can take a matrix $B_\alpha \in \operatorname{End}(E_\alpha)\tilde{g}_i^{1-m_i}$ such that $B_\alpha|_{(2m_i-1)\tilde{t}_i}(e_j|_{(2m_i-1)\tilde{t}_i}) = (\int \mu^{(i)}_j)e_j|_{(2m_i-1)\tilde{t}_i}$ for each $0 \leq j \leq r-1$. Here note that $\mu^{(i)}_j$ has no residue part, and so $\int \mu^{(i)}_j$ is single valued. We have

$$\begin{aligned} &(A_\alpha|_{(2m_i-1)\tilde{t}_i}B_\alpha|_{(2m_i-1)\tilde{t}_i} - B_\alpha|_{(2m_i-1)\tilde{t}_i}A_\alpha|_{(2m_i-1)\tilde{t}_i})(e_j|_{(2m_i-1)\tilde{t}_i}) \\ &\quad = A_\alpha|_{(2m_i-1)\tilde{t}_i}\Big(\Big(\int \mu^{(i)}_j\Big)e_j|_{(2m_i-1)\tilde{t}_i}\Big) \\ &\qquad - B_\alpha|_{(2m_i-1)\tilde{t}_i}\big((\tilde{g}_i^{m_i}\nu^{(i)}_j)e_j|_{(2m_i-1)\tilde{t}_i}\big) \\ &\quad = \Big(\int \mu^{(i)}_j\Big)A_\alpha|_{(2m_i-1)\tilde{t}_i}(e_j|_{(2m_i-1)\tilde{t}_i}) - (\tilde{g}_i^{m_i}\nu^{(i)}_j)B_\alpha|_{(2m_i-1)\tilde{t}_i}(e_j|_{(2m_i-1)\tilde{t}_i}) \\ &\quad = \Big(\int \mu^{(i)}_j\Big)(\tilde{g}_i^{m_i}\nu^{(i)}_j)(e_j|_{(2m_i-1)\tilde{t}_i}) - (\tilde{g}_i^{m_i}\nu^{(i)}_j)\Big(\int \mu^{(i)}_j\Big)(e_j|_{(2m_i-1)\tilde{t}_i}) = 0. \end{aligned}$$

This means that $A_\alpha B_\alpha - B_\alpha A_\alpha \in \tilde{g}_i^{m_i}\operatorname{End}(E_\alpha)$. We define

$$C_\alpha := \tilde{g}_i^{m_i}\frac{\partial B_\alpha}{\partial \tilde{g}_i} + A_\alpha B_\alpha - B_\alpha A_\alpha \in \operatorname{End}(E_\alpha).$$

Then we have $(A_\alpha + \epsilon C_\alpha)\tilde{g}_i^{-m_i}\, d\tilde{g}_i|_{m_i\tilde{t}_i}(e_j|_{m_i\tilde{t}_i}) = (\nu_j^{(i)} + \epsilon\mu_j^{(i)})(e_j|_{m_i\tilde{t}_i})$. We put

$$\tilde{A}_\alpha := (A_\alpha + \epsilon C_\alpha)\tilde{g}_i^{-m_i}\, d\tilde{g}_i + B_\alpha\, d\epsilon,$$

and define a connection $\nabla_\alpha : E_\alpha \to E_\alpha \otimes \tilde{\Omega}^1$ by

$$\nabla_\alpha\Big(\sum_{j=0}^{r-1} a_j e_j\Big) := \sum_{j=0}^{r-1} da_j \otimes e_j + \sum_{j=0}^{r-1} a_j \tilde{A}_\alpha(e_j)$$

for $a_j \in \mathcal{O}_{W_\alpha}$, where $\tilde{\Omega}^1$ is the subsheaf of $\Omega^1_{\mathcal{C}_{V_k}\times_U \operatorname{Spec}\mathcal{O}_U[\epsilon]/V_k}(D)$ locally generated by $\tilde{g}_i^{-m_i}\, d\tilde{g}_i$ and $\tilde{g}_i^{1-m_i}\, d\epsilon$. Then ∇_α is a flat connection, that is, $\nabla_\alpha \circ \nabla_\alpha = 0$. We define a local parabolic structure $\{(l_\alpha)_j^{(i)}\}$ by $(l_\alpha)_j^{(i)} = \langle e_{r-1}|_{m_i\tilde{t}_i}, \dots, e_j|_{m_i\tilde{t}_i}\rangle$. So we obtain a triple $(E_\alpha, \nabla_\alpha, \{(l_\alpha)_j^{(i)}\})$ which satisfies $\nabla_\alpha|_{m_i\tilde{t}_i}((l_\alpha)_j^{(i)}) \subset (l_\alpha)_j^{(i)} \otimes \tilde{\Omega}^1$ for any i,j and $(\tilde{\nabla}_\alpha|_{m_i\tilde{t}_i} - (\nu_j^{(i)} + \epsilon\mu_j^{(i)}))((l_\alpha)_j^{(i)}) \subset (l_\alpha)_{j+1}^{(i)} \otimes \Omega^1_{\mathcal{C}_{V_k}[\epsilon]/V_k[\epsilon]}(D_{V_k}[\epsilon])$ for any i,j, where $\mathcal{C}_{V_k}[\epsilon] = \mathcal{C}_{V_k} \times_U \operatorname{Spec}\mathcal{O}_U[\epsilon]$, $D_{V_k}[\epsilon] = D_{V_k} \times_U \operatorname{Spec}\mathcal{O}_U[\epsilon]$, and $\tilde{\nabla}_\alpha$ is the relative connection induced by ∇_α.

We call $(\mathcal{E}, \nabla_{\mathcal{E}}, \{(l_{\mathcal{E}})_j^{(i)}\})$ a horizontal lift of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ with respect to v if

(1) $\mathcal{E}$ is a vector bundle on $\mathcal{C}_{V_k} \times_U \operatorname{Spec}\mathcal{O}_U[\epsilon]$,

(2) $\mathcal{E}|_{m_i\tilde{t}_i} = (l_{\mathcal{E}})_0^{(i)} \supset \cdots \supset (l_{\mathcal{E}})_r^{(i)} = 0$ is a filtration by subbundles for $i = 1, \dots, n$, and

(3) $\nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes \tilde{\Omega}^1$ is a connection satisfying

(a) $\nabla_{\mathcal{E}}|_{m_i\tilde{t}_i}((l_{\mathcal{E}})_j^{(i)}) \subset (l_{\mathcal{E}})_j^{(i)} \otimes \tilde{\Omega}^1$ for any i,j,

(b) the curvature $\nabla_{\mathcal{E}} \circ \nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes \tilde{\Omega}^2$ is zero,

(c) $(\tilde{\nabla}_{\mathcal{E}}|_{m_i\tilde{t}_i} - (\nu_j^{(i)} + \epsilon\mu_j^{(i)})\mathrm{id})((l_{\mathcal{E}})_j^{(i)}) \subset (l_{\mathcal{E}})_{j+1}^{(i)} \otimes \Omega^1_{\mathcal{C}_{V_k}[\epsilon]/V_k[\epsilon]}(D_{\mathcal{C}_{V_k}[\epsilon]})$ for any i,j, where $\tilde{\nabla}_{\mathcal{E}}$ is the relative connection induced by $\nabla_{\mathcal{E}}$, and

(d) $(\mathcal{E}, \tilde{\nabla}_{\mathcal{E}}, \{(l_{\mathcal{E}})_j^{(i)}\}) \otimes \mathcal{O}_U[\epsilon]/(\epsilon) \cong (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$.

Note that $(E_\alpha, \nabla_\alpha, \{(l_\alpha)_j^{(i)}\})$ is a local horizontal lift and that the obstruction class for the existence of a global horizontal lift lies in $\mathbf{H}^2(\mathcal{F}^\bullet)$, where

$$\mathcal{F}^0 := \big\{u \in \mathcal{E}nd(\tilde{E}) \;\big|\; u|_{m_i\tilde{t}_i}\big(\tilde{l}_j^{(i)}\big) \subset \tilde{l}_j^{(i)} \text{ for any } i,j\big\},$$

$$\mathcal{F}^1 := \left\{ u \in \mathcal{E}nd(\tilde{E}) \otimes \overline{\Omega}^1 \;\middle|\; \begin{array}{l} u|_{m_i\tilde{t}_i}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \otimes \overline{\Omega}^1 \\ \text{for any } i,j, \text{ and the image of} \\ \tilde{l}_j^{(i)} \hookrightarrow \tilde{E}|_{m_i\tilde{t}_i} \xrightarrow{u|_{m_i\tilde{t}_i}} \tilde{E}|_{m_i\tilde{t}_i} \otimes \overline{\Omega}^1 \\ \longrightarrow \tilde{E}|_{m_i\tilde{t}_i} \otimes \Omega^1_{\mathcal{C}_{V_k}/V_k}(D_{V_k}) \\ \text{lies in } l_{j+1}^{(i)} \otimes \Omega^1_{\mathcal{C}_{V_k}/V_k}(D_{V_k}) \text{ for any } i,j \end{array} \right\},$$

$$\mathcal{F}^2 := \big\{u \in \mathcal{E}nd(\tilde{E}) \otimes \tilde{\Omega}^2 \;\big|\; u|_{m_i\tilde{t}_i}\big(l_j^{(i)}\big) \subset \tilde{l}_{j+1}^{(i)} \otimes \tilde{\Omega}^2 \text{ for any } i,j\big\},$$

$$d^0 : \mathcal{F}^0 \ni u \mapsto \tilde{\nabla} \circ u - u \circ \tilde{\nabla} + u\, d\epsilon \in \mathcal{F}^1,$$
$$d^1 : \mathcal{F}^1 \ni \omega + a\, d\epsilon \mapsto d\epsilon \wedge \omega + (\tilde{\nabla} \circ a - a \circ \tilde{\nabla}) \wedge d\epsilon \in \mathcal{F}^2.$$

Here $\overline{\Omega}^1 = \Omega^1_{\mathcal{C}_{V_k}/V_k}(D_{V_k}) \oplus \mathcal{O}_{\mathcal{C}_{V_k}}\, d\epsilon$. We can easily check that the complex $\mathcal{F}^\bullet$ is exact and so $\mathbf{H}^2(\mathcal{F}^\bullet) = 0$. So there is a horizontal lift $(\mathcal{E}, \nabla_{\mathcal{E}}, \{(l_{\mathcal{E}})^{(i)}_j\})$ of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\})$. (In fact, we can see that a horizontal lift is unique because of $\mathbf{H}^1(\mathcal{F}^\bullet) = 0$.) $(\mathcal{E}, \nabla_{\mathcal{E}}, \{(l_{\mathcal{E}})^{(i)}_j\})$ determines an algebraic vector field $\Psi'(v) \in H^0(V_k, (\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}}})_{V_k})$. We can see that $\Psi'(v)$ descends to an algebraic vector field $\overline{\Psi'}(v) \in H^0((\pi'_{2,\boldsymbol{\nu}_{\mathrm{res}}})^{-1}(U), \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}}})$. By construction we have $\overline{\Psi'}(v) = \Psi(v)$, that is, Ψ is algebraic. $\square$

REMARK 6.1

The algebraic splitting in (38) also defines an algebraic differential system on the phase space $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}}$:

$$\boldsymbol{\theta}'_{\boldsymbol{\nu}_{\mathrm{res}}} = \boldsymbol{\theta}'_{\mathbf{p}} := \Psi\big((\pi'_{2,\boldsymbol{\nu}_{\mathrm{res}}})^*(\Theta_{T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}})\big) \subset \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}}}, \tag{39}$$

which coincides with $\boldsymbol{\theta}_{\boldsymbol{\nu}_{\mathrm{res}}} = \boldsymbol{\theta}_{\mathbf{p}}$ on $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}}$. It seems natural to expect that $\boldsymbol{\theta}'_{\boldsymbol{\nu}_{\mathrm{res}}}$ also satisfies the geometric Painlevé property when $\boldsymbol{\nu}_{\mathrm{res}}$ is nonresonant and irreducible; that is, the condition for simpleness for $\boldsymbol{\nu}$ (or $\boldsymbol{\nu}_{\mathrm{top}}$) may not be necessary. If we will fix a nonsimple $\boldsymbol{\nu}_{\mathrm{top}}$ and vary the other data in $T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}$, we can show the geometric Painlevé property for the vector fields $\boldsymbol{\theta}'_{\boldsymbol{\nu}_{\mathrm{res}}}$ from Theorem 5.1.

Now we show that geometric Painlevé property of a differential system $\boldsymbol{\theta}_{\boldsymbol{\nu}_{\mathrm{res}}}$ on $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}}$ implies that the analytic or classical Painlevé property of differential system holds as follows (cf. [9], [8]). Assume that on an affine Zariski-open subset U of $T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}$ we have algebraic coordinates $T_1, \dots, T_l$ of U where $l = l(g, n, (m_i), \mathbf{p}) = \dim T^\circ_{\boldsymbol{\nu}_{\mathrm{res}}}$. Then we may also consider them as a coordinate system on $U \cap T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}$. Then we can see that the differential systems $\boldsymbol{\theta}'_{\boldsymbol{\nu}_{\mathrm{res}}}$ on the phase space $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/U}$ over U are generated by the following algebraic vector fields:

$$\boldsymbol{\theta}'_{\boldsymbol{\nu}_{\mathrm{res}}} = \{\theta'_1, \dots, \theta'_l\}, \quad \text{where } \theta'_i = \Psi\Big(\frac{\partial}{\partial T_i}\Big).$$

These vector fields naturally commute to each other, and by using affine algebraic coordinate charts of $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/U}$ we may write these vector fields explicitly and define algebraic partial differential equations on $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/U}$. Restricting these vector fields on the phase space $M^{\boldsymbol{\alpha}}_{\mathcal{D}/\mathcal{C}/U \cap T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}}$ over $U \cap T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}$, we obtain the vector fields $\boldsymbol{\theta}_{\boldsymbol{\nu}_{\mathrm{res}}} = \{\theta_1, \dots, \theta_l\}$ which are equivalent to the isomonodromic flows defined in Theorem 6.1. Hence $\boldsymbol{\theta}_{\boldsymbol{\nu}_{\mathrm{res}}}$ can be written in partial algebraic differential equations with the independent variables $T_1, \dots, T_n$. Since all the solutions of $\boldsymbol{\theta}_{\boldsymbol{\nu}_{\mathrm{res}}}$ are in the isomonodromic flows, the solutions stay in the phase space over $U \cap T^{\circ,s}_{\boldsymbol{\nu}_{\mathrm{res}}}$. This means that all solutions can be arranged in a coordinate chart after the rational transformations of algebraic coordinates of the fibers. So the movable singularities of the associated differential equations are only poles, which implies the analytic Painlevé property.

REMARK 6.2

Jimbo, Miwa, and Ueno [13] gave explicit isomonodromic differential systems in the case of $C = \mathbf{P}^1$.

REMARK 6.3

Even if $\boldsymbol{\nu}_{\mathrm{res}}$ is resonant or reducible, we can define the Riemann–Hilbert correspondence $\mathbf{RH}_{\boldsymbol{\nu}_{\mathrm{res}}}$ under the condition that $\boldsymbol{\nu}$ is generic. We expect that the Riemann–Hilbert correspondence $\mathbf{RH}_{\boldsymbol{\nu}_{\mathrm{res}}}$ is a *proper* surjective bimeromorphic analytic map on each fiber of every closed point of $T^{\circ}_{\boldsymbol{\nu}_{\mathrm{res}}}$. If we can show this fact, we can define an isomonodromic differential system and show its geometric Painlevé property.

6.2. Relations to the classical Painlevé equations

Painlevé [18] and [19] and Gambier [6] classified the second-order rational algebraic ordinary differential equations which may have analytic Painlevé property into 6 types, $P_J, J = I, \ldots, VI$. We call these equations classical Painlevé equations. However they did not give the proof of the Painlevé property for classical Painlevé equations.

Okamoto introduced a one-parameter family of algebraic surfaces associated to each type of classical Painlevé equation (see [17]) on which the Painlevé equation has horizontal separated solutions at least locally. A surface appearing as a fiber in the Okamoto's family is called Okamoto's space of initial conditions. It has a nice compactification S, which is a smooth rational projective surface, whose anticanonical divisor $-K_S = Y = \sum_{i=1}^{s} n_i Y_i$ is an effective normal crossing divisor, and the space of initial conditions can be given as $S \setminus Y_{\mathrm{red}}$. It satisfies the condition $-K_S \cdot Y_i = Y \cdot Y_i = 0$ for all $i, 1 \leq i \leq s$. We call such a pair (S, Y) where S is a smooth projective rational surface and $Y \in |-K_S|$ with the above condition an Okamoto–Painlevé pair (see [25], [23], [24]). In [25], [23], and [24], Okamoto–Painlevé pairs (S, Y) are classified into 8 types corresponding to the affine Dynkin diagrams of types $D_k^{(1)}, 4 \leq k \leq 8, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$. Moreover, one can show that such pairs (S, Y) have a special one-parameter deformation, and one can derive the classical Painlevé equations from the special deformations of Okamoto–Painlevé pairs.

In [10] and [11], we proved that the Okamoto–Painlevé pair of type $D_4^{(1)}$ which corresponds to Painlevé VI equation P_{VI} can be obtained by the moduli space of stable $\boldsymbol{\nu}$-parabolic connections of rank 2 and degree -1 over $\mathbf{P}^1$ with 4-regular singular points. Since it was known that Painlevé VI equations can be obtained as isomonodromic differential equations, so we can prove the Painlevé property for P_{VI} in [10] and [11].

One can classify types of regular or irregular singularities of parabolic connections of rank 2 on $\mathbf{P}^1$ whose isomonodromic differential equations give the

Table 1

Dynkin	Painlevé equations	m_0	m_1	m_∞	m_t	$\dim \mathcal{P}$
$D_4^{(1)}$	P_{VI}	1	1	1	1	4
$D_5^{(1)}$	P_V	1	1	2	-	3
$D_6^{(1)}$	deg $P_V = P_{III}(D_6^{(1)})$	1	1	1+1/2	-	2
$D_6^{(1)}$	$P_{III}(D_6^{(1)})$	2	-	2	-	2
$D_7^{(1)}$	$P_{III}(D_7^{(1)})$	1+ 1/2	-	2	-	1
$D_8^{(1)}$	$P_{III}(D_8^{(1)})$	1+1/2	-	1+ 1/2	-	0
$E_6^{(1)}$	P_{IV}	1	-	3	-	2
$E_7^{(1)}$	$P_{II}(FN) = P_{II}$	1	-	1+3/2	-	1
$E_7^{(1)}$	P_{II}	-	-	4	-	1
$E_8^{(1)}$	P_I	-	-	1+5/2	-	0

classical Painlevé equations of 8 types (see [13], [20]). In Table 1, we list the types of singularities of linear connections of rank 2 by specifying the orders m_i of singularities at 4 points of $\mathbf{P}^1$: $i = 0, 1, \infty, t \neq 0, 1, \infty$. When $m_i = -$, it indicates that there are no singularities at the point, and when m_i is a half integer, it indicates that the connection has a ramified irregular singularity with Katz invariant $m_i - 1$. Moreover, $\mathcal{P}$ is the space of formal monodromies as in the previous subsection.

From Table 1, we can see that the following 5 types depending on the parameter $\mathbf{p} \in \mathcal{P}$ correspond to the rank 2 connections with regular or unramified irregular singularities:

$$
\begin{aligned}
&P_{VI}(D_4^{(1)})_{\mathbf{p}}, \qquad P_V(D_5^{(1)})_{\mathbf{p}}, \qquad P_{III}(D_6^{(1)})_{\mathbf{p}}, \\
&P_{IV}(E_6^{(1)})_{\mathbf{p}}, \qquad P_{II}(E_7^{(1)})_{\mathbf{p}}.
\end{aligned} \tag{40}
$$

As a corollary of Theorem 6.1, we have the following.

THEOREM 6.3

Classical Painlevé equations of above 5 types in (40) have the geometric Painlevé property as well as the analytic Painlevé property if the parameter $\mathbf{p} \in \mathcal{P}$ is nonresonant and irreducible.

Proof

It is easy to check that each classical Painlevé equation listed above coincides with our isomonodromic flows $\boldsymbol{\theta}_{\mathbf{p}}$ on a Zariski open set of our family of the moduli space of the parabolic connections of the type above (cf. [13] or [20]). Then by Theorem 6.1 classical Painlevé equations satisfy the geometric Painlevé property. □

REMARK 6.4

In the case of $P_{VI}(D_4^{(1)})_{\mathbf{p}}$, the geometric Painlevé property holds even for resonant and reducible parameter $\mathbf{p} \in \mathcal{P}$ (cf. [10], [11]). Actually, if all singularities are regular, the result of Inaba [8] implies that the corresponding isomonodromic differential systems $\boldsymbol{\theta}_{\mathbf{p}}$ have the geometric Painlevé property even for resonant or reducible parameters $\mathbf{p} \in \mathcal{P}$.

REMARK 6.5

In [20], explicit families of connections corresponding to each type in Table 1 are given as well as isomonodromic differential equations for these families. However these connections only cover a Zariski dense open set of our moduli spaces. So it is not enough to show the Painlevé property for classical Painlevé equations. Moreover, even when $C = \mathbf{P}^1, d = 0$, constructions of moduli spaces by using only the trivial bundle do not give a whole moduli space of ours because of the existence of a jumping locus of the bundle type.

REMARK 6.6

In [20], one can see the all of the explicit equations corresponding to the moduli spaces $\mathcal{R}(\boldsymbol{\nu}_{\mathrm{res}})$ of generalized monodromy data for ten types in Table 1. These equations are all cubic equations in three variables x_1, x_2, x_3 with the coefficients in parameters in $\mathcal{P}$. In the case of $P_{VI}(D_4^{(1)})_{\mathbf{p}}$, the equation is given classically by Fricke and Klein (cf. [9], [20]).

Acknowledgments. Saito thanks Marius van der Put for the hospitality and many discussions on the generalized monodromy data of irregular singularities during his visits in Groningen. Both authors thank the referees for pointing out mistakes about the multiplicity of singular lines in the first version of this paper and for other useful suggestions.

References

[1] D. G. Babbitt and V. S. Varadarajan. *Local Moduli for Meromorphic Differential Equations*, Astérisque **169-170**, Soc. Math. France, Montrouge, 1989. MR 1014083.

[2] C. L. Bermer and D. S. Sage, *Moduli spaces of irregular singular connections*, preprint, arXiv:1004.4411v2 [math.AG].

[3] O. Biquard and P. Boalch, *Wild non-abelian Hodge theory on curves*, Compos. Math. **140** (2004), 179–204. MR 2004129. DOI 10.1112/S0010437X03000010.

[4] P. Boalch, *Symplectic manifolds and isomonodromic deformations*. Adv. Math. **163** (2001), 137–205. MR 1864833. DOI 10.1006/aima.2001.1998.

[5] A. A Bolibruch, S. Malek, and C. Mitschi, *On the generalized Riemann–Hilbert problem with irregular singularities*, Expo. Math. **24** (2006), 235–272. MR 2250948. DOI 10.1016/j.exmath.2005.11.003.

[6] B. Gambier, *Sur les équations différentiells du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes*, Acta Math. **33** (1910), 1–55. MR 1555055. DOI 10.1007/BF02393211.

[7] T. Hausel and F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties*, with an appendix by Nicholas M. Katz, Invent. Math. **174** (2008), 555-624. MR 2453601. DOI 10.1007/s00222-008-0142-x.

[8] M. Inaba, *Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence*, to appear in J. Algebraic Geom., preprint, arXiv:0602004v2 [math.AG].

[9] M. Inaba, K. Iwasaki, and M.-H. Saito, "Dynamics of the sixth Pailevé equations" in *Théories asymptotiques et équations de Painlevé*, Sémin. Congr. **14**, Soc. Math. France, Paris, 2006, 103–167.

[10] ———, *Moduli of stable parabolic connections, Riemann–Hilbert correspondence and geometry of Painlevé equation of type VI, I*, Publ. Res. Inst. Math. Sci. **42** (2006), 987–1089. MR 2289083.

[11] ———, "Moduli of stable parabolic connections, Riemann–Hilbert correspondence and geometry of Painlevé equation of type VI, II" in *Moduli Spaces and Arithmetic Geometry*, Adv. Stud. Pure Math. **45**, Math. Soc. Japan, Tokyo, 2006, 387–432. MR 2310256.

[12] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, II*, Physica D **2** (1981), 407–448. MR 0625446. DOI 10.1016/0167-2789(81)90021-X.

[13] M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I: General theory and τ-function*, Physica D **2** (1981), 306–352. MR 0630674. DOI 10.1016/0167-2789(81)90013-0.

[14] B. Malgrange, "Sur les déformations isomonodromiques, I: Singularités régulières" in *Mathematics and Physics (Paris, 1979/1982)*, Progr. Math. **37**, Birkhäuser, Boston, 401–426. MR 0728431.

[15] ———, "Sur les déformations isomonodromiques, II: Singularités irrégulières" in *Mathematics and Physics (Paris, 1979/1982)*, Progr. Math. **37**, Birkhäuser, Boston, 1983, 427–438. MR 0728432.

[16] T. Miwa, *Painlevé property of monodromy preserving equations and the analyticity of τ functions*, Publ. Res. Inst. Math. Sci. **17** (1981), 709–721. MR 0642657. DOI 10.2977/prims/1195185270.

[17] K. Okamoto, *Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Espaces des conditions initiales*, Japan. J. Math. (N.S.) **5** (1979), 1–79. MR 0614694.

[18] P. Painlevé, *Mémoire sur les équations différentielles dont l'integrale generale est uniforme*, Bull. Soc. Math. France **28**, (1900), 206–261.

[19] ———, *Sur les équations différentielles du second ordre à points critiques fixes*, C. R. Acad. Sci. Paris **143** (1906), 1111–1117.

[20] M. van der Put and M.-H. Saito, *Moduli spaces for linear differential equations and the Painlevé equations*, Ann. Inst. Fourier (Grenoble) **59** (2009), 2611–2667. MR 2649335.

[21] M. van der Put and M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren Math. Wiss. **328**, Springer, Berlin, 2003. MR 1960772.

[22] C. Sabbah, *Isomonodromic Deformations and Frobenius Manifolds: An Introduction*, Universitext, Springer, London; EDP Sciences, Les Ulis, France, 2007. MR 2368364.

[23] M.-H. Saito and T. Takebe, *Classification of Okamoto–Painlevé Pairs*, Kobe J. Math. **19** (2002), 21–55. MR 1980990.

[24] M.-H. Saito, T. Takebe, and H. Terajima, *Deformation of Okamoto–Painlevé pairs and Painlevé equations*, J. Algebraic Geom. **11** (2002), 311–362. MR 1874117. DOI 10.1090/S1056-3911-01-00316-2.

[25] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), 165–229. MR 1882403. DOI 10.1007/s002200100446.

[26] Y. Sibuya, *Perturbation of linear ordinary differential equations at irregular singular points*, Funkcial. Ekvac. **11** (1968), 235–246. MR 0243171.

[27] ———, *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, Transl. Math. Monogr. **82**, Amer. Math. Soc., Providence, 1990. MR 1084379.

Inaba: Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto, 606-8502, Japan; inaba@math.kyoto-u.ac.jp

Saito: Department of Mathematics, Graduate School of Science, Kobe University, Kobe, Rokko, 657-8501, Japan; mhsaito@math.kobe-u.ac.jp